5. Tunneling in quantum field theory

Generalize calculation from Euclidean path integral from QM to QFT.

Lagrangian (Minkowski):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - U(\Phi)$$

Let $U(\Phi)$ have a false vacuum v_+
and a true vacuum v_-

Euclidean equation of motion:

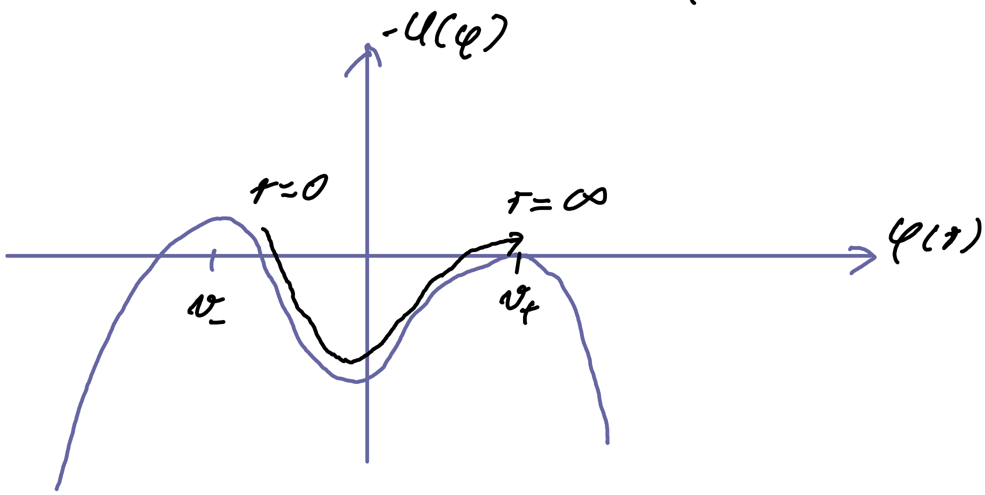
$$-\partial^2 \varphi(x) + \frac{\partial U(\varphi(x))}{\partial \varphi(x)} = 0$$

Bounce configuration:

$$\varphi(x_4 \to \pm\infty) = v_+$$
$$\dot{\varphi}(x_4 = 0) = 0$$

Assuming spherical symmetry, the equation of motion is

$$-\partial_r^2 \varphi(r) + \overbrace{\frac{d-1}{r}}^{\to 3} \partial_r \varphi(r) + \frac{\partial U(\varphi(r))}{\partial \varphi(r)} = 0$$



To solve for the spherical symmetric bounce, find the initial condition $\varphi(r=0)$ so that we comply with $\varphi(r \to \infty) = v_+$. The additional condition $\varphi'(r)|_{r=0}$ follows from the requirement of avoiding a singularity in the equations of motion. Note that this implies for $U(v_+)=0$ that $-U(\varphi(r=0)) > 0$.
The "effective friction" term $\frac{3}{r} \partial_r \varphi(r)$ damps the rolling of the field for $r \to 0$.
When there are no analytic solutions, the initial condition can be efficiently determined through interval bisection according to under/overshoots.

Euclidean action

$$B = \int d^4x \left[\frac{1}{2} \left(\frac{d\varphi}{dr} \right)^2 + U(\varphi) \right]$$

$$= 2\pi^2 \int_0^\infty r^3 dr \left[\frac{1}{2} \left(\frac{d\varphi}{dr} \right)^2 + U(\varphi) \right]$$

This is the saddle point value — to determine the partition function we need to consider the fluctuations.

Given the spherical symmetry of the bounce, we can express the eigenmodes as

$$\phi_n^{j\{l\}}(x) = \phi_n^j(r)\, Y_{j\{l\}}(\hat{e}_x)$$

where $\{l\} = \{l_1, l_2, \dots, l_{d-2}\}$

$\longrightarrow$

Eigenvalue equation in $d=4$ dimensions:

$$\left[-\frac{d^2}{dr^2}-\frac{3}{r}\frac{d}{dr}+\frac{j(j+2)}{r^2}+U''(\varphi)\right]\phi_n^j=\lambda_n^j\,\phi_n^j$$

