Saalburg Lectures on Worldline QFT

PLAN

1. GW FROM LINEARIZED GRAVITY

2. WEFT & GRAV. 2-BODY PROBLEM : PN VS. PM EXPANSION

3. WORLDLINE QFT APPROACH

   - DIAGRAMMATICS
   - OBSERVABLES
   - CLASSICAL PHYSICS FROM QFT

4. BOUNDARY CONDITIONS & IN-IN FORMALISM

5. SPINNING BHs AS SUPERPARTICLES

6. WAVEFORM ; WAVE MEMORY @ LO

7. DEFLECTION

8. FEYNMAN INTEGRATION TECHNIQUES.

1. MOTIVATION

→ SLIDES

2. GW FROM LINEARIZED GRAVITY

Einstein-Hilbert action: $$S_{E-H} = \frac{1}{16\pi G}\int d^dx \sqrt{g}\, R$$

mostly minus metric, G: Newton's constant $\kappa = \sqrt{32\pi G}$

$R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$ R. Curvature

$R = g^{\mu\nu} R_{\mu\nu}$ R. Scalar

$$R^{\mu}{}_{\nu\alpha\beta} = 2\left(\partial_{[\alpha}\Gamma^{\mu}{}_{\beta]\nu} - \Gamma^{\mu}{}_{\kappa[\alpha}\Gamma^{\kappa}{}_{\beta]\nu}\right)$$

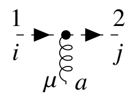
$$\Gamma^{\mu}{}_{\alpha\beta} = g^{\mu\nu}\left(\partial_{(\beta}g_{\alpha)\nu} - \frac{1}{2}\partial_\nu g_{\alpha\beta}\right)$$

Perturbative (quantum) gravity $$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$$

Linearized action:

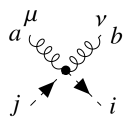
$$S^{\text{lin}}_{EH} = \int d^dx \Big[\partial_\alpha h\, \partial_\beta h^{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} - \frac{1}{2}(\partial_\alpha h)^2 + \frac{1}{2}(\partial_\gamma h_{\alpha\beta})^2\Big] + \text{total derivatives} + \mathcal{O}(h^3)$$

FOR DETAILS SEE: (Scattering amps in QFT; Badger, Henn, P, Zoia)



$$= \mathrm{i}g(T_R^a)_j{}^i(p_2 - p_1)^\mu \,. \tag{1.72}$$

with all momenta outgoing, and a four-point interaction originating from the term quartic in the fields in (1.70),



$$= \mathrm{i}g^2(T_R^b)_i{}^k\,(T_R^a)_k{}^j\eta^{\mu\nu} \,. \tag{1.73}$$

Note that in the case of an adjoint scalar field ($R = A$) one replaces $(T_A^a)_b{}^c = -\mathrm{i}\sqrt{2}\, f^{abc}$ in the above expressions.

1.6 Perturbative quantum gravity

The second fundamental theory of nature is Einstein's theory of gravity. Here we want to discuss its perturbative quantisation. It is famously known to be a non-renormalisable theory, which excludes it as a fundamental *quantum* field theory of nature in its present form. Yet, the modern viewpoint on the non-renormalisability of Einstein's gravity is to understand it as an effective quantum field theory valid for energy scales below the Planck mass, see e.g. [6]. In this setting graviton scattering amplitudes can be performed, one needs to include higher mass order counter terms order by order in the loop expansion. Doing so physical quantities may be extracted. In this fashion systematic quantum corrections to Newton's potential, studies in a perturbative weak gravity (post-Newtonian or post-Minkowskian) approach to the gravitational two body problem for bound and scattering scenarios, or cosmological scenarios have been addressed. Moreover, the study of graviton scattering amplitudes has led to uncover many surprising connections to scattering amplitudes in non-Abelian gauge theory, which we will address as well.

Let us now discuss the perturbative quantisation of Einstein's theory. We assume the reader to be familiar with classical general relativity. The gravitational field is given the metric $g_{\mu\nu}(x)$. The minimal coupling of gravity to matter emerges by replacing the flat-space Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}(x)$ in the Lagrangians. This works fine for bosonic fields, while fermions need a special treatment.

> Einstein-Hilbert Lagrangian.

The dynamics of gravity is dictated by the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = \frac{2}{\kappa^2}\sqrt{-g}\, R \,, \tag{1.74}$$

where $g = \det(g_{\mu\nu})$ and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar built from the Ricci tensor $R_{\mu\nu}$ that describes the curvature of space-time. The gravitational coupling constant κ has inverse mass dimension one in four dimensions (in general D we have $[\kappa] = (D-2)/2$). It is related to Newton's gravitational constant G via $\kappa^2 = 32\pi G$.

The Ricci tensor is defined by

$$
\begin{aligned}
R_{\mu\nu} &= \partial_\mu \Gamma^\rho{}_{\rho\nu} - \partial_\rho \Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\rho\nu} - \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\mu\nu}\,, \\
\Gamma^\rho{}_{\mu\nu} &= \tfrac{1}{2} g^{\rho\kappa}\left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right)\,,
\end{aligned}
\tag{1.75}
$$

with the affine connection $\Gamma^\rho{}_{\mu\nu}$. In perturbative quantum gravity we assume a weak gravitational field: the metric is flat on which small fluctuations propagate. These are given by the *graviton field* $h_{\mu\nu}(x)$. Therefore, we write the metric as

$$
g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x)\,. \tag{1.76}
$$

In the classical theory the graviton field $h_{\mu\nu}$ represents gravitational waves.[6] We now insert this expression for the metric into the Einstein-Hilbert action, and perform a power series expansion in powers of κ and the gravitational field. This is a weak field expansion. Let us gather the various building blocks in this expansion. For the inverse metric one has

$$
g^{\mu\nu}(x) = \eta^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha{}^\nu + O\left(\kappa^3\right)\,. \tag{1.77}
$$

From now on we raise and lower indices with the flat Minkowski metric $\eta_{\mu\nu}$. The further quantities entering $\mathcal{L}_{\mathrm{EH}}$ take the following forms up to cubic order in κ:

$$
\begin{aligned}
\sqrt{-g} &= 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8}(h^2 - 2\, h^{\alpha\beta} h_{\alpha\beta}) + O\left(\kappa^3\right)\,, \\
\Gamma^\rho_{\mu\nu} &= \frac{\kappa}{2}(\partial_\mu h^\rho{}_\nu + \partial_\nu h^\rho{}_\mu - \partial^\rho h_{\mu\nu}) - \frac{\kappa^2}{2} h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + O\left(\kappa^3\right)\,, \\
R &= \kappa(\partial^2 h - \partial^\alpha \partial^\beta h_{\alpha\beta}) - \frac{\kappa^2}{2}\Big[h^{\alpha\beta}(\partial^2 h_{\alpha\beta} + \partial_\alpha \partial_\beta h - 2\, \partial_\rho \partial_\alpha h^\rho{}_\beta) + \partial_\alpha h\, \partial_\beta h^{\alpha\beta} \\
&\quad - (\partial_\alpha h)^2 + \tfrac{1}{2}\partial_\gamma h_{\alpha\beta}\, \partial^\gamma h^{\alpha\beta} - \partial_\alpha h_{\gamma\beta}\, \partial^\beta h^{\gamma\alpha} + \text{total derivatives}\Big] + O\left(\kappa^3\right)\,,
\end{aligned}
\tag{1.78}
$$

where $h := h^\alpha{}_\alpha$. Inserting these expansions into the Einstein-Hilbert Lagrangian (1.74) yields to leading order in κ the expression

$$
\begin{aligned}
\mathcal{L}_{\mathrm{EH}} = {} & \partial_\alpha h\, \partial_\beta h^{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\, \partial^\beta h^{\alpha\gamma} - \tfrac{1}{2}(\partial_\alpha h)^2 + \tfrac{1}{2}(\partial_\gamma h_{\alpha\beta})^2 \\
& + \text{total derivatives} + O\left(\kappa, h^3\right)\,.
\end{aligned}
\tag{1.79}
$$

---

[6] In fact the quantum field theory methods to be discussed may also be applied to this case in their classical limit. This has proven to be a very efficient approach, see e.g. [7, 8, 9].

These quadratic terms in $h_{\mu\nu}$ give rise to the kinetic term for the graviton. The omitted infinite series of higher powers in κ gives rise to the graviton self interactions. They take the schematic form

$$\mathcal{L}_{\text{EH,int}} = \sum_{n=1}^{\infty} \kappa^n \left[\partial^2 h^{n+2}\right] , \tag{1.80}$$

where the term in brackets simply denotes the order in derivatives and fields encountered in this expansion. In general one finds all possible tensor structures. Hence, the Feynman rules for perturbative quantum gravity have vertices of *all* multiplicities. Yet, in a computation to a given order in κ only a finite number of vertices enter, as the power of κ of a vertex grows with its multiplicity.

Gravity is invariant under general coordinate transformations, which take the infinitesimal form

$$x^\mu \longrightarrow x^\mu + \xi^\mu(x) \tag{1.81}$$

with an arbitrary space-time dependent vector $\xi^\mu(x)$. Under these coordinate transformations the graviton field transforms as[7]

$$\delta h_{\mu\nu} = 2\, h_{\sigma(\mu}\partial_{\nu)}\xi^\sigma + \xi^\sigma \partial_\sigma h_{\mu\nu} + \frac{2}{\kappa}\, \partial_{(\mu}\xi_{\nu)} \,. \tag{1.82}$$

Just as in Yang-Mills theory, this local invariance necessitates a gauge fixing in oder not to “overcount” in the path-integral over $h_{\mu\nu}$ through the Fadeev-Popov procedure. As our transformation freedom lies in an arbitrary space-time vector $\xi^\mu(x)$, we need to gauge fix four components of $h_{\mu\nu}$. A popular and convenient choice is the de Donder gauge:

$$G_\mu = \partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h = 0\,, \tag{1.83}$$

that we shall also employ. Note that this is the linearised version (in κ) of the harmonic coordinate choice $g^{\mu\nu}\Gamma^\rho{}_{\mu\nu} = 0$, frequently used in general relativity. The gauge fixing term to be added to the Lagrangian takes the form[8]

$$\mathcal{L}_{\text{GF}} = G_\mu G^\mu = \partial^\nu h_{\mu\nu}\, \partial^\rho h^\mu{}_\rho + \tfrac{1}{4}(\partial_\mu h)^2 - \partial^\nu h_{\mu\nu}\, \partial^\mu h \,. \tag{1.84}$$

Adding this to $\mathcal{L}_{\text{EH}}$ then cancels the first two terms in eq. (1.79) and yields a nice, invertible quadratic term:

$$\begin{aligned} \mathcal{L}_{\text{EH}}|_{h^2} + \mathcal{L}_{\text{GF}} &= -\tfrac{1}{2} h_{\alpha\beta}\, \partial^2 h_{\alpha\beta} + \tfrac{1}{4}\, h\, \partial^2 h \\ &= -\tfrac{1}{2} h_{\alpha\beta} \underbrace{\left[\eta^{\alpha(\gamma}\eta^{\delta)\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\right]}_{=I^{\alpha\beta,\gamma\delta}} \partial^2 h_{\gamma\delta} \,. \end{aligned} \tag{1.85}$$

[7] Recall that we symmetrise with unit weight $a^{(\mu}b^{\nu)} := (a^\mu b^\nu + a^\nu b^\mu)/2$.

