6. Archetypical QFT example: Quasi-degenerate quartic potential

$$U(\Phi) = \frac{1}{2} m_\Phi^2 \Phi^2 + \frac{g}{3!}\Phi^3 + \frac{\lambda}{4!}\Phi^4 + U_0$$

where $m_\Phi^2 = -\mu^2 < 0$

With

$$v = \sqrt{\frac{6\mu^2}{\lambda}} \qquad \bar{v} = \frac{3g}{2\lambda} \longrightarrow v_\pm = \pm v \sqrt{1 + \frac{\bar{v}^2}{v^2}} - \bar{v}$$

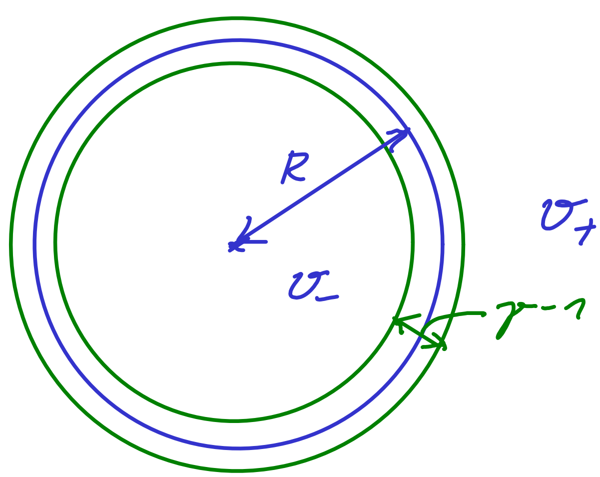
$$\varepsilon = \frac{U(v_+) - U(v_-)}{2} = \frac{g v^3}{6}\left(1 + \frac{\bar{v}^2}{v^2}\right)^{\frac{3}{2}}$$

The quasi-degenerate case occurs when $\frac{g^2}{\mu^2} \ll \frac{8\lambda}{3}$. Above terms can then readily be expanded and approximated. Further,

$$U(v_+) - U_0 \approx U(v) - U_0 \approx -\frac{3}{2}\frac{\mu^4}{\lambda} + \sqrt{6}\, g \left(\frac{\mu^2}{\lambda}\right)^{\frac{3}{2}}$$

$$\longrightarrow U_0 = \left(\frac{\mu v}{2}\right)^2 - \frac{g v^3}{6}$$

Thin wall bubble



Bounce

$$-\frac{d^2}{dr^2}\varphi - \frac{3}{r}\frac{d}{dr}\varphi + U'(\varphi) = 0$$

$\longrightarrow$ For $g=0$, this is solved by the kink

$$\varphi(r) = v \tanh\left(\gamma(r-R)\right) = v u$$

where $\gamma = \frac{\mu}{\sqrt{2}}$, and note $\frac{dr}{du} = \frac{1}{\gamma}\frac{1}{1-u^2}$

The parameter R is determined by extremizing the action, which we decompose into surface tension and latent heat:

$$B[\varphi] = \int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + U(\varphi)\right]$$

$$\approx 2\pi^2 \int_0^{R+\Delta R} r^3\, dr \left[\frac{1}{2}(\partial_r\varphi)^2 + U(\varphi)\right]$$

$$= \underbrace{2\pi^2 \int_0^{R-\Delta R} r^3\, dr\; U(v_-)}_{\text{latent heat}} + \underbrace{2\pi^2 \int_{R-\Delta R}^{R+\Delta R} r^3\, dr \left(\frac{d\varphi}{dr}\right)^2}_{\text{surface tension}}$$

where $\gamma^{-1} \ll \Delta R \ll R$ $\longrightarrow$ Thin wall approximation

The latent heat can readily be evaluated as

$$2\pi^2 \frac{R^4}{4}\left(-\frac{1}{3} g v^3\right)$$

As for the surface tension

$$2\pi^2 \int_{R-\Delta R}^{R+\Delta R} r^3 \, dr \left(\frac{d\varphi}{dr}\right)^2 \approx 2\pi^2 R^3 \int_{-v}^{v} d\varphi \, \frac{d\varphi}{dr}$$

$$= 2\pi^2 R^3 \int_{-v}^{v} d\varphi \, \frac{1}{\sqrt{2}} \mu v \left(1 - \frac{\varphi^2}{v^2}\right) = \frac{8\pi^2}{3\sqrt{2}} R^3 \mu v^2$$

