Universität Bremen

Summer Term 2019
Volker Perlick

Black Holes

Worksheet 11

Problem 1: Consider a Kerr black hole with $0 \le a^2 < m^2$.

(a) Calculate the length of an r line in the equatorial plane from r_+ to r_0 where $r_+ < r_0 < \infty$. Consider the limit $a^2 \to m^2$.

(b) Investigate if the 2-surface $(t = \text{constant}, \vartheta = \pi/2, r > r_+)$ can be embedded into Euclidean 3-space,

$$g^{(2)} = g_{rr} dr^2 + g_{\varphi\varphi} d\varphi^2 = dZ^2 + dR^2 + R^2 d\varphi^2$$

with embedding functions $Z(r)$ and $R(r)$. Consider the limit $a^2 \to m^2$.

Problem 2: Determine all circular lightlike geodesics in a Kerr spacetime with $0 < a^2$. Distinguish circular lightlike geodesics in the equatorial plane and off the equatorial plane.

Hint: Circular geodesics satisfy $\dot{r} = 0$, $\ddot{r} = 0$, $\dot{\vartheta} = 0$ and $\ddot{\vartheta} = 0$.

Problem 3: Two observers are orbiting an extreme Kerr black hole, $a^2 = m^2$, in co-rotating geodesic motion in the equatorial plane, one at radius r_1 and the other at radius r_2. During a coordinate time interval Δt, the proper time interval that elapses for the observer at r_1 is $\Delta\tau_1$ and that for the observer at r_2 is $\Delta\tau_2$. Calculate the ratio $\Delta\tau_2/\Delta\tau_1$ and consider the limiting case $r_1 \to r_{\text{ISCO}} = m$ and $r_2 \to \infty$. Compare with the result of Problem 2 of Worksheet 4.

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Solutions to Worksheet 11

Problem 1:

(a) From the Kerr metric we read that the length of an arc from r_1 to r_2 on an r line is

$$\ell = \int_{r_1}^{r_2} \sqrt{g_{rr}}\, dr = \int_{r_1}^{r_2} \sqrt{\frac{\rho^2}{\Delta}}\, dr \, .$$

We consider a Kerr black hole with $0 < a^2 < m^2$ and we assume that $r_+ = m + \sqrt{m^2 - a^2} < r_1 < r_2$ and $\vartheta = \pi/2$. Then $\sqrt{\rho^2} = r > 0$ and

$$\Delta = r^2 + a^2 - 2mr = (r - r_-)(r - r_+) > 0$$

on the integration interval. We find

$$\ell = \int_{r_1}^{r_2} \frac{r\, dr}{\sqrt{(r - r_-)(r - r_+)}} = \Big[\sqrt{(r - r_-)(r - r_+)} + (r_+ + r_-) \ln \Big| \sqrt{r - r_-} + \sqrt{r - r_+} \Big| \Big]_{r_1}^{r_2}$$

$$= \sqrt{(r_2 - r_-)(r_2 - r_+)} \; - \; \sqrt{(r_1 - r_-)(r_1 - r_+)} \; + \; 2\, m \ln \left| \frac{\sqrt{r_2 - r_-} + \sqrt{r_2 - r_+}}{\sqrt{r_1 - r_-} + \sqrt{r_1 - r_+}} \right| .$$

In the limit $r_1 \to r_+$ we find

$$\ell = \Big[\sqrt{(r_2 - r_{)}(r_2 - r_+)} \; + \; 2\, m \ln \left| \frac{\sqrt{r_2 - r_-} + \sqrt{r_2 - r_+}}{\sqrt{2\sqrt{m^2 - a^2}}} \right| .$$

As long as $a^2 < m^2$, this length is finite. However, in the limit $a^2 \to m^2$ it becomes infinite, i.e., for an extreme black hole the horizon is infinitely far away from an observer in the domain of outer communication if the distance is measured in the hypersurface $t =$ constant along an r line. This can be achieved, e.g., by laying out meter sticks along an r-line. By contrast, we know already that particles that fall towards the black hole reach the horizon in a finite proper time. This is equally true for an extreme Kerr black hole as for a non-extreme one. So the question of whether the horizon of an extreme black hole has a finite or an infinite distance from an outside observer is rather subtle. In part (b) we will shed some more light on this question.

