

**SPONTANEOUS SYMMETRY BREAKING**

**&**

**NAMBU - GOLDSTONE BOSONS**

# REMINDER: PHYSICAL STATE

- **Pure state** : normalized state vector  $|\psi\rangle \in \mathcal{H}$
- **Mixed state** : **density operator**

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad \text{such that } p_i > 0, \sum_i p_i = 1$$

→  $\rho$  is Hermitian & positive-semidefinite

→  $\text{tr } \rho = 1$

→ average of observables :  $\langle A \rangle_\rho = \sum_i p_i \langle \psi_i | A | \psi_i \rangle = \text{tr}(\rho A)$

- All density operators of a system span a convex set with the density operators of pure states on the boundary.

# REMINDER : SYMMETRY

- Wigner's theorem of QM : any symmetry represented by a unitary or antiunitary operator on  $\mathcal{H}$

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$\rho \longrightarrow U\rho U^\dagger$$

- Here exclusively continuous symmetry : Lie groups.
- Symmetry of a quantum system :  $G = \{g \mid U(g)H = HU(g)\}$   
Hamiltonian 

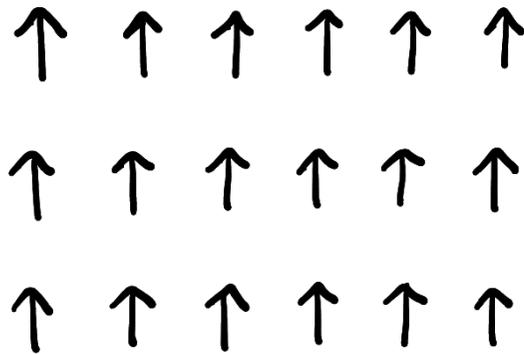
# ISOTROPY GROUP

- Terminology : isotropy group = stabilizer group = little group
  - Definition :  $H_\rho = \{ g \in G \mid U(g)\rho U(g)^\dagger = \rho \}$
  - Spontaneous symmetry breaking :  $H_\rho$  is a proper subgroup of  $G$
  - How to detect SSB :
    - ↳ There must be an observable  $A$  such that  $\langle A \rangle_{U\rho U^\dagger} \neq \langle A \rangle_\rho$
    - ↳ This observable must itself break the symmetry!
- Proof by contradiction :  $\langle A \rangle_{U\rho U^\dagger} = \text{tr}(U\rho U^\dagger A) = \text{tr}(\rho U^\dagger A U) = \text{tr}(\rho A) = \langle A \rangle_\rho$

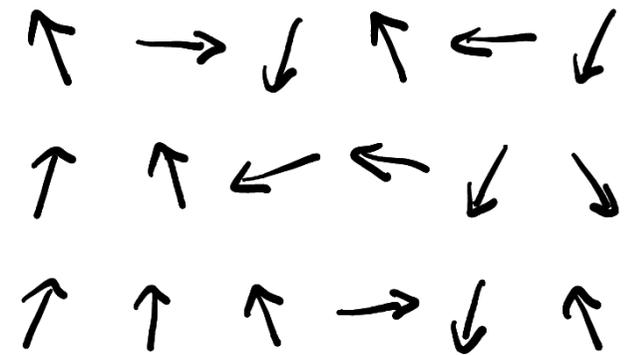
# SSB AND THERMODYNAMICS

Thermal equilibrium :  $\rho_{\beta} = \frac{1}{Z} e^{-\beta H}$  ... invariant under all the symmetries of  $H$ !

Ferromagnets :  $T < T_{\text{Curie}}$



$T > T_{\text{Curie}}$



order parameter for SSB

How is this possible at all?

# EXTERNAL PERTURBATIONS

- Pick set of observables transforming in a nontrivial representation of  $G$ :

$$A^i \xrightarrow{g} U(g)^\dagger A^i U(g) = R(g)^i_j A^j$$

- Add the observables to the Hamiltonian:

$$H(\lambda) = H - \lambda_i A^i \quad , \quad \rho_{\beta, \lambda} = \frac{1}{Z(\lambda)} e^{-\beta H(\lambda)}$$

- $G$ -invariance of the partition function:

$$Z(\lambda) = \text{tr} [U(g)^\dagger e^{-\beta H(\lambda)} U(g)] = \text{tr} e^{-\beta U(g)^\dagger H(\lambda) U(g)} = \text{tr} e^{-\beta [H - \lambda_i R(g)^i_j A^j]}$$

$$g \in G : \lambda_i \rightarrow \lambda_j R(g)^j_i$$

# FREE ENERGY

- Definition :  $Z(\lambda) = e^{-\beta F(\lambda)}$
- Use for calculation of averages of observables :

$$\langle A^i \rangle_{\rho_{\beta, \lambda}} = \text{tr}(\rho_{\beta, \lambda} A^i) = - \frac{\partial F(\lambda)}{\partial \lambda_i}$$

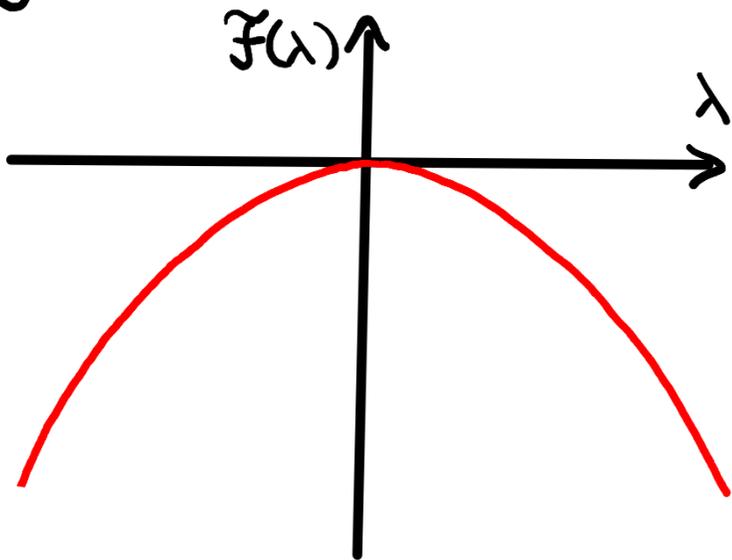
- **Analyticity in a finite volume** : can Taylor-expand

$$F(\lambda) = F(0) + \frac{\partial F(\lambda)}{\partial \lambda_i} \Big|_{\lambda=0} \lambda_i + \mathcal{O}(\lambda^2) \quad \longrightarrow \quad \lim_{\lambda \rightarrow 0} \langle A^i \rangle_{\rho_{\beta, \lambda}} = 0$$

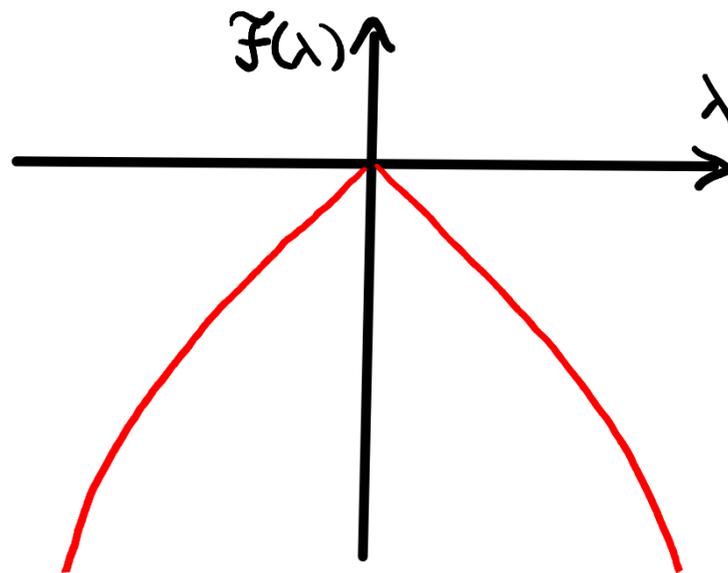
**The order parameter for SSB must vanish in any finite system in equilibrium !**

# THERMODYNAMIC LIMIT

Taking the limit  $V \rightarrow \infty$  may change the analytic properties of  $\mathcal{F}(\lambda) = \frac{F(\lambda)}{V}$ !



no SSB :  $\lim_{\lambda \rightarrow 0} \langle A \rangle = 0$



SSB :  $\lim_{\lambda \rightarrow 0} \langle A \rangle \neq 0$

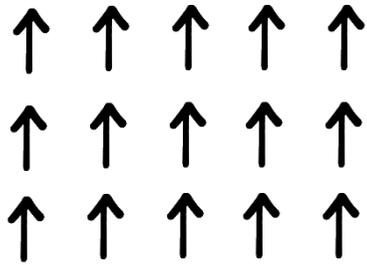
Hallmark of SSB :  $\lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \langle A \rangle_{\beta, \lambda} \neq \lim_{V \rightarrow \infty} \lim_{\lambda \rightarrow 0} \langle A \rangle_{\beta, \lambda}$

# MORALS : WHAT IS SSB ?

- Thermodynamically stable states carrying nonzero **order parameter**.
- Multiple **degenerate equilibrium states** connected by symmetry transformations.
- Enhanced sensitivity of the equilibrium state to **external perturbations**.

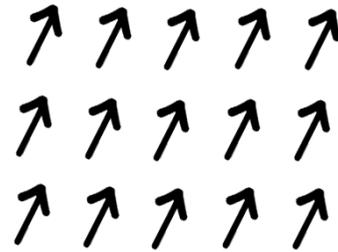
# FLUCTUATIONS OF ORDER PARAMETER

vacuum



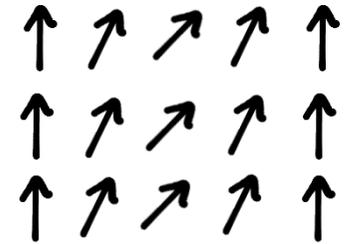
$$E=0$$

rotated vacuum



$$E=0$$

excitation



$$E>0$$



Nambu-Goldstone boson :

propagating fluctuation of the order parameter such that

$$\lim_{\vec{p} \rightarrow \vec{0}} E(\vec{p}) = 0$$

Energy required to create the fluctuation goes to zero with increasing wavelength.

