Modern Methods for Scattering Amplitudes

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1 Exercise session 3 - $gg \to H$ the standard way

In this exercise we consider the production of a Higgs boson in gluon-gluon annihilation at 1 loop in QCD. We write

$$g(p_1) + g(p_2) \to H(q)$$

with $p_1^2 = p_2^2 = 0$ and $q^2 = 2p_1 \cdot p_2 = m_H^2$. The quark running in the loop has mass m_t. We assume that the gluons are in the cyclic axial gauge

$$\epsilon_1 \cdot p_1 = \epsilon_2 \cdot p_2 = \epsilon_1 \cdot p_2 = \epsilon_2 \cdot p_1 = 0\,.$$

First of all, it is easy to write down explicit expressions from the Feynman diagrams and see that

$$S = \delta_{a_1 a_2} g_H \alpha_S\, A \tag{1}$$

with

$$A = \int \frac{d^D k}{(2\pi)^D} \left[\frac{\mathrm{Tr}\left[(\not{k} + m_t)\not{\epsilon}_1(\not{k} - \not{p}_1 + m_t)\not{\epsilon}_2(\not{k} - \not{p}_1 - \not{p}_2 + m_t)\right]}{D_1 D_2 D_3} + \{p_1 \leftrightarrow p_2, \epsilon_1 \leftrightarrow \epsilon_2\} \right], \tag{2}$$

with

$$D_1 = k^2 - m_t^2, \quad D_2 = (k-p_1)^2 - m_t^2\,, \quad D_3 = (k - p_1 - p_2)^2 - m_t^2$$

and the swap $\{p_1 \leftrightarrow p_2\}$ has to be applied also on the propagators!

Shifting in the second term $k \to -k + p_1 + p_2$, you should be able to see that the second diagram gives exactly the same contribution, such that we only consider

$$A = 2\int \frac{d^D k}{(2\pi)^D} \left[\frac{\mathrm{Tr}\left[(\not{k} + m_t)\not{\epsilon}_1(\not{k} - \not{p}_1 + m_t)\not{\epsilon}_2(\not{k} - \not{p}_1 - \not{p}_2 + m_t)\right]}{D_1 D_2 D_3} \right]. \tag{3}$$

To perform the traces, note that only the terms proportional to m_t and m_t^3 contribute, since all traces with odd numbers of γ matrices vanish. We have

$$\begin{aligned}
&\mathrm{Tr}\left[(\not{k} + m_t)\not{\epsilon}_1(\not{k} - \not{p}_1 + m_t)\not{\epsilon}_2(\not{k} - \not{p}_1 - \not{p}_2 + m_t)\right] = \\
&= m\mathrm{Tr}\left[\not{\epsilon}_1(\not{k} - \not{p}_1)\not{\epsilon}_2(\not{k} - \not{p}_1 - \not{p}_2)\right] + m\mathrm{Tr}\left[\not{k}\not{\epsilon}_1\not{\epsilon}_2(\not{k} - \not{p}_1 - \not{p}_2)\right] \\
&+ m\mathrm{Tr}\left[\not{k}\not{\epsilon}_1(\not{k} - \not{p}_1)\not{\epsilon}_2\right] + m^3\mathrm{Tr}\left[\not{\epsilon}_1\not{\epsilon}_2\right] \\
&= 4m\left[4(\epsilon_1 \cdot k)(\epsilon_2 \cdot k) - \frac{m_H^2}{2}(\epsilon_1 \cdot \epsilon_2) - D_2(\epsilon_1 \cdot \epsilon_2)\right]
\end{aligned} \tag{4}$$

and D_2 is the second propagator defined above!

