3. Tunneling from Euclidean path integrals

The approximate solution to the Schrödinger equation is cumbersome but confirms an intuitive picture one may have about tunneling. It leaves open the questions of

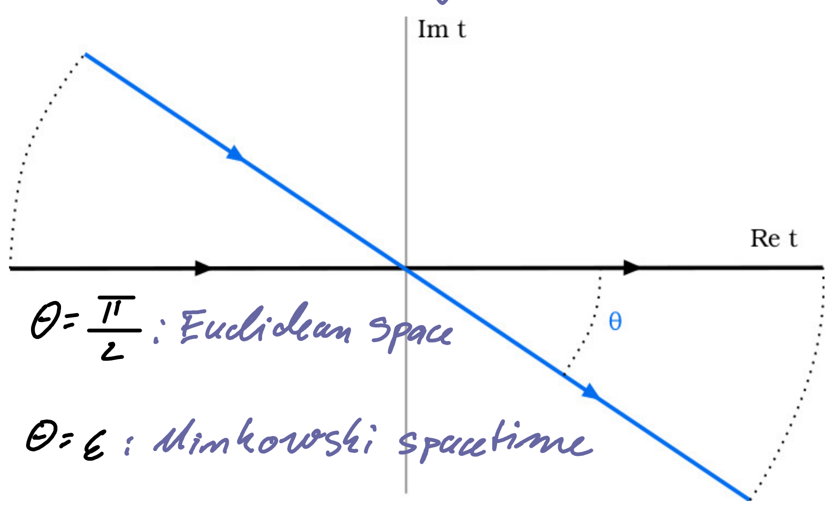
- how to go beyond the leading approximation
- how to treat tunneling in QFT, where infinitely many degrees of freedom couple and a functional Schrödinger approach appears hopeless.

Functional approach: Expansion around saddle points (classical solutions)

However, there is no classical tunneling path in real time.

It turns out, that such saddles exist for imaginary time.

$$t \to t\, e^{-i\theta}$$



Analytic continuation of the action:

$$S_\theta[x] = e^{-i\theta} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \left[\frac{1}{2} \left(\frac{dx}{dt} \right) e^{2i\theta} - V(x) \right]$$

Path integral:

$$Z = \langle x_f |\, e^{-iHT e^{-i\theta}} \,| x_i \rangle = \int Dx(t)\, e^{iS_\theta[x]}$$

We have identified here the amplitude with the partition function Z.

For the range of interest $\varepsilon < \Theta < \frac{\pi}{2}$, we project in the limit $T \to \infty$ the position eigenstate $|x_+\rangle$ on the false ground state as we will thus isolate the state with lowest $Re[E]$. (For calculations in Minkowski spacetime where this feature may not be desired, one must set $\varepsilon = 0$.)

Formal goal:

- Compute the Euclidean amplitude $\langle x_+ | e^{-HT} | x_+ \rangle$ in the limit $T \to \infty$.
- From this, extract the imaginary part of the vacuum energy, leading to the decay rate

Saddle point expansion:

Consider solutions to the Euclidean equations of motion that observe the boundary conditions $x(\tau \to \pm\infty) = x_+$.

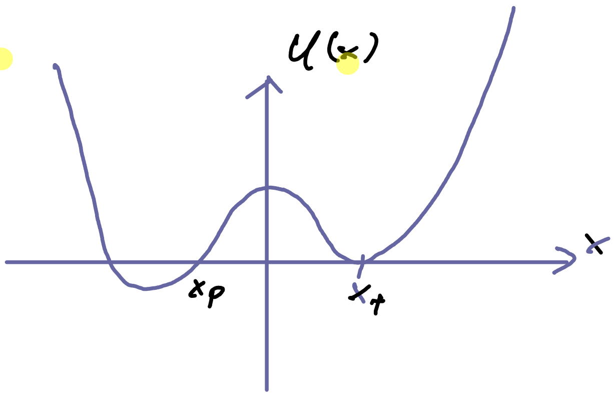
Recall that the Euclidean equations of motion correspond to an upside-down potential.

