4. One-loop determinants

Many phenomenological estimates content themselves with the exponential factor. This may be OK in order to check the cosmological stability (hence consistency) of particular models. However, for the calculations of e.g. backgrounds of stochastic gravitational waves, this may not be good enough. Also, as a matter of principle, we would like to know how to practically compute the one-loop determinants and how to systematically go beyond that approximation.

We discuss here two principal methods:

1. Gel'fand Yaglom theorem
Used in a number of phenomenological studies. Not clear how to go beyond one loop.
2. Green's function/ resolvent method
Allows to systematically go beyond one-loop order, in particular if loop resummation is necessary such as in the Standard Model.

Gel'fand - Yaglom method

Consider

$$(-\partial_\tau^2 + W(\tau))\,\psi(\tau) = \lambda\,\psi(\tau)$$

Let $\psi_\lambda(\tau)$ a solution on $\tau \in [-\frac{T}{2};\frac{T}{2}]$ satisfying the boundary conditions

$$\psi_\lambda(-\tfrac{T}{2}) = 0$$

$$\partial_\tau \psi_\lambda(\tau)\Big|_{\tau=-\frac{T}{2}} = 1$$

Next, consider the eigenvalue equation

$$(-\partial_\tau^2 + W(\tau))\,\psi_{\lambda_n}(\tau) = \lambda_n\,\psi_{\lambda_n}(\tau)$$

where $\psi_{\lambda_n}(\pm\frac{T}{2}) = 0$

$$\longrightarrow \det(-\partial_\tau^2 + W(\tau)) = \prod_n \lambda_n$$

The Gel'fand-Yaglom theorem then states that

$$\frac{\det[-\partial_\tau^2 + W^R(\tau) - \lambda]}{\det[-\partial_\tau^2 + W^S(\tau) - \lambda]} = \frac{\psi_\lambda^R(\frac{T}{2})}{\psi_\lambda^S(\frac{T}{2})}$$

We give the argument leading to this relation for discrete spectra, while it apparently applies to spectra with continuum pieces as well.
On the left hand side, we have zeroes for each $\lambda = \lambda_n^R$ and simple poles for each $\lambda = \lambda_n^S$.

By construction, the same holds for the right-hand side. Furthermore, for λ going to infinity in any direction but the positive real axis, the solutions tend to the same exponentially growing/decaying function, that is the ratio goes to one in that limit. Hence both sides agree as meromorphic functions in λ.

Note that for a matrix operator, $\psi_\lambda\left(\frac{T}{2}\right)$ is given by its characteristic polynomial. The determinant is then the constant term, i.e. the characteristic polynomial for $\lambda=0$.

Application to the bounce:

For the normalizing determinant, choose
$W^R(\tau) = V''(x_+) = m^2$
Advantage: Since also the (multi-) bounces sit at x_+ for almost all τ, ground-state energy is readily subtracted this way.
$\longrightarrow$

$$\psi_0^R(\tau) = \frac{1}{m} \sinh\left[m\left(\tau+\frac{T}{2}\right)\right]$$

$$\longrightarrow \psi_0^R\left(\frac{T}{2}\right) \sim \frac{1}{2m} e^{mT} \text{ for large } T$$

For the bounce,

$$W^S(\tau) = V''(\bar{x}(\tau))$$

For the primed determinant, take out zero eigenvalue.
Note: For finite T, this eigenvalue λ_1 is not exactly zero. So we can divide, then take $T \to \infty$.

To find $\psi_0^{(2)}$, consider the equation

$$[-\partial_\tau^2 + V''(x_B(\tau))]\, \psi(\tau) = 0$$

One basis solution is of course the translational mode:

$$x^{(1)}(\tau) = \frac{1}{\sqrt{B}} \frac{d\bar{x}(\tau)}{d\tau} \sim \pm \frac{A}{\sqrt{m}} e^{-m|\tau|} \quad \text{for } \tau \to \pm\infty .$$

Recall that when solving the Schrödinger equation, we had fixed $\dfrac{d\bar{x}(\tau)}{d\tau} \sim \sqrt{B} \dfrac{A}{\sqrt{m}} e^{-m\tau}$ for $\tau \to \infty$.

