HOLOGRAPHIC ASPECTS OF BLACK HOLES

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Lectures at the 27th "Saalburg" Summer School

Outline (revised)

Lecture 1: Motivation
Classical preliminaries

Lecture 2: Black hole thermodynamics

Lecture 3: Hawking effect

Lecture 4: Information paradox
Black hole complementarity

Classical preliminaries

Action for metric: $g_{\mu\nu}$ signature $(-+++)$

$$S = S_{EH} + S_{matter} + S_{maxwell} + \ldots$$

higher-order terms
R^2, $R_{\mu\nu}R^{\mu\nu}$, ...

$$S_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(R - 2\Lambda)$$

Work in $D=4$, then $\kappa^2 = 8\pi G_N$ has units of $(\text{length})^2$

Cosmological constant: $\Lambda = \begin{cases} 0 & \text{asymptotically flat} \\ -\frac{3}{L^2} & \text{— " — AdS} \\ +\frac{3}{\ell^2} & \text{— " — dS} \end{cases}$

Leave S_{matter} unspecified for now.

$$S_{Maxwell} = -\frac{1}{4e^2}\int d^4x \sqrt{-g}\, F^2$$

$$F = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \leftarrow \quad \begin{array}{l}\text{gauge potential} \\ A_\mu = (\phi, \vec{A})\end{array}$$

Note: $D=2$ toy models (JT, CGHS, ...) are often useful.

⊕ simplified regularization + renormalization

⊕ back-reaction of quantized matter

$\longrightarrow$ <u>Semi-classical gravity</u>

⊖ no local d.o.f.'s, conformally flat, etc.

Vacuum Einstein eq.s $(F_{\mu\nu} = 0)$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

AdS - Schwarzschild solution: $\Lambda = -\frac{3}{L^2}$

$$ds^2 = -f_k(r)\, dt^2 + \frac{dr^2}{f_k(r)} + r^2 ds_k^2$$

$$f_k(r) = \frac{r^2}{L^2} + k - \frac{2m}{r} \leftarrow \text{b.h. mass}$$

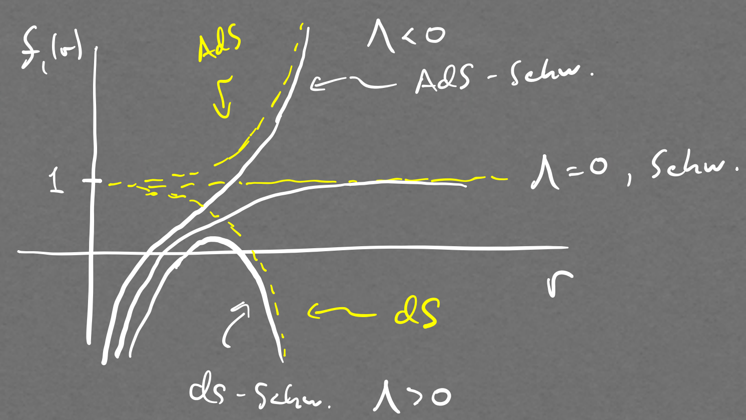
$$k \in \{1, 0, -1\}$$

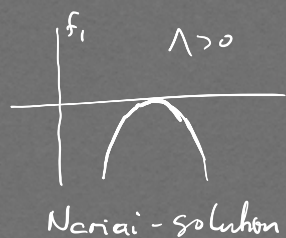
$$ds_k^2 = \frac{d\rho^2}{1-k\rho^2} + \rho^2 d\varphi^2 = \begin{cases} d\theta^2 + \sin^2\theta\, d\varphi^2 & k=+1 \\ d\rho^2 + \rho^2 d\varphi^2 & k=0 \\ d\chi^2 + \sinh^2\chi\, d\varphi^2 & k=-1 \end{cases}$$

Note: $L \to \infty$ gives $\Lambda = 0$ Schwarzschild sol.

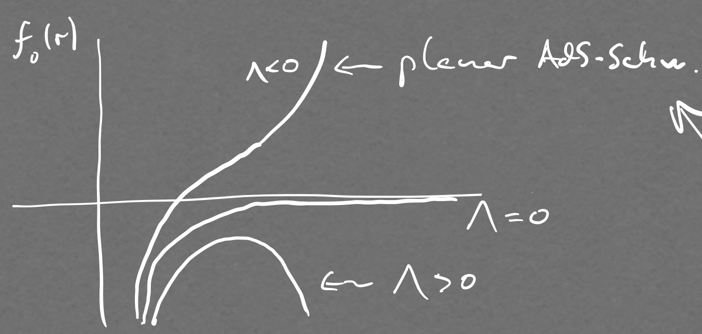
$L \to i\ell$ gives $\Lambda > 0$ dS-Schwarzschild sol.