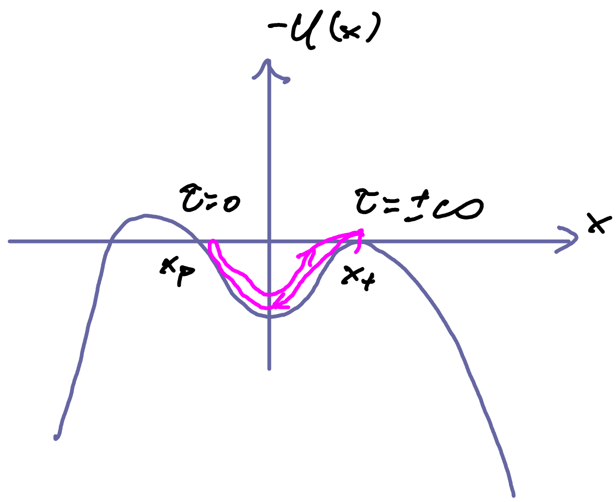
$\longrightarrow$ There is a trivial solution $x(\tau) = x_+ = \text{const}$.

$\longrightarrow$ There are bounces and multibounces

The bounce

This solution moves from x_t to x_p and back. We can identify it with $\overline{x}(\tau)$ previously used in the computation of the WKB integral. (There it was defined in the interval $\tau \in [0;\infty]$ where $\overline{x}(0) = x_p$. Apparently, we can extend this to $\tau \to -\infty$.)





A suppression of the tunneling rate by $e^{-\frac{B}{\hbar}}$ is expected — so it makes sense that the bounce also is central to the path integra approach.

Recall nonetheless that we are after the energy of the lowest state. So we need to compare with e^{-ET} which only makes sense when summing over additional saddle points.
It also turns out that in order to reproduce the leading result from the Schrödinger equation, we need to include one-loop determinants.

One-loop determinants

Recall:

$$\left.\begin{aligned} L_E &= \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V \\ S_E &= \int_{-\infty}^{\infty} d\tau\, L_E \end{aligned}\right\}$$
Euclidean Lagrangian & action, $S_E[\bar{x}] = B$ for single bounce

Equation of motion

$$\frac{d^2}{d\tau^2}\bar{x} - \frac{\partial V(\bar{x})}{\partial \bar{x}} = 0$$

Denote the partition function for the single-bounce saddle by Z_B. $\longrightarrow$

$$Z_B = \int_{\bar{x}} \mathcal{D}x \; e^{-S_E[x]}$$

The $\bar{x}$ at the integral indicates that we consider here perturbative fluctuations about $\bar{x}$.

$\longrightarrow$

$$Z_B = \int \mathcal{D}\delta x \; e^{-S_E[\bar{x}] - \frac{1}{2}\int d\tau\, \delta x(\tau)\, G^{-1}(\bar{x};\tau)\, \delta x(\tau) + \cdots}$$

Since

$$\delta^2 S_E = \frac{1}{2}\int d\tau \left\{ (\delta\dot{x})^2 + V''(\bar{x})(\delta x)^2 \right\}$$

$$= \frac{1}{2}\int d\tau\, \delta x(\tau) \underbrace{\left\{ -\frac{d^2}{d\tau^2} + V''(\bar{x}) \right\}}_{=: G^{-1}(\bar{x};\tau)} \delta x(\tau)$$

Carrying out the path integral to quadratic order gives the tentative result:

$$Z_B \stackrel{?}{=} \mathcal{N} \, e^{-S_E(\bar{x})} \frac{1}{\sqrt{\det G^{-1}(\bar{x})}}$$

where $\mathcal{N}$ is a normalization to be cancelled or fixed by some reference quantity.

<u>Zero modes / collective coordinates</u>

The question mark is here because the determinant is ill defined:

The theory specified by L_E (or its Minkowski-counterpart) is invariant under time translations — the bounce breaks this symmetry spontaneously $\longrightarrow$ There is a Goldstone mode, i.e. a zero eigenvalue of $G^{-1}(\bar{x})$.

$\longrightarrow$ Zero mode is given by the generator of time translations:

$$x^{(1)}(\tau) = \frac{1}{\sqrt{B}} \frac{d\bar{x}}{d\tau}$$

To see this explicitly, consider the derivative of the equation of motion

$$\frac{d}{d\tau}\left(\frac{d^2}{d\tau^2}\bar{x} - \frac{\partial V(\bar{x})}{\partial \bar{x}} \right) = 0 = \left(\frac{d^2}{d\tau^2} - \frac{\partial^2 V(\bar{x})}{\partial \bar{x}^2} \right) \frac{d\bar{x}}{d\tau}$$

The normalization has already been proved in the part on the WKB approximation.

So the determinant formula is not quite correct since we have treated a flat direction like a Gaußian integral.

To correctly evaluate the path integral, recall that we formally expand

$$\delta x(\tau) = \sum_n c_n \, x^{(n)}(\tau)$$

$\longrightarrow$ Path integral measure

$$\mathcal{D}\delta x = \prod_n \frac{1}{\sqrt{2\pi}} \, dc_n$$

Now consider fluctuations about $\bar{x}(\tau-\tau_0)$.