[8] In analogy to the gauge theory discussion around eqs. (1.62), with suitable choice of gauge-fixing parameter $\xi = -1/2$.

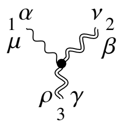
1.7 Feynman rules for perturbative quantum gravity

Going to momentum space and inverting the differential operator $I^{\alpha\beta\gamma\delta}\partial^2$ of eq. (1.85) leads us to the graviton propagator

$\alpha\beta$  $\gamma\delta$

$$= \frac{\mathrm{i}\, P_{\alpha\beta,\gamma\delta}}{p^2 + \mathrm{i}0} \qquad \text{with} \quad P_{\alpha\beta,\gamma\delta} = \eta_{\alpha(\gamma}\eta_{\delta)\beta} - \tfrac{1}{D-2}\eta_{\alpha\beta}\eta_{\gamma\delta}\,. \tag{1.86}$$

One indeed verifies that $I^{\alpha\beta,\gamma\delta}\, P_{\gamma\delta,\rho\kappa} = \delta^{\alpha}_{(\rho}\delta^{\beta}_{\kappa)}$. The graviton self-interaction vertices take an involved structure due to a proliferation of indices. For example, we exhibit the three-graviton vertex [10]

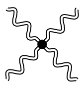
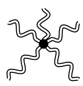
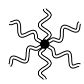


$$\begin{aligned} &= \mathrm{i}\kappa\, \mathrm{sym}[\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\gamma}) \\ &- \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\rho\gamma}) \\ &+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\rho}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\rho}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\rho\gamma}) \\ &+ P_3(k_{1\rho}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\rho}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\rho}) \\ &+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\rho}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\nu\alpha}\eta_{\rho\beta}\eta_{\mu\gamma})]\,, \end{aligned} \tag{1.87}$$

where "sym" means symmetrisation in the index pairs $(\mu\alpha)$, $(\nu\beta)$ and $(\rho\gamma)$. The symbol P_n denotes the symmetrisation in the momentum-index combinations $(k_1\mu\alpha, k_2\nu\beta, k_3\rho\gamma)$ associated with the three legs and results in n distinct terms. For example the first term above evaluates to

$$\begin{aligned} \mathrm{sym}\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\gamma}) =& \tfrac{1}{2}(k_1\cdot k_2\eta_{\mu(\nu}\eta_{\beta)\alpha}\eta_{\rho\gamma} + k_2\cdot k_3\eta_{\nu(\rho}\eta_{\gamma)\beta}\eta_{\mu\alpha} \\ &+ k_3\cdot k_1\eta_{\rho(\mu}\eta_{\alpha)\gamma}\eta_{\nu\beta})\,. \end{aligned} \tag{1.88}$$

The higher-point vertices take the schematic structure

 $\sim \kappa^2 k^2\,,$  $\sim \kappa^3 k^2\,,$  $\sim \kappa^4 k^2\,,$ $\quad \ldots \qquad (1.89)$

and grow considerably in size. E.g. the four-graviton vertex consists of 60 distinct terms, see [10] for its explicit form. Through the Fadeev-Popov procedure one also picks up a ghost sector. The local symmetry transformations are now the general coordinate transformations given in eq. (1.81). Hence, the gravity ghosts carry a vector index: $b^\nu(x)$ and $\bar{b}^\mu(x)$. The ghost contribution to the Lagrangian takes the form

$$\mathcal{L}_{\mathrm{GH}} = -\bar{b}^\mu\left(\kappa\frac{\delta G_\mu}{\delta\xi^\nu}\right)b^\nu\,. \tag{1.90}$$

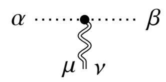
From the de Donder gauge-fixing function of eq. (1.83) one deduces the differential operator in the ghost sector

$$\kappa\frac{\delta G_\mu}{\delta\xi^\nu} = \eta_{\mu\nu}\partial^2+\kappa\left[\partial^\rho h_{\mu\nu}\partial_\rho+\partial^\rho h_{\nu\rho}\partial_\mu+\partial^\rho(\partial_\nu h_{\mu\rho})-\partial_\mu h_{\nu\rho}\partial^\rho-\tfrac{1}{2}\partial_\mu(\partial_\nu h)\right], \tag{1.91}$$

where the first term gives rise to the kinetic term of the ghost fields, yielding the propagator

$$\alpha \cdots\cdots\cdots \beta \;=\; \frac{\mathrm{i}\,\eta_{\alpha\beta}}{p^2+\mathrm{i}0}\,. \tag{1.92}$$

The remaining terms yield a graviton-ghost-anti-ghost interaction vertex,



(1.93)

However, ghosts will play no role in the modern approaches to scattering amplitudes developed in these lecture notes. Therefore we do not need to spell out this involved vertex here.

1.8 Spinor-helicity formalism for massless particles

In this section we will introduce a formalism that efficiently captures the kinematical data of the scattering states in the S-matrix: the momenta and polarisations of the scattered particles. The spinor-helicity variables allow one to express this data (momenta and helicities) for a massless particle in a uniform object thereby guaranteeing the on-shell conditions. In fact, the scattering amplitudes involving massless scalars, fermions, gluons, photons and gravitons take very compact forms in these variables.

The starting point is to rewrite the four-momentum p^μ as a bi-spinor $p^{\dot\alpha\alpha}$ that we may represent as a 2×2 matrix

$$p^\mu \longrightarrow p^{\dot\alpha\alpha} = \bar\sigma_\mu^{\dot\alpha\alpha}\,p^\mu = \begin{pmatrix} p^0+p^3 & p^1-\mathrm{i}p^2 \\ p^1+\mathrm{i}p^2 & p^0-p^3 \end{pmatrix}, \tag{1.94}$$

where $\bar\sigma_\mu^{\dot\alpha\alpha} = (\mathbb{1}, +\boldsymbol{\sigma})$, cf. exercise 1.1. The determinant of this matrix is given by

$$\det\bigl(p^{\dot\alpha\alpha}\bigr) = \bigl(p^0\bigr)^2 - \mathbf{p}^2 = p^2\,, \tag{1.95}$$

where $\mathbf{p} = (p^1, p^2, p^3)$ is the spatial momentum vector. If we put the momentum p^μ on the mass shell, i.e. $p^2 = m^2$, we see that the determinant equals m^2. Hence, there is a distinction between the massive and massless case. In the massive case the hermitian 2×2 matrix $p^{\dot\alpha\alpha}$ has rank 2 and may be decomposed into the sum of two outer products of commuting Weyl spinors,

$$p^{\dot\alpha\alpha} = \tilde\lambda^{\dot\alpha}\lambda^\alpha + \tilde\mu^{\dot\alpha}\mu^\alpha\,, \tag{1.96}$$

Gauge freedom: General coord. transformations, need for gauge fixing: $\delta h_{\mu\nu} = 2\partial_{(\mu}\varepsilon_{\nu)}(x) \Leftrightarrow \delta A_\mu = \partial_\mu \lambda(x)$

De Donder gauge: $G_\mu = \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h = 0$

A la Fadeev-Popov: $S_{gf} = \frac{1}{2}\int d^dx\, \eta_{\sigma\delta}\, G^\sigma G^\delta$

$$\Rightarrow S_{GR} = S_{E\text{-}H} + S_{gf} = -\frac{1}{2} h_{\mu\nu} I^{\mu\nu\gamma\sigma} \Box h_{\gamma\sigma} + S_{INT}$$

$$S_{INT} = \kappa\left[\partial^2 h^3\right] + \kappa^2\left[\partial^2 h^4\right] + \dots$$


where $$I^{\alpha\beta\gamma\delta} = \eta^{\alpha(\gamma}\eta^{\delta)\beta} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}$$

(1) Show that $I \cdot P = \mathbb{1}$

$\Box$ Inverse: $$P^{\alpha\beta\gamma\delta} = \eta^{\alpha(\gamma}\eta^{\delta)\beta} - \frac{1}{d-2}\eta^{\alpha\beta}\eta^{\gamma\delta}$$

$$S = \eta^{\alpha(\gamma}\eta^{\delta)\beta} - \frac{1}{D}\eta^{\alpha\beta}\eta^{\gamma\delta} \qquad T = \eta^{\alpha\beta}\eta^{\gamma\delta}$$

Coupling to matter: $S = S_{GR} + S_{MATTER}$

E.O.M: $$\frac{\delta S}{\delta g_{\mu\nu}(x)} = \frac{\delta(S_{GR} + S_{MATTER})}{\delta g_{\mu\nu}(x)} = 0$$

$$\Rightarrow \Box h_{\mu\nu}(x) = \kappa^2 T_{\mu\nu} = -\kappa^2 \frac{\delta(S_{INT} + S_{MATTER})}{\delta h_{\mu\nu}}$$

a) Vacuum, weak grav. fields: $\Box h_{\mu\nu}(x) = 0$

$$\Rightarrow \quad h_{\mu\nu}(x) = \varepsilon_{\mu\nu}(k)\, e^{-ik\cdot x} \quad \text{with} \quad k^2 = 0$$

gravitational wave, polarizations $\varepsilon_{\mu\nu} \to \{\varepsilon_+, \varepsilon_\times\}$

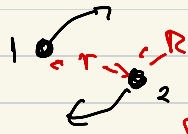
helicity: $++$ $\quad --$

b) Local source, of length scale ℓ. For field wavefcn
$|\vec{x}| \gg \{\ell, \ell^2\omega, 1/\omega\}$ (wave zone)

$$\kappa h_{\mu\nu}(x) = \frac{4G}{|\vec{x}|} \int_\omega P_{\mu\nu\alpha\beta}\, \tau^{\alpha\beta} e^{-ik\cdot x} + \mathcal{O}(1/x^2) \qquad k^\mu = (\omega, \omega\hat{x})$$