$k=1$





$k=0$



applied AdS/CFT

Add radial electric field:

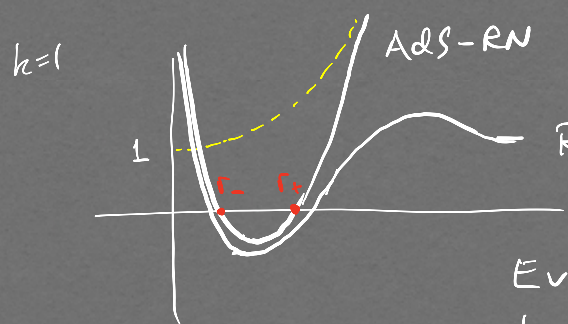
$$A_t = \frac{e}{\varkappa} h(r) \quad , \quad A_i = 0$$

Charged b.h. solution

$$h(r) = \mu - \frac{Q}{r}$$ ← b.h. charge

↖ chemical potential

$$\boxed{\Lambda < 0} \quad f_k(r) = \frac{r^2}{L^2} + k - \frac{2m}{r} + \frac{Q^2}{r^2}$$



Event horizon: $r = r_+$

Inner horizon: $r = r_-$

Single valued gauge field at $r = r_+$

$$h(r_+) = 0 \quad \text{i.e.} \quad \mu = \frac{Q}{r_+}$$

Applied AdS/CFT: Planar AdS-RN solution is dual to boundary QFT with conserved U(1) charge at finite T and finite μ near a quantum critical point.

Kruskal-Szekeres extension:

$$ds^2 = f_k(r)\left(-dt^2 + dr_*^2\right) + r(r_*)^2 \, ds_k^2$$

tortoise coordinate: $r_* = \int^r \frac{dr'}{f_k(r')}$ ← inverse defines $r(r_*)$

$$f_k(r) = f_k'(r_H)(r - r_H) + \ldots \quad \text{as} \quad r \to r_H$$

$$\to \quad r_* = \frac{1}{f_k'(r_H)} \log\left(\frac{r}{r_H} - 1\right) + H(r)$$

↑ regular at $r = r_H$

Write $v = t + r_*$, $u = t - r_*$

→ near-horizon metric takes the form

$$ds^2 \simeq f'(r_H)(r - r_H)(-dv\,du) + r_H^2\, ds_k^2$$

$\xi = \frac{1}{2} f_k'(r_H)$

↑ surface gravity

$$= 2\xi r_H e^{-2\xi H(r_H)} e^{2\xi r_*}(-dv\,du) + r_H^2 ds_k^2$$

$$= 2\xi r_H e^{-2\xi H(r_H)} e^{\xi(v-u)}(-dv\,du) + r_H^2 ds_k^2$$

$$= \frac{2r_H}{\xi} e^{-2\xi H(r_H)}(-dV\,dU) + r_H^2 ds_k^2$$

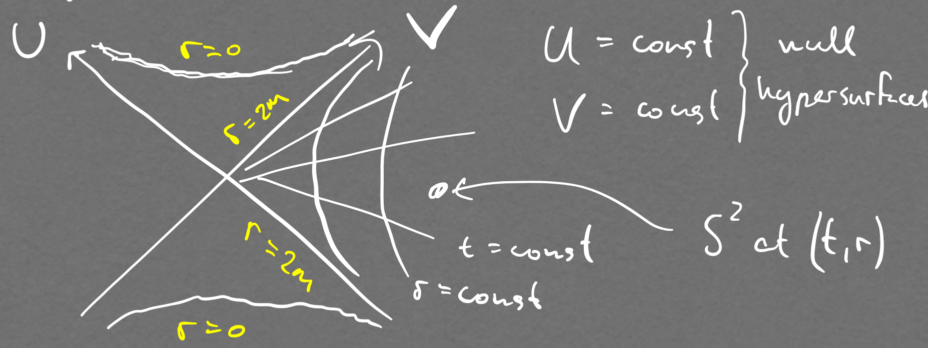
with $V = e^{\xi v}$, $U = -e^{-\xi u}$ null Kruskal coordinates

$r \to r_H$ amounts to $U \to 0$ or $V \to 0$

Metric is non-singular in UV-coordinates

Exercise 1: Work out Kruskal-Szekeres extension for the special case of $D=4$ Schwarzschild black hole ($\Lambda = 0$, $k=+1$ with $f_1(r) = 1 - \frac{2M}{r}$) and verify that $\xi = \frac{1}{4M}$ and $ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dV\,dU + r^2 d\Omega^2$.

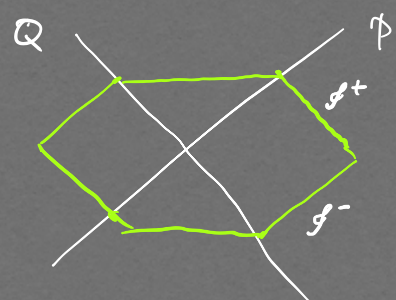
Kruskal-diagram:



It's easy to read off causal relationships

Carter-Penrose diagram: (conformal diagram)

$$\tan Q = U \quad , \quad \tan P = V$$



$P \to \frac{\pi}{2} \Leftrightarrow V \to \infty$

etc..