Putting this back in the amplitude, we can write

$$A = 2\left\{\int \frac{d^Dk}{(2\pi)^D}\left[\frac{4(\epsilon_1\cdot k)(\epsilon_2\cdot k)}{D_1D_2D_3}\right] - \epsilon_1\cdot\epsilon_2\left[\frac{m_H^2}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1D_2D_3} + \int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1D_3}\right]\right\} \tag{5}$$

1.1 The tensor integral

To compute the tensor integral, let's write

$$\int \frac{d^Dk}{(2\pi)^D}\frac{(\epsilon_1\cdot k)(\epsilon_2\cdot k)}{D_1D_2D_3} = \epsilon_{1\mu}\epsilon_{2\nu}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{D_1D_2D_3} \tag{6}$$

Using Bose symmetry, we can impose that the result should be symmetric under tensor decompositon,

$$\int \frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{D_1D_2D_3} = F_1 g^{\mu\nu} + F_2(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu) + F_3(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \tag{7}$$

Clearly, with our gauge choice only the first form factor can contribute

$$\epsilon_{1\mu}\epsilon_{2\nu}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{D_1D_2D_3} = \epsilon_1\cdot\epsilon_2 F_1 \tag{8}$$

The projector that selects F_1 is easy to compute, it reads

$$P^{(1)}_{\mu\nu} = \frac{1}{D-2}\left[g_{\mu\nu} - \frac{2}{m_H^2}\left(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}\right)\right]. \tag{9}$$

Note that this is slightly different that what we did in class, but entirely equivalent. Try it out, also in the standard way with the projector that we build together! Applying the projector and using symmetry relations to relate equal integrals, we can write

$$F_1 = I(1,1,1)\left(\frac{8m_t^2}{D-2} - m_H^2\right) + \frac{8I(1,-1,1)}{(D-2)m_H^2} - \frac{8I(1,0,0)}{(D-2)m_H^2} + \left(\frac{8}{D-2} - 2\right)I(1,0,1) \tag{10}$$

Now let's consider the rank 1 integral

$$\begin{aligned} I(1,-1,1) &= \int\frac{d^Dk}{(2\pi)^D}\frac{(k-p_1)^2 - m_t^2}{D_1D_3} = \int\frac{d^Dk}{(2\pi)^D}\frac{(k^2 - 2k\cdot p_1 - m_t^2)}{D_1D_3} \\ &= \int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_3} - \int\frac{d^Dk}{(2\pi)^D}\frac{2k\cdot p_1}{D_1D_3} \end{aligned} \tag{11}$$

Now in the second integrals, perform the shift $k \to -k + p_1 + p_2$, which maps $D_1 \leftrightarrow D_3$ and gives at the numerator

$$\int\frac{d^Dk}{(2\pi)^D}\frac{k\cdot p_1}{D_1D_3} = -\int\frac{d^Dk}{(2\pi)^D}\frac{k\cdot p_1 + m_H^2/2}{D_1D_3} \tag{12}$$

such that solving for $k\cdot p_1$ and substituing in the original integral, we get

$$\int\frac{d^Dk}{(2\pi)^D}\frac{D_2}{D_1D_3} = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1} - \frac{m_H^2}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{D_1D_3} \tag{13}$$

or

$$I(1,-1,1) = I(1,0,0) - \frac{m_H^2}{2}I(1,0,1)\,.$$

Substituting this into (10) and using $D = 4 - 2\epsilon$ we finally obtain

$$F_1 = -\left(m_H^2 - \frac{4m_t^2}{1-\epsilon}\right)I(1,1,1) + \frac{2\epsilon}{1-\epsilon}I(1,0,1) \tag{14}$$

1.2 The master integrals

We can derive a system of canonical differential equations for this problem. There are three master integrals to consider

$$I(1,0,0)\,,\quad I(1,0,1)\,,\quad \text{and}\quad I(1,1,1)$$

We are interested in the differential equations in m_H^2, the only non-trivial variable. We have seen that the differential operator can be written as

$$\frac{\partial}{\partial m_H^2} = \frac{1}{2m_H^2}\left(p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu}\right). \tag{15}$$