# RELATIVISTIC SCALAR FIELDS

● Model Lagrangian :  $\mathcal{L} = \frac{1}{2} \delta_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi)$

● Invariance (both kinetic term and potential) under representation R of group G:

$$\phi^i \rightarrow R(g)^i_j \phi^j$$

$$V(R(g)\phi) = V(\phi)$$

finite

$$\delta \phi^i = i \epsilon^A R(Q_A)^i_j \phi^j$$

$$\frac{\partial V(\phi)}{\partial \phi^i} R(Q_A)^i_j \phi^j = 0$$

infinitesimal

● Order parameter :  $\langle \phi^i \rangle$

● Unbroken subgroup :

$$H = \{ h \in G \mid R(h)^i_j \langle \phi^j \rangle = 0 \}$$

Index notation :

● unbroken :  $\alpha, \beta, \dots$

● broken :  $a, b, \dots$

● any :  $A, B, \dots$

# RELATIVISTIC SCALAR FIELDS

Differentiate once more the invariance condition :

$$\frac{\partial^2 V(\phi)}{\partial \phi^i \partial \phi^j} R(Q_A)^j_k \phi^k + \frac{\partial V(\phi)}{\partial \phi^j} R(Q_A)^j_i = 0$$



evaluate at ground state

$$\left. \frac{\partial^2 V(\phi)}{\partial \phi^i \partial \phi^j} \right|_{\phi = \langle \phi \rangle} R(Q_A)^j_k \langle \phi^k \rangle = 0$$

- Each  $R(Q_A)^j_i \langle \phi^i \rangle$  is a zero mode of the mass matrix.
- The number of linearly independent zero modes is (at least)  $\dim G - \dim H$ .

# RELATIVISTIC NG BOSONS

Main features :

- Masless scalar particles .
- Total expected number :  $\dim G - \dim H$  .
- Created from the ground state by the action of the broken symmetry generators .

# GOING NONRELATIVISTIC

## Assumptions :

- Spacetime symmetry contains at least **continuous spacetime translations and spatial rotations**. Remains unbroken.
- Internal symmetry  $G$ .

## Changes & new features :

- **NG bosons = gapless quasiparticles**,  $\lim_{\vec{p} \rightarrow \vec{0}} E(\vec{p}) = 0$ .
- Dispersion relation  $E(\vec{p}) \sim |\vec{p}|^n$ ,  $n \geq 1$  for  $\vec{p} \rightarrow \vec{0}$ .
- **# NG fields =  $\dim G - \dim H$** , but **# NG modes may be smaller!**

# CANONICAL CONJUGATION OF NG FIELDS

- One NG field  $\pi^a$  for each broken generator  $Q_a$ : parameterize the fluctuations  $\phi^i - \langle \phi^i \rangle$  around ground state ( $\pi^a = 0$ ).
- Action of  $G$  on NG fields:  $\pi^a \rightarrow \pi^a + \epsilon^a + \epsilon^a_b \pi^b + \frac{1}{2} \epsilon^a_{bc} \pi^b \pi^c + \dots$
- Nonrelativistic spacetime admits operators linear in time derivatives:

$$\mathcal{L} = \frac{1}{2} \rho_{ab} \pi^a \partial_0 \pi^b + \dots$$

reduces # modes in spectrum!

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- Nonrelativistic spacetime admits operators linear in time derivatives:

$$\mathcal{L} = \frac{1}{2} p_{ab} \pi^a \partial_0 \pi^b + \dots$$

but what is this?

# CHARGE COMMUTATOR MATRIX

- Noether's theorem :  $\delta S = \int d^D x J_A^\mu(x) \partial_\mu \epsilon^A(x)$
  - $\mathcal{L} = \frac{1}{2} \rho_{ab} \pi^a \partial_0 \pi^b + \dots \longrightarrow J_a^0 = -\rho_{ab} \pi^b + \dots$
  - Symmetry transformation of charge density :
    - $\longrightarrow J_a^0 = -\rho_{ab} \pi^b + \dots \longrightarrow -\rho_{ab} (\pi^b + \epsilon^b) + \dots$
    - $\longrightarrow J_a^0 \longrightarrow e^{i\epsilon^A Q_A} J_a^0 e^{-i\epsilon^A Q_A} = J_a^0 + i\epsilon^A [Q_A, J_a^0] + \dots$
- $$\rho_{ab} = i \langle [J_a^0, Q_b] \rangle = i \lim_{V \rightarrow \infty} \frac{\langle [Q_a, Q_b] \rangle}{V}$$

# SPECTRUM OF NG BOSONS

# paired fields :  
rank  $\rho$



$$E(\vec{p}) \sim |\vec{p}|^2$$

for  $\vec{p} \rightarrow \vec{0}$



$\frac{1}{2}$  rank  $\rho$  of  
type-B NG modes

Example : ferromagnets ,  $G \simeq SU(2)$

spontaneous magnetization

$$\langle S_3 \rangle = -i \langle [S_1, S_2] \rangle \neq 0$$

$$\text{rank } \rho = 2$$

1 type-B mode : magnon

# SPECTRUM OF NG BOSONS

$\dim G - \dim H$   
NG fields

$n_A = \dim G - \dim H - \text{rank } \rho$   
type-A NG bosons  
 $E(\vec{p}) \sim |\vec{p}|$

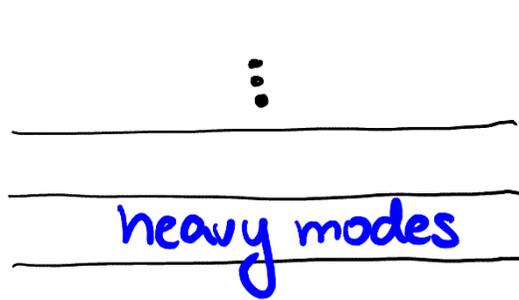
$n_B = \frac{1}{2} \text{rank } \rho$   
type-B NG bosons  
 $E(\vec{p}) \sim |\vec{p}|^2$

Total # of NG bosons :  $n_A + n_B = \dim G - \dim H - \frac{1}{2} \text{rank } \rho$

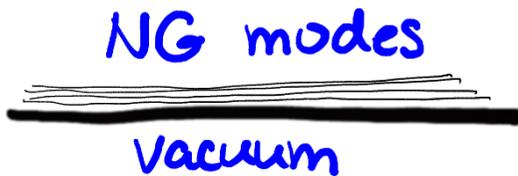
# NONLINEAR REALIZATION OF SYMMETRY

# MOTIVATION

spectrum



} UV (high-energy) physics



} IR (low-energy) physics

It should be possible to describe IR physics using NG fields alone!

# EXAMPLE

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

$U(1)$  symmetry  $\phi \rightarrow e^{i\epsilon} \phi$

ground state :  $\phi^* \phi = \frac{m^2}{2\lambda} = \frac{v^2}{2}$  , choose  $\langle \phi \rangle = \frac{v}{\sqrt{2}}$

parameterization of fluctuations :  $\phi(x) = \frac{1}{\sqrt{2}} [v + \psi(x) + i\pi(x)]$

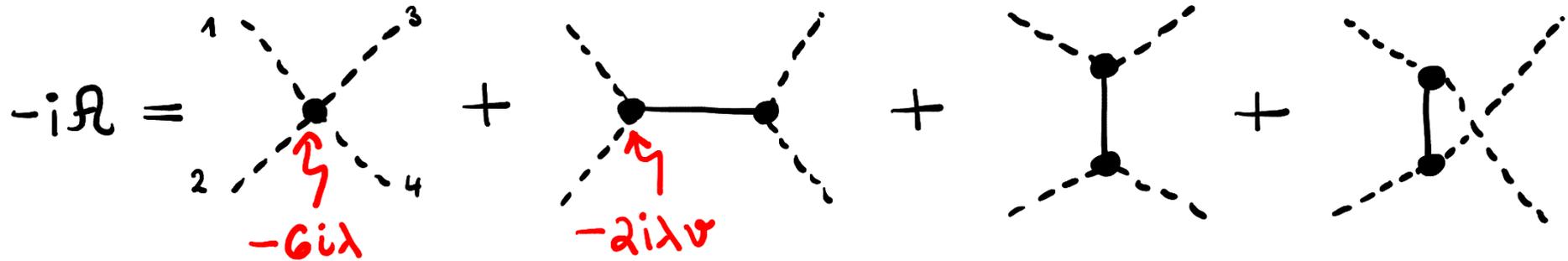
expand the Lagrangian :  $\mathcal{L} = \mathcal{L}_{\text{bilin}} + \mathcal{L}_{\text{int}}$

$\frac{1}{2}(\partial_\mu \psi)^2 - m^2 \psi^2 + \frac{1}{2}(\partial_\mu \pi)^2$

$-\lambda v \psi (\psi^2 + \pi^2) - \frac{\lambda}{4} (\psi^2 + \pi^2)^2$

# EXAMPLE

Sample scattering process :  $\pi\pi \rightarrow \pi\pi$



$$\mathcal{A} = 6\lambda + 4\lambda m^2 \left( \frac{1}{s-2m^2} + \frac{1}{t-2m^2} + \frac{1}{u-2m^2} \right)$$



low-energy limit :  
expand in powers of  $s, t, u$

$$\mathcal{A} = -\frac{\lambda}{2m^4} (s^2 + t^2 + u^2) + \dots$$

Adler zero :  $\lim_{\text{any } \vec{p}_i \rightarrow \vec{0}} \mathcal{A} = 0$

# EXAMPLE

Choose a different parameterization :  $\phi(x) = \frac{e^{i\pi(x)/v}}{\sqrt{2}} [v + \psi(x)]$

Why this one? Simple action of symmetry  $\pi \rightarrow \pi + \epsilon\sigma, \psi \rightarrow \psi$

Modified interaction Lagrangian :  $\mathcal{L}_{int} = -\lambda v \psi^3 - \frac{\lambda}{4} \psi^4 + \left(\frac{\psi}{v} + \frac{\psi^2}{2v^2}\right) (\partial_\mu \pi)^2$

$$-i\mathcal{L} = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$

Diagram 1: A vertex with two external dashed lines (momenta  $p$  and  $q$ ) and two internal solid lines. A red arrow points to the vertex with the label  $-\frac{2i}{v} p \cdot q$ .