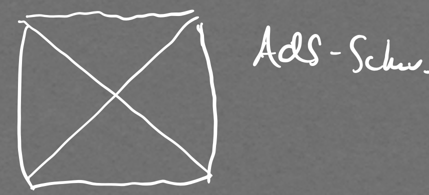
Maximally extended Schwarzschild spacetime

AdS-Schwarzschild solution: $\quad k=+1$

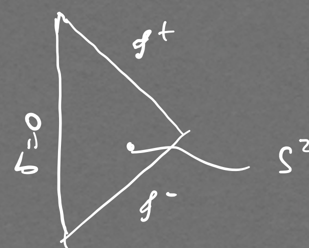
$$r_* = \int^r \frac{dr'}{\frac{r'^2}{L^2} + 1 - \frac{2M}{r}} \quad \text{finite as } r \to \infty$$

$\to$ boundary at finite value of UV

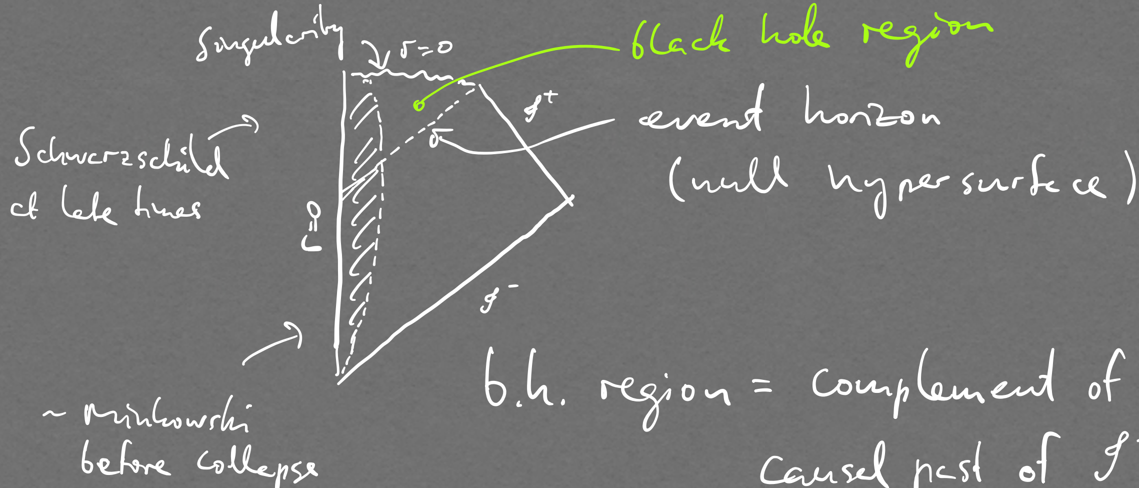
Conformal diagram:



Minkowski spacetime:



Gravitational collapse:



b.h. region = complement of the causal past of $\mathscr{I}^+$

event horizon = boundary of b.h. region

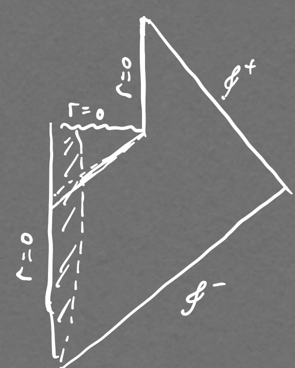
points on event horizon are null or spacelike separated but <u>not</u> timelike separated

Hawking effect (lecture 3)

$\rightarrow$ b.h. emits radiation

Back-reaction effect is small if $M \gg M_{pl}$

but important on long enough time scales



$\leftarrow$ semi-classical black hole spacetime

Analytic description available in $D=2$

RST model.

Black hole thermodynamics

Consider a $k=1$ RN black hole of mass M and charge Q in asymptotically flat ($\Lambda=0$) space time.

B.H. absorbs a particle of mass δM and charge δQ

$\rightarrow$ event horizon moves to $r_H + \delta r_H$

$$0 = f_{M+\delta M,\, Q+\delta Q}(r_H + \delta r_H)$$

$$= 1 - \frac{2(M+\delta M)}{r_H + \delta r_H} + \frac{(Q+\delta Q)^2}{(r+\delta r_H)^2}$$

$$= \underbrace{f_{M,Q}(r_H)}_{=0} + f'_{M,Q}(r_H)\,\delta r_H - \frac{2\,\delta M}{r_H} + \frac{2Q\,\delta Q}{r_H^2} + O(\delta x)^2$$

$$\rightarrow \quad \delta M = \underbrace{\tfrac{1}{2} f'(r_H)}_{\xi}\ \underbrace{r_H\,\delta r_H}_{\frac{1}{8\pi}\delta A_H} + \underbrace{\frac{Q}{r_H}}_{\mu}\,\delta Q$$

Kerr-Newman

$$\rightarrow \quad \delta M = \frac{\xi}{8\pi}\,\delta A_H + \mu\,\delta Q + \Omega\,\delta J$$

1^{st} law of b.h. dynamics

Comparison with $dE = T\,dS + \mu\,dQ + \Omega\,dJ$ suggests $T \propto \xi$, $S \propto A_H$

Definition of surface gravity:

Killing vector K^μ:

$K^\mu \nabla_\mu K^\nu = \xi K^\nu$ on Killing horizon

null hypersurface where the norm of K^μ vanishes

(c.f. $\frac{\partial}{\partial t}$ on event horizon)

0^{th} law: ξ is constant on event horizon of a stationary b.h.