2. WEFT & gravitational 2-body problem

Consider 2 BHs or NSs

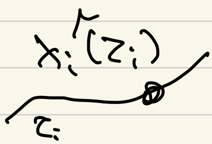


$r \gg R$

$R \sim 2GM$

model as point particles:

Trajectories



$$S_{MATTER} = \sum_{i=1}^{2} S_i \qquad S_i = -m_i \int ds_i = -m_i \int_{-\infty}^{\infty} d\tau_i \sqrt{g_{\mu\nu}\, \dot{x}_i^\mu \dot{x}_i^\nu}$$

Polyakov trick: Introduce einbein e:

$$S_i = -m_i \int d\tau \sqrt{\dot{x}^2} \quad \Leftrightarrow \quad \tilde{S}_i = -\frac{m}{2} \int d\tau \left(e^{-1} \dot{x}^2 + e \right)$$

EOM for e: $\quad \dfrac{\delta \tilde{S}_i}{\delta e} = 0 = 1 - \dfrac{1}{e^2} \dot{x}^2 \Rightarrow e = \sqrt{\dot{x}^2}$

Proper time gauge $\quad e = 1 \quad \Leftrightarrow \quad \boxed{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1}$

EFT:

$$S = -\sum_{i=1}^{2} \frac{m_i}{2} \int d\tau_i \Big(g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + \underset{\text{DROP}}{\underset{\uparrow}{1}} \Big) + S_{GR} + \left(\begin{array}{c} \text{finite size /} \\ \text{spin} \\ \text{corrections} \end{array} \right)$$

First finite size (tidal) effects:

$$S_T = m_i \int d\tau_i \left[C_{E^2,i} E_{\mu\nu}^2 + C_{B^2,i} B_{\mu\nu}^2 \right] + R^3\text{-terms}$$

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta} \dot{x}^\alpha \dot{x}^\beta \qquad B_{\mu\nu} = \frac{1}{2} R_{\mu\alpha}{}^{\delta\sigma} \varepsilon_{\delta\sigma\nu\beta} \dot{x}^\alpha \dot{x}^\beta$$

Electric & magnetic curvatures. C_{E^2} & C_{B^2} "Love numbers"

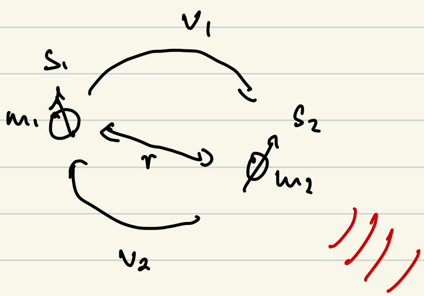
For BHs: $\quad C_{E^2} = C_{B^2} = 0$!

E.O.M.:

$$\frac{\delta S}{\delta h_{\mu\nu}} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{K^2}{8} T_{\mu\nu}$$ EINSTEIN'S EQS.

$$\frac{\delta S}{\delta x^{\mu}_{i}} = 0 \Rightarrow \ddot{x}^{\mu}_{i} + \Gamma^{\mu}{}_{\nu\rho} \dot{x}^{\rho}_{i} \dot{x}^{\nu}_{i} = 0$$ GEODESIC EQS.

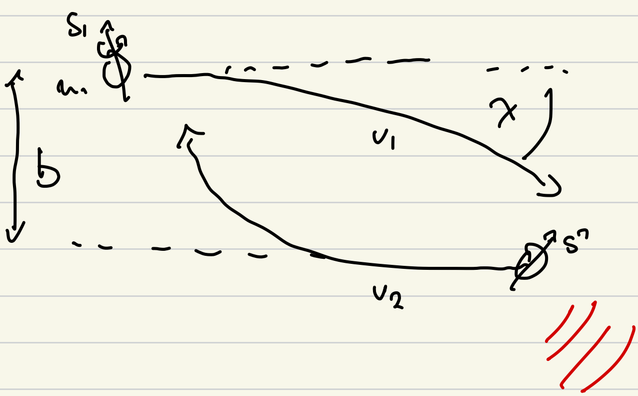
2 BODY PROBLEM: BOUND / UNBOUND — JUST AS IN NEWTON CASE



VIRIAL THEOREM:

$$\left\langle \frac{GM}{r} \right\rangle \sim \langle v^2 \rangle \ll 1 \qquad c = 1$$

→ post-Newtonian expansion (non-relativistic)



Scattering

$$\frac{Gm}{b} \ll 1$$

but exact in v

post-Minkowskian exp.

WE FOCUS ON SCATTERING Scenario

TRADITIONAL APPROACH: Solve E.O.M. perturbatively in G !

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \cdot \tau + z_i^\mu(\tau) \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\tilde{z}_i^\mu = \sum_{n\geq 0} G^n \, z_i^{(n)\mu}(\tau) \qquad \tilde{h}_{\mu\nu} = \sum_{n\geq 0} G^n \, h_{\mu\nu}^{(n)}(\tau)$$

INTERESTING OBSERVABLES:

- FAR FIELD WAVEFORM $\qquad \lim_{r\to\infty} h_{\mu\nu} = \dfrac{f_{\mu\nu}(t-r, \theta, \varphi)}{r} + \mathcal{O}(\tfrac{1}{r^2})$

- IMPULSE (change of momentum):

$$\Delta p_i^\mu = m_i \dot{x}_i^\mu \Big|_{\tau=-\infty}^{\tau=+\infty} = m_i \int_{-\infty}^{\infty} d\tau \, \ddot{x}_i(\tau)$$

3. Worldline Quantum Field Theory

Streamline this by quantization!

$$S = \sum_i \left(-\frac{m_i}{2} \int d\tau \, g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \text{[spin \& tidal effect]} \right)$$

$$+ \frac{1}{16\pi G} \int d^4x \sqrt{-g} \, R$$

Background field expansion: $\quad x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)$

$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x)$

$$\langle \mathcal{O} \rangle_{WQFT} = \int D[h,z]\; e^{\frac{i}{\hbar} S[h,z]}$$

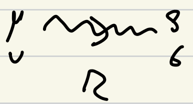
$\xrightarrow{\hbar \to 0}$ CLAIM: Tree-level one-point functions

solve class. E.O.M

$\Rightarrow$
$$\langle z_i^\mu(\tau) \rangle_{TREE} = z^\mu_{i,\,CLASS}(\tau)$$
$$\langle h_{\mu\nu}(x) \rangle_{TREE} = h^{CLASS}_{\mu\nu}(x)$$

FEYNMAN DIAGRAMMATIC EXPANSION:

PROPAGATORS: MOMENTUM SPACE $\quad h_{\mu\nu}(x) = \int_k e^{-ik\cdot x} h_{\mu\nu}(k)$

- Graviton $\langle h_{\mu\nu}(k)\, h_{\rho\sigma}(-k) \rangle =$  $= \dfrac{i\, P_{\mu\nu\rho\sigma}}{k^2}$

- Worldline deflection: $S^{(0)} = -\frac{m}{2} \int d\tau \left(\dot{x}^\mu(\tau)\, \dot{x}^\nu(\tau)\, \eta_{\mu\nu} + 1 \right)$

$$= -\int d\tau \Big(m + m\, \underbrace{v\cdot \dot{z}}_{\text{DROP.}} + \frac{m}{2} \dot{z}^\mu(\tau)\, \dot{z}^\nu(\tau)\, \eta_{\mu\nu} \Big)$$

$\Rightarrow \langle z^\mu(\omega)\, z^\nu(-\omega) \rangle = -\dfrac{i}{m} \dfrac{\eta_{\mu\nu}}{\omega^2}$

(F.T. $x^\mu(\tau) = \int_\omega e^{-i\omega\tau} x^\mu(\omega)$)

* IS DROPPING OK?

$$-m\int_{-\infty}^{\infty} dz\, V\cdot\dot{Z}(z) = -m\, V_\mu \big(\underbrace{Z^\mu(+\infty)}_{\neq 0} - \underbrace{Z^\mu(-\infty)}_{=0}\big)$$

more later...

VERTICES:

$$\delta\!\!\!{}^{-}(\omega) = 2\pi\,\delta(\omega) \qquad \int_\omega := \int_{-\infty}^{\infty}\frac{d\omega}{2\pi} \qquad \int_q := \int_{-\infty}^{\infty}\frac{d^dq}{(2\pi)^d}$$

$$\delta\!\!\!{}^{-}(q) = (2\pi)^d\,\delta^{(d)}(q)$$

$$S_{INT} = -\frac{m}{2}\kappa\int_{-\infty}^{\infty} dz\; h_{\mu\nu}(x(z))\;\dot{x}^\mu(z)\,\dot{x}^\nu(z)$$

$$= -\frac{m\kappa}{2}\int dz\; h_{\mu\nu}[x(z)]\left(V^\mu V^\nu + 2V^\mu\dot{Z}^\nu(z) + \dot{Z}^\mu(z)\dot{Z}^\nu(z)\right)$$

But:

$$h_{\mu\nu}[x(z)] = \int_q e^{iq\cdot(b+Vz+Z(z))}\; h_{\mu\nu}(-q)$$

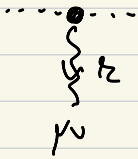
$$= \int_q \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_q e^{iq\cdot(b+V\cdot z)}\,[q\cdot Z(z)]^n\, h_{\mu\nu}(-q)$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{q,\omega_1\ldots\omega_n} e^{iq\cdot b}\, e^{i(q\cdot V+\sum_{i=1}^n\omega_i)z}\left(\prod_{i=1}^n q\cdot Z(-\omega_i)\right) h_{\mu\nu}(-q).$$

INSERT THIS TO PRODUCE GRAVITON-Z^n VERTICES:

n=0:

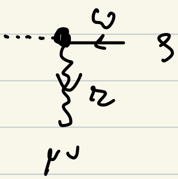
$$S_{INT}\Big|_{z^0} = -\frac{mk}{2}\int_q e^{iq\cdot b}\,\delta(q\cdot v)\,h_{\mu\nu}(-q)\,v^\mu v^\nu$$



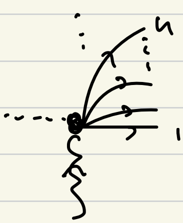
$$= -i\,\frac{mk}{2}\,e^{iq\cdot b}\,\delta(q\cdot v)\,v^\mu v^\nu$$

n=1:

$$S_{INT}\Big|_{z^1} = -i\,\frac{mk}{2}\int_{q,\omega} e^{iq\cdot b}\,\delta(q\cdot v+\omega)\,h_{\mu\nu}(-q)\,z^\rho(-\omega)\left(2\omega\, v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu q_\rho\right)$$



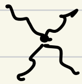
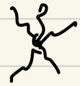
$$= \frac{mk}{2}\,e^{iq\cdot b}\,\delta(q\cdot v+\omega)\left(2\omega\, v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu q_\rho\right)$$

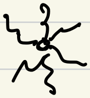


$$\sim mk\,e^{iq\cdot b}\,\delta\Big(q\cdot v+\sum_i \omega_i\Big)$$

[polynomial in ω_i, q of degree n]

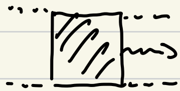
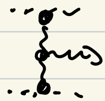
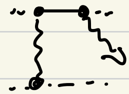
BULK GRAVITON VERTICES

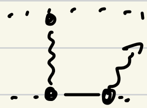
 $\sim \kappa q^2$  $\sim \kappa^2 q^2$  $\sim \kappa^3 q^2$

 $\sim \kappa^4 q^2$ ...