(b) We want to satisfy the equation

$$g_{rr} dr^2 + g_{\varphi\varphi} d\varphi^2 = dZ^2 + dR^2 + R^2 d\varphi^2 \, .$$

If we insert the metric coefficients, noting that we consider the equatorial plane, we find

$$\frac{r^2}{\Delta} dr^2 + \Big(r^2 + a^2 + \frac{2ma^2}{r} \Big) d\varphi^2 = \big(Z'(r)^2 + R'(r)^2 \big) dr^2 + R(r)^2 d\varphi^2 \, .$$

Comparing coefficients of $d\varphi^2$ yields

$$R(r)^2 = r^2 + a^2 + \frac{2ma^2}{r}\,, \qquad 2R(r)R'(r) = 2r - \frac{2ma^2}{r^2}$$

and thus

$$R'(r)^2 = \frac{\left(r - \dfrac{ma^2}{r^2}\right)^2}{r^2 + a^2 + \dfrac{2ma^2}{r}}\,.$$

Comparing coefficients of dr^2, and using our result for $R'(r)$, yields

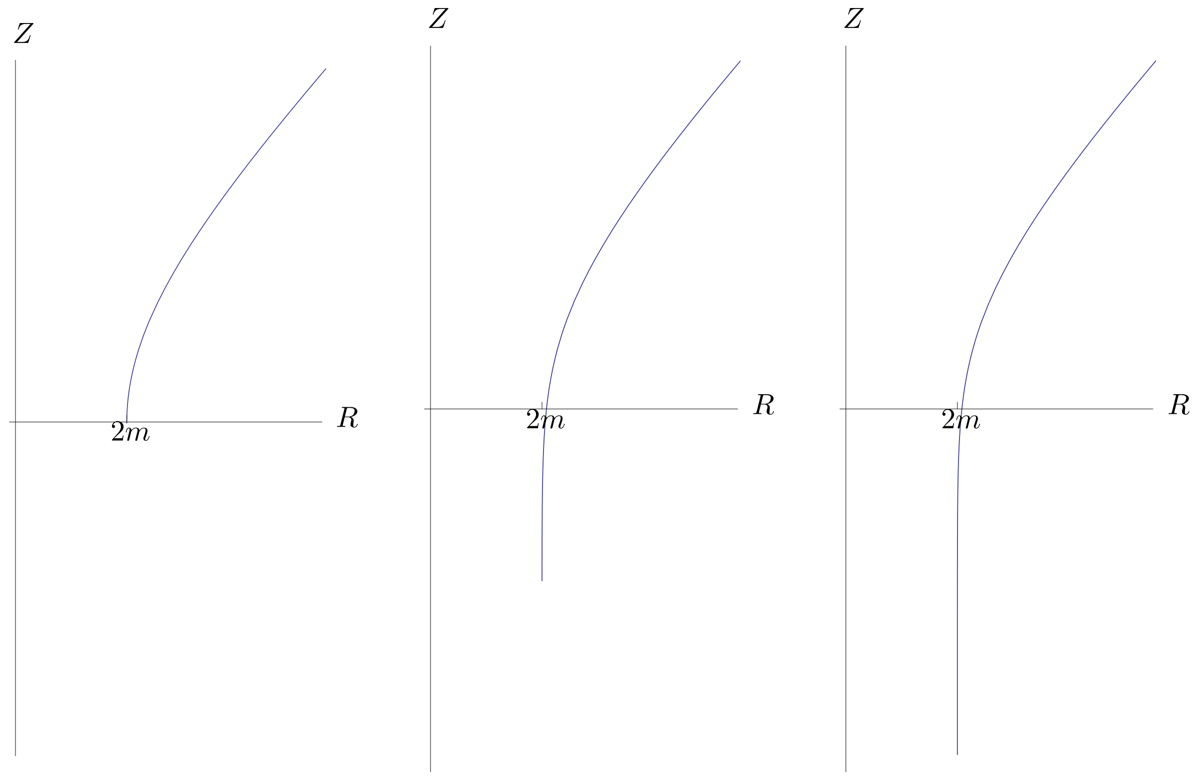
$$Z'(r)^2 = \frac{r^2}{\Delta} - R'(r)^2 = \frac{r^2}{r^2 + a^2 - 2mr} - \frac{\left(r - \dfrac{ma^2}{r^2}\right)^2}{r^2 + a^2 + \dfrac{2ma^2}{r}}$$