Then we take the other linearly independent solution $\tilde{x}^{(1)}(\tau)$ that we normalize such that the Wronskian is

$$x^{(1)} \partial_\tau \tilde{x}^{(1)} - \tilde{x}^{(1)} \partial_\tau x^{(1)} = 2A^2$$

Hence $\tilde{x}^{(1)}(\tau) \sim \dfrac{A}{\sqrt{m}} e^{m|\tau|}$ as $\tau \to \pm\infty$

With this piece of information we construct the solution that satisfies the boundary conditions

$$\psi_\lambda\left(-\frac{T}{2}\right) = 0, \qquad \partial_\tau \psi_\lambda(\tau)\Big|_{\tau = -\frac{T}{2}} = 1$$

$\longrightarrow$

$$\psi_0^S(\tau) = -\frac{1}{2\sqrt{m}\,A}\left(e^{m\frac{T}{2}} x^{(1)}(\tau) + e^{-m\frac{T}{2}} \tilde{x}^{(1)}(\tau)\right)$$

$$\Longrightarrow \quad \psi_0^S\left(\frac{T}{2}\right) = -\frac{1}{m}$$

To find λ_1, we put this small eigenvalue in the game and expand

$$\psi_{\lambda_1}(\tau) = \psi_0^S(\tau) + \delta\psi_{\lambda_1}(\tau)$$

$\longrightarrow$

$$\left(-\partial_\tau^2 + V''(\bar{x}(\tau))\right)\delta\psi_{\lambda_1}(\tau) = \lambda_1\,\psi_0^S(\tau)$$

$\tilde{x}^{(1)}\left(\frac{T}{2}\right) \sim \frac{A}{\sqrt{m}}\, e^{m\frac{T}{2}}$ $\qquad$ $x^{(1)}\left(\frac{T}{2}\right) \sim \frac{A}{\sqrt{m}}\, e^{-m\frac{T}{2}}$

$\longrightarrow$

$$\psi_{\lambda_1}(\tau) = \psi_0^S(\tau) - \frac{\lambda_1}{2A^2}\int_{-\frac{T}{2}}^{\tau} d\tau' \left(\tilde{x}^{(1)}(\tau)\,x^{(1)}(\tau') - x^{(1)}(\tau)\,\tilde{x}^{(1)}(\tau')\right)\psi_0^S(\tau')$$

(We have used the theorem leading to this relation already when we constructed the false ground state wave function.)

$\longrightarrow$

$$\psi_{\lambda_1}\left(\frac{T}{2}\right) = -\frac{1}{m} + \frac{\lambda_1}{4mA^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau' \left(e^{mT}\left(x^{(1)}(\tau')\right)^2 - e^{-mT}\left(\tilde{x}^{(1)}(\tau')\right)^2\right)$$

Since $x^{(1)}$ is normalized, we arrive at

$$\psi_{\lambda_1}\left(\frac{T}{2}\right) \approx -\frac{1}{m} + \frac{\lambda_1}{4mA^2}\, e^{mT} \stackrel{!}{=} 0 \quad \Longrightarrow \quad \lambda_1 = 4A^2 e^{-mT}$$

$\longrightarrow$

$$\frac{\det'\left[-\partial_\tau^2 + V''(\bar{x})\right]}{\det\left[-\partial_\tau^2 + V''(x_+)\right]} = \frac{\psi_0^S\left(\frac{T}{2}\right)}{\lambda_1\, \psi_0^R\left(\frac{T}{2}\right)} = \frac{-\frac{1}{m}}{4A^2 e^{-mT}\, \frac{1}{2m}\, e^{mT}}$$

$$= -\frac{1}{2A^2}$$

Note that this result is negative because of the presence of the negative eigenvalue. For instanton transitions the answer is positive.

Substituting this into the formula for the decay rate we obtain

$$\Gamma = \sqrt{\frac{B}{2\pi}}\, e^{-B}\, \mathrm{Im} \sqrt{\frac{\det G^{-1}(x_+)}{\det' G^{-1}(\bar{x})}} = \sqrt{\frac{B}{\pi}}\, A\, e^{-B}$$

in agreement with the result from solving the Schrödinger equation.

Resolvent method

The resolvent is the function $G(\tau,\tau';s)$ that solves the following equation:

$$\left[-\frac{d^2}{d\tau^2}+V''(\bar{x}(\tau))+s\right]G(\tau,\tau';s)=\delta(\tau-\tau')$$

For $s=0$, the resolvent coincides with the Green's function.