OBSERVABLES:

WAVEFORM

$\langle h_{\mu\nu} \rangle =$  $=$  $+$ 

$+$  $+ \mathcal{O}(G^{5/2})$

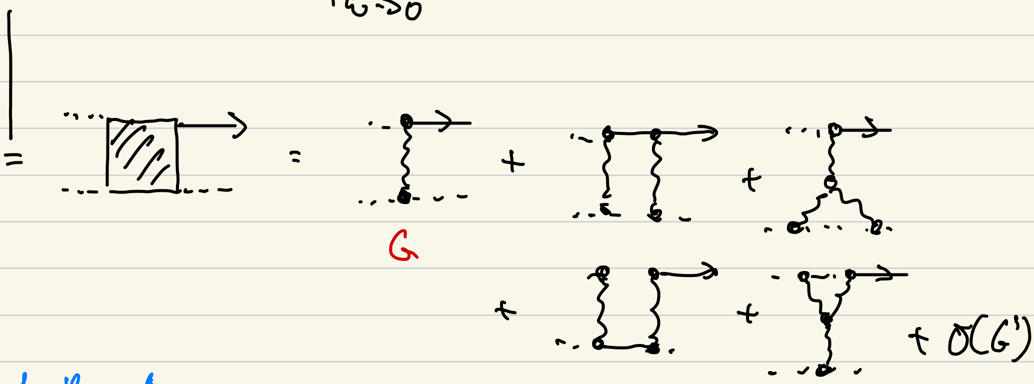
$G^{3/2}$

IMPULSE

$$\Delta p_i^\mu = m_i \langle \dot{x}_i^\mu \rangle \Big|_{\tau=-\infty}^{\tau=+\infty} = m_i \int_{-\infty}^{\infty} d\tau \, \langle \ddot{x}_i(\tau) \rangle$$

$$= m_i \int d\tau \, \frac{d^2}{d\tau^2} \langle z_i^\mu(\tau) \rangle \overset{\text{F.T.}}{=} - m_i \, \omega^2 \langle z_i^\mu(\omega) \rangle \Big|_{\omega \to 0}$$

NB:
$$\int dz \int d\omega\, \partial_z^2 e^{i\omega\cdot z} Z(\omega) = \int d\omega\, (-\omega^2)\, Z(\omega) \int dz\, e^{i\omega z}$$
$$= -\omega^2 Z(\omega)\Big|_{\omega\to 0}$$



G

G^2

State of the art

Claim: Tree-level one-point functions solve class. E.O.M

PROOF:

Let $S[\phi_A]$ with fields $\phi_A(x_A) = \{h_{\mu\nu}(x), z^\mu(z)\}$

and coordinates $x_A = \{x^\mu, z\}$

Partition function

$$Z[J_A] = \int D[\phi_A] \exp\left[\frac{i}{\hbar}\left(S[\phi_A] + \sum_A \int dx_A\, J_A(x_A)\, \phi_A(x_A)\right)\right]$$

WITH PHYSICAL SOURCE ($\hat{=}$ BACKGROUND) $\quad Q = \{ b_i^r, V_i^r \}$

For simplicity take:

$$S[\phi, Q] = \frac{1}{2} \int d^4x \, \partial_\mu \phi \, \partial^\mu \phi \; + \; S_{INT}[\phi, Q]$$

E.O.M:

$$\partial^2 \phi_{CLASS} = \left. \frac{\delta S_{INT}[\phi, Q]}{\delta \phi(x)} \right|_{\phi = \phi_{CLASS}}$$

PATH INTEGRAL QUANTIZATION: GENERATING FUNCTIONAL

$$e^{\frac{i}{\hbar} W[J]} = \int \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar} \left(S(\phi, Q) + \int d^4x \, J(x) \, \phi(x) \right] \right.$$

ONE POINT FUNCTION:

$$\langle \hat{\phi}(x) \rangle = \left. \frac{\delta W}{\delta J(x)} \right|_{J=0}$$

EFFECTIVE ACTION: LEGENDRE TRANSFORM OF $W[J]$

$$\Gamma[\phi] = \frac{i}{\hbar} \int d^4x \, J(x) \, \phi(x) - W[J]$$

① CENTRAL QFT RESULT: E.O.M OF EFFECTIVE ACTION ARE SOLVED BY ONE-POINT FUNCTION:

$$\boxed{\left. \frac{\delta \Gamma[\phi]}{\delta \phi(x)} \right|_{\phi(x) = \langle \hat{\phi}(x) \rangle} = 0}$$

② $\hbar$-EXPANSION ($\hat{=}$ LOOP EXPANSION) OF $\Gamma[\phi]$:

$$\Gamma[\phi] = \frac{1}{2}\int d^4x\, \partial_\mu \phi\, \partial^\mu \phi + S_{INT}[\phi, Q] + \mathcal{O}(\hbar)$$

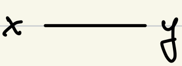
$$\Rightarrow \quad \boxed{\phi_{CLASS}(x) = \lim_{\hbar \to 0} \langle \hat{\phi}(x) \rangle = \langle \hat{\phi}(x) \rangle_{TREE}}$$

EXAMPLE:

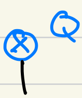
$$S_{INT}[\phi, Q] = \int d^4x\, Q(x)\, \phi(x)$$

E.O.M.: $\Box_x \phi(x) = Q(x)$

FEYNMAN RULES IN X-SPACE:

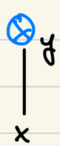


$$= D_F(x-y) = \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = \int d^4k \frac{1}{k^2 + i0}$$



$$= Q(x)$$

$$\langle \hat{\phi}(x) \rangle_{TREE} =$$



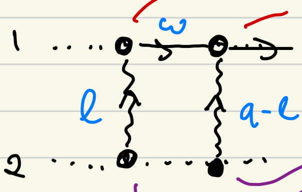
$$= \int d^4y\, D_F(x-y)\, Q(y)$$

INDEED SOLVES E.O.M.:

$$\Box_x \langle \hat{\phi}(x) \rangle_{TREE} = \int d^4y\, \underbrace{\Box_x D_F(x-y)}_{\delta^{(4)}(x-y)}\, Q(y) = Q(x) \quad \checkmark$$

LOOPS IN TREE DIAGRAMS !

Example:



$$= \int_{q,\ell,\omega} \frac{\delta(\omega - \ell\cdot v_1)\ \delta(\omega + (q-\ell)\cdot v_1)\ \delta(\ell\cdot v_2)\ \delta[(q-\ell)\cdot v_2]}{(\omega+i0)^2\ \ell^2\ (\ell-q)^2}\, e^{-iq\cdot b} (\dots)$$

$$= \int_q \delta(q\cdot v_1)\ \delta(q\cdot v_2)\ e^{-iq\cdot b} \underbrace{\int_\ell \frac{\delta(\ell\cdot v_2)}{(\ell\cdot v_1 + i0)^2\ \ell^2\ (\ell-q)^2}}_{f(q^2,\ v_1\cdot v_2)}$$

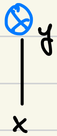
Due to hybrid QFT structure: 4+4+1 integrations
4 δ-functions

GENERAL STRUCTURE FOR IMPULSE @ n-PM order:

$$I_n = \int_q \delta(q\cdot v_1)\ \delta(q\cdot v_2)\ e^{-iq\cdot b} \int_{\ell_1 \dots \ell_{n-1}} \frac{\text{num}}{D_1 \dots D_j}\ \delta(\ell_1\cdot v_*) \dots \delta(\ell_{n-1}\cdot v_*)$$

4. BOUNDARY CONDITIONS & IN-IN FORMALISM

SOLUTION IN EXAMPLE CORRECT, YET WOULD WANT

$$\langle \hat{\phi}(x) \rangle_{\text{TREE}} = $$  $$= \int d^4y \, D_R(x-y) \, Q(y)$$

WANT RETARDED PROPAGATOR IN CLASSICAL PHYSICS!

RECALL:

$$D_R(x-y) = \Theta(x^0 - y^0) \, {}_0\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle_0$$

$$D_F(x-y) = {}_0\langle 0 | T(\hat{\phi}(x) \hat{\phi}(y)) | 0 \rangle_0$$

$$= \Theta(x^0 - y^0) \, {}_0\langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle_0$$

$$+ \Theta(y^0 - x^0) \, {}_0\langle 0 | \hat{\phi}(y) \hat{\phi}(x) | 0 \rangle_0$$

$$D_R(x-y) = D_F(x-y) - \underbrace{D_-(x-y)}_{= {}_0\langle 0 | \hat{\phi}(y) \hat{\phi}(x) | 0 \rangle_0}$$

STANDARD PATH INTEGRAL $\hat{=}$ IN-OUT FORMALISM

SOLUTION TO BOUNDARY VALUE PROBLEM NOT AN INITIAL VALUE PROBLEM!