$$\uparrow$$

$$\frac{d\varphi}{dr} = \sqrt{2U(\varphi)} \qquad\qquad = \frac{8\pi^2}{3\sqrt{2}} R^3 \mu \, 6 \frac{\mu^2}{\lambda} = \frac{16\pi^2}{\sqrt{2}\lambda} R^3 \mu^3$$

$$U(\varphi) = \frac{\mu^2 v^2}{4} \left(1 - \frac{\varphi^2}{v^2}\right)^2$$

$\Longrightarrow$ Bounce action:

$$B = -\frac{1}{6} \pi^2 R^4 g v^3 + \frac{8\pi^2}{3\sqrt{2}} R^3 \mu v^2$$

Critical radius:

$$\frac{\partial B}{\partial R} = 0 \Longrightarrow R = \frac{8\pi^2}{\sqrt{2}} \mu v^2 \, \frac{3}{2} \, \frac{1}{\pi^2} \, \frac{1}{g v^3} = \frac{12}{\sqrt{2}} \, \frac{\mu}{g v}$$

$$= 2 \, \frac{\sqrt{3\lambda}}{g}$$

$\longrightarrow$ back in B:

$$B\left(R = 2 \frac{\sqrt{3\lambda}}{g}\right) = \frac{8\pi^2 R^3 g^3}{\lambda}$$

Negative mode:

Since B is maximal for the critical radius R, the negative mode is associated with dilatations, the negative mode is given by:

$$\phi_0^{j=0} = \frac{1}{\sqrt{B}} \partial_r \varphi = - \frac{1}{\sqrt{B}} \partial_R \varphi$$

$\longrightarrow$ Eigenvalue:

$$\lambda_0 = \frac{\delta^2 B}{\delta(\phi_0^{j=0})^2} = \frac{1}{B} \frac{\delta^2 B}{\delta(\partial_R \varphi)^2} = \frac{1}{B} \frac{\partial B}{\partial R^2} = - \frac{3}{R^2}$$

Other way to see this:
At the bubble wall, this mode is of the same form as the translational mode, where we know that

$$\left[-\frac{d^2}{dr^2} - \frac{3}{r} \frac{d}{dr} + \frac{3}{r^2} + U''(\varphi) \right] \partial_r \varphi = 0$$

$\longrightarrow$ with $\frac{3}{r^2} \longrightarrow \frac{3}{R^2}$ we confirm the negative eigenvalue to be $\lambda_0 = - \frac{3}{R^2}$

Solving for the radial Green's functions:

Set $d=4$ and $r=R$ in the centrifugal term of the general expression, neglect $\frac{1}{r}\frac{d}{dr}$-derivative

$\longrightarrow$

$$\left[-\frac{d^2}{dr^2}+\frac{j(j+2)}{R^2}+U''(\varphi)+s\right]G_j(\varphi;r,r')=\frac{1}{R^3}\delta(r-r')$$

We have also reintroduced here the parameter s so that we can readily apply this to the resolvent as well.

Given the tanh profile of the kink, the differential operator corresponds to the one in the Pöschl-Teller problem. The solutions take a particularly simple form in the coordinate

$u=\tanh(\gamma(r-R))$.

cf. Lifschitz.xoj for x_+-Green's function

$$\gamma=\frac{\mu}{\sqrt{2}}$$

Further, define

$$G(u,u',m)=R^3G_j(\varphi;r,r')$$

$$m=2\sqrt{1+\frac{j(j+2)}{2\mu^2R^2}+\frac{s}{2\mu^2}}$$

$$=\sqrt{4+\frac{2j(j+2)}{2\mu^2R^2}+\frac{2s}{\mu^2}}$$

$$m=2\sqrt{1+\frac{j(j+2)}{4\gamma^2R^2}+\frac{s}{4\gamma^2}}$$

$$n=2$$

$\longrightarrow$

$$\left[\frac{d}{du}(1-u^2)\frac{d}{du}-\frac{m^2}{1-u^2}+n(n+1)\right]G(u,u',m)=-\frac{1}{\gamma}\delta(u-u')$$

This is the associated Legendre equation, and the solution therefore can be written in terms of the associated Legendre function $P_n^m(u)$ and $Q_n^m(u)$.