2^{nd} law (area theorem):

Null energy condition (NEC)

$T_{\mu\nu} k^\mu k^\nu \geq 0$ for all null k^μ

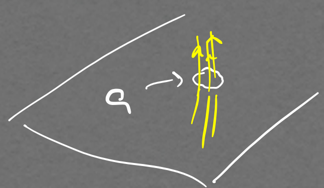
matter energy momentum tensor

"For matter satisfying NEC the area of spatial cross sections of the event horizon is non-decreasing"

Sketch of proof:

Event horizon is generated by by null geodesics

Consider a patch of a spatial cross section



$$\frac{da}{d\lambda} = \theta a$$

parameter along geodesics

expansion is the fractional change in a along bundle of geodesics

area theorem $\Longleftrightarrow$ $\theta \geq 0$

Raychaudhuri's equation for geodesic deviation

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{3} - \underbrace{\sigma_{\mu\nu}\sigma^{\mu\nu}}_{\text{shear}} - R_{\mu\nu} k^\mu k^\nu$$

Einstein eq. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \varkappa^2 T_{\mu\nu}$

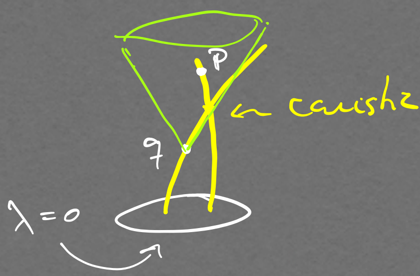
$\rightarrow$ $R_{\mu\nu} k^\mu k^\nu = \varkappa^2 T_{\mu\nu} k^\mu k^\nu \geq 0$ by NEC

It follows that $\frac{d\theta}{d\lambda} \leq -\frac{\theta^2}{3}$ i.e. $\frac{d}{d\lambda}\left(\frac{1}{\theta}\right) \geq \frac{1}{3}$

If $\theta(\lambda=0) = \theta_0 < 0$ then $\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta_0} + \frac{\lambda}{3}$ for all $\lambda > 0$

Then $\frac{1}{\theta} = 0$ for some $\lambda < -\frac{3}{\theta_0}$ i.e. $\boxed{\theta \to -\infty}$

caustic



p is inside future light-cone of q but there are no time like separated events on event horizon.

It follows that $\Theta \geq 0$ for all λ.

Bekenstein (1973): Thermodynamic interpretation of b.h. laws

$\rightarrow$ identify $\xi \propto T$, $A_H \propto S$

Hawking (1975): Black holes emit thermal radiation

$$T_H = \frac{\xi}{2\pi} = \frac{1}{8\pi M} \quad \text{Schwarzschild}$$

So that $S_{BH} = \frac{1}{4} A_H$ ← area in Planck units

Note: Hawking emission leads to $\frac{dA_H}{d\lambda} < 0$

$\rightarrow$ Generalized 2^{nd} law including the entropy of Hawking radiation

Large black holes ($M \gg M_{pl}$) evolve slowly

Stefan-Boltzmann law:

$$T = \frac{1}{8\pi M}$$

$$\frac{dM}{dt} \propto - A_H T^4 \sim M^2 \frac{1}{M^4}$$

$$\rightarrow \; t_f - t_i \propto \int_{M_i}^{M_f} dM \, M^2 \propto M_i^3 - \underbrace{M_f^3}_{O(M_{pl})^3} \simeq M_i^3$$

Black hole lifetime (Schwarzschild) $\tau_{b.h.} \propto \left(\frac{M_i}{M_{pl}}\right)^3 t_{pl}$

putting in numbers $\tau_{b.h.} \simeq 10^{71} s \left(\frac{M}{M_\odot}\right)^3$

Parametrically longer than other b.h. time scales:

light-crossing time $\sim M$

scrambling time $\sim M \log M$

$\longrightarrow$ can work on static background if time scales of problem are $\ll \tau_{bh}$

$\rightarrow$ Euclidean approach [to b.h. thermodynamics]

Euclidean AdS-Schw. b.h.

$$ds^2 = f_k(r)\, d\tau^2 + \frac{dr^2}{f_k(r)} + r^2 ds_k^2$$

$$f_k(r) = \frac{r^2}{L^2} + k - \frac{2m}{r}$$

Obtained from real time solution via $t \to i\tau$

Near-horizon region $f_k(r) = \underbrace{f_k'(r_H)}_{=2\xi}(r-r_H) + \ldots$ as $r \to r_H$

write $r = r_H + \frac{\xi}{2}\rho^2$

$\to ds^2 = \rho^2 \xi^2 d\tau^2 + d\rho^2 + r_H^2 ds_k^2$ for small ρ

$r = r_H$ is the origin in (ρ, τ) plane

Euclidean geometry is smooth if $\xi\tau$ is the polar angle

$\to \tau \simeq \tau + \beta$ with $\beta^{-1} = \frac{\xi}{2\pi}$ Hawking temperature