Diagram 2: A vertex with two external dashed lines and two internal solid lines forming a loop.

Diagram 3: A vertex with two external dashed lines and two internal solid lines forming a loop, with a different internal structure than Diagram 2.

NG field  $\pi$  only interacts through derivatives : makes Adler zero manifest!

# EXAMPLE

Low-energy EFT for  $\pi$  : must preserve symmetry under  $\pi \rightarrow \pi + \epsilon v$

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_\mu \pi)^2 + c_4 [(\partial_\mu \pi)^2]^2 + \dots$$



$$-i\mathcal{A} = \text{diagram}$$

The diagram shows a central black dot with four dashed lines extending outwards, representing a four-point interaction vertex.

$$\mathcal{A} = -2c_4 (s^2 + t^2 + u^2)$$

Matching to the UV model :  $c_4 = \frac{\lambda}{4m^4}$

# MORAL LESSONS

- Physical observables must be independent of the choice of parameterization of the fields.
- We can make the parameterization work for us : include NG fields via coordinate-dependent symmetry transformation.
- Need a general approach to constructing low-energy EFT for any (Abelian or non-Abelian) symmetry!

# PLAN OF ATTACK

- This lecture : generic symmetry transformation for SSB  
"nonlinear realization", "coset construction", "induced representation", "Callan-Coleman-Wess-Zumino (CCWZ) construction"
- Next lecture : construction of effective actions  
for NG bosons

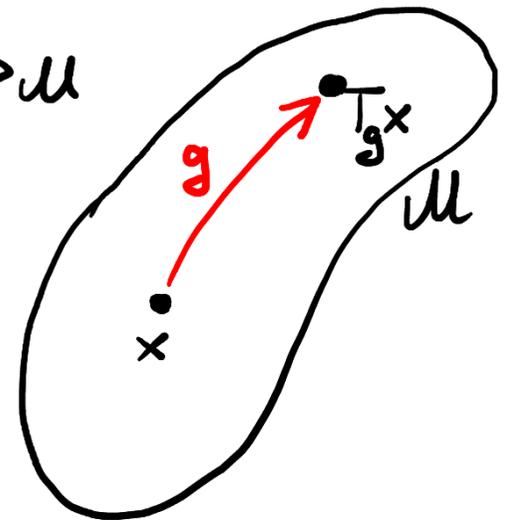
# ACTION OF GROUPS ON MANIFOLDS

Generalizes the concept of group representations from vector spaces to manifolds.

**Action of  $G$  on  $\mathcal{M}$** : set of smooth invertible maps  $T_g: \mathcal{M} \rightarrow \mathcal{M}$

such that

- $T_e = \text{id}$
- $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$
- $T_{g^{-1}} = (T_g)^{-1}$



Intuition: think of  $\mathcal{M}$  as a target space from which a set of fields takes value.

Then the action of  $G$  on  $\mathcal{M}$  describes **pointlike symmetry transformations**,

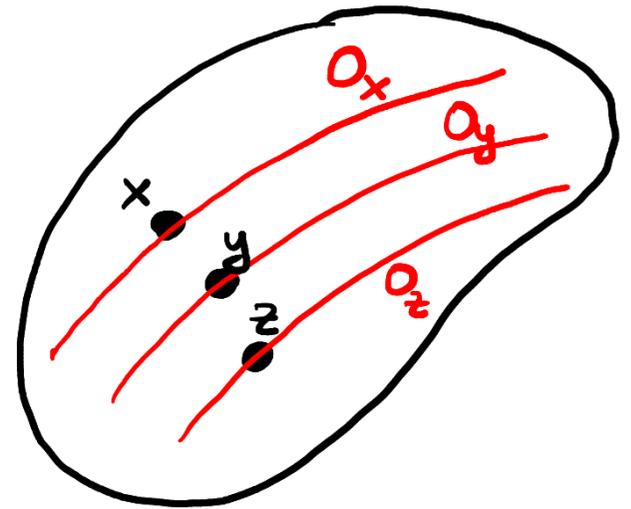
$$\phi^i \rightarrow \phi'^i(\phi, g).$$

# ACTION OF GROUPS ON MANIFOLDS

- **Orbit** of a point  $x \in \mathcal{M}$  :  $O_x = \{T_g x \mid g \in G\}$

The whole manifold is a disjoint union of a set of orbits. That is, for any  $x, y \in \mathcal{M}$ ,

$$O_x = O_y \quad \text{or} \quad O_x \cap O_y = \{\}$$



- **Isotropy group (stabilizer, little group)** of a point  $x \in \mathcal{M}$  :

$$H_x = \{h \in G \mid T_h x = x\}$$

All points on the same orbit have mutually conjugate isotropy groups :

$$H_{T_g x} = g H_x g^{-1}$$

# HOMOGENEOUS / COSET SPACES $G/H$

## Geometric definition

Manifold carrying action of  $G$   
that consists of a single orbit.

$$H_x \approx H \quad \forall x \in \mathcal{U}$$

- Examples :
- $ISO(n)/SO(n) \approx \mathbb{R}^n$
  - $SO(n+1)/SO(n) \approx S^n$
  - $U(n)/U(n-1) \approx S^{2n-1}$

## Group-theoretic definition

Left coset of  $H$  in  $G$  :  $gH = \{gh \mid h \in H\}$ .

The group  $G$  is a disjoint union of cosets:

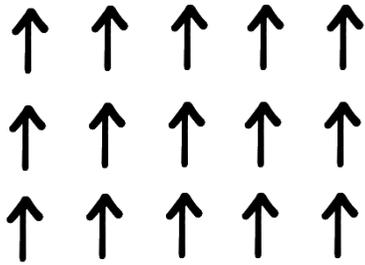
$$g_1H = g_2H \quad \text{or} \quad g_1H \cap g_2H = \{\} \quad \forall g_1, g_2 \in G$$

$G/H$  = set of all cosets of  $H$  in  $G$

# WHY COSET SPACES ?

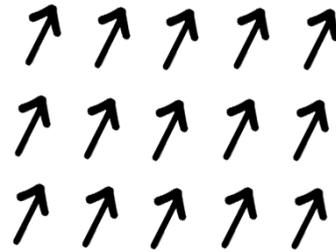
Reminder : NG boson = coordinate-dependent broken symmetry transformation

vacuum



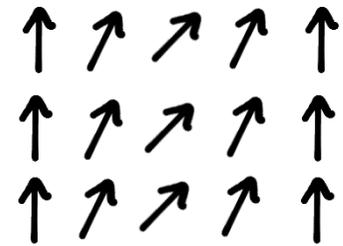
$|0\rangle$

rotated vacuum



$g|0\rangle$

excitation



$g(x)|0\rangle$

defined by  $g$  modulo right multiplication with  $h \in H$

Expect the low-energy EFT for NG bosons to be one of  $G/H$ -valued NG fields!

# ACTION OF $G$ ON $G/H$

- Step 1 : choose for each coset in  $G/H$  a unique **representative**  $U(\pi)$

local coordinates on  $G/H$   
(NG fields)

Technical assumptions :

↳  $U(\pi)$  is a smooth function of  $\pi^a$  near the origin  $\pi^a = 0$

↳ The trivial coset  $H$  is represented by  $U(0) = e$

↳ The adjoint action of  $H$ ,  $U(\pi) \rightarrow h U(\pi) h^{-1}$ , defines a linear transformation of  $\pi^a$

- Step 2 : for any  $g \in G$  and  $U(\pi)$ , find the unique decomposition  
 $g U(\pi) = U(\pi'(\pi, g)) h(\pi, g)$ , where  $g \in G$ ,  $h(\pi, g) \in H$

- Step 3 : **nonlinear realization of  $G$  on the coset space**

$$U(\pi) \xrightarrow{g} U(\pi'(\pi, g)) = g U(\pi) h(\pi, g)^{-1}$$

# MAURER-CARTAN FORM

It turns out (see next lecture) that we don't need  $U(\pi)$  explicitly to construct effective actions: remember that NG fields always enter with derivatives!

$$\omega(\pi) = -i \underbrace{U(\pi)^{-1} dU(\pi)}$$

↑ takes values in the Lie algebra of  $G$

How to compute this: choose exponential parameterization  $U(\pi) = e^{i\pi^a Q_a}$

- $e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$
- $e^{-A} d e^A = \int_0^1 dt e^{-tA} (dA) e^{+tA}$

# MAURER-CARTAN FORM

$$\omega(\pi) = -i U(\pi)^{-1} dU(\pi)$$

Why is this useful?