SCATTERING: IN-STATE → OUT-STATE

4. BOUNDARY CONDITIONS & IN-IN FORMALISM

STANDARD PATH INTEGRAL: IN-OUT FORMALISM

[Galley, Tiglio] [Jordan]

- Hamiltonian formalism: Time evolution operator $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$

Background

$$U_J(T,T') = \mathcal{T}\exp\left[\frac{i}{\hbar}\int_{T'}^{T} dt \int d^3x \left\{\hat{H}_{\text{int}}[\phi_0(\mathbf{x},t), Q(\mathbf{x},t)] + J(x)\,\phi_0(\mathbf{x},t)\right\}\right]$$

Free fields

- Heisenberg picture: $\phi_H(\mathbf{x},t) = U_0(-\infty,t)\,\phi_0(\mathbf{x},t) U_0(t,-\infty)$

- Path integral representation:

$$\langle 0|U_J(\infty,-\infty)|0\rangle = \int [D\phi]\exp\left[\frac{i}{\hbar}\left(S[\phi,Q] + \int d^4x\, J(x)\,\phi(x)\right)\right] = \exp\left[\frac{i}{\hbar}W[J]\right]$$

Boundary cond: $\phi(t = \pm\infty) = 0$

- One-point function:

$$= U_0(t,-\infty)\,\phi_H(\mathbf{x},t) U_0(-\infty,t)$$

$$\langle\phi_H(\mathbf{x},t)\rangle_{\text{in-out}} = \frac{\delta W[J]}{\delta J(\mathbf{x},t)}\bigg|_{J=0} = \langle 0|\,U_0(\infty,t)\,\phi_0(\mathbf{x},t)\,U_0(t,-\infty)\,|0\rangle$$

$$= \underbrace{\langle 0|\,U_0(\infty,-\infty)}_{{}_{\text{out}}\langle 0|}\,\phi_H(\mathbf{x},t)\,\underbrace{|0\rangle}_{|0\rangle_{\text{in}}}{}_{\cancel{\text{in-out}}} = {}_{\text{out}}\langle 0|\phi_H(\mathbf{x},t)|0\rangle_{\text{in}}$$

IN-IN (SCHWINGER-KELDYSH) FORMALISM

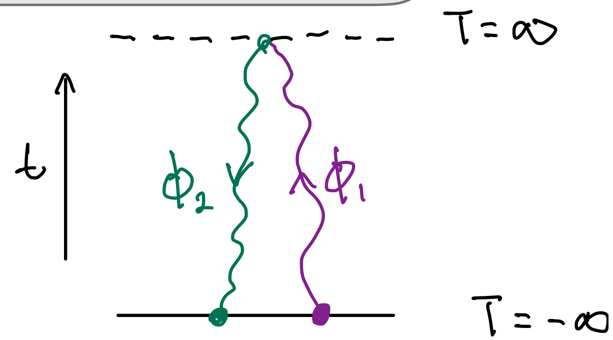
[Galley,Tiglio] [Jordan]

- Standard path integral yields $\langle \phi_H(x) \rangle_{\mathrm{in-out}} = {}_{\mathrm{out}}\langle 0 | \phi_H(x) | 0 \rangle_{\mathrm{in}}$ but want

$$\langle \phi_H(x) \rangle_{\mathrm{in-in}} = {}_{\mathrm{in}}\langle 0 | \phi_H(x) | 0 \rangle_{\mathrm{in}} = \langle 0 | U(-\infty, t)\, \phi_0(t, \mathbf{x})\, U(t, -\infty)\, | 0 \rangle$$

TRUE VEV OF $\phi_H(x)$?

$\Rightarrow$ need two evolution operators: Double the fields!



$$Z[J_1, J_2] = \langle 0 |\, U_{J_1}(-\infty, \infty)\, U_{J_1}(\infty, -\infty)\, | 0 \rangle$$

$$= \int D[\phi_1, \phi_2] \exp \left[\frac{i}{\hbar} \left(S[\phi_1] - S[\phi_2] + \int_x J_1(x)\, \phi_1(x) - J_2(x)\, \phi_2(x) \right) \right]$$

Boundary conditions:

$$\phi_1(t = +\infty, \mathbf{x}) = \phi_2(t = +\infty, \mathbf{x})$$
$$\phi_1(t = -\infty, \mathbf{x}) = \phi_2(t = -\infty, \mathbf{x}) = 0$$

$$\frac{1}{Z[0,0]} \frac{\delta Z[J_1, J_2]}{\delta J_1(x)} \bigg|_{J_i = 0} = \langle \Phi_H(x) \rangle_{\mathrm{in-in}}$$

Free theory:

$$\hat{\mathcal{U}}_J^{[0]}(T',T) = T\exp\left[\frac{i}{\hbar}\int_T^{T'} dt \int d^3x \left(J(x)\,\hat{\phi}(x)\right)\right]$$

We now have a propagator matrix:

$$\langle \phi_A(x)\,\phi_B(y)\rangle = \left.\frac{\delta^2 W^{(0)}[J_1,J_2]}{\delta J_2(y)\,\delta J_1(x)}\right|_{J_A(x)=0} = \left.\frac{\delta^2 \langle 0|\,\mathcal{U}^{(0)}_{J_2}(-\infty,\infty)\,\mathcal{U}^{(0)}_{J_1}(\infty,-\infty)|0\rangle}{\delta J_2(y)\,\delta J_1(x)}\right|_{J_A=0}$$

$$\langle \phi_1(x)\,\phi_1(y)\rangle = \underbrace{\langle 0|\,\mathcal{T}\,\phi(x)\,\phi(y)\,|0\rangle}_{D_F(x,y)} \qquad \langle \phi_1(x)\,\phi_2(y)\rangle = \overbrace{\langle 0|\,\phi(y)\,\phi(x)\,|0\rangle}^{D_-(x,y)}$$

$$\Rightarrow \langle \phi_2(x)\,\phi_1(y)\rangle = \langle 0|\,\phi(x)\,\phi(y)\,|0\rangle \to D_+(x,y) \qquad \langle \phi_2(x)\,\phi_2(y)\rangle = \langle 0|\,\mathcal{T}^*\phi(x)\phi(y)|0\rangle$$

$D_D(x,y)$ (anti-time order)

$$\Rightarrow D_{AB}(x,y) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & D_F(x,y) & D_-(x,y) \\ 2 & D_+(x,y) & D_D(x,y) \end{array}$$

$A,B = 1,2$

Go to Keldysh basis:

$$\phi_- = \phi_1 - \phi_2 \qquad J_- = J_1 - J_2$$

$$\phi_+ = \frac{1}{2}(\phi_1 + \phi_2) \qquad J_+ = \frac{1}{2}(J_1 + J_2)$$

$$\Rightarrow \langle \phi_a(x)\,\phi_b(y)\rangle = \begin{array}{c|cc} & + & - \\ \hline + & \frac{1}{2}D_H(x,y) & D_R(x,y) \\ - & -D_A(x,y) & 0 \end{array}$$

with "Hadamard" function $D_H(x,y) = \langle 0|\,\{\phi(x),\phi(y)\}\,|0\rangle$.

GENERATING FUNCTIONAL OF IN-IN THEORY IN KELDYSH BASIS:

$$e^{\frac{i}{\hbar}W[J_+,J_-]} = \int \mathcal{D}[\phi_+,\phi_-] \exp\left[\frac{i}{\hbar}\left(S[\phi_+ + \tfrac{1}{2}\phi_-] - S[\phi_+ - \tfrac{1}{2}\phi_-]\right) + \frac{i}{\hbar}\int d^4x \left(J_+\phi_- + J_-\phi_+\right)\right]$$

The TRUE VEV OF $\hat{\phi}_H(x)$ MAY NOW BE COMPUTED AS:

$$\langle \hat{\phi}_H(t,\vec{x})\rangle_{IN-IN} = \langle \phi_+(t,\vec{x})\rangle\Big|_{J_+=J_-=0} = \frac{\delta W[J_+,J_-]}{\delta J_-}\Big|_{J_+=J_-=0}$$

$$= \frac{1}{Z}\int \mathcal{D}[\phi_+,\phi_-]\, \phi_+(t,\vec{x}) \exp\left[\frac{i}{\hbar}\left(S[\phi_+ + \tfrac{1}{2}\phi_-] - S[\phi_+ - \tfrac{1}{2}\phi_-]\right)\right]$$

Note that $\langle \phi_-(t,\vec{x})\rangle\Big|_{J_\pm=0} = 0$ AS $\langle \phi_1\rangle\Big|_{J_\pm=0} = \langle \phi_2\rangle\Big|_{J_\pm=0}$.

IMPORTANTLY IN-IN EFFECTIVE ACTION:

$$\Gamma[\langle\phi_+\rangle,\langle\phi_-\rangle] = W[J_+,J_-] - \int d^4x \left(J_-\langle\phi_+\rangle + J_+\langle\phi_-\rangle\right)$$

E.O.M:

$$0 = \frac{\delta \Gamma[\langle\phi_+\rangle,\langle\phi_-\rangle]}{\delta\langle\phi_-\rangle}\Bigg|_{\langle\phi_-\rangle=0,\ \langle\phi_+\rangle=\phi_{CLASS},\ J_\pm=0}$$

@ TREE-LEVEL:

$$\Gamma[\langle\phi_+\rangle,\langle\phi_-\rangle] = \langle\phi_-\rangle \cdot \underbrace{\left(\frac{\delta S}{\delta\phi}\right)_{\phi\to\langle\phi_+\rangle}}_{\text{CLASSICAL E.O.M}} + \mathcal{O}(\langle\phi_-\rangle^3) + \mathcal{O}(\hbar)$$

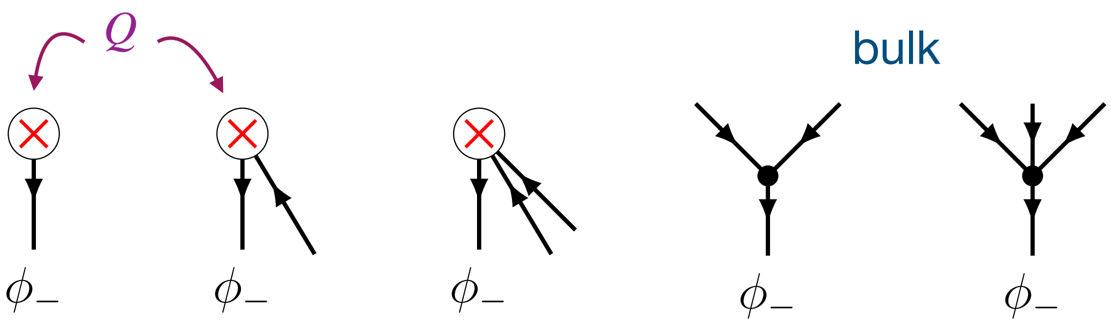
ONE POINT FUNCTIONS @ TREE-LEVEL:

- Vertices including background field Q

$$S_{\text{int}}[\phi; Q] \to \phi_- \left(\frac{\delta S_{\text{int}}[\phi, Q]}{\delta \phi} \right) \Bigg|_{\phi \to \phi_+} + \mathcal{O}(\phi_-^3)$$

$Q_- = 0$

$Q_+ = Q$

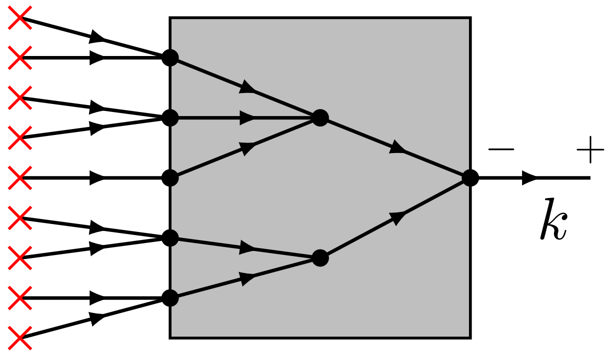


→ needs loops in 1PT Function!

