28th “Saalburg” Summer School
Course “String-inspired methods and the worldline formalism”
Christian Schubert, 29.8. - 2.9.2022

Exercises: part 3

Exercise 1: Higgs production by gluon fusion

Here we calculate the leading contribution to the process of Higgs production by gluon fusion in the SM, which is the one mediated by a top-quark loop. We denote the gluon momenta by $k_{1,2}$, the Higgs momentum by $-k_3$. In the worldline formalism this amplitude can be written as (F. Bastianelli, O. Corradini, J.P. Edwards, D.G.C. McKeon and C. Schubert, “Unified worldline treatment of Yukawa and axial couplings”, in preparation):

$$\Gamma(k_1,\varepsilon_1;k_2,\varepsilon_2;k_3) = (-2)(ig)^2(2m_t g_H)\int_0^T \frac{dT}{T}\,\mathrm{e}^{-m_t^2 T}(4\pi T)^{-2}\langle V_1^g V_2^g V_3^H\rangle \qquad (1)$$

with vertex operators

$$V_{1,2}^g = T^{a_{1,2}}\int_0^T d\tau_{1,2}(\varepsilon_{1,2}\cdot\dot{x}_{1,2} - i\psi_{1,2}\cdot f_{1,2}\cdot\psi_{1,2})\,\mathrm{e}^{ik_{1,2}\cdot x_{1,2}} \qquad (2)$$

$$V_3^H = \int_0^T d\tau_3\,\mathrm{e}^{ik_3\cdot x_3} \qquad (3)$$

(where $f_i^{\mu\nu} = k_i^\mu\varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$) and Wick contraction rules

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle = -G_B(\tau,\tau') \qquad (4)$$

$$\langle \psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2}G_F(\tau,\tau') \qquad (5)$$

Since this amplitude is finite we have set $D = 4$.

1. Perform the Wick contractions, not yet using any on-shell conditions.
2. Remove terms involving $\ddot{G}_B$ by partial integration.
3. Show that the on-shell amplitude vanishes for opposite gluon helicities.
4. Calculate the on-shell amplitude for the equal-helicity case.

Solution:

1.

$$\langle V_1^g V_2^g V_3^H \rangle = \mathrm{tr}(T^{a_1}T^{a_2})\Big\{\ddot{G}_{12}\varepsilon_1\cdot\varepsilon_2 - \Big(\dot{G}_{12}\varepsilon_1\cdot k_2 + \dot{G}_{13}\varepsilon_1\cdot k_3\Big)\Big(\dot{G}_{21}\varepsilon_2\cdot k_1 + \dot{G}_{23}\varepsilon_2\cdot k_3\Big) - G_{F12}^2\frac{1}{2}\mathrm{tr}(f_1f_2)\Big\}\,\mathrm{e}^{G_{12}k_1\cdot k_2 + G_{13}k_1\cdot k_3 + G_{23}k_2\cdot k_3} \tag{6}$$

2.

$$\langle V_1^g V_2^g V_3^H \rangle \to \mathrm{tr}(T^{a_1}T^{a_2})\Big\{\Big(\dot{G}_{12}^2 - G_{F12}^2\Big)\frac{1}{2}\mathrm{tr}(f_1f_2) + \frac{1}{2}\dot{G}_{12}\varepsilon_1\cdot\varepsilon_2\Big(\dot{G}_{23}k_2\cdot k_3 - \dot{G}_{13}k_1\cdot k_3\Big) - \Big(\dot{G}_{12}\varepsilon_1\cdot k_2 + \dot{G}_{13}\varepsilon_1\cdot k_3\Big)\Big(\dot{G}_{21}\varepsilon_2\cdot k_1 + \dot{G}_{23}\varepsilon_2\cdot k_3\Big)\Big\} \times \mathrm{e}^{G_{12}k_1\cdot k_2 + G_{13}k_1\cdot k_3 + G_{23}k_2\cdot k_3} \tag{7}$$

3. For opposite helicities, using spinor helicity with reference momenta $r_{1,2} = k_{2,1}$ leads to

$$\varepsilon_1\cdot k_2 = \varepsilon_2\cdot k_1 = \varepsilon_1\cdot\varepsilon_2 = 0 \tag{8}$$

and thus to the vanishing of (7) after eliminating k_3 through momentum conservation.