Split  $\omega(\pi)$  into "unbroken" and "broken" components:

$$\omega = \omega_{\parallel} + \omega_{\perp}$$

$\omega^A Q_A$        $\omega^{\alpha} Q_{\alpha}$        $\omega^a Q_a$

$$\omega^A(\pi) = \omega_a^A(\pi) d\pi^a$$

Action of  $G$  on the MC form:

$$\omega_{\parallel}(\pi) \xrightarrow{g} \omega_{\parallel}(\pi') = h(\pi, g) \omega_{\parallel}(\pi) h(\pi, g)^{-1} - i h(\pi, g) dh(\pi, g)^{-1} \dots \text{gauge field of } H$$

$$\omega_{\perp}(\pi) \xrightarrow{g} \omega_{\perp}(\pi') = h(\pi, g) \omega_{\perp}(\pi) h(\pi, g)^{-1} \dots \text{covariant derivative of NG fields}$$

# SYMMETRY TRANSFORMATION OF NG FIELDS

$$U(\pi)^{-1} U(\pi') = U(\pi)^{-1} g U(\pi) h(\pi, g)^{-1} \left\{ \begin{array}{l} \text{apply this to an infinitesimal } G\text{-transformation} \\ g \approx e + i\epsilon^A Q_A, \quad h(\pi, g) \approx e + i\epsilon^A K_A^\alpha(\pi) Q_\alpha \\ \delta\pi^a(\pi, g) \approx \epsilon^A \xi_A^a(\pi) \end{array} \right.$$

Killing vector field on  $G/H$

Consistency conditions :

$$\begin{aligned} V_A^\alpha(\pi) &= \xi_A^a(\pi) \omega_a^\alpha(\pi) + K_A^\alpha(\pi) \\ V_A^a(\pi) &= \xi_A^b(\pi) \omega_b^a(\pi) \end{aligned}$$

defined by

$$U(\pi)^{-1} Q_A U(\pi) = V_A^B(\pi) Q_B$$

MC form  $\omega_\perp$  defines a basis on the cotangent space to  $G/H$ , dual to the set of (broken) Killing vectors

# SYMMETRY TRANSFORMATION OF NG FIELDS

Proof : ●  $U(\pi)^{-1}U(\pi') - e = U(\pi)^{-1}[U(\pi') - U(\pi)] = i\omega_a^A(\pi)Q_A \delta\pi^a$   
 $= \underline{i\epsilon^A \xi_A^a(\pi) \omega_a^B(\pi) Q_B}$

●  $U(\pi)^{-1}gU(\pi)h(\pi,g)^{-1} = [e + i\epsilon^A \nu_A^B(\pi)Q_B][e - i\epsilon^A k_A^\alpha(\pi)Q_\alpha]$   
 $\approx \underline{e + i\epsilon^A \nu_A^B(\pi)Q_B - i\epsilon^A k_A^\alpha(\pi)Q_\alpha}$

Altogether :  $\nu_A^B(\pi)Q_B = \xi_A^a(\pi)\omega_a^B(\pi)Q_B + k_A^\alpha(\pi)Q_\alpha$

unbroken  
B

broken B

$\nu_A^\alpha(\pi) = \xi_A^a(\pi)\omega_a^\alpha(\pi) + k_A^\alpha(\pi)$

$\nu_A^a(\pi) = \xi_A^b(\pi)\omega_b^a(\pi)$

# SYMMETRIC COSET SPACES

$G/H$  is symmetric iff the Lie algebra of  $G$  admits an automorphism  $P$  such that:

$$P([Q_A, Q_B]) = [P(Q_A), P(Q_B)] \quad , \quad \underbrace{P(Q_\alpha) = Q_\alpha, \quad P(Q_a) = -Q_a}$$

guarantee commutation

relations of the type

$$[Q_\alpha, Q_\beta] = i f_{\alpha\beta}^\gamma Q_\gamma$$

$$[Q_\alpha, Q_b] = i f_{\alpha b}^c Q_c$$

$$[Q_a, Q_b] = i f_{ab}^\gamma Q_\gamma$$

- Examples :
- $ISO(n)/SO(n) \cong \mathbb{R}^n$  ...  $P$  is spatial inversion in  $\mathbb{R}^n$
  - $SO(n+1)/SO(n) \cong S^n$  ...  $P$  is inversion of one of the coordinate axes
  - $G_L \times G_R / G_V$  ... chiral group,  $P$  exchanges  $L \leftrightarrow R$

# SYMMETRIC COSET SPACES

Choose the coset representative so that  $P(U(\pi)) = U(\pi)^{-1}$ , e.g.  $U(\pi) = e^{i\pi^a Q_a}$ .

$$U(\pi') = g U(\pi) h(\pi, g)^{-1}$$

↓ apply P & invert

$$U(\pi') = h(\pi, g) U(\pi) P(g)^{-1}$$

define new variable  $\Sigma(\pi) = U(\pi)^2$   
 $\Sigma(\pi) \xrightarrow{g} \Sigma(\pi'(\pi, g)) = g \Sigma(\pi) P(g)^{-1}$   
 $\Sigma(\pi)$  transforms linearly under the whole G!

How about the MC form?

Use P to project out the (un)broken components:  $\omega_{||} = \frac{1}{2} [\omega + P(\omega)]$

$$\omega_{\perp} = \frac{1}{2} [\omega - P(\omega)]$$

Direct relation between  $\Sigma$  &  $\omega_{\perp}$ :  $\omega_{\perp} = -\frac{i}{2} \bar{U}^{\dagger} d\Sigma \bar{U}^{\dagger} = +\frac{i}{2} U d\Sigma^{\dagger} U$

# SYMMETRIC COSET SPACES

Proof :  $\omega = -i\bar{u}'du \rightarrow \omega_{\perp} = -\frac{i}{2}(\bar{u}'du - u d\bar{u}') = -\frac{i}{2}(\bar{u}'du + du\bar{u}')$

$$= -\frac{i}{2}\bar{u}'(duu + udu)\bar{u}'$$
$$= \underline{\underline{-\frac{i}{2}\bar{u}'d\Sigma\bar{u}'}}$$

Alternative expression :  $d\Sigma = -\Sigma d\bar{\Sigma}'\Sigma$

↓

$$\omega_{\perp} = +\frac{i}{2}\bar{u}'\Sigma d\bar{\Sigma}'\Sigma\bar{u}' = \underline{\underline{+\frac{i}{2}u d\bar{\Sigma}'u}}$$

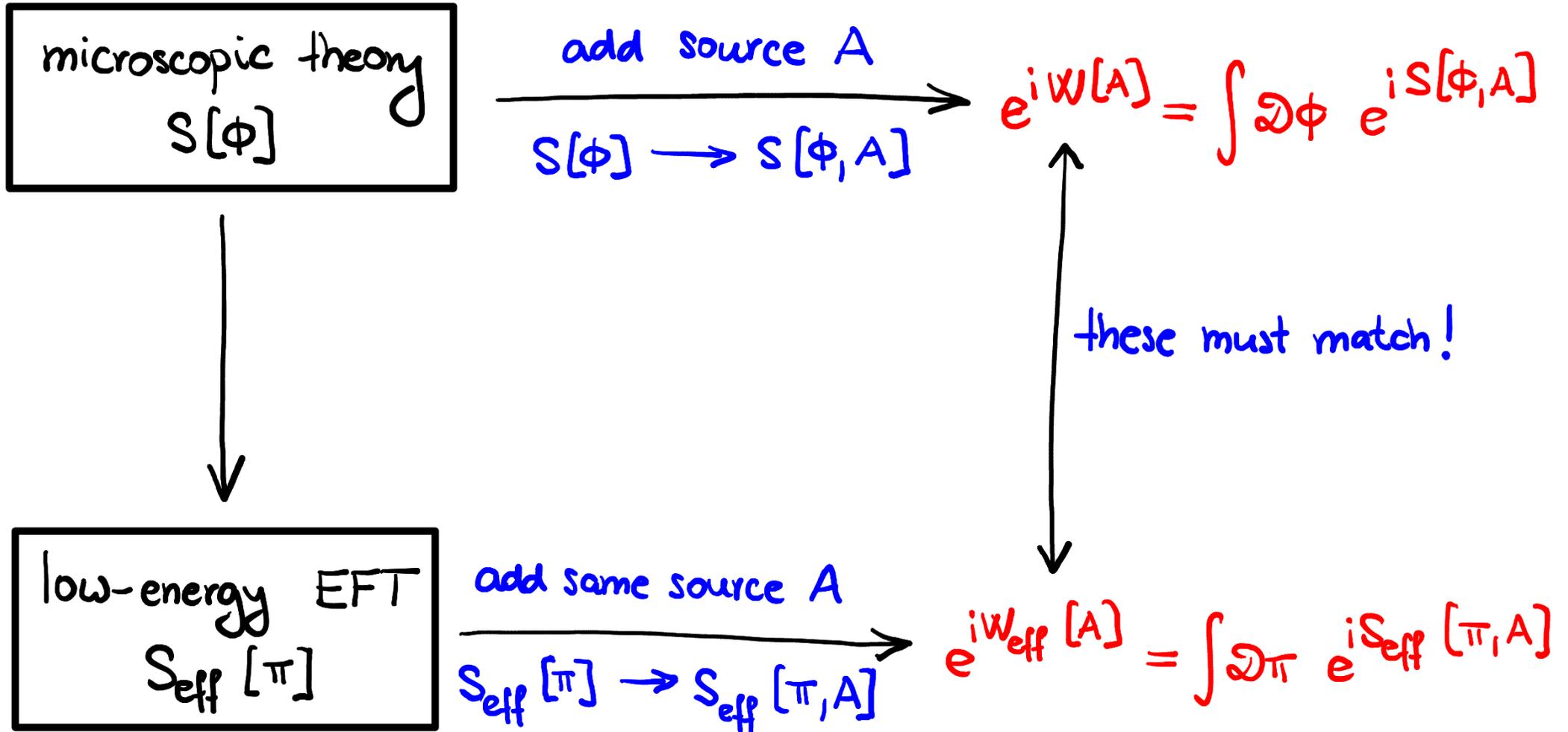
LOW-ENERGY EFFECTIVE THEORY

FOR SSB

# GOAL

- Want to construct a low-energy EFT for  $G/H$ -valued NG fields.
- We know the action of  $G$  on the NG fields  $\pi^a$ .
- What is the most general  $G$ -invariant action?
- Only interested in low energies  $\longrightarrow$  derivative expansion.