Translational zero modes: Take derivative of equation of motion

$$\frac{d}{dr}\left[-\frac{d^2}{dr^2}\varphi-\frac{3}{r}\frac{d}{dr}\varphi+U'(\varphi)\right]=0$$

$$=\left[-\frac{d^2}{dr^2}-\frac{3}{r}\frac{d}{dr}+\frac{3}{r^2}+U''(\varphi)\right]\frac{d\varphi}{dr}=0$$

$\Longrightarrow \phi_0^{j=1}=\frac{1}{\sqrt{B}}\frac{d\varphi}{dr}$ is the zero eigenmode

To show the normalization, consider the scale transformation

$$\varphi(x)\longrightarrow\varphi((1+\varepsilon)x)$$

$$\Longrightarrow \partial_\mu\varphi((1+\varepsilon)x)=\frac{1}{1+\varepsilon}\partial_\mu\varphi(x)\approx\partial_\mu\varphi(x)-\varepsilon\,\partial_\mu\varphi(x)$$

$$U(\varphi((1+\varepsilon)x))\approx U(\varphi(x))+U'(\varphi(x))\,\varepsilon x_\mu\,\partial_\mu\varphi(x)$$

$$=U(\varphi(x))+\frac{\partial U(\varphi(x))}{\partial x_\mu}\,\varepsilon\,x_\mu$$

In the action, we can integrate partially and can replace

$$U(\varphi((1+\varepsilon)x)) \to U(\varphi(x)) - 4\varepsilon\, U(\varphi(x))$$

The boundary terms can be neglected because $U(\varphi(x)) \longrightarrow U(v_+) = 0$ for $|x| \longrightarrow \infty$.

Since φ is a saddle, the action is stationary under the scale transformation such that we obtain

$$\int d^4x \left(\partial_\mu\varphi\,\partial_\mu\varphi + 4\,U(\varphi)\right) = 0$$

$$\Longrightarrow B = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\,\partial_\mu\varphi + U(\varphi)\right) = \frac{1}{4}\int d^4x\, \partial_\mu\varphi\,\partial_\mu\varphi$$

Hence, $\int d^4x\, \partial_\xi\varphi\,\partial_\xi\varphi \Big|_{\substack{\text{no summation}\\ \text{over } \xi}} = B$ so that the normalization factor is $\frac{1}{\sqrt{B}}$.

We can now generalize the QM result as

$$\frac{\Gamma}{V} = \left(\frac{B}{2\pi}\right)^2 e^{-B}\ \mathrm{Im}\ \sqrt{\frac{\det G^{-1}(v_+)}{\det^{(4)} G^{-1}(\varphi)}}$$

Here, $\det^{(4)}$ implies that we have omitted the four translational zero modes. Integration over the pertaining collective coordinates yields the factor VT. The factor T cancels with the one in ET, V appears on the LHS of above expression so that we have the correct dimension of a decay rate per unit volume.

The determinants can be in principle evaluated for each j using the Gel'fand Yaglom theorem. In order not to brick the possibility of going beyond one-loop order and to potentially carry out resummation such as it is necessary for classically scale invariant potentials (with the Standard Model an important speimen), we pursue here the method of resolvents and Green's functions.

That is, we need to solve

$$(-\Delta^{(d)} + U''(\varphi))\, G^{(d)}(\varphi; x, x') = \delta^{(d)}(x-x')$$

where

$$\Delta^{(d)} = r^{1-d}\partial_r r^{d-1}\partial_r + \Delta_{S^{d-1}}$$

$\longrightarrow$ Separation ansatz

$$G^{(d)}(\varphi; x, x') = \sum_{j\{\ell\}} G_j^{(d)}(\varphi; r, r')\, Y^*_{j\{\ell\}}(\hat{e}_{r'})\, Y_{j\{\ell\}}(\hat{e}_r)$$

where

$$\Delta_{S^{d-1}} Y_{j\{\ell\}} = -j(j+d-2)\, Y_{\{\ell\}}$$

$\longrightarrow$

$$\left[-r^{1-d}\frac{d}{dr} r^{d-1}\frac{d}{dr} + \frac{j(j+d-2)}{r^2} + U''(\varphi)\right] G_j^{(d)}(\varphi; r, r') = r^{1-d}\,\delta(r-r')$$

Since the $\{l\}$ are degenerate, it is possible to carry out the sum over the hyperspherical harmonics. In $d=4$ one obtains

$$G^{(4)}(\varphi; x, x') = \frac{1}{2\pi^2} \sum_{j=0}^{\infty} (j+1)\, U_j(\cos\vartheta)\, G_j^{(4)}(\varphi; r, r')$$

where U_j are the Cebyshev polynomials. In $d=3$ one would obtain the familiar Legendre polynomials instead.