$$= \frac{r^2\left(r^2 + a^2 + \dfrac{2ma^2}{r}\right) - \left(r - \dfrac{ma^2}{r^2}\right)^2\left(r^2 + a^2 - 2mr\right)}{\left(r^2 + a^2 - 2mr\right)\left(r^2 + a^2 + \dfrac{2ma^2}{r}\right)}$$

$$= \frac{(r^2 + a^2)\left(\cancel{r^2} - \cancel{r^2} + \dfrac{2ma^2}{r} - \dfrac{m^2a^4}{r^4}\right) + 2mr\left(a^2 + r^2 - \dfrac{2ma^2}{r} + \dfrac{m^2a^4}{r^4}\right)}{\left(r^2 + a^2 - 2mr\right)\left(r^2 + a^2 + \dfrac{2ma^2}{r}\right)}$$

$$= \frac{\Delta\,\dfrac{ma^2}{r^4}\left(2r^3 - ma^2\right) + 2mr\left(a^2 + r^2\right)}{\Delta\left(r^2 + a^2 + \dfrac{2ma^2}{r}\right)}\,.$$

As $r > r_+$ implies $2r^3 > ma^2$, this expression is manifestly positive, so the surface $(t = \text{constant}, \vartheta = \pi/2, r > r_+)$ can be isometrically embedded into Euclidean 3 space. For $a = 0$ we get the Flamm paraboloid, recall Worksheet 2. With increasing a^2, the "neck" becomes longer and longer until in the limiting case $a^2 \to m^2$ it is actually infinitely long. (Note that $Z'(r)^2 \to \infty$ for $r \to m$ if $a^2 = m^2$.) This is in agreement with our finding from part (a) that in the extreme case the horizon is infinitely far away if measured along an r line. The pictures on the next page show the embedding diagram for $a = 0$ (left), $a = 0.999m$ (middle) and $a = 0.999999m$ (right). Note that in the ambient Euclidean space the horizon is always at $R = 2m$.



Problem 2:

Circular lightlike geodesics are in particular spherical lightlike geodesics, i.e., they satisfy $\dot{r} = 0$ and $\ddot{r} = 0$. We know from the lectures that these two conditions imply

$$c^2 \frac{K}{E^2} = \frac{4r^2 \Delta}{(r-m)^2} \,, \qquad \text{(S1)}$$

$$ac\frac{L}{E} = r^2 + a^2 - \frac{2r\Delta}{r-m} \,, \qquad \text{(S2)}$$

see eq. (364) and (365) in the lecture notes. The ϑ component of lightlike geodesics satisfies

$$\rho^4 \dot{\vartheta}^2 = K - \Big(\frac{L}{\sin\vartheta} - \frac{a}{c} \sin\vartheta \, E \Big)^2 \,,$$

see eq. (357) with $\varepsilon = 0$. For circular lightlike geodesics we must have $\dot{\vartheta} = 0$ and $\ddot{\vartheta} = 0$, hence

$$0 = K - \Big(\frac{L}{\sin\vartheta} - \frac{a}{c} \sin\vartheta \, E \Big)^2 \,,$$

$$0 = -2 \Big(\frac{L}{\sin\vartheta} - \frac{a}{c} \sin\vartheta \, E \Big) \Big(\frac{-L\cos\vartheta}{\sin^2\vartheta} - \frac{a}{c} \cos\vartheta \Big) \,.$$