# EFFECTIVE FIELD THEORY



# WHAT BACKGROUND?

global continuous symmetry



conserved (Noether) current



background gauge invariance  
of  $S[\Phi, A]$ ,  $S_{\text{eff}}[\pi, A]$

couple this to classical  
gauge fields

encodes all Ward identities  
including possible anomalies

# BACKGROUND GAUGE INVARIANCE

- Gauge fields :  $A_\mu = A_\mu^A Q_A$
- Gauge transformation :  $A_\mu \xrightarrow{g} T_g A_\mu = g A_\mu g^{-1} + i g \partial_\mu g^{-1}$   
 $U(\pi) \xrightarrow{g} U(\pi'(g)) = g U(\pi) h(\pi, g)^{-1}$
- Gauge invariance trick :  $S_{\text{eff}}[\pi, A] \xrightarrow{g=U(\pi)^{-1}} S_{\text{eff}}[0, T_{U(\pi)^{-1}} A]$

NG & gauge fields only enter the action

together through  $T_{U(\pi)^{-1}} A_\mu = i U(\pi)^{-1} (\partial_\mu - i A_\mu) U(\pi) !$



# GAUGED MAURER-CARTAN FORM

$$\Omega(\pi, A) = -T_{U(\pi)}^{-1} A = -i U(\pi)^{-1} (d - iA) U(\pi)$$

Same transformation properties as the ungauged MC form  $\omega(\pi)$ :

$$\Omega_{\parallel}(\pi, A) \xrightarrow{g} \Omega_{\parallel}(\pi', A') = h(\pi, g) \Omega_{\parallel}(\pi, A) h(\pi, g)^{-1} - i h(\pi, g) dh(\pi, g)^{-1}$$

$$\Omega_{\perp}(\pi, A) \xrightarrow{g} \Omega_{\perp}(\pi', A') = h(\pi, g) \Omega_{\perp}(\pi, A) h(\pi, g)^{-1}$$

Now look for the effective action as  $S_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}] = \int dx \mathcal{L}_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}]$

minus gauge field of H

covariant adjoint scalar field of H

# ACTION RECONSTRUCTION

The effective Lagrangian can always be written in the form

$$\mathcal{L}_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}] = \mathcal{L}_{\text{inv}}[\Omega_{\parallel}, \Omega_{\perp}] + \mathcal{L}_{\text{CS}}[\Omega_{\parallel}]$$

strictly  
G-invariant



Chem-Simons Lagrangian  
quasi-invariant  
depends on  $\Omega_{\parallel}$  only



# ACTION RECONSTRUCTION

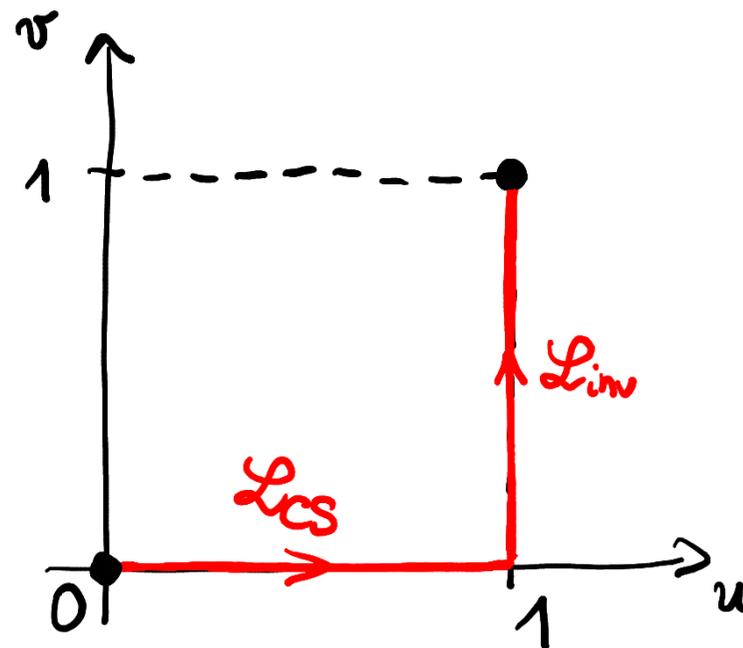
Sketch of proof: define  $J_\alpha^\mu = \frac{\delta S_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}]}{\delta \Omega_{\mu}^\alpha}$ ,  $\Sigma_a^\mu = \frac{\delta S_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}]}{\delta \Omega_{\mu}^a}$

both of these transform covariantly under H!

$$\frac{\partial S_{\text{eff}}[u\Omega_{\parallel}, v\Omega_{\perp}]}{\partial u} = \int d^D x \Omega_{\mu}^\alpha J_\alpha^\mu [u\Omega_{\parallel}, v\Omega_{\perp}]$$

$$\frac{\partial S_{\text{eff}}[u\Omega_{\parallel}, v\Omega_{\perp}]}{\partial v} = \int d^D x \Omega_{\mu}^a \Sigma_a^\mu [u\Omega_{\parallel}, v\Omega_{\perp}]$$

$$S_{\text{eff}}[\Omega_{\parallel}, \Omega_{\perp}] = \int d^D x \int_0^1 du \Omega_{\mu}^\alpha J_\alpha^\mu [u\Omega_{\parallel}, 0] + \int d^D x \int_0^1 dv \Omega_{\mu}^a \Sigma_a^\mu [\Omega_{\parallel}, v\Omega_{\perp}]$$



# INVARIANT LAGRANGIANS

Basic building blocks :

- $\Omega_\mu^a$  ... covariant scalar under H
- $F_{\mu\nu}^\alpha = -\partial_\mu \Omega_\nu^\alpha + \partial_\nu \Omega_\mu^\alpha + f_{\beta\delta}^\alpha \Omega_\mu^\beta \Omega_\nu^\delta$  ... field strength of  $-\Omega_\parallel$
- their covariant derivatives built using  $-\Omega_\parallel$

$$D_\mu \Omega_\nu^a = \partial_\mu \Omega_\nu^a + i [\Omega_{\parallel\mu}, \Omega_{\perp\nu}]^a = \partial_\mu \Omega_\nu^a - f_{ab}^a \Omega_\mu^a \Omega_\nu^b$$

$$D_\mu F_{\nu\lambda}^\alpha = \partial_\mu F_{\nu\lambda}^\alpha + i [\Omega_{\parallel\mu}, F_{\nu\lambda}]^\alpha = \partial_\mu F_{\nu\lambda}^\alpha - f_{\beta\delta}^\alpha \Omega_\mu^\beta F_{\nu\lambda}^\delta$$

## EXAMPLE - TRIVIAL

Lorentz-invariant theories with  $G/H \simeq U(1)/\{\pm 1\}$

- no  $H \rightarrow$  no  $\Omega_{||}$
- $U(1) = e^{i\pi} \rightarrow \Omega_{\perp}(\pi, A) = d\pi - A$
- no  $\Omega_{||} \rightarrow D_{\mu} \Omega_{\perp \nu} = \partial_{\mu} \Omega_{\perp \nu} = \partial_{\mu} (\partial_{\nu} \pi - A_{\nu})$

$\mathcal{L}_{\text{eff}}$  is an arbitrary Lorentz-invariant function of  $\partial_{\mu} \pi - A_{\mu}$  and its derivatives :

$$\mathcal{L}_{\text{eff}}[\pi, A] = \frac{1}{2} (\partial_{\mu} \pi - A_{\mu})^2 + c_4 [(\partial_{\mu} \pi - A_{\mu})^2]^2 + \dots$$

# EXAMPLE - MORE SOPHISTICATED

Lorentz-invariant theories with arbitrary G/H

Lowest-order Lorentz-invariant Lagrangian built out of  $\Omega_{\perp}$ :

$$\mathcal{L}_{\text{eff}}[\pi, A] = \frac{1}{2} g_{ab} \Omega_{\mu}^a \Omega^{b\mu} + \dots$$



H-invariant coupling

$$f_{ca}^c g_{cb} + f_{ab}^c g_{ac} = 0$$

## EXAMPLE - MORE SOPHISTICATED

$$\mathcal{L}_{\text{eff}}[\pi, A] = \frac{1}{2} g_{ab} \Omega_{\mu}^a \Omega^{b\mu} + \dots$$

What is this in terms of the  $\pi^a, A_{\mu}^A$  fields?

$$\begin{aligned} \Omega_{\mu}^a(\pi, A) &= -i [U(\pi)^{-1} \partial_{\mu} U(\pi)]^a - [U(\pi)^{-1} A_{\mu} U(\pi)]^a = \omega_b^a(\pi) \partial_{\mu} \pi^b - v_A^a(\pi) A_{\mu}^A \\ &= \omega_b^a(\pi) \left[ \partial_{\mu} \pi^b - A_{\mu}^A \xi_A^b(\pi) \right] = \omega_b^a(\pi) D_{\mu} \pi^b \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\text{eff}}[\pi, A] &= \frac{1}{2} g_{ca} \omega_a^c(\pi) \omega_b^d(\pi) D_{\mu} \pi^a D^{\mu} \pi^b + \dots \\ &= \frac{1}{2} g_{ab}(\pi) D_{\mu} \pi^a D^{\mu} \pi^b + \dots \end{aligned}$$