Following the procedure of matching and normalization to the δ-function, we obtain

$$G(u,u';m)=\frac{1}{2\gamma m}\left[\theta(u-u')\left(\frac{1-u}{1+u}\right)^{\frac{m}{2}}\left(\frac{1+u'}{1-u'}\right)^{\frac{m}{2}}\left(1-3\,\frac{(1-u)(1+m+u)}{(1+m)(2+m)}\right)\right.$$
$$\left.*\left(1-3\,\frac{(1-u')(1-m+u')}{(1-m)(2-m)}\right)+(u\leftrightarrow u')\right]$$

It is possible (but not necessary) to solve the spectrum analytically. For each j, the modes with

$\lambda=\frac{j(j+2)-3}{R^2}$ and $\lambda=3\gamma^2+\frac{j(j+2)-3}{R^2}$ are discrete,

and there is a continuum for $\lambda\geq 4\gamma^2\frac{j(j+2)-3}{R^2}$.

For the normalization, we need the false-vacuum Green's function (i.e. for $\varphi=v=\text{const.}$), which is

$$G_F(u,u';m)=\frac{1}{2\gamma m}\left[\theta(u-u')\left(\frac{1-u}{1+u}\right)^{\frac{m}{2}}\left(\frac{1+u'}{1-u'}\right)^{\frac{m}{2}}+(u\leftrightarrow u')\right]$$

To compute the determinant, we need the coincident limits

$$G(u,u,m)=\frac{1}{2\gamma m}\left[1+3(1-u^2)\sum_{n=1}^{2}\frac{(-1)^n(n-1-u^2)}{m^2-n^2}\right]$$

$$G_F(u,u,m)=\frac{1}{2\gamma m}$$

For $j=0,1$, we identify in $G(u,u,m)$ the simple poles for $s\to 0$, $m^2 \to 4$ with the negative and the translational mode (note that we have taken $\frac{1}{R}\frac{d}{d\tau}\to 0$ in the computation of the Green's function so that the negative mod yields a singularity). To obtain G' we therefore drop the term with the leading $\frac{1}{s}$ behaviour. For $j=0,1$, we therefore have

$$G'(u,u,m) = \frac{1}{2\gamma m}\left[1+3(1-u^2)\left(\sum_{n=1}^{2}\frac{(-1)^n(n-1-u^2)}{m^2-n^2} - \frac{m}{2}\frac{(1-u^2)}{m^2-4}\right)\right]$$

We could have constructed G' directly as the Green's function for the subspace without the zero mode, as discussed previously. That method can also be implemented numerically. Here, we choose however to make use of the analytic structures to avoid this laborious step.

$\longrightarrow$

$$G'(u,u,m) - G_F(u,u,m) = -\frac{3}{4\gamma m}\frac{(1-u^2)(1+3u^2+2u^2m-m^2+u^2m^2)}{m(1-m^2)(2+m)}$$

$\longrightarrow$

$$I(s) = \int_{-1}^{1} d\tau\,[G'-G_F] = \int_{-1}^{1}\frac{du}{\gamma(1-u^2)}[G'-G_F] = \frac{m^2-m-3}{\gamma^2 m(1-m^2)(2+m)}$$

$\longrightarrow$

$$PV\int_0^{\Lambda^2} ds\, I(s) = -\log\frac{48\gamma^2}{\Lambda^2} = -\log\frac{24\mu^2}{\Lambda^2}$$

As remarked earlier, there is an extra factor of Λ^2 in the determinants that is to be discarded. This result can also be obtained from the Gel'fand-Yaglom theorem.

$$\frac{\det' G^{-1}_{j=0}}{\det G^{-1}_{F\,j=0}}\left(\frac{\det' G^{-1}_{j=1}}{\det G^{-1}_{F\,j=1}}\right)^4 = (24\mu^2)^{-5}$$

Note that we differ here from e.g. 1501.07466 where this factor is taken to be $(\mu^2)^5$ as these are the lowest eigenvalues of G_F. Since these are part of a continuum, there is apparently a measure problem invalidating that argument.