These two equations can be rewritten as

$$0 = c^2 \frac{K}{E^2} - \frac{1}{a^2 \sin^2\vartheta} \Big(\frac{acL}{E} - a^2 \sin^2\vartheta \Big)^2 \,, \qquad \text{(C1)}$$

$$0 = \Big(\frac{acL}{E} - a^2 \sin^2\vartheta \Big) \cos\vartheta \Big(\frac{acL}{E} + a^2 \sin^2\vartheta \Big) \,. \qquad \text{(C2)}$$

A circular lightlike geodesic exists at coordinates (r, ϑ) if (C1) and (C2) are satisfied, with the constants of motion given by (S1) and (S2). As the right-hand side of (C2) is a product of three factors, we have to distinguish three cases.

Case A: $\dfrac{acL}{E} - a^2 \sin^2\vartheta = 0$.

Then (C1) requires $K = 0$ and (S1) can be satisfied only if $r\Delta = 0$. With the latter result, (S2) yields $acL/E = r^2 + a^2$. Equating the two expressions for acL/E gives $r^2 + a^2 = a^2 \sin^2\vartheta$ which is the equation for the ring singularity. As the latter is not part of the spacetime, Case A gives no circular lightlike geodesics.

Case B: $\cos\vartheta = 0$.

This will give us the circular lightlike geodesics in the equatorial plane, $\vartheta = \pi/2$. In this case, (C1) requires

$$0 = c^2 \frac{K}{E^2} - \frac{1}{a^2}\Big(\frac{acL}{E} - a^2 \Big)^2 .$$

Inserting (S1) and (S2) yields

$$0 = \frac{4r^2\Delta}{(r-m)^2} - \frac{1}{a^2}\Big(r^2 - \frac{2r\Delta}{r-m} \Big)^2 ,$$

$$0 = 4\cancel{r^2} a^2 \Delta - \Big(r^{\cancel{2}}(r-m) - 2\cancel{r}\,\Delta \Big)^2 .$$

Here dividing by r was allowed because, with our assumption $\cos\vartheta = 0$, we have $r \neq 0$ everywhere except at the ring singularity. The last equation reduces to

$$0 = \cancel{4a^2\Delta} - r^2(r-m)^2 + 4r(r-m)\Delta - 4\Delta(r^2 + \cancel{a^2} - 2mr) ,$$

$$0 = r^{\cancel{2}}(r^2 - 2mr + m^2)^2 - 4\Delta\Big(\cancel{r}\,(r-m) - \cancel{r}\,(r-2m) \Big) ,$$

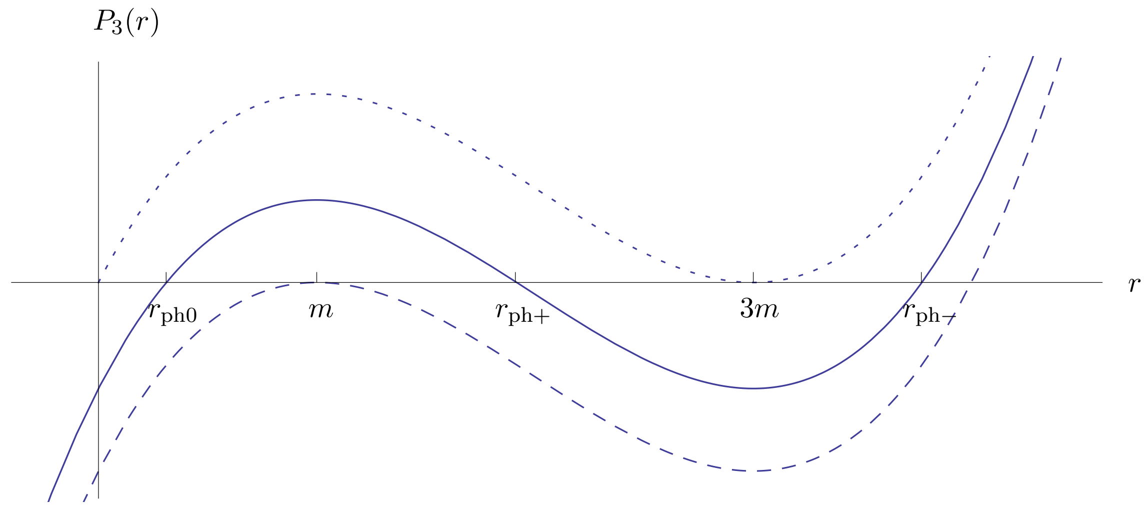
$$0 = r^3 - 2mr^2 + m^2 r - 4(r^2 + a^2 - 2mr)m ,$$