Note: Proper distance around Euclidean time circle depends on radial position

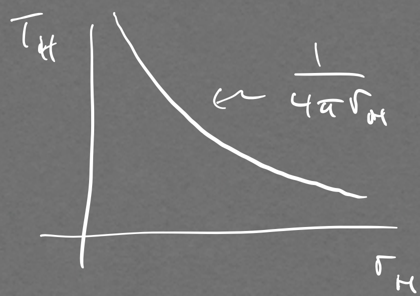
$$\beta(r) = \frac{2\pi}{\xi}\sqrt{f_k(r)}$$

$\to$ local temperature for fiducial observer

$$T_{fid}(r) = \frac{T_H}{\sqrt{f_k(r)}} \to \infty \quad \text{as } r \to r_H$$

Tempting to interpret as a b.h. in thermal equilibrium with heat bath but not always possible

$k=1,\ \Lambda=0$ Schwarzschild solution



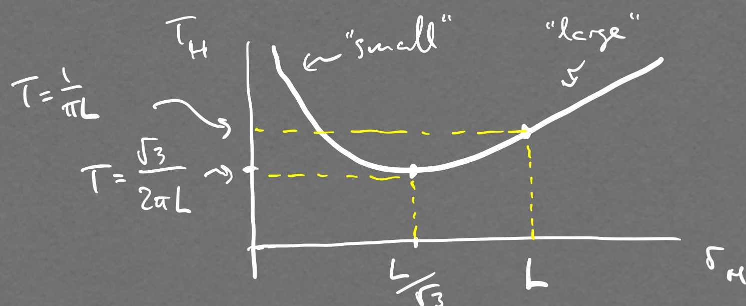
$$C = \frac{dM}{dT} < 0$$

negative specific heat!

Jeans instability - uniform thermal radiation is unstable to gravitational collapse

Situation is better for $\Lambda<0$, AdS-Schw.

$$T_H = \frac{1}{4\pi}\left(\frac{k}{r_H} + \frac{3r_H}{L^2}\right)$$



Partition function $Z(\beta) = \int [Dg] \, e^{-S_E[g]}$

Euclidean path integral is ill defined

Consider a formal classical limit anyway and look for Euclidean saddle points with period β in τ

$$\longrightarrow \quad Z(\beta) \simeq e^{-S_E[g_*]}$$

dominant saddle pt.

$$S_E = -\frac{1}{2\kappa^2} \int d^4x \sqrt{g} \left(R + 6 \right) \quad \circledast$$

set $L = 1$

Candidate saddle pt's

(1) Euclidean AdS (with $k = 1$)

$$ds^2 = (r^2 + 1) \, d\tau^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega^2$$

can have $\tau \simeq \tau + \beta$ with any β

(2) Euclidean AdS - Schw.

Both (1) and (2) have the form

$$ds^2 = f(r) \, d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

Inserting into ⊛ gives

$$S_E = -\frac{1}{2\kappa^2}\int d\tau\, dr\, d\theta\, d\phi\; r^2 \sin\theta \left[6 - \frac{1}{r^2}\left(r^2 f'' + 4rf' + 2f - 2\right)\right]$$

Two problems: (i) 2^{nd} derivative of f in S.

(ii) S_E is divergent

Both are solved by boundary terms:

$$S_b = \lim_{r_0\to\infty} \frac{1}{2\kappa^2} \int_{r=r_0} d^3x \sqrt{\gamma}\left(-2K + 4 + {}^{(2)}R\right)$$

γ_{ij} induced metric on $r=r_0$ hypersurface

K trace of 2^{nd} fundamental form $K_{ij} = \frac{1}{2}(\nabla_i n_j + \nabla_j n_i)$

outward directed unit normal

${}^{(3)}R$ Ricci scalar of γ_{ij}

$$\sqrt{\gamma} = r^2\sqrt{f}\sin\theta \;;\quad {}^{(3)}R = \frac{2}{r^2}\,,\quad K = \frac{rf' + 4f}{2r\sqrt{f}}$$

Bulk action includes the term

$$\frac{1}{2\kappa^2}\int d\tau\, dr\, 4\pi r^2 f'' = -\frac{4\pi}{\kappa^2}\int d\tau\, dr\, rf' + \frac{2\pi}{\kappa^2}\int d\tau\, r^2 f'\Big|_{r_H}^{r_0}$$

Remaining bulk terms in S_E cancel on-shell

Divergent parts (when $r_0\to\infty$) of boundary terms cancel giving a finite result

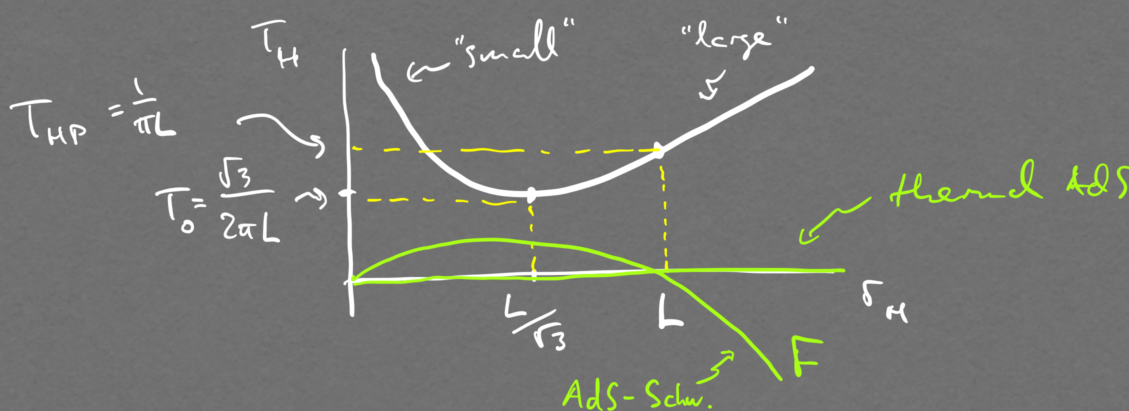
$$S_E + S_b = \frac{8\pi}{\kappa^2}\int d\tau\,(\underbrace{M - TS})$$