- One point function:

@ TREE-LEVEL

$$\langle \phi(k) \rangle_{\text{in-in}} =$$



Only retarded propagators contribute!

UPSHOT:

USE STANDARD IN-OUT FEYNMAN RULES WITH RETARDED PROPS

4. CONNECTION TO (WORLDLINE) SCATTERING AMPLITUDES

- USE WORLDLINE (FEYNMAN-SCHWINGER) REPRESENTATION OF PROPAGATOR:

SCALAR FIELD: $$(\Box + m^2)\, G(x,x') = \delta^{(D)}(x-x')$$ GREEN'S FUNCTION $G(x,x')$

$$G(x,x') = \int d^4p \, e^{ip\cdot(x-x')} \frac{i}{p^2-m^2} = \int_0^\infty ds \int d^4p \, e^{i(p^2-m^2)\cdot s} \, e^{ip\cdot(x-x')} =$$

$$= \int_0^\infty ds \, e^{-ism^2} \langle x | \, e^{is\hat{p}^2} \, | x' \rangle$$

Feynman P.I.

$$= \int_0^\infty ds \, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x \; \exp\left[-i \int_0^s d\sigma \left(\frac{1}{4} \eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right) \right]$$

Coupling to gravity? $g_{\mu\nu}(x)$?

Graviton-dressed scalar propagator

Massive complex scalar $\phi(x)$ interacting with gravity:

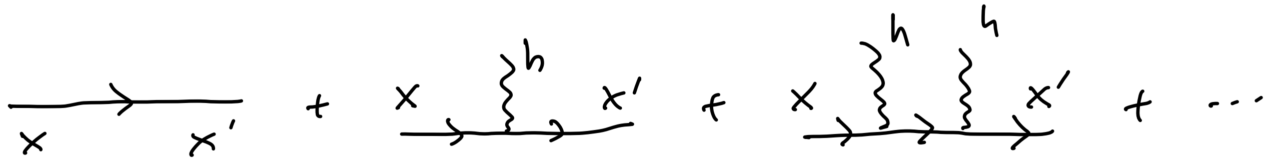
$$S = \int d^D x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi - \xi R \phi^\dagger \phi \right)$$

non-minimal coupling

$$\left(\nabla_\mu \nabla^\mu + m^2 + \xi R \right) G(x,x') = \sqrt{-g}\, \delta^{(D)}(x-x')$$

Weak-field approximation

$$G(x,x') =$$



This gravitationally-dressed Green's function has a worldline path integral form

[Bastianelli, van Nieuwenhuizen][De Witt][Bekenstein,Parker]

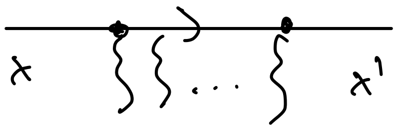
$$G(x,x') \sim \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x \exp\left[-i \int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \left(\xi - \frac{1}{4} \right) R \right) \right]$$

⊕ "LEE-YANG" GHOSTS

Worldline action familiar from Polyakov formulation.

This provides a link : QFT $\longleftrightarrow$ worldline.

LSZ-REDUCTION: PUTTING THE SCALAR LEGS ON-SHELL [Mogull,Plefka,Steinhoff]

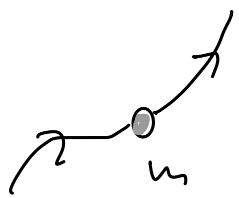
$G(x,x') =$ 

$$\sim \int_0^\infty ds\; e^{-i s m^2} \int \mathcal{D}[z,h]\, \exp\left[-i \int_0^s d\sigma \left(\tfrac{1}{4}\dot{x}^2 + (\xi - \tfrac{1}{6}) R\right) + S_{E.H}\right]$$

CLAIM: LSZ reduction $\hat{=}$ Drop s integral & take $s \to \infty$

$$(\Box_x - m^2)(\Box_{x'} - m^2)\, G(x,x')$$

WORLDLINE EFT!

$$\sim \int \mathcal{D}h\; e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau \left(\frac{1}{2}\dot{x}^2 + m^2\right) + \int d^4x \sqrt{-g}\, R} \;\hat{=}$$ 

From the S-matrix to the worldline

$$G(x,x') = \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}[x,a,b,c] \exp\left[-i \int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu}\left(\dot{x}^\mu \dot{x}^\nu + \overbrace{a^\mu a^\nu + b^\mu c^\nu}^{\text{non-propagating}}\right) + \left(\xi - \frac{1}{4}\right) R\right)\right]$$

How does the dressed propagator help us analyze S-matrices? Use the 2-point function:

$$G_i(x,x') = \int \mathcal{D}\phi_i\, \phi_i(x)\, \phi_i^\dagger(x')\, e^{iS_i}$$

$$S_{EH} = -2 m_{Pl}^{D-2} \int d^D x \sqrt{-g}\, R$$

Insert into a time-ordered correlator:

$$\langle \Omega | T\{\phi_1(x_1)\phi_1^\dagger(x_1')\phi_2(x_2)\phi_2^\dagger(x_2')\} | \Omega \rangle \quad \leftarrow \text{SCATTERING AMPLITUDE}$$

$$= \int \mathcal{D}[h_{\mu\nu}, \phi_1, \phi_2]\, \phi_1(x_1)\phi_1^\dagger(x_1')\, \phi_2(x_2)\, \phi_2^\dagger(x_2')\, e^{i(S_{EH} + S_1 + S_2)}$$

$$= \int \mathcal{D}h_{\mu\nu}\, G_1(x_1, x_1')\, G_2(x_2, x_2')\, e^{iS_{EH}} \quad \leftarrow \text{EFT}$$

(WITH QUANTIZED $X^\mu(\tau)$)

Argument neglects scalar loops

$\Rightarrow$ $\hbar \to 0$ LIMIT NECESSARY ? ↯ IN-IN FORMULATION

4. SUSY IN THE SKY WITH GRAVITONS

Adding spin DOF.

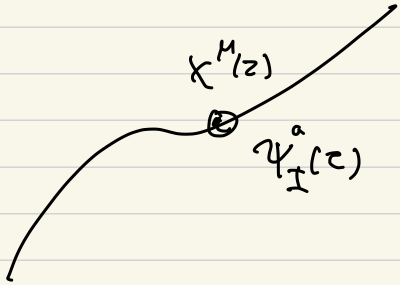
Starting point: Free wave equation for spin S particle in flat space-time

$$(\Box - m^2)\, G(x,x') = 0 \quad \rightarrow \quad \text{higher spin}$$

Described by N-extended superparticle $(S = N/2)$

WORLDLINE FIELDS: $X^\mu(\tau)$

$\psi^a_I(\tau) \quad i = 1, \dots, N$



$\underline{N=2}$ $\bar{\psi}^\mu, \psi_\mu, X^\mu, p_\mu$

POISSON BRACKETS: $\{X^\mu, p_\nu\}_{P.B.} = \delta^\mu_\nu$ $\{\psi_\mu, \bar{\psi}^\nu\}_{P.B.} = -i\,\delta^\nu_\mu$

$H = \frac{1}{2} p_\mu p^\mu$ $Q = p_\mu \psi^\mu$ $\bar{Q} = p_\mu \bar{\psi}^\mu$ $J = \psi^\mu \bar{\psi}_\mu - 2$

FOR CONV.

$\{Q, \bar{Q}\}_{P.B.} = -2iH$; $\{J, Q\}_{P.B.} = iQ$; $\{J, \bar{Q}\} = -i\bar{Q}$

SUPERFIELD $p_\mu = i\frac{\partial}{\partial x^\mu}$ $\bar{\psi}_\mu = \frac{\partial}{\partial \psi^\mu}$

$$\Phi(x,\psi) = F(x) + F_\mu(x)\psi^\mu + \frac{1}{2} F_{\mu_1\mu_2} \psi^{\mu_1}\psi^{\mu_2}$$
$$+ \dots + \frac{1}{d!} F_{\mu_1\dots\mu_D} \psi^{\mu_1}\dots\psi^{\mu_D}$$

CONSTRAINTS: $Q\Phi = 0$ $\bar{Q}\Phi = 0$ $J\Phi = 0$

$$J\Phi = (\psi^\mu\bar{\psi}_\mu - 2)\,\psi^{\mu_1}\dots\psi^{\mu_r} = (r-2)\,\psi^{\mu_1}\dots\psi^{\mu_r}$$

$\Rightarrow$ $\underline{r=2}$

$$Q\,\frac{1}{2} F_{\mu_1\mu_2}\psi^{\mu_1}\psi^{\mu_2} = \partial_\mu F_{\mu_1\mu_2}\psi^\mu\psi^{\mu_1}\psi^{\mu_2} \overset{!}{=} 0$$

$\Rightarrow$ $\partial_{[\mu} F_{\nu\rho]} = 0$ BIANCHI

$$\bar{Q}\ \frac{1}{2} F_{\mu_1\mu_2}\, \psi^{\mu_1}\psi^{\mu_2} = \partial^\mu F_{\mu\nu}\, \psi^\nu \quad \Rightarrow \quad \partial^\mu F_{\mu\nu} = 0$$

E.O.M

Describes Spin 1 (massless). Massive through dim. reduction from $D = 4+1$.

Choosing $2 \to p \Rightarrow F_{\mu_1 \dots \mu_{p+1}}$ form.