We have already derived a canonical basis for the first two integrals (the tadpole is trivial of course). The triangle turns out to be already canonical, modulo a small rescaling. In our notation (for Minkowski kinematics) the basis reads

$$\begin{aligned} f_1 =& \epsilon\,(m^2)^\epsilon I(2,0,0) \\ f_2 =& \epsilon\, m_H^2 (m^2)^\epsilon \sqrt{1-\frac{4m^2}{m_H^2}} I(1,0,2) \\ f_3 =& \epsilon\, m_H^2 (m^2)^\epsilon I(1,1,1). \end{aligned} \tag{16}$$

where we have rescaled $(m^2)^\epsilon$ and used the standard integration measure, such that the canonical tadpole reads

$$f_1 = (m^2)^\epsilon \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m_t^2} = 1 + \frac{\pi^2\epsilon^2}{12} + \mathcal{O}(\epsilon^3)\,.$$

We derive the differential equations in the Landau variable defined by the equation

$$\frac{m^2}{-m_H^2} = \frac{x}{(1-x)^2}. \tag{17}$$

We see that they are in canonical form:

$$\partial_x \mathbf{f} = \epsilon\left(\frac{\mathbf{a}}{x} + \frac{\mathbf{b}}{x+1}\right)\mathbf{f} \tag{18}$$

where $\mathbf{a}, \mathbf{b}$ are numerical matrices (*please double check these numbers, there could be minor typos*)

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The equations can now be solved straightforwardly in the variable x giving

$$f_2 = \epsilon \log(x) + \epsilon^2\left(-2\mathrm{Li}_2(-x) + \frac{\log^2(x)}{2} - 2\log(x+1)\log(x)\right) \tag{19}$$

$$f_3 = -\frac{1}{2}\epsilon^2 \log^2(x) \tag{20}$$

where

$$\mathrm{Li}_2(-x) = -\int_0^x \frac{\log(1+t)}{t}. \tag{21}$$

We have fixed here the boundary condition as described below.

2 Boundary condition

We did not have much time to discuss about how to fix the boundary conditions. This small discussion should help. The idea is the following: from the usual cut rules, we know that both the bubble f_2 and the triangle f_3 have no branch cut at $m_H^2 = 0$, but instead can only have one at $m_H^2 = 4m_t^2$. In fact, both by direct calculation, or by inspecting the differential equations, one can see that

$$
\begin{aligned}
I(1,0,1)|_{m_H^2 \to 0, x \to 1} &= -\frac{(\epsilon-1)\mathrm{I}(0,0,1)}{m^2} \\
I(1,1,1)|_{m_H^2 \to 0, x \to 1} &= \frac{(\epsilon-1)\epsilon \mathrm{I}(0,0,1)}{2\left(m^2\right)^2}\,.
\end{aligned} \tag{22}
$$

The easiest way to see this, is from the integral representation. For example in the case of the bubble one finds

$$
\begin{aligned}
I(1,0,1)|_{m_H^2 \to 0} &= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 - m_t^2)\left((k-q)^2 - m_t^2\right)}|_{q^2 = m_H^2 \to 0} \\
&= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 - m_t^2)^2} = I(2,0,0)
\end{aligned} \tag{23}
$$

By reducing the tadpole, we find the first of eqs. (22). If this is not obvious (indeed in Minkowski kinematics $q^2 = 0$ does not imply that $q^\mu = 0$!), you can see this by Wick rotating to euclidean kinematics, taking the limit $q^2 \to 0$ which now implies $q^\mu \to 0$, and then rotating back! Another way to see this, is from the Feynman parameter representation, where only q^2 appears.

Similarly one finds

$$
\begin{aligned}
I(1,1,1)|_{m_H^2 \to 0} &= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 - m_t^2)\left((k-p_1)^2 - m_t^2\right)\left((k-p_1-p_2)^2 - m_t^2\right)}|_{q^2 = m_H^2 \to 0} \\
&= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 - m_t^2)^3} = I(3,0,0)
\end{aligned} \tag{24}
$$

and by reducing the tadpole we find the second of eqs. (22).