free energy

Exercise 2: Verify that divergences cancel in the on-shell action with boundary terms and the free energy has the value $F = M - TS$

$$F = m - TS$$
$$= \frac{r_H}{4}\left(k - r_H^2\right) \quad \leftarrow \quad k=1$$

Hawking-Page transition (with factors of L reinstated)



$T < T_0$: thermal AdS is only saddle pt.

$T > T_0$: thermal AdS small b.h. large b.h.

dominates for $T_0 < T < T_{HP}$ dominates for $T > T_{HP}$

QFT in curved spacetime: (in brief)

Minimally coupled scalar field

$$S_\phi = -\frac{1}{2}\int_M d^4x\,\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2 + \frac{1}{6}R\phi^2\right)$$

for conformal coupling

Klein-Gordon eq: $(\nabla^2 - m^2)\phi = 0$

Assume $M = \mathbb{R}\times\Sigma$ ← Cauchy surfaces

and choose local coordinates $(x^0, \vec{x})$

K-G inner product: $(f,g) = \int_\Sigma d\Sigma^\mu\, j_\mu(f,g)$

← complex valued fcn's

$$d\Sigma^\mu = d^3x\,\sqrt{h}\,n^\mu$$

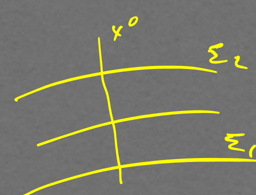
← future directed unit normal to Σ

induced metric on Σ

$$j_\mu(f,g) = -i\left(f\,\partial_\mu g^* - (\partial_\mu f)\,g^*\right)$$

K-G equation $\longrightarrow$ $\nabla^\mu j_\mu = 0$

and it follows that $(f,g)_{\Sigma_1} = (f,g)_{\Sigma_2}$



One finds $(f,g)^* = -(f^*,g^*) = (g,f)$ and $(f,f^*) = 0$

Note: K-G inner product is <u>not</u> positive definite

Mode expansion: $\phi(x) = \sum_i \left(a_i u_i(x) + a_i^\dagger u_i^*(x)\right)$

$(u_i, u_j) = \delta_{ij} \qquad (u_i^*, u_j^*) = -\delta_{ij} \qquad (u_i, u_j^*) = 0$

Canonical quantization: $[a_i, a_j^\dagger] = \delta_{ij}$

Vacuum state: $a_i |0\rangle = 0 \quad \forall\, i$

Fock space basis $\left\{ \left(\frac{1}{n_1! n_2! \dots}\right)^{1/2} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots |0\rangle \right\}$

Minkowski spacetime: $\mathcal{M} = \mathbb{R}^{1,3}$

$$u_{\vec{k}} = \frac{1}{\sqrt{2\omega(2\pi)^3}} e^{i\vec{k}\cdot x - i\omega x^0}$$

$\omega = \sqrt{\vec{k}^2 + m^2}$

$\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^i}$ generate symmetries of $\mathbb{R}^{1,3}$

(subset of Poincaré transformation)

that leave $|0\rangle$ invariant.

General curved background:

No symmetry, no preferred coordinates

$\rightarrow$ choice of positive frequency modes u_i is not unique

Different set of orthonormal modes $\tilde{u}_i(x)$

$$\phi(x) = \sum_i \left(\tilde{a}_i \tilde{u}_i(x) + \tilde{a}_i^\dagger \tilde{u}_i^*(x)\right)$$

$\tilde{a}_i |\tilde{0}\rangle = 0 \quad \forall i$ defines a new vacuum

Bogolubov transformation:

$$\tilde{u}_i = \sum_j \left(\alpha_{ij} u_j + \beta_{ij} u_j^*\right) \leftarrow \beta_{ij} = 0 \text{ in flat spacetime}$$

where $\alpha_{ij} = (\tilde{u}_i, u_j) \qquad \beta_{ij} = -(\tilde{u}_i, u_j^*)$

and $a_i = \sum_j \left(\alpha_{ji} \tilde{a}_j + \beta_{ji}^* \tilde{a}_j^\dagger\right)$

If $\beta_{ij} \neq 0$ then $a_i |\tilde{0}\rangle \neq 0$

and $\langle \tilde{0} | N_i | \tilde{0}\rangle = \langle \tilde{0} | a_i^\dagger a_i | \tilde{0}\rangle = \sum_j |\beta_{ij}|^2$