FIRST ORDER FORM OF ACTION:

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + i \bar{\psi}_\mu \dot{\psi}^\mu - e H - i \bar{\chi} Q - i \chi \bar{Q} - a J \right]$$

$$= \int d\tau \left[p \cdot \dot{x} + i \bar{\psi} \cdot \dot{\psi} - e \frac{1}{2} p^2 - i \bar{\chi}\, p \cdot \psi - i \chi\, p \cdot \bar{\psi} - a (\psi \cdot \bar{\psi} + c) \right]$$

Eliminate p by inserting algebraic e.o.m:

$$p^\mu = \frac{1}{e} (\dot{x}^\mu - i \bar{\chi} \psi^\mu - i \chi \bar{\psi}^\mu)$$

$$\Rightarrow \quad S = \int dt \left[\frac{1}{2} e^{-1} (\dot{x}^\mu - i \bar{\chi} \psi^\mu - i \chi \bar{\psi}^\mu)^2 + i \bar{\psi} \cdot \dot{\psi} - a (\bar{\psi} \cdot \psi c) \right]$$

SUSY IN THE SKY WITH GRAVITONS

*Jakobsen, **GM**, Plefka, Steinhoff* ***JHEP 01*** *(2022)*

- First quantised theory of a **spin-N/2 particle** in a **flat background (D=4+1)**:

· spin.

$$S = -\int \mathrm{d}\tau \left[p_M \dot{x}^M + \frac{i}{2}\psi^A_\alpha \dot{\psi}^B_\alpha \eta_{AB} - e\,H - i\chi^\alpha Q_\alpha - \frac{1}{2} f_{\alpha\beta} M^{\alpha\beta} \right]$$

Hamil formula

gravitinos

$$H = \frac{1}{2}p^2\,, \quad Q_\alpha = p\cdot\psi_\alpha\,, \quad M_{\alpha\beta} = i\psi_\alpha\cdot\psi_\beta\,. \quad \{x^M, p_N\} = \delta^M_N\,, \quad \{\psi^A_\alpha, \psi^B_\beta\} = -i\delta_{\alpha\beta}\,\eta^{AB}\,,$$

- Theory enjoys an **N-SUSY algebra** (α=1,2,…,N):

$$\{Q_\alpha, Q_\beta\} = -2i\delta_{\alpha\beta} H\,, \quad \{H, Q_\beta\} = \{H, M_{\alpha\beta}\} = 0\,,$$
$$\{M_{\alpha\beta}, Q_\gamma\} = -2\delta_{\gamma[\alpha}Q_{\beta]}\,, \quad \{M_{\alpha\beta}, M^{\gamma\delta}\} = -4\delta_{[\alpha}{}^{[\gamma} M_{\beta]}{}^{\delta]}\,.$$

- Coupling to a **curved background** possible up to **N=2** (spin-1) [Bastianelli, Benincasa, Giombi '05]:

$$Q = \psi^a\, e^\mu_a(x)\,\pi_\mu \qquad \pi_\mu = p_\mu - i\omega_{\mu ab}\bar{\psi}^a\psi^b$$

$$\{Q, \bar{Q}\} = -2i\Big[\underbrace{\frac{1}{2}(g^{\mu\nu}\pi_\mu\pi_\nu - m^2 - R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d)}_{H}\Big]$$

GLOBAL SUSY ACTION

Jakobsen, GM, Plefka, Steinhoff **Phys. Rev. Lett.** *128 (2022)*

- Gauge fix action by setting **e=1**, Lagrange multipliers to zero:

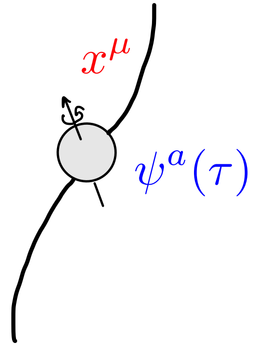
$$S_{\mathrm{BH/NS}} = -m\int d\tau\Big[\tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + i\bar{\psi}D_\tau\psi + \tfrac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d + C_E R_{a\mu b\nu}\dot{x}^\mu\dot{x}^\nu\bar{\psi}^a\psi^b\,\bar{\psi}\cdot\psi\Big]$$

spin degrees of freedom *neutron star term*

- Theory now enjoys a **global SUSY:**

$$\delta x^\mu = ie^\mu_a(\bar{\epsilon}\psi^a + \epsilon\bar{\psi}^a)\,,$$

$$\delta\psi^a = -\epsilon e^a_\mu\,\dot{x}^\mu - \delta x^\mu\omega_\mu{}^a{}_b\psi^b$$



$$S^{\mu\nu} = -2i\bar{\psi}^{[\mu}\psi^{\nu]} = \epsilon^{\mu\nu\rho\sigma}p_\rho a_\sigma$$

- Symmetries imply **conserved charges**:

$$\dot{x}^2 = 1 + R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d \qquad \bar{\psi}\cdot\psi = s$$

Conserved spin length

$$p\cdot\psi = p\cdot\bar{\psi} = 0 \quad\Longrightarrow\quad p_\mu S^{\mu\nu} = 0$$

Covariant SSC

- Neutron star term **preserves SUSY up to O(S²)**.

SPINNING WQFT FEYNMAN RULES

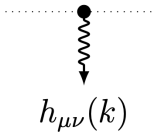
- Inclusion of spin requires **extended Feynman rules**:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$$

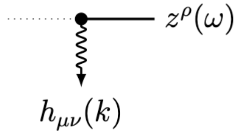
$$x_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + z_i^\mu(\tau_i)$$

$$\psi_i^\mu(\tau_i) = \Psi_i^\mu + \psi_i'^\mu(\tau_i)$$

$$\mathcal{S}^{\mu\nu} = -2i\bar{\Psi}^{[\mu}\Psi^{\nu]}$$

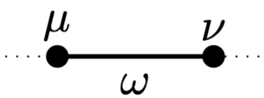


$$= -i\frac{m\kappa}{2}e^{ik\cdot b}\bar{\delta}(k\cdot v) \times \left(v^\mu v^\nu + i(k\cdot\mathcal{S})^{(\mu}v^{\nu)} - \frac{1}{2}(k\cdot\mathcal{S})^\mu(k\cdot\mathcal{S})^\nu + \frac{C_E}{2}v^\mu v^\nu(k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k)\right),$$

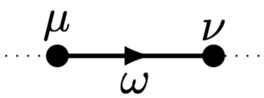


$$= \frac{m\kappa}{2}e^{ik\cdot b}\bar{\delta}(k\cdot v+\omega) \times \left(2\omega v^{(\mu}\delta_\rho^{\nu)} + v^\mu v^\nu k_\rho + i(k\cdot\mathcal{S})^{(\mu}(k_\rho v^{\nu)} + \omega\delta_\rho^{\nu)}) + \frac{1}{2}k_\rho(k\cdot\mathcal{S})^\mu(\mathcal{S}\cdot k)^\nu + \frac{C_E}{2}\left(\left(2\omega v^{(\mu}\delta_\rho^{\nu)} + v^\mu v^\nu k_\rho\right)(k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k) - \omega^2 k_\rho(\mathcal{S}\cdot\mathcal{S})^{\mu\nu} + 2\omega^2(k\cdot\mathcal{S}\cdot\mathcal{S})^{(\mu}\delta_\rho^{\nu)}\right)\right)$$

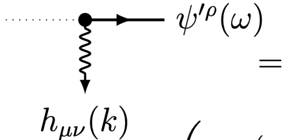
- Propagators:



$$= -i\frac{\eta^{\mu\nu}}{m(\omega+i\epsilon)^2},$$



$$= -i\frac{\eta^{\mu\nu}}{m(\omega+i\epsilon)},$$

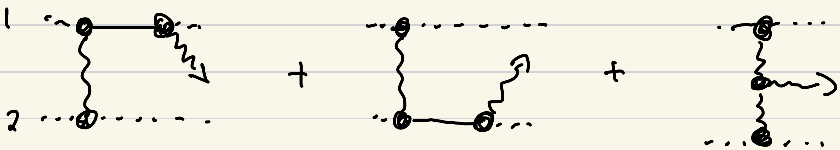


$$= -im\kappa e^{ik\cdot b}\bar{\delta}(k\cdot v+\omega) \times \left(k_{[\rho}\delta_{\sigma]}^{(\mu}(v^{\nu)} - i(\mathcal{S}\cdot k)^{\nu)}) + iC_E\left(v^{(\mu}k_\lambda + \omega\delta_\lambda^{(\mu}\right)\left(v^{\nu)}k_{[\rho} + \omega\delta_{[\rho}^{\nu)}\right)\mathcal{S}^\lambda{}_{\sigma]}\right)\bar{\Psi}^\sigma.$$

- Equivalent to solving **Mattison-Papapetrou-Dixon (MPD)** EoMs.
- Combine worldline modes into a "superfield": $Z_i = \{z_i, \psi_i'\}$

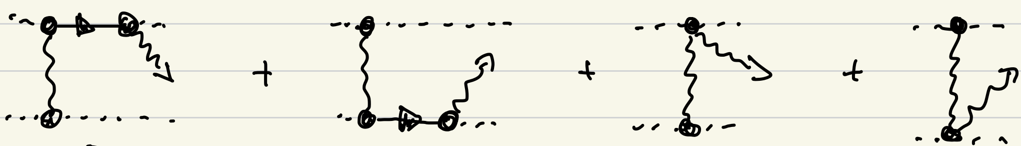
5. GRAVITATIONAL BREMSSTRAHLUNG WAVEFORM

3 DIAGRAMS CONTRIBUTE (w/o SPIN):



(a) + (b) + (c)

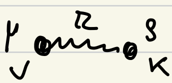
INCLUDING SPIN WE'D HAVE



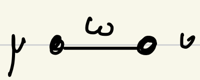
$(\bar{a})$ + $(\bar{b})$ + (d) + (e)

We focus on spinless case. In E&M only (a) + (b).