4. For equal gluon helicities, we use the same reference momenta, which still leads to the vanishing of $\varepsilon_1\cdot k_2$ and $\varepsilon_2\cdot k_1$, but now $\varepsilon_1\cdot\varepsilon_2$ survives. Using the further on-shell relations

$$k_1^2 = k_2^2 = 0,\; k_3^2 = -m_H^2, \tag{9}$$

and momentum conservation we have

$$k_1\cdot k_3 = k_2\cdot k_3 = -k_1\cdot k_2 = \frac{m_H^2}{2} \tag{10}$$

With these relations (7) simplifies to

$$\Big\{\Big\} = \varepsilon_1\cdot\varepsilon_2 m_H^2\Big[\frac{1}{2}\Big(\dot{G}_{12}^2 - G_{F12}^2\Big) + \frac{1}{4}\dot{G}_{12}\Big(\dot{G}_{23} - \dot{G}_{13}\Big)\Big], \tag{11}$$

$$G_{12}k_1\cdot k_2 + G_{13}k_1\cdot k_3 + G_{23}k_2\cdot k_3 = \Big(G_{13} + G_{23} - G_{12}\Big)\frac{m_H^2}{2}$$

Rescaling $\tau_i = Tu_i$, eliminating the T-integral, and introducing the ratio $R \equiv \frac{m_H^2}{4m_t^2}$, we have

$$\begin{aligned}
\Gamma(k_1,\varepsilon_1^\pm;k_2,\varepsilon_2^\pm;k_3) &= \frac{\alpha_s}{\pi} g_H m_t^{-1} m_H^2 \mathrm{tr}(T^{a_1}T^{a_2})\varepsilon_1^\pm \cdot \varepsilon_2^\pm \, I(R) \\
I(R) &= \int_0^1 du_1 \int_0^1 du_2 \int_0^1 du_3 \frac{\frac{1}{2}\Big(\dot G_{12}^2 - G_{F12}^2\Big) + \frac{1}{4}\dot G_{12}\Big(\dot G_{23} - \dot G_{13}\Big)}{1 - 2\Big(G_{13} + G_{23} - G_{12}\Big)R}
\end{aligned} \tag{12}$$

We use the translation invariance to set $u_3 = 0$, and the symmetry $1 \leftrightarrow 2$ to write $\int_0^1 du_1 \int_0^1 du_2 = 2\int_0^1 du_1 \int_0^{u_1} du_2$. Writing out the worldline Green's functions yields

$$I(R) = 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{3(u_1-u_2) - 2(u_1-u_2)^2}{1 - 4u_2(1-u_1)R} \tag{13}$$

MATHEMATICA (version 13.0) is able to perform this double integral (in the opposite order and under the assumption that $R < 1$, which is fulfilled for the actual values of m_t and m_H), giving the result

$$I(R) = \frac{1}{R} - \frac{1-R}{2R^2}\Big[\mathrm{Li}_2\Big(2R + 2i\sqrt{R(1-R)}\Big) + \mathrm{Li}_2\Big(2R - 2i\sqrt{R(1-R)}\Big)\Big] \tag{14}$$

Finally, one calculates

$$\varepsilon_1^+ \cdot \varepsilon_2^+ = \frac{[12]}{\langle 12\rangle}, \quad \varepsilon_1^- \cdot \varepsilon_2^- = \frac{\langle 12\rangle}{[12]} \tag{15}$$

so that $|\varepsilon_1^+ \cdot \varepsilon_2^+| = |\varepsilon_1^- \cdot \varepsilon_2^-| = 1$.
For quarks $\mathrm{tr}(T^{a_1}T^{a_2}) = \frac{1}{2}\delta^{a_1a_2}$, so our final result for the absolute value of the matrix element becomes

$$|\Gamma(k_1,\varepsilon_1^\pm;k_2,\varepsilon_2^\pm;k_3)| = \delta^{a_1a_2}\frac{\alpha_s}{2\pi} g_H m_t^{-1} m_H^2 \, I(R) \tag{16}$$

To identify it with the form usually given in the literature, we can use $g_H = m_t\sqrt{\sqrt{2}G_F}$ and the identity

$$\begin{aligned}
&\frac{1}{R} - \frac{1-R}{2R^2}\Big[\mathrm{Li}_2\Big(2R + 2i\sqrt{R(1-R)}\Big) + \mathrm{Li}_2\Big(2R - 2i\sqrt{R(1-R)}\Big)\Big] \\
&= \tau_t\Big[1 + (1-\tau_t)\arcsin^2 1/\sqrt{\tau_t}\Big]
\end{aligned} \tag{17}$$

where $\tau_t = 1/R$ (MATHEMATICA seems to be unable to verify this identity analytically, so I had to check it numerically).