Exercise 3:

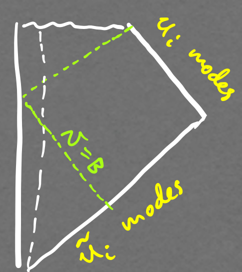
(a) Establish the following properties of the K-G norm:

$$(f, g)^* = -(f^*, g^*) = (g, f) \quad ; \quad (f, f^*) = 0$$

(b) Starting from the Bogolubov transformation verify the relations

$$\alpha_{ij} = (\tilde{u}_i, u_j) \; ; \quad \beta_{ij} = -(\tilde{u}_i, u_j^*) \; ; \quad a_i = \sum_j \alpha_{ji} \tilde{a}_j + \beta_{ji}^* \tilde{a}_j^\dagger$$

Hawking effect:



Scalar field in $|\tilde{0}\rangle$ state for $\tilde{u}_i$ modes at $\mathcal{I}^-$

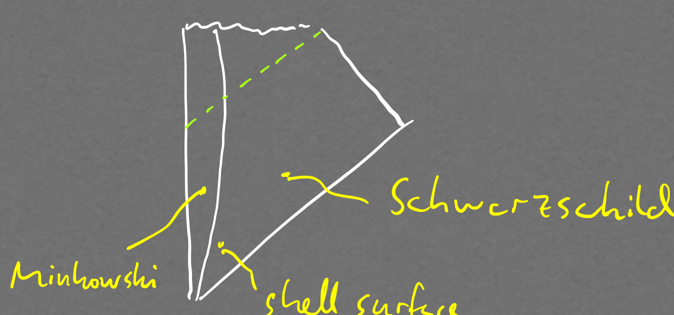
Want to calculate $\beta_{ij} = -(\hat{u}_i, u_j^*)$

Evolve u_i mode back from $\mathcal{I}^+$ to $\mathcal{I}^-$ reflecting at $r=0$ (take $m=0$ for simplicity)

$\rightarrow$ only has support for $v < B$

Main contribution to overlap integral comes from $v \rightarrow B$, i.e. close to horizon

Collapsing shell model (Birrell & Davies: QFT in curved space)



Solve K-G equation outside and inside

Match solutions across shell

Spherical symmetry: $u_\omega = \frac{1}{r} R_{\omega \ell}(r) \, Y_{\ell m}(\theta,\phi) \, e^{-i\omega t}$

Radial equation outside shell $(m=0)$

$$\frac{d^2 R_{\omega\ell}}{dr_*^2} + \left(\omega^2 - \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)\left(1 - \frac{2M}{r}\right) R_{\omega\ell} = 0$$

Outgoing mode at $\mathscr{I}^+$: $u_\omega \simeq \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u}$

$u = t - r_*$

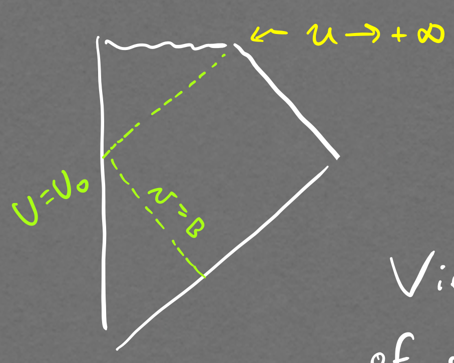
Ingoing mode at $\mathscr{I}^-$: $\tilde{u}_{\omega'} \simeq \frac{1}{\sqrt{4\pi\omega'}} e^{-i\omega v}$

$v = t + r_*$

Matching condition across shell gives

$$U_0 - U = e^{-\xi u} \qquad \text{for } u \to \infty \text{ (late times)}$$

surface gravity



i.e. $u = \frac{1}{\xi} \log(U_0 - U)$

At $r=0$ we have $V = U$

View the outgoing mode as reflection of an ingoing mode

$$u_\omega \propto e^{-i\omega \frac{1}{\xi} \log(U_0 - V)}$$

Assume large shell mass $M \gg M_{pl}$

$\rightarrow$ spacetime curvature outside horizon is small

Note: $R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} = \dfrac{48 M^2}{r^6} \xrightarrow[r \to 2M]{} \dfrac{3}{4M^4}$

Then $V = v + \text{const}$ and $u_\omega \propto \begin{cases} e^{-i\frac{\omega}{\xi}\log\left(\frac{B-v}{A}\right)} & v < B \\ 0 & v > B \end{cases}$

with B and A constants

Finally, evaluate overlap integral on a Cauchy surface near $\mathscr{I}^-$:

$$\beta_{\omega'\omega} = -\left(\tilde{u}_{\omega'}, u_\omega^*\right) \leftarrow \begin{cases} \tilde{u}_{\omega'} = \dfrac{1}{\sqrt{4\pi\omega'}} e^{-i\omega' v} \\ u_\omega = \dfrac{1}{\sqrt{4\pi\omega}} e^{-\frac{i\omega}{\xi}\log\left(\frac{B-v}{A}\right)} \end{cases}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{B} dv\, e^{-i\omega' v} \left(\frac{B-v}{A}\right)^{i\omega/\xi}$$

exercise

$$= \frac{e^{iX}}{2\pi\sqrt{\omega\omega'}} e^{-\frac{\pi}{2}\frac{\omega}{\xi}} \Gamma\left(1 + \frac{i\omega}{\xi}\right) \quad (*)$$