$$0 = r^3 - 6mr^2 + 9m^2 r - 4a^2 m := P_3(r) .$$

For $0 < a^2 < m^2$, the polyniomial $P_3(r)$ has three real zeros r_{ph0}, $r_{\mathrm{ph+}}$ and $r_{\mathrm{ph-}}$ where

$$0 < r_{\mathrm{ph0}} < m < r_{\mathrm{ph+}} < 3m < r_{\mathrm{ph-}} < 4m ,$$

see the plot (solid) below. The limiting cases $a = 0$ (dotted) and $a^2 = m^2$ (dashed) are also shown. The three circular lightlike geodesics can be seen in the picture of the photon region as well, see p. 99 of the lecture notes: They are situated at the intersection of the boundary of the photon region with the equatorial plane. All three are on the boundary of the blue region, i.e., they are unstable. Inspection of $\dot{\varphi}/\dot{t}$ shows that the one at $r_{\mathrm{ph+}}$ is co-rotating whereas the other two are counter-rotating. In the naked-singularity case, $a^2 > m^2$, there is only one counter-rotating photon orbit at $r_{\mathrm{ph-}} > 4m$.



Case C: $\dfrac{acL}{E} + a^2 \sin^2\vartheta = 0$.

In this case (C1) reduces to

$$0 = c^2 \frac{K}{E^2} + \frac{E}{acL}\left(\frac{2acL}{E}\right)^2 ,$$

$$0 = c^2 \frac{K}{E^2} + \frac{4acL}{E} . \qquad \text{(F1)}$$

The assumption of Case C can, thus, be rewritten as

$$\sin^2\vartheta = \frac{-cL}{aE} = \frac{c^2 K}{4a^2 E^2}$$

and, with (C1),

$$\sin^2\vartheta = \frac{r^2 \Delta}{(r-m)^2 a^2} . \qquad \text{(H1)}$$

This equation determines the ϑ coordinate of off-equatorial circular lightlike geodesics if the corresponding r coordinate is known.

For determining the possible values of r we further evaluate (F1) with the help of (S1) and (S2),

$$0 = \frac{\cancel{4}\, r^2 \Delta}{(r-m)^2} + \cancel{4}\left(r^2 + a^2 - \frac{2r\Delta}{r-m} \right) ,$$

$$0 = r^2 \Delta + (\Delta + 2mr)(r-m)^2 - 2r\Delta(r-m) ,$$

$$0 = \Delta\big(\cancel{r^2} + \cancel{r^2} - \cancel{2mr} + m^2 - \cancel{2r^2} + \cancel{2mr} \big) + 2mr(r-m)^2 ,$$

$$0 = \Delta\, m + 2r(r-m)^2 . \qquad \text{(F2)}$$

This allows us to rewrite (H1) as

$$\sin^2\vartheta = \frac{-2r^3}{m\, a^2} . \qquad \text{(H2)}$$

This equation clearly shows that off-equatorial circular lightlike geodesics exist only at negative r values. From (F2) we find