$G$ -invariant metric on  $G/H$

# EXAMPLE - MORE SOPHISTICATED

$$\mathcal{L}_{\text{eff}}[\pi, A] = \frac{1}{2} g_{ab}(\pi) D_\mu \pi^a D^\mu \pi^b + \dots$$

The geometric interpretation may help us find  $\mathcal{L}_{\text{eff}}$  explicitly in concrete cases:

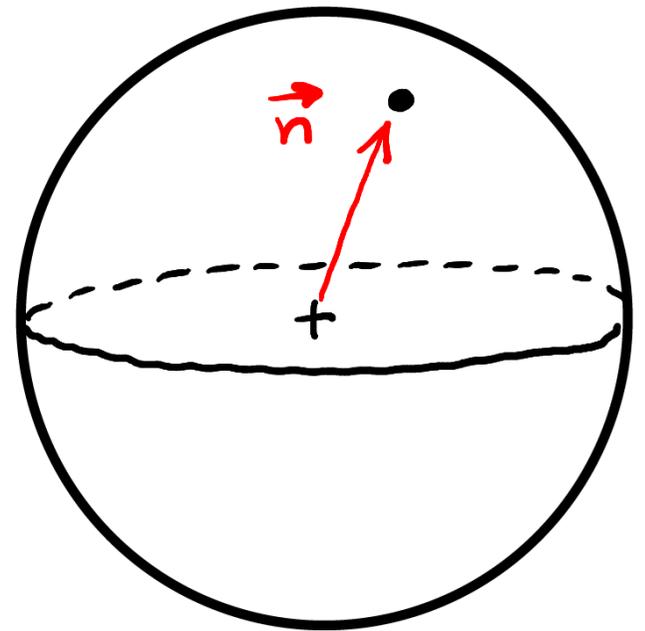
$$G/H \simeq SO(3)/SO(2) \simeq S^2$$

Unique  $SO(3)$ -invariant metric on the sphere:

$$g(\pi) = \delta_{ij} dn^i(\pi) \otimes dn^j(\pi)$$

scale  
of SSB

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} v^2 D_\mu \vec{n} \cdot D^\mu \vec{n} + \dots$$



# CHERN-SIMONS LAGRANGIANS

$\mathcal{L}_{CS}[\Omega_{11}]$  is only quasi-invariant, but  $J_{\alpha}^{\mu}[\Omega_{11}] = \frac{\delta S_{CS}[\Omega_{11}]}{\delta \Omega_{\mu}^{\alpha}}$  is H-covariant!

Steps to reconstruct  $\mathcal{L}_{CS}[\Omega_{11}]$ :

● construct  $J_{\alpha}^{\mu}[\Omega_{11}]$  using  $F_{\mu\nu}^{\alpha}$  and its covariant derivatives

● impose H-covariance and  $D_{\mu} J_{\alpha}^{\mu} = \partial_{\mu} J_{\alpha}^{\mu} - f_{\alpha\beta}^{\gamma} \Omega_{\mu}^{\beta} J_{\gamma}^{\mu} = 0$

required for gauge invariance of  $S_{CS}[\Omega_{11}]$

● reconstruct the Lagrangian from

$$\mathcal{L}_{CS}[\Omega_{11}] = \int_0^1 du \Omega_{\mu}^{\alpha} J_{\alpha}^{\mu}[u\Omega_{11}]$$

# CHERN-SIMONS LAGRANGIANS

Lowest-order solution : constant current  $J_\alpha^\mu[\Omega_{11}] = -\delta_0^\mu e_\alpha + \dots$

$$\mathcal{L}_{CS}[\pi_1 A] = -e_\alpha \Omega_0^\alpha + \dots$$

H-invariant coupling  
 $f_{\alpha\beta}^\gamma e_\gamma = 0$

- Some properties :
- $e_\alpha = \lim_{V \rightarrow \infty} \frac{\langle Q_\alpha \rangle}{V} \dots$  unbroken charge density in the ground state
  - possible  $e_\alpha$  are in one-to-one correspondence with  $U(1)$  factors of  $H$  (for compact  $H$ )

# NONRELATIVISTIC EFFECTIVE ACTIONS

- Assumptions :
- unbroken symmetry under spacetime translations and **spatial rotations**,  $d \geq 3$  spatial dimensions
  - internal symmetry under **G that can be gauged**

Derivative expansion :

$$\mathcal{L}_{\text{eff}} = \sum_{s,t \geq 0} \mathcal{L}_{\text{eff}}^{(s,t)}$$

↖ # spatial derivatives  
↙ # temporal derivatives

$$\mathcal{L}_{\text{eff}}^{(0,1)} = -e_A \omega_a^A(\pi) \dot{\pi}^a + e_A v_B^A(\pi) A_0^B$$

$$\mathcal{L}_{\text{eff}}^{(2,0)} = -\frac{1}{2} g_{cd} \omega_a^c(\pi) \omega_b^d(\pi) \vec{D}\pi^a \cdot \vec{D}\pi^b$$

$$\mathcal{L}_{\text{eff}}^{(0,2)} = \frac{1}{2} \bar{g}_{cd} \omega_a^c(\pi) \omega_b^d(\pi) D_0 \pi^a D_0 \pi^b$$

$\left\{ \begin{array}{l} e_a \dots \text{strictly invariant} \\ e_a \dots \text{quasi-invariant} \end{array} \right.$

# EXAMPLE : FERROMAGNETS

● For simplicity, drop the gauge fields  $A_{\mu}^A$  ( $A_0^A = \text{external magnetic field!}$ ).

● Discard  $\mathcal{L}_{\text{eff}}^{(0,2)}$  ...  $\mathcal{L}_{\text{eff}}^{(0,1)}$  will dominate at low energies.

● Already know this :  $\mathcal{L}_{\text{eff}}^{(2,0)} = -\frac{1}{2} v^2 \delta^{ij} \partial_i \vec{n} \cdot \partial_j \vec{n}$ .  
↑ spin stiffness

●  $\mathcal{L}_{\text{eff}}^{(0,1)} = -e_3 \omega_a^3(\pi) \dot{\pi}^a = 2i m \text{Tr} \left[ \frac{\tau_3}{2} U(\pi)^{-1} \partial_0 U(\pi) \right]$  we  $U(\pi) = e^{\frac{i}{2} \pi^a \tau_a$   
↑ magnetization density  $\underline{m}$

$= 2i m \int_0^1 d\lambda \text{Tr} \left[ \frac{\tau_3}{2} U(\lambda\pi)^{-1} \left( \frac{i}{2} \dot{\pi}^a \tau_a \right) U(\lambda\pi) \right] = -\frac{m}{2} \dot{\pi}^a \int_0^1 d\lambda \text{Tr} \left[ \tau_a U(\lambda\pi) \tau_3 U(\lambda\pi)^{-1} \right]$   
↑  $N(\lambda\pi) = \vec{n}(\lambda\pi) \cdot \vec{e}$

$= -m \int_0^1 d\lambda \dot{\pi}^a n_a(\lambda\pi) \simeq m \int_0^1 d\lambda \pi^a \dot{n}_a(\lambda\pi)$  ↑ modulo total time derivative

## EXAMPLE : FERROMAGNETS

$$\mathcal{L}_{\text{eff}}^{(0,1)} \stackrel{12}{=} m \int_0^1 d\lambda \underbrace{\pi^2 \dot{n}_a(\lambda\pi)}$$

$$\vec{\pi} \cdot \dot{\vec{n}} = (\vec{n} \times \vec{\pi}) \cdot (\dot{\vec{n}} \times \dot{\vec{n}})$$

$$\begin{aligned} \partial_\lambda N(\lambda\pi) &= i \left[ \vec{\pi} \cdot \frac{\dot{\vec{n}}}{2}, N(\lambda\pi) \right] \\ &= -(\vec{\pi} \times \dot{\vec{n}}) \cdot \vec{e} \end{aligned}$$

$$\partial_\lambda \vec{n}(\lambda\pi) = \dot{\vec{n}}(\lambda\pi) \times \vec{\pi}$$

Final result for the effective Lagrangian :

$$\mathcal{L}_{\text{eff}} = m \int_0^1 d\lambda \partial_\lambda \vec{n}(\lambda\pi) \cdot [\vec{n}(\lambda\pi) \times \dot{\vec{n}}(\lambda\pi)] - \frac{1}{2} \delta_{ij} \partial_i \vec{n} \cdot \partial_j \vec{n} + \dots$$

# CONCLUDING REMARKS

- In  $d=2$  spatial dimensions, another term is allowed in  $\mathcal{L}_{\text{eff}}^{(2,0)}$ :  
$$\mathcal{L}_{\text{eff}}^{(2,0)} \ni -\frac{1}{2} b_{cd} \omega_a^c(\pi) \omega_b^d(\pi) \epsilon^{ij} D_i \pi^a D_j \pi^b, \quad b_{ab} = -b_{ba}$$
- Chern-Simons Lagrangians exist only at odd orders in derivatives; classified explicitly up to order 3.
- The assumption that  $G$  can be gauged is not innocuous. While  $\mathcal{L}_{\text{lin}}$  can always be gauged, gauging of  $\mathcal{L}_{\text{CS}}$  may be obstructed.  
Example :  $G/H \simeq U(1) \times U(1) / \mathbb{Z}_2 \simeq T^2$ ,  $\mathcal{L}_{\text{eff}}^{(0,1)} \sim \epsilon_{ab} \pi^a \partial_0 \pi^b$

Ask for more details!