Expanding bubbles

The Euclidean solutions

$$-\partial^2 \varphi(x) + \frac{\partial U(\varphi(x))}{\partial \varphi(x)} = 0$$

also solve the Minkowski-space equation motion

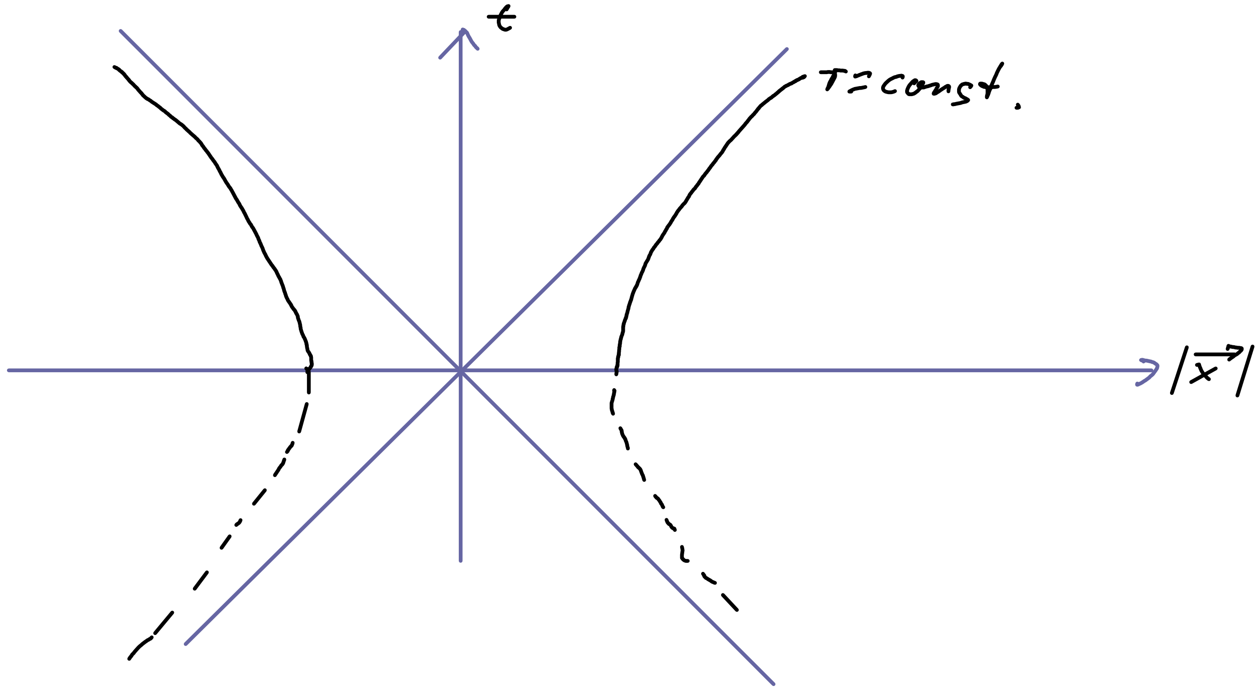
$$\frac{\partial^2}{\partial t^2} \varphi(\vec{x}, t) - \vec{\nabla}^2 \varphi(\vec{x}, t) + \frac{\partial U(\varphi(\vec{x}, t))}{\partial \varphi(\vec{x}, t)} = 0$$

where

$$\varphi(\vec{x}, t) = \varphi\left(r = \sqrt{\vec{x}^2 - t^2}\right)$$

follows from analytic continuation $t \longrightarrow -it$.

Hence the bounce tells us what the bubble looks like outside of the light cone:



As indicated, the solution also includes a contracting phase. Where the transition between contraction and expansion occurs is frame-dependent. It would therefore be interesting to understand how different observers perceive the quantum nucleation of classical bubbles.