$$0 = (r^2 + a^2 - 2mr)\, m + 2r(r^2 - 2mr + m^2)\,,$$

$$0 = mr^2 + ma^2 - \cancel{2m^2r} + 2r^3 - 4mr^2 + \cancel{2m^2r}\,,$$

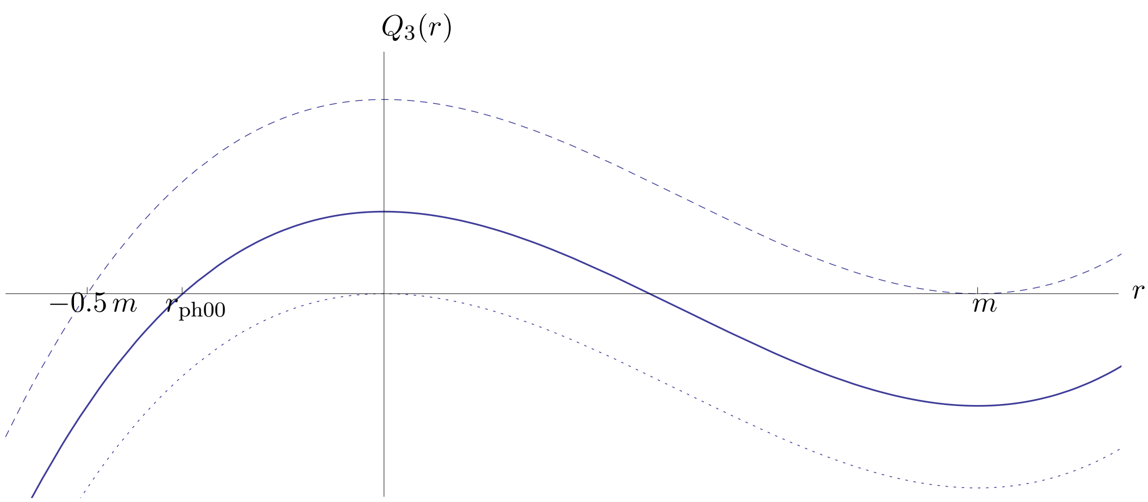
$$0 = 2r^3 - 3mr^2 + ma^2 =: Q_3(r)\,.$$

With this result (H2) can be rewritten as

$$\sin^2\vartheta = \frac{-3mr^2 + ma^2}{a^2 m} = \frac{-3r^2}{a^2} + 1\,,$$

$$\cos^2\vartheta = \frac{3r^2}{a^2}\,, \qquad \text{(H3)}$$

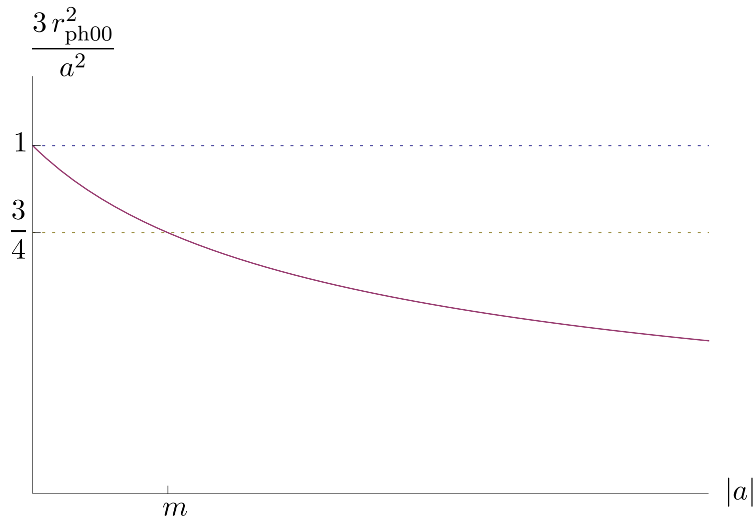
For $0 < a^2 < m^2$, the polynomial Q_3 has exactly one negative zero $r = r_{\rm ph00}$, see the plot below (solid). Again, the limiting cases $a = 0$ (dotted) and $a^2 = m^2$ (dashed) are also shown. We have $-0.5\,m < r_{\rm ph00} < 0$ with $r_{\rm ph00} \to 0$ for $a \to 0$ and $r_{\rm ph00} \to -0.5\,m$ for $a^2 \to m^2$.