The solution can be constructed directly — an approach that can also be implemented numerically and analytically (in case such solutions to the ordinary differential equation exist).

Let $\left[-\frac{d^2}{d\tau^2}+V''(\bar{x}(\tau))+s\right]f^{<,>}(\tau)=0$

where $\left.\begin{array}{l} f^{<}(\tau\to-\infty)=0 \\ f^{>}(\tau\to\infty)=0 \end{array}\right\}$ This behaviour can only be imposed for $s\neq 0$ because there is a translational zero mode otherwise!

$$\longrightarrow G(\tau,\tau';s)=\vartheta(\tau-\tau')A^{>}(\tau')f^{>}(\tau)+\vartheta(\tau'-\tau)A^{<}(\tau')f^{<}(\tau)$$

Continuity:

$$A^{>}(\tau)f^{>}(\tau)=A^{<}(\tau)f^{<}(\tau)$$

Normalization to δ-function:

$$A^{>}(\tau)\frac{d}{d\tau}f^{>}(\tau)-A^{<}(\tau)\frac{d}{d\tau}f^{<}(\tau)=-1$$

Together:

$\longrightarrow$

$$f^> = -\frac{f^<}{f^<\frac{d}{d\tau}f^> - f^>\frac{d}{d\tau}f^<} = -\frac{f^<}{W[f^<,f^>]}$$

$$f^< = -\frac{f^>}{f^<\frac{d}{d\tau}f^> - f^<\frac{d}{d\tau}f^>} = -\frac{f^>}{W[f^<,f^>]}$$

$$G(\tau,\tau';s) = -\frac{1}{W[f^<,f^>]}\Big(\vartheta(\tau-\tau')\, f^>(\tau)\, f^<(\tau') + \vartheta(\tau'-\tau)\, f^<(\tau)\, f^>(\tau')\Big)$$

To find the functional determinant, we note the spectral representation:

$$G(\tau,\tau';s) = \sum_n \frac{\psi_n(\tau)\,\psi_n^*(\tau')}{\lambda_n+s}$$

where

$$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau))\right]\psi_n(\tau) = \lambda_n\,\psi_n(\tau)$$

which can be verified directly:

$$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) + s\right] G(\tau,\tau';s) = \sum_n \frac{\lambda_n+s}{\lambda_n+s}\,\psi_n(\tau)\,\psi_n^*(\tau')$$
$$= \delta(\tau-\tau')$$

where the last identity is the completeness relation.

$\longrightarrow$

$$\int_0^{\Lambda^2} ds \int_{-\infty}^{\infty} d\tau\, G(\tau,\tau;s) = \sum_n \log(\lambda+s)\Big|_{s=0}^{s=\Lambda^2} = -\sum_n \log \frac{\lambda_n}{\lambda_n+\Lambda^2}$$

$\longrightarrow$

$$\lim_{\Lambda\to\infty} \int_0^{\Lambda^2} ds \int_{-\infty}^{\infty} d\tau\, G(\tau,\tau;s) = -\overbrace{\mathrm{tr}\log}^{=\log\det}\left[-\frac{d^2}{d\tau^2}+V''(\bar{x})\right]$$

$$+\lim_{\Lambda\to\infty} \sum_n \log \Lambda^2$$

The way we have written the limits here is apparently not accurate. When normalizing the determinant e.g. by the false vacuum determinant (and, in QFT, UV regularization) we can however obtain convergent sums in n and cancellations of the sums over the $\log \Lambda^2$.

Taking out zero modes

We still need to restrict to the subspace of nonzero modes. To do so, define

$$G_\perp(\tau,\tau';s) = \sum_{\substack{n\\ \lambda_n\neq 0}} \frac{\psi_n(\tau)\,\psi_n^*(\tau')}{\lambda_n+s}$$

$\longrightarrow$

$$\left[-\frac{d^2}{d\tau^2}+V''(\bar{x}(\tau))+s\right] G_\perp(\tau,\tau';s) = \delta(\tau-\tau') - \sum_{\substack{n\\ \lambda_n=0}} \psi_n(\tau)\,\psi_n^*(\tau')$$

The function $G_\perp$ can be constructed in the same fashion as G, just that $f^{<>}$ are now constructed from a homogeneous equation. This way, the boundary conditions can be satisfied even though there is a zero mode.