$\leftarrow$ phase factor

$$\rightarrow |\beta_{\omega'\omega}|^2 = \frac{e^{-\pi\omega/\xi}}{4\pi^2\omega\omega'} \left|\Gamma\left(1+\frac{i\omega}{\xi}\right)\right|^2$$

$$= \frac{1}{2\pi\xi\omega'} \frac{1}{e^{\frac{2\pi}{\xi}\omega} - 1} \quad (**)$$

Exercise 4:

Starting from the overlap integral for $\beta_{\omega\omega'}$ derive (*) and (**)

Notes: (1) $T_H = \frac{\xi}{2\pi} = \frac{1}{8\pi M} = 6 \cdot 10^{-8} K \left(\frac{M_\odot}{M}\right)$

(2) $\int d\omega' |\beta_{\omega'\omega}|^2$ is log divergent!

$\rightarrow$ back-reaction, b.h. evaporates

(3) Dominant contribution to overlap integral comes from $u \rightarrow \infty$ region very close to event horizon

$\rightarrow$ trans-Planckian modes

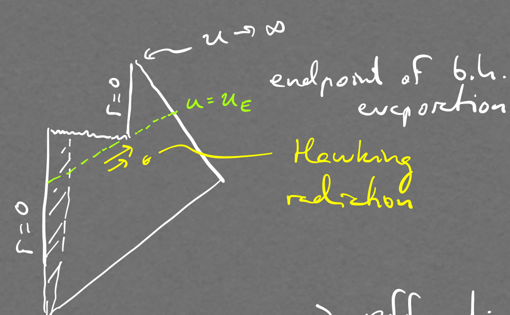
Addressed by infalling lattice model of Corley and Jacobson (1997)

(4) Correction to spectrum due to mode propagation in Schwarzschild geometry outside b.h.
$\rightarrow$ gray body factors

(5) See Hawking's 1975 CMP paper for additional subtleties of calculation

(6) QFT on maximally extended Schwarzschild spacetime (eternal b.h.)
- elegant mathematical formulation
- results depend on choice of vacuum (Boulware, Unruh, Hartle-Hawking)
- HH vacuum appears to observer outside b.h. as a mixed state described by a thermal density matrix at $T = T_H$

<u>Semi-classical b.h. evolution:</u>



Want to describe evolution of matter + geometry including back-reaction effects

$\rightarrow$ effective field theory (EFT)

$$S = S_{EH} + S_{matter} + \int d^4x \sqrt{-g} \Big[c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \ldots + \frac{d_1}{\Omega^2} R^2 + \ldots \Big]$$

Ω is a UV cutoff scale below which EFT is valid

All terms consistent with symmetries and locality are included in EFT action

Expect higher order terms to be suppressed as long as "$|R_{\mu\nu\lambda\sigma}|$" $\ll \Omega^2$

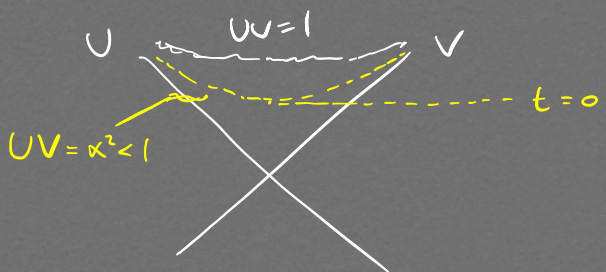
Example: Schw. b.h. w. $M = M_\odot$ and $\Omega = 1\,\mathrm{TeV}$

$$\longrightarrow \frac{R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}}{\Omega^4} \simeq 10^{-44} \text{ at event horizon}$$

The <u>extrinsic</u> curvature of all Cauchy surfaces must also remain small compared to cutoff scale

$\longrightarrow$ nice slices

Simple construction for eternal Schw. b.h. in Kruskal-Szekerez coordinates:



$t=0$ slice:

$UV = \alpha^2 < 1$ if $V < U$

$U + V = 2\alpha$ if $V > U$

(smooth the join at $U = V$)

$t \neq 0$ slices: $UV = \alpha^2$ if $V < e^{\frac{t}{2m}} U$

$e^{\frac{t}{4m}} U + e^{-\frac{t}{4m}} V = 2\alpha$ if $V > e^{\frac{t}{2m}} U$

3-momentum of infalling particle that starts at rest outside b.h. remains small in local rest frame of nice slice

Note: The 3-momentum diverges in Schwarzschild as $r \to 2M$

Nice slice Hamiltonian

$$i \frac{\partial}{\partial t_{NS}} |\psi\rangle = H_{NS} |\psi\rangle$$

H_{NS} is time dependent near b.h.

$\longrightarrow$ particle production at $O\left(\frac{M_{pl}^2}{M}\right)$ energies

$\longrightarrow$ Hawking radiation

Formulation of the information paradox implicitly assumes existence of nice slices and validity of local EFT.