The ϑ coordinate of a circular lightlike geodesic at $r_{\rm ph00}$ has to satisfy, by (H3),

$$\cos\vartheta_{\rm ph00} = \pm\sqrt{\frac{3r_{\rm ph00}^2}{a^2}}\,.$$

As this equation shows, there are two such geodesics, situated symmetrical with respect to the equatorial plane. In the limit $a^2 \to m^2$ the angle $\vartheta_{\rm ph00} - \pi/2$ approaches $\pm\pi/3$, i.e., $\cos^2\vartheta_{\rm ph00}$ approaches $3/4$; in the limit $a \to 0$, the angle $\vartheta_{\rm ph00} - \pi/2$ approaches $\pm\pi/2$, i.e. $\cos^2\vartheta_{\rm ph00}$ approaches 0 as can be verified with the rule of Bernoulli-l'Hôpital. The two off-equatorial circular photon orbits are also present in the naked-singularity case, $a^2 > m^2$.



The off-equatorial circular lightlike geodesics can be seen in the picture of the photon region on p. 99: They are situated where the blue region is tangential to a sphere about the origin ($r = -\infty$).

We summarise our observations in the following way: In a Kerr black-hole spacetime with $0 < a^2 < m^2$ there are five circular lightlike geodesics: Three in the equatorial plane and two off the equatorial plane. One equatorial and both off-equatorial circular lightlike geodesics are hidden behind the horizon for an outside observer. In the spacetime of a Kerr naked singularity, $a^2 > m^2$, only one counter-rotating equatorial and the two off-equatorial photon orbits survive.

Problem 3:

We write $\dot{t} = dt/d\tau$ where t is coordinate time and τ is proper time. For a particle orbiting at radius r in the equatorial plane we have, by equation (341) from the lecture notes,

$$r^2\,\dot{t} = \frac{\Big((r^2+a^2)r^2 + 2mra^2\Big)E - 2mracL}{c^2\Delta}\,.$$

We consider an extreme black hole where we assume, without loss of generality, that a is positive. (This can always be achieved by a coordinate transformation $\varphi \to -\varphi$.)Then $a = m$ and the equation for $\dot{t}$ reduces to

$$r^2\,\dot{t} = \frac{E}{c^2(r-m)^2}\left(r^4 + m^2r^2 + 2m^3r - 2m^2r\frac{cL}{E}\right).$$

For a particle in circular geodesic motion in the equatorial plane, the expressions for the constants of motion L and E in terms of the radius coordinate were given in the lecture notes, see eqs. (382) and (383). For $a = m$ these equations specify to

$$\frac{cL}{E} = \frac{\pm\sqrt{mr^3}\,(r-m)^2 - m^4 - m^2r(3r-4m)}{r(r-2m)^2 - m^3}\,,$$

$$\frac{E^2}{c^4} = \frac{r^3(r-3m)(r-2m)^2 - m^3r^2(3r-5m) \pm 2m^2\sqrt{mr^3}\,(r-m)^2}{\big(r(r-2m)^2 - m^3\big)^2}\,.$$

As we assume co-rotating motion, we choose the upper sign in the following. The expressions for cL/E and E^2/c^4 can be simplified by dividing numerator and denominator by an appropriate factor:

$$\frac{cL}{E} = \frac{\sqrt{mr}(r+m) + m(r-m)}{r - m + \sqrt{mr}}$$

$$\frac{E^2}{c^4} = \frac{r\big(r + 2\sqrt{mr}\big)}{(r - m + \sqrt{mr})^2}\,.$$

(I have let MATHEMATICA do this, but it is not too difficult to verify the result by hand.) Inserting cL/E into the expression for $\dot{t}$ yields