Planar wall approximation

Since $R \gg \gamma^{-1}$, we may as well approximate

$$\frac{j(j+2)}{R^2} \approx k$$

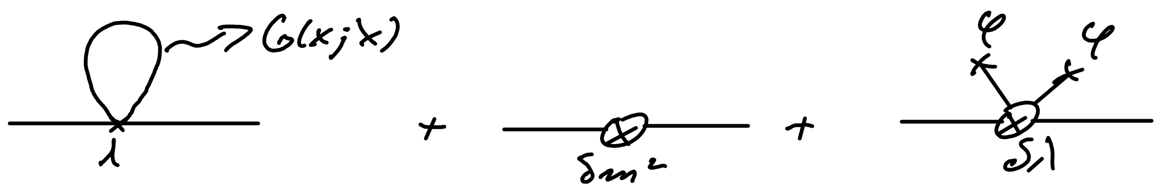
where k is the momentum parallel to the wall. In particular we neglect corrections due to its curvature.

Numerically, it is well possible to work without the thin wall/planar wall approximation, as it has been carried out e.g. for the Fubini–Lipatov instanton that is a first approximation to the decay of the metastable Standard Model vacuum.

In the planar wall approximation, we hence replace

$$\frac{1}{2\pi^2}\sum_{j=1}^{\infty}(j+1)\,U_j(\cos\vartheta=1)=\frac{1}{2\pi^2}\sum_{j=1}^{\infty}(j+1)^2=\frac{R^3}{2\pi^2}\int_0^{\infty}k^2\,dk$$

Given that we are considering Φ^4 QFT, these momentum integrals are ultraviolet divergent and require renormalization.
To see how to do this, notice that $\frac{\lambda}{2}G(x,x)$ gives rise to a tadpole insertion



There is a mass-correction δm^2 independent of φ as well as a correction $\delta\lambda$ to the coupling $\propto\varphi^2$
In addition, there is also a wave-function correction $\propto(\partial_r\varphi)^2$ that we do not need to consider here because it is UV finite.

Recall here that in our coordinates

$$\varphi(u)=v\,u=\sqrt{\frac{6\mu^2}{\lambda}}\,u=\sqrt{\frac{12\gamma^2}{\lambda}}\,u$$

The coincident Green's function in the planar wall approximation

$$G(\varphi; x, x) = \frac{1}{2\pi^2} \sum_{j=0}^{\infty} \underbrace{(j+1)\, u_j(1)}_{=(j+1)^2} \; \underbrace{G_j(\varphi; r, r)}_{\to \frac{1}{R^3} G(u,u,m)}$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} k^2 dk \; G(u,u,m)$$

Regulating with a cutoff K, we obtain the tadpole

$$\Pi(u) = \frac{\lambda}{2} \frac{1}{2\pi^2} \int_0^{K} k^2 dk \;\; G(u,u,m)$$

$$= \frac{\lambda}{16\pi^2} K^2 - \frac{\lambda}{16\pi^2} \gamma^2 (1 - 3u^2) \log \frac{\gamma^2}{K^2} + \frac{\lambda}{16\pi^2} \gamma^2 \left(2 - \sqrt{3}\, \pi\, u^2 (1-u^2)\right)$$

Noting $u^2 \gamma^2 = \frac{\lambda}{12} \varphi^2$

$\longrightarrow$ Counter terms (minimal subtraction)

$$\delta m^2 = -\frac{\lambda}{16\pi^2} K^2 + \frac{\lambda}{16\pi^2} \gamma^2 \log \frac{\gamma^2}{K^2}$$

$$\delta\lambda = -\frac{3}{8\pi^2} \frac{1}{12} \lambda^2 \log \frac{\gamma^2}{K^2}$$

$\longrightarrow$

$$\Pi^{ren}(u) = \Pi(u) + \delta m^2 + \frac{1}{2} \delta\lambda \; \frac{12 u^2 \gamma^2}{\lambda}$$

$$= \frac{\lambda}{16\pi^2} \gamma^2 \left(2 - \sqrt{3}\, \pi\, u^2 (1-u^2)\right)$$

Functional determinant ($j>1$ part):

$$\log \sqrt{\frac{\det G_F^{-1}}{\det G^{-1}}}$$

$$= \frac{1}{2}\frac{1}{2\pi^2}\int_0^K k^2 dk \int_0^{\lambda^2} ds \; 2\pi^2 R^3 \int_{-1}^{1} du \, \frac{1}{\gamma(1-u^2)} \left[G(u,u,m) - G_F(u,u,m) \right]$$

$$= 2\pi^2 R^3 \left(\frac{3}{4\pi^2}\gamma K^2 + \frac{3}{4\pi^2}\gamma^3 \log \frac{\gamma^2}{K^2} + \frac{\gamma^3}{4\sqrt{3}\,\pi} \right)$$