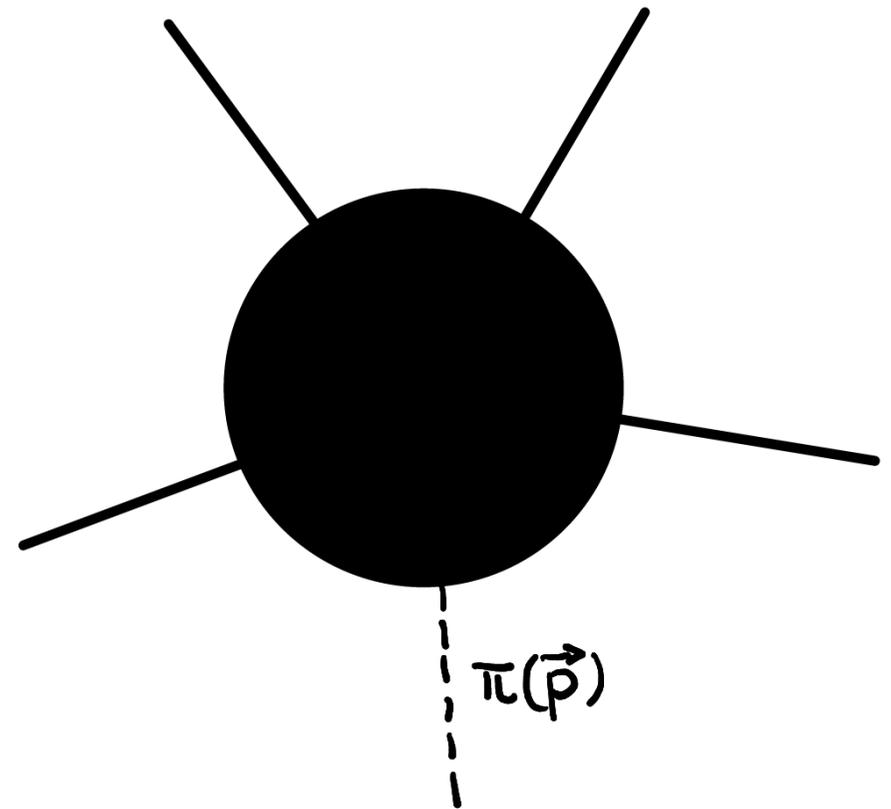
SELECTED FURTHER ASPECTS  
OF SSB

ADLER ZERO

# NAIVE ARGUMENT

NG bosons interact weakly because they are derivatively coupled.

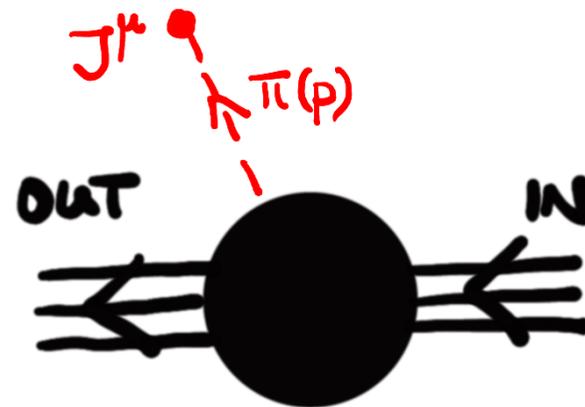
- For broken  $U(1)$ , every NG field carries a derivative ✓
- For non-Abelian symmetry, interactions are dominated by operators with an arbitrary number of NG fields and just two derivatives!



vanishes for  $\vec{p} \rightarrow \vec{0}$  ?

# LESS NAIVE ARGUMENT

- Assumptions :
- for simplicity, stay relativistic
  - conserved current  $J^\mu$  that couples to the NG state



$$\langle \text{out} | J^\mu(0) | \text{in} \rangle = \underbrace{\langle 0 | J^\mu(0) | \pi(\vec{p}) \rangle}_{i p^\mu F} \frac{i}{p^2} \langle \text{out} + \pi(\vec{p}) | \text{in} \rangle + R^\mu(p)$$

remainder regular in the on-shell limit  $p^2 \rightarrow 0$

Current conservation  $\longrightarrow$   $\langle \text{out} + \pi(\vec{p}) | \text{in} \rangle = \frac{1}{F} p_\mu R^\mu(p)$

this is still valid for any on-shell  $p^\mu$

# LESS NAIVE ARGUMENT

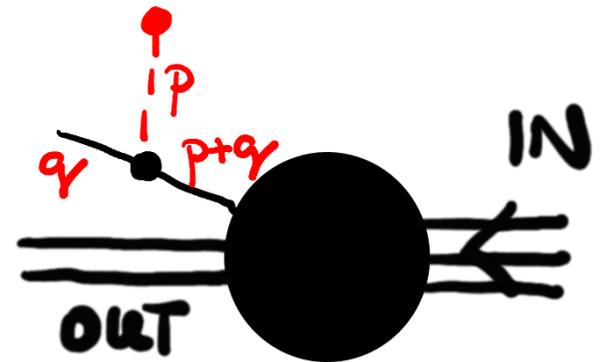
**Adler zero** :  $\lim_{\vec{p} \rightarrow \vec{0}} \langle \text{out} + \pi(\vec{p}) | \text{in} \rangle = \lim_{\vec{p} \rightarrow \vec{0}} \frac{1}{F} p_\mu R^\mu(\vec{p}) = 0$

**Watch out!**

- We assumed that  $R^\mu(\vec{p})$  is regular as  $p^2 \rightarrow 0$
- now we also need that  $R^\mu(\vec{p})$  is regular as  $\vec{p} \rightarrow \vec{0}$

**Possible exception** : if  $J^\mu$  contains terms quadratic in NG fields, the state  $\pi(\vec{p})$  can be inserted in the IN or OUT state

$\langle \text{out} | J^\mu(0) | \text{in} \rangle$  now contains  $\frac{i}{p^2} \frac{i}{(p+q)^2}$ , which carries additional singularity as  $p^2 \rightarrow 0$  (not  $p^2 \rightarrow 0$ )



# THE SAD TRUTH

The occasional violation of Adler zero is not a bug but a feature.

Recall relativistic coset theories:  $\mathcal{L}_{\text{eff}} = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b$

On-shell amplitudes must be invariant under field redefinitions  $\rightarrow$  can only depend on  $g_{ab}(\pi)$  through the Riemann tensor and its covariant derivatives.

$$\mathcal{A}_n \sim \nabla_{a_5} \dots \nabla_{a_n} R_{a_1 a_2 a_3 a_4}$$

Adler zero iff all covariant derivatives of  $R_{abcd}$  vanish

Adler zero iff  $G/H$  is symmetric

# TOPOLOGICAL ASPECTS OF SSB

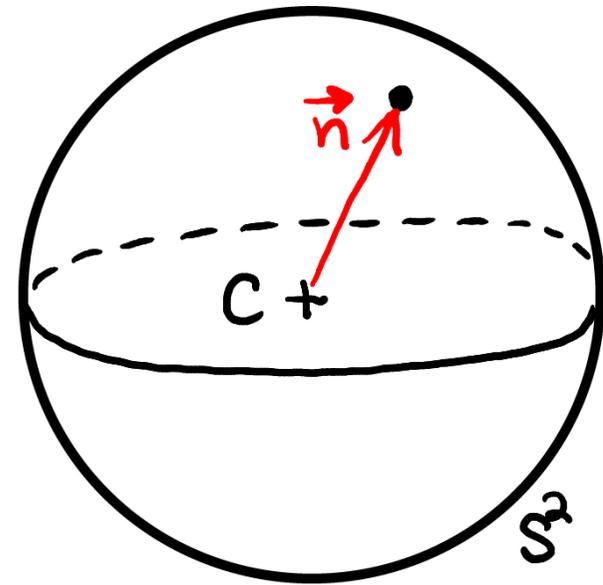
# FERROMAGNETS REVISITED

$$\mathcal{L}_{\text{eff}} = m \int d\lambda \partial_\lambda \vec{n}(\lambda\pi) \cdot [\vec{n}(\lambda\pi) \times \dot{\vec{n}}(\lambda\pi)] - \frac{v^2}{2} \delta^{ij} \partial_i \vec{n} \cdot \partial_j \vec{n}$$

$$\equiv c_a(\pi) \dot{\pi}^a \quad \text{where} \quad c_a(\pi) = -e_A \omega_a^A(\pi)$$

Quasi-invariance of  $\mathcal{L}_{\text{eff}}^{(a_i)}$   $\longleftrightarrow$  action of  $G$  changes

$c_a(\pi)$  by a gauge transformation  $c_a(\pi) \rightarrow c_a(\pi) + \partial_a \chi(\pi)$ .