$$\begin{aligned}
r^2\,\dot{t} &= \frac{E}{c^2(r-m)^2}\left(r^4 + m^2r^2 + 2m^3r - 2m^2r\frac{\sqrt{mr}(r+m) + m(r-m)}{r-m+\sqrt{mr}}\right) \\
&= \frac{E}{c^2(r-m)^2}\left(r^4 + m^2r^2 + \cancel{2m^3r} - 2m^2r\left\{\frac{\sqrt{mr}\,r}{r-m+\sqrt{mr}} + \cancel{m}\right\}\right) \\
&= \frac{E\,r^2\left(\big(r^2+m^2\big)(r-m) + \big(r^2 + \cancel{m^2}\big)\sqrt{mr} - \cancel{2}m^2\sqrt{mr}\right)}{c^2(r-m)^2\,\big(r-m+\sqrt{mr}\big)} \\
&= \frac{E\,r^2\left(\big(r^2+m^2\big)\cancel{(r-m)} + \sqrt{mr}\,(r+m)\,\cancel{(r-m)}\right)}{c^2(r-m)^{\cancel{2}}\,\big(r-m+\sqrt{mr}\big)}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{t}^2 &= \frac{E^2\left(r^2 + m^2 + \sqrt{mr}\,(r+m)\right)^2}{c^4(r-m)^2\,\big(r-m+\sqrt{mr}\big)^2} \\
&= \frac{r\,\big(r+2\sqrt{mr}\big)\left(r^2 + m^2 + \sqrt{mr}\,(r+m)\right)^2}{(r-m)^2\,\big(r-m+\sqrt{mr}\big)^4}\,.
\end{aligned}$$

As the radius is a constant, integration over a coordinate time interval Δt yields, for the observer at r_1 and r_2, respectively,

$$\left(\frac{\Delta t}{\Delta\tau_1}\right)^2 = \frac{r_1\big(r_1 + 2\sqrt{mr_1}\big)\left(r_1^2 + m^2 + \sqrt{mr_1}\,(r_1+m)\right)^2}{(r_1-m)^2\,\big(r_1 - m + \sqrt{mr_1}\big)^4}\,,$$

$$\left(\frac{\Delta t}{\Delta\tau_2}\right)^2 = \frac{r_2\big(r_2 + 2\sqrt{mr_2}\big)\left(r_2^2 + m^2 + \sqrt{mr_2}\,(r_2+m)\right)^2}{(r_2-m)^2\,\big(r_2 - m + \sqrt{mr_2}\big)^4}\,,$$

hence

$$\left(\frac{\Delta\tau_2}{\Delta\tau_1}\right)^2 = \frac{r_1\big(r_1 + 2\sqrt{mr_1}\big)\left(r_1^2 + m^2 + \sqrt{mr_1}\,(r_1+m)\right)^2(r_2-m)^2\,\big(r_2 - m + \sqrt{mr_2}\big)^4}{r_2\big(r_2 + 2\sqrt{mr_2}\big)\left(r_2^2 + m^2 + \sqrt{mr_2}\,(r_2+m)\right)^2(r_1-m)^2\,\big(r_1 - m + \sqrt{mr_1}\big)^4}\,.$$

For $r_2 \to \infty$ we have

$$\frac{\Delta t}{\Delta \tau_2} \to 1\,,$$

whereas for $r_1 \to r_{\mathrm{ISCO}} = m$ we have

$$\frac{\Delta t}{\Delta \tau_1} \to \infty\,,$$

hence

$$\frac{\Delta \tau_2}{\Delta \tau_1} \to \infty\,.$$

This demonstrates that an observer on a co-rotating stable geodesic orbit can undergo arbitrarily large gravitational time-dilation effects with respect to an observer who stayed farther away from the black hole. This is what happened to the astronauts in the movie *Interstellar* on Miller's planet. Recall from Worksheet 4 that in a stable orbit around a Schwarzschild black hole no strong gravitational time-dilation effects can occur. Also note that for counter-rotating stable orbits the gravitational time-dilation effects in the field of an extreme Kerr black hole are even weaker than for a Schwarzschild black hole.