Contributions from counter terms

$$2\pi^2 R^3 \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2}\delta m^2(\varphi^2 - v^2) + \frac{\delta\lambda}{4!}(\varphi^4 - v^4) \right)$$

$$= 2\pi^2 R^3 \int_{-1}^{1} \frac{du}{1-u^2}\frac{1}{\gamma} \left(\frac{1}{2}\delta m^2 \, 12\,(u^2-1)\frac{\gamma^2}{\lambda} + \frac{\delta\lambda}{24} \, \underbrace{144(u^4-1)}_{=\int_{-1}^{1} du \frac{u^4-1}{u^2-1} = \frac{8}{3}}\frac{\gamma^4}{\lambda^2} \right)$$

$$= 2\pi^2 R^3 \left(12\gamma\,\delta m^2 - 6\,\delta\lambda\,\gamma^4 \cdot \frac{8}{3} \right)$$

$$= 2\pi^2 R^3 \left(-\frac{3}{4\pi^2}\gamma K^2 + \frac{3}{4\pi^2}\log\frac{\gamma^2}{K^2} - \frac{3}{2\pi^2}\log\frac{\gamma^2}{K^2} \right)$$

$$\longrightarrow \log^{(\mathrm{ren})} \sqrt{\frac{\det G_F^{-1}}{\det G^{-1}}} = 2\pi^2 R^3 \frac{\gamma^3}{4\sqrt{3}\,\pi}$$

In total:

$$\frac{\Gamma}{V} = \left(\frac{B}{2\pi}\right)^2 e^{-B} \sqrt{\frac{R^2}{3}} \sqrt{\frac{\det G_F^{-1}}{\det^{(5)} G^{-1}}}^{\,(\text{ren})}$$

$$= \left(\frac{4\pi R^3 \gamma^3}{\lambda}\right)^2 e^{-\frac{8\pi^2 R^3 \gamma^3}{\lambda}} \quad \underbrace{2^{10} 3^2 \sqrt{3}}_{=48^{\frac{5}{2}}} \gamma^5 \frac{R}{\sqrt{3}} \quad e^{\frac{2\pi^2 R^3 \gamma^3}{4\sqrt{3}\pi}}$$

↪ from negative mode

We have used here the fact that the normalized zero modes in the thin-wall approximation are $\frac{1}{\sqrt{B}} \frac{d\varphi}{d\tau}$ since the equation of motion for the bounce agrees with the QM case for $R \longrightarrow \infty$.
Further, $\det^{(5)}$ is a short hand notation for the negative and the four zero modes being deleted.

Correction to the bounce:

Given the renormalized tadpole, we can find a one-loop improved bounce that satisfies the equation of motion

$$\left[\frac{d^2}{d\tau^2} + \mu^2 - \frac{\lambda}{2}\varphi^2 - \Pi^{\text{ren}} \varphi^2\right](\varphi + \delta\varphi) = 0$$

$\Longrightarrow$

$$\left[\frac{d^2}{d\tau^2} + \mu^2 - \frac{\lambda}{2}\varphi^2\right]\delta\varphi = \Pi^{\text{ren}} \varphi$$

We can solve this with the help of the Green's function for $k=0$, i.e. $m=2$.

$$\delta\varphi(u) = \int_{-1}^{1} \frac{du'}{\gamma(1-u')}\, G(u,u',m=2)\, \Pi^{ren}(u')\, \varphi(u')$$

While $G(u,u',m)$ is singular for $m=2$, it turns out that this only affects the contributions even in u'. Since $\Pi(u')$ and the integration measure are even in u' and $\varphi(u')$ is odd, the integral can nonetheless be carried out. After some calculation, this leads to

$$\delta\varphi(u) = \frac{3\lambda v}{16\pi^2}\,\frac{\pi}{\sqrt{3}}\,\frac{1-u^2}{8}\left(\log\frac{1+u}{1-u} - \frac{4}{3}u\right)$$

One can obtain from $\varphi + \delta\varphi$ improved Green's functions. Getting a self-consistent system of improved bounce & Green's functions amounts to a one-loop resummation. This is of relevance for models with radiative symmetry breaking or with (approximate) classical scale invariance.