Alternative derivation of the determinant from the resolvent

Let $\mathcal{O}$ be a Hermitian differential operator (in one dimension and the variable t, for simplicity).

Then, consider

$$(\mathcal{O}+s)\, f_k(s;t) = \lambda_k(s)\, f_k(s;t)$$

Since

$$(\mathcal{O}+s+\Delta s)\, f_k(s;t) = (\lambda_k(s)+\Delta s)\, f_k(s;t)$$

$$\Longrightarrow \left.\frac{\partial \lambda_k(s;t)}{\partial s}\right|_{s=0} = 1$$

Next, take

$$g(s) = \prod_{k=0}^{\infty} \lambda_k(s) = \det(\mathcal{O}+s)$$

$$\Longrightarrow \frac{d}{ds} \log g(s) = \frac{d}{ds} \log \det(\mathcal{O}+s) = \frac{1}{g(s)} \frac{d}{ds} g(s)$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{d\lambda_k(s)}{ds}}_{=1} \frac{1}{\lambda_k(s)} = \int dt \; G(t,t;s)$$

This is again the resolvent formula for the determinant.

Direct argument for equivalence of resolvent method and Gel'fand Yaglom method

Consider

$$(\mathcal{O}+s)\,\psi(s;t) = 0$$

with boundary conditions

$$\psi(s;-\tfrac{T}{2}) = 0$$

$$\frac{d}{dt}\psi(s;-\tfrac{T}{2}) = 1$$

With $\varphi(s;t) = \frac{d}{ds}\psi(s;t) \Longrightarrow$

$$(\mathcal{O}+s)\,\varphi(s;t) + \psi(s;t) = 0$$

Now we recall once again the procedure for constructing solutions to an inhomogeneous differential equation that we used in order to construct the wave function for the tunneling problem:

A particular solution to

$$u''(t) + p(t)\,u'(t) + q(t)\,u(t) = f(t)$$

is given by

$2u_2' u_1 f + u_1' u_2 f$

$-2u_1' u_2 f - u_1 u_2' f = (u_1 u_2' - u_2 u_1') f$

$$u(t) = u_2(t) \int_{-\frac{T}{2}}^{t} dt' \, \frac{u_1(t') f(t')}{W[u_1(t'), u_2(t')]} - u_1(t) \int_{-\frac{T}{2}}^{t} dt' \, \frac{u_2(t') f(t')}{W[u_1(t'), u_2(t')]}$$

where $u_{1/2}$ are solutions to the homogeneous equation.

For the present problem, one of the solutions to the homogeneous equation is given by $\underbrace{\varphi(s;t)}_{\to u_1}$.

For the additional solution, we choose

$$(\Theta + s)\underbrace{\tilde{\varphi}(s;t)}_{\to u_2} = 0$$

with boundary conditions

$$\tilde{\varphi}(s;\tfrac{T}{2}) = 0$$

$$\frac{d}{dt}\tilde{\varphi}(s;\tfrac{T}{2}) = 1$$

$\longrightarrow$

$$\varphi(s;t) = \varphi(s;t) \int_{-\frac{T}{2}}^{t} dt' \, \frac{\tilde{\varphi}(s;t')\,\psi(s;t')}{W[\varphi(s;t'), \tilde{\varphi}(s;t')]} - \tilde{\varphi}(s;t) \int_{-\frac{T}{2}}^{t} dt' \, \frac{\varphi(s;t')\,\psi(s;t')}{W[\varphi(s;t'), \tilde{\varphi}(s;t')]}$$

$\longrightarrow$

$$\varphi(s;\tfrac{T}{2}) = \varphi(s;\tfrac{T}{2}) \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \, \frac{\tilde{\varphi}(s;t)\,\psi(s;t)}{W[\varphi(s;t), \tilde{\varphi}(s;t')]}$$

$$= \psi(s;\tfrac{T}{2}) \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, G(t,t;s)$$

where we have identified our construction of the resolvent.

$$\Longrightarrow$$

$$\frac{d}{ds} \log \psi(s;\tfrac{T}{2}) = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, G(t,t;s) = \frac{d}{ds} \log \det(O+s)$$

$$\Longrightarrow \quad \psi(0;\tfrac{T}{2}) = \det(O) * \text{const.}$$

The const. is cancelled when considering ratios of determinants.