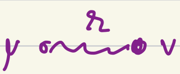
Recall Feynman rules:



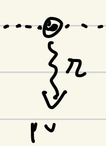
$$= \frac{i P_{\mu\nu\rho\sigma}}{(k^0+i0)^2-\vec{k}^2}$$



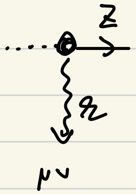
$$= -\frac{i}{m}\frac{\eta^{\mu\nu}}{(\omega+i0)^2}$$



$$= \frac{i\eta_{\mu\nu}}{(k^0+i0)^2-\vec{k}^2}$$ (EM)



$$= -i\frac{m\kappa}{2}\, e^{ik\cdot b}\,\delta(k\cdot v)\begin{cases} v^\mu v^\nu & \text{GR} \\ v^\mu \dfrac{e}{m\kappa} & \text{EM} \end{cases}$$

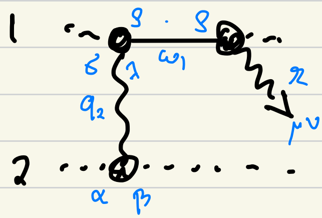


$$= \frac{m\kappa}{2} e^{ik\cdot b} \delta(k\cdot v + \omega) \begin{cases} 2\omega v^{(\mu} \delta^{\nu)}_{\rho} + v^\mu v^\nu k_\rho & \text{GR} \\ \left(\omega \eta^{\mu\rho} + v^\mu k^\rho\right) \frac{Q}{m\kappa} & \text{EM} \end{cases}$$

Then diagram (a) takes the form:

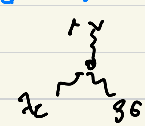
$$k^2 \langle h_{\mu\nu}(k)\rangle\Big|_{(a)} = -\frac{m_1 m_2 \kappa^3}{8} \int_{q_1 q_2} \mu_{12}(k) \frac{P_{\sigma\lambda,\alpha\beta} v_2^\alpha v_2^\beta}{(\omega_1 + i0)^2 \left[(q_2^0 + i0)^2 - \vec{q}_2^{\,2}\right]}$$

$$\left(2\omega_1 v_1^{(\mu} \delta^{\nu)}_{\rho} - v_1^\mu v_1^\nu k_\rho\right)\left(2\omega_1 v_1^{(\sigma} \eta^{\lambda)\rho} - v_1^\sigma v_1^\lambda q_2^\rho\right)$$



$$\mu_{12}(k) = e^{i(q_1\cdot b_1 + q_2\cdot b_2)} \delta(q_1\cdot v_1)\, \delta(q_2\cdot v_2)\, \delta(k - q_1 - q_2)$$

Diagram (b) follows via $1 \leftrightarrow 2$. Diagram (c) needs the 3-graviton vertex:



$$k^2 \langle h_{\mu\nu}(k)\rangle\Big|_{(c)} = -\frac{m_1 m_2 \kappa^3}{8} \int_{q_1, q_2} \mu_{12}(k) \frac{V_{(3)}^{(\mu\nu)(\rho\sigma)(\lambda\kappa)} P_{\rho\sigma\, v_1 v_1} P_{\lambda\kappa, v_2 v_2}}{\left[(q_1^0 + i0)^2 - \vec{q}_1^{\,2}\right]\left[(q_2^0 + i0)^2 - \vec{q}_2^{\,2}\right]}$$

Let us focus on (a):

PROBLEM 1:

Consider electromagnetism:

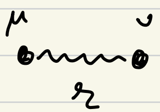
$$S = -\sum_{i=1}^{2} \int d\tau_i \left(\frac{m_i}{2} \dot{x}_i^2 - q_i A_\mu(x) \dot{x}_i^\mu \right) - \frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

$$+ S_{g.f.}$$

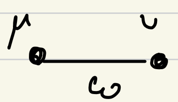
1) Go to Feynman gauge: $S_{g.f.} = \frac{1}{2\xi} \int (\partial \cdot A)^2$, i.e. show that the bulk part takes the form:

$$S_{BULK} = \frac{1}{16\pi} \int d^4x \, A_\mu(x) \, \Box \, A^\mu(x)$$

2) Argue the propagators take the form

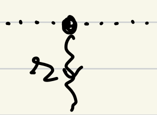


$$\langle A^\mu(k) \, A^\nu(-k) \rangle = \frac{i \eta_{\mu\nu}}{k^2}$$

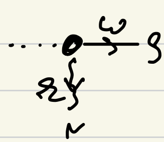


$$\langle Z^\mu(\omega) \, Z^\nu(-\omega) \rangle = -\frac{i}{m} \frac{\eta_{\mu\nu}}{\omega^2}$$

3) Find the interaction vertex:



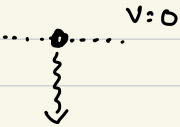
$$\sim q \, e^{ik\cdot b} \, \delta(k\cdot v) \, v^\mu$$



$$\sim q \, e^{ik\cdot b} \, \delta(k\cdot v + \omega) \left[\omega \eta^{\mu\rho} + v^\mu k^\rho \right]$$

4) Look at a single charge at rest at the origin

Show that

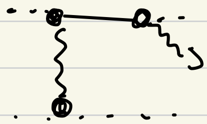


$$= \langle A_\mu(x)\rangle = \delta_{\mu 0}\,\frac{q}{4\pi|\vec{x}|}$$

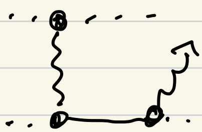
For this you will need the F.T. of $\int d^3k \, \frac{1}{|\vec{k}|^2}\, e^{-i\vec{k}\cdot\vec{x}}$

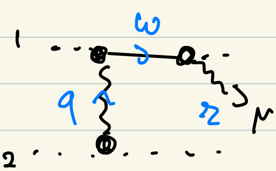
6) Compute the LO Bremsstrahlung!

$\langle A_\mu(k)\rangle =$



$+$





$$= \int_{q,\omega} \frac{\delta(v_2\cdot q)\,\delta(v_1\cdot q-\omega)\,\delta(\omega - v_1\cdot k)}{m_1\, q^2\, \omega^2\, k^2}$$

$$\times\, e^3 \left[e^{iqb_2}\, v_2^{\mu'}\right]\left[e^{-iq\cdot b_1}\left(\omega\,\eta^{\mu'\beta} - v_1^{\mu'} q^\beta\right)\right]$$

$$\left[e^{ik\cdot b_1}\left(-\omega\,\eta^{\mu\beta} + v_1^{\mu} k^\beta\right)\right]$$

NUMERATOR : $q^3 e^{iqb} e^{i\ell\cdot b_1}\left(\omega v_2^\sigma - v_1\cdot v_2\, q^\sigma\right)\left(-\omega \eta^{\mu\sigma} + v_1^\mu \ell^\sigma\right)$

$$= e^3 e^{iq b} e^{i\ell\cdot b_1}\left(-\omega^2 v_2^\mu + \omega\, v_2\cdot\ell\; v_1^\mu + \omega\, v_1\cdot v_2\, q^\mu - v_1\cdot v_2\; q\cdot\ell\, v_1^\mu\right)$$

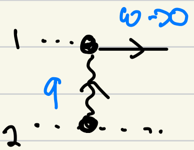
$\omega = v_1\cdot\ell$

$$= e^3 e^{iq\cdot b} e^{i\ell\cdot b_1}\Big[-(v_1\cdot\ell)^2 v_2^\mu + (v_1\cdot k)(v_2\cdot k)\, v_1^\mu + (v_1\cdot\ell)\gamma\, q^\mu - \gamma\, q\cdot\ell\; v_1^\mu\Big]$$

INTEGRAL:

$$I_1^\mu = e^3 \int_q \frac{\delta(v_2\cdot q)\,\delta[v_1\cdot(q-\ell)]}{\ell^2\, q^2}\, e^{iq\cdot b_{21}}\, e^{i\ell\cdot b_1}\left[-v_2^\mu + \frac{v_2\cdot\ell}{v_1\cdot\ell} v_1^\mu + \frac{\gamma}{v_1\cdot\ell} q^\mu - \gamma \frac{q\cdot\ell}{v_1\cdot\ell^2} v_1^\mu\right]$$

5) COMPUTE THE LO IMPULSE !

$\Delta p_1^\mu =$ 

$$= \int_q \left(e_2\, e^{iq\cdot b_2}\,\delta(q\cdot v_2)\, v_2^\mu\right)\frac{i\eta^{\mu\nu}}{q^2}\left(e_1\, e^{-iq\cdot b_1}\,\delta(q\cdot v_1-\omega)\left[\omega\cdot\eta^{\nu\sigma} - v_1^\nu q^\sigma\right]\right)_{\omega\to 0}$$

$$= e_1 e_2 \int_q e^{iq\cdot b} \frac{\delta(q\cdot v_1)\,\delta(q\cdot v_2)}{q^2} (-i)\, v_1\cdot v_2 \; q^\beta$$

$$b = b_2 - b_1$$

Need F.T.:

$$\frac{\partial}{\partial b^\beta} \int \frac{d^4 q}{(2\pi)^4} \frac{\delta(q\cdot v_1)\,\delta(q\cdot V_2)}{q^2} e^{iq\cdot b} = \frac{b^\beta}{|\vec{b}|} \frac{1}{2\pi}$$

Go to rest frame of particle 1: $v_1 = (1,0,0,0)$ $\quad \gamma = \frac{1}{\sqrt{1-v^2}}$

$v_2 = (\gamma, \gamma v, 0, 0)$

$$= \int \frac{d^3 q}{(2\pi)^2} \frac{\delta(q_1 \gamma v)}{-\vec{q}^{\,2}} e^{-i\vec{q}\cdot\vec{b}} = \frac{-1}{\gamma v} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{\vec{q}^{\,2}} e^{-i\vec{q}\cdot\vec{b}}$$

(NOW 2D INTEGRAL)

TIP:

$$I_n^d = \int \frac{d^d q}{(2\pi)^d} \frac{1}{|\vec{q}|^n} e^{-i\vec{q}\cdot\vec{b}} = \Omega_n^d \frac{1}{|\vec{b}|^{d-n}}$$

WITH $\quad \Omega_n^d = \dfrac{1}{2^n \sqrt{\pi}^{\,d}} \dfrac{\Gamma\left(\frac{d-n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$

Here:

$d=2 \quad n=2 \qquad \Gamma(0)=\Gamma(1)=1$

$d=2-2\varepsilon \qquad d-n=-2\varepsilon$

$$I_2^{2-2\varepsilon} = \frac{1}{4\pi^{1-\varepsilon}} \frac{\Gamma(-\varepsilon)}{1} |\vec{b}|^{2\varepsilon} \approx -\frac{\log|\vec{b}|}{2\pi} + \frac{1}{\varepsilon} + \text{const}$$

We find

$\Rightarrow$

$$\Delta p_1^\mu = \frac{v_1 \cdot v_2}{\gamma v} \underbrace{\frac{e_1 e_2}{4\pi}} \frac{b^\mu}{|\vec{b}|^2} = \frac{e_1 e_2}{4\pi v} \frac{b^\mu}{|\vec{b}|^2}$$