- $\partial_a c_b(\pi) - \partial_b c_a(\pi)$  is gauge-invariant (i.e.  $G$ -invariant)
- $\partial_a c_b - \partial_b c_a \longleftrightarrow$  magnetic field of a monopole at the center  $C$
- $c_a \longleftrightarrow$  vector potential of the monopole's magnetic field

# FERROMAGNETS REVISITED

$$\mathcal{L}_{\text{eff}} = m \int d\lambda \partial_\lambda \vec{n}(\lambda\pi) \cdot [\vec{n}(\lambda\pi) \times \dot{\vec{n}}(\lambda\pi)] - \frac{v^2}{2} \delta_{ij} \partial_i \vec{n} \cdot \partial_j \vec{n}$$

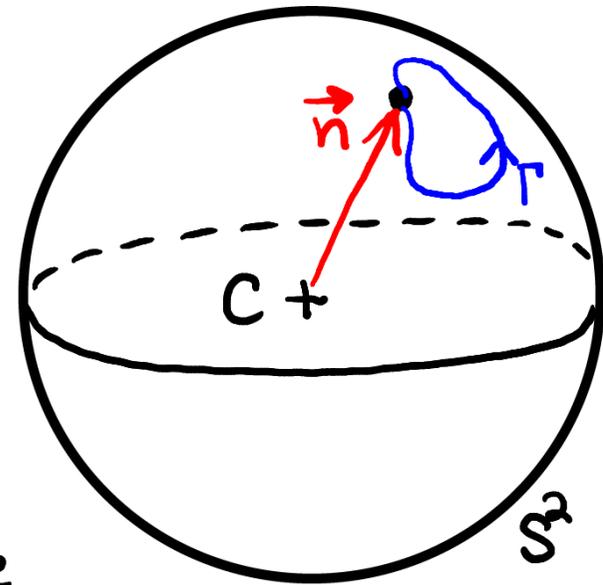
$$\equiv c_a(\pi) \dot{\pi}^a \quad \text{where} \quad c_a(\pi) = -e_A \omega_a^A(\pi)$$

Consequences :

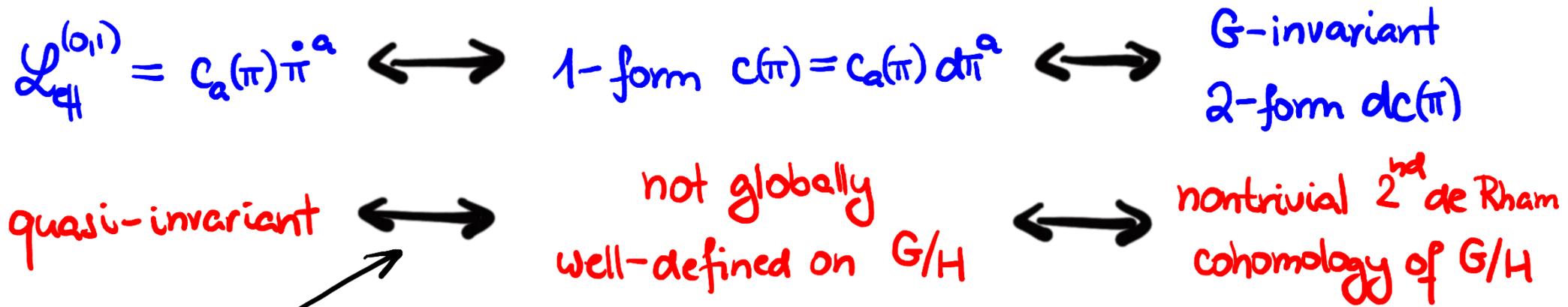
- $c_a(\pi)$  and therefore  $\mathcal{L}_{\text{eff}}^{(0,1)}$  is not well-defined globally on the coset space

- in finite volume,  $\underline{m}$  is quantized  $\leftrightarrow$  quantization of spin
- $c_a(\pi)$  generates a **Berry phase** when the ground state is adiabatically dragged by an external magnetic field

Berry phase  $\sim$  magnetic flux through  $\Gamma$



# GENERAL MORALS



This equivalence only holds if:

- $G$  is compact & connected
- $H$  is closed & connected

## Consequences :

### classical physics

- EM remains OK
- some observables such as EM tensor ill-defined

### quantum physics

- only  $e^{iS}$  has to be well-defined
- quantization of the coupling in  $c(\pi)$

# FURTHER TOPOLOGICAL ASPECTS

- **Wess-Zumino-Witten terms** : in  $D$  dimensions, classified by  $(D+1)$ -th de Rham cohomology of  $G/H$  ; even-dimensional terms closely related to chiral anomalies.
- **Emergent generalized topological conservation laws** : classified by de Rham cohomology of  $G/H$ .
- **Topological defects** : classified by the homotopy groups of  $G/H$ .

NO-GO THEOREMS

FOR SSB

# COLEMAN THEOREM

- Original version (1973) :
- $D=1+1$  spacetime dimensions
  - Lorentz-invariant systems at  $T=0$

Continuous global symmetry cannot be spontaneously broken (no NG bosons).

Reason: IR divergence of two-point correlation function of NG bosons.

- Recent generalization :
- type-A NG bosons,  $E \sim |\vec{p}|^n$ : only in  $D \geq n+2$
  - type-B NG bosons,  $E \sim |\vec{p}|^{2n}$ : in any  $D \geq 2$

# HOHENBERG - MERMIN - WAGNER THEOREM

Original version (1966-1967): ●  $D=1+1$  or  $2+1$  spacetime dimensions.

● Superfluids (H) and (anti)ferromagnets (MW) at  $T>0$ .

Continuous global symmetry cannot be spontaneously broken (no NG bosons).

Reason: IR divergence of two-point correlation function of NG bosons.

Recent generalization: ● type-A NG bosons,  $E \sim |\vec{p}|^n$ : only in  $D \geq 2n+2$

● type-B NG bosons,  $E \sim |\vec{p}|^{2n}$ : only in  $D \geq 2n+2$

# LANDAU-PEIERLS INSTABILITY

- Assumptions :
- inhomogeneous order parameter with one-dimensional modulation
  - continuous spatial rotational invariance,  $D = 2+1$  or  $3+1$

Transverse IR fluctuations destroy the order parameter at any  $T > 0$ .

- Technical reason :
- again IR divergence of two-point correlation function.
  - anisotropic dispersion relation  $E^2 \sim \# \vec{p}_{\parallel}^2 + \# \vec{p}_{\perp}^4$

# Vafa-Witten Theorem

- Assumptions :
- Lorentz invariance,  $T=0$
  - vector-like gauge theory with vanishing  $\theta$ -angle

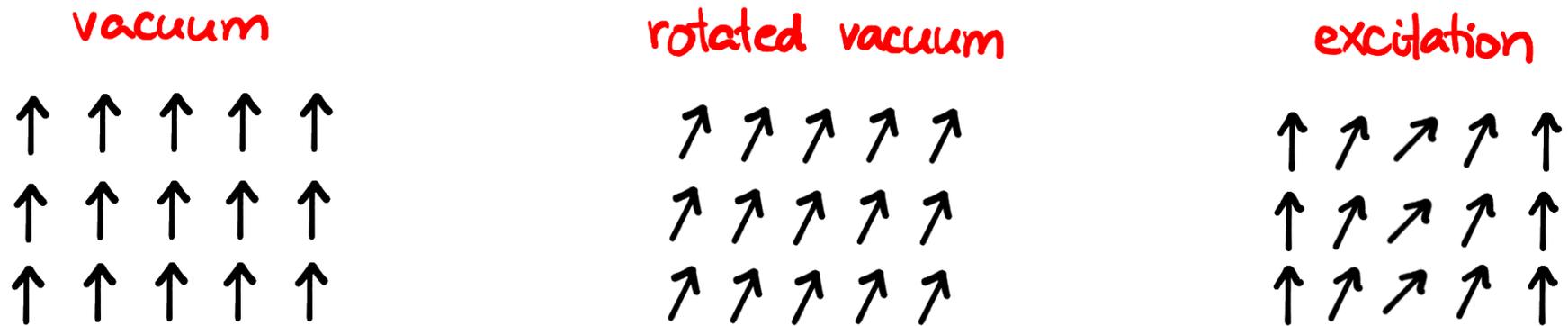
Vector-like continuous global symmetry (such as baryon number or isospin in QCD) cannot be spontaneously broken.

- Technical ingredients :
- prove exponential decay of two-point function
  - needs positivity of Dirac determinant  
(absence of sign problem)

SPONTANEOUS BREAKING  
OF SPACETIME SYMMETRY

# REDUNDANCY OF ORDER PARAMETER FLUCTUATIONS

Recall that NG bosons are local fluctuations of the order parameter!



The local forms of some coordinate-dependent symmetry transformations may coincide  $\longrightarrow$  corresponding fluctuations are redundant.

# REDUNDANCY OF ORDER PARAMETER FLUCTUATIONS

## Free massless scalar

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2$$

- global symmetry:

$$\phi(x) \rightarrow \phi(x) + a + b_\mu x^\mu$$

- local versions coincide:

$$b_\mu(x) \leftrightarrow a(x) = b_\mu(x) x^\mu$$

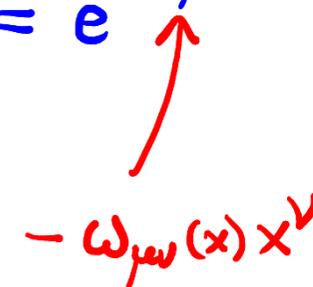
## Relativistic scalars in general

- relation between Noether charges:

$$J^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$$

- redundancy of local transformations:

$$e^{-\frac{i}{2} \omega_{\mu\nu}(x) J^{\mu\nu}} = e^{-i \epsilon_\mu(x) P^\mu}$$



$-\omega_{\mu\nu}(x) x^\nu$

# COUNTING OF NG FIELDS

- symmetry transformation :  $\delta\phi^i(x) = \epsilon^A F_A^i[\phi, x]$
- redundancy of order parameter fluctuations :  $\epsilon^A(x) F_A^i[\langle\phi\rangle, x] = 0$

# independent NG fields =

$\dim G/H - \#$  independent solutions to



# EXAMPLES

system	broken symmetries	# broken generators	# NG modes
massless scalar	$\phi \rightarrow \phi + a + b_\mu x^\mu$	$1 + D$	1
Crystal	spatial translations & rotations	$d + \frac{d(d-1)}{2}$	$d$

# CHANGES TO COSET CONSTRUCTION

- Include all **nonlinearly realized** (not necessarily spontaneously broken) symmetries in  $U(\pi)$ .
- One NG field for each nonlinearly realized symmetry.
- Eliminate redundant fields in a way consistent with the symmetry: **inverse Higgs mechanism**.
- The MC form now includes a  **$G$ -covariant spacetime vielbein**.