$\longrightarrow$ Decompose trajectory as

$$x(\tau) = \bar{x}(\tau-\tau_0) + c_1 \, x^{(1)}(\tau-\tau_0) + x'(\tau-\tau_0)$$

where x' contains the fluctuations perpendicular to $x^{(1)}$.

In particular, we can decompose

$$\int \mathcal{D}x = \int \mathcal{D}x' \, \frac{dc_1}{\sqrt{2\pi}}$$

Then we play the Faddeev-Popov trick:

$$1 = \int d\tau_0 \, |\partial_{\tau_0} h(\tau_0)| \, \delta(h(\tau_0))$$

where

$$h(\tau_0') = \int d\tau \; x(\tau) \, \partial_\tau \bar{x}(\tau-\tau_0')$$

and in particular

$$h(\tau_0' = \tau_0) = \sqrt{B}\, c_1$$

so that

$$\delta(h(\tau_0)) = \frac{\delta(c_1)}{\sqrt{B}}$$

Further,

$$\partial_{\tau_0'} h(\tau_0') \Big|_{\tau_0' = \tau_0} = - \int d\tau\, x(\tau)\, \partial_\tau^2 \overline{x}(\tau - \tau_0)$$

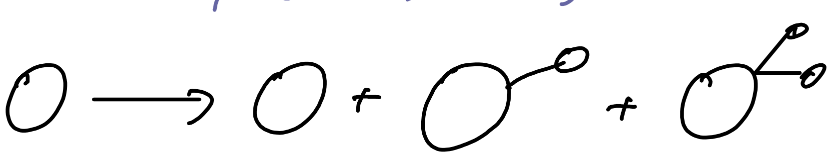
$$= \underbrace{\int d\tau \left[\partial_\tau \overline{x}(\tau - \tau_0)\right]^2}_{= B} - \int d\tau\, x'(\tau - \tau_0)\, \partial_\tau^2 \overline{x}(\tau - \tau_0)$$

Altogether:

$$\int \mathcal{D}x = \sqrt{\frac{B}{2\pi}} \int \mathcal{D}x' \int d\tau_0 \left[1 - \frac{1}{B} \int d\tau\, x'(\tau - \tau_0)\, \partial_\tau^2 \overline{x}(\tau - \tau_0)\right]$$

$$= T \sqrt{\frac{B}{2\pi}} \int \mathcal{D}x' \left(1 - \frac{1}{B} \int d\tau\, x'(\tau)\, \partial_\tau^2 \overline{x}(\tau)\right)$$

$\longrightarrow$ Tadpole insertions



etc.

Provided we stay at the level of quadratic fluctuations, the short approximation is:

$$dx(\tau)\Big|_{\text{translation}} = x^{(1)}(\tau)\,dc_1 = \frac{d\bar{x}(\tau)}{d\tau}\,d\tau = \sqrt{B}\,x^{(1)}(\tau)\,d\tau$$

$$\Longrightarrow \frac{dc_1}{\sqrt{2\pi}} \longrightarrow \sqrt{\frac{B}{2\pi}}\,d\tau$$

Here terms from translating contributions of different shape than the leading $\bar{x}(\tau)$ are neglected — OK to one-loop order

This technique of handling Goldstone flat directions can generally be applied to solitons in QFT. The variable τ_0 (such as corresponding ones) is called a <u>collective coordinate.</u>

<u>Negative mode</u>

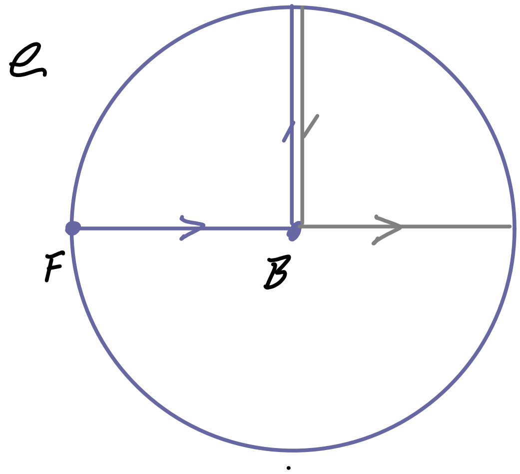
The evaluation of the determinant is not fully clear yet. This is because $x^{(1)}(\tau)$ has a single node at $\tau = 0$. By the node theorem, there must be a lower, i.e. negative mode $x^{(0)}(\tau)$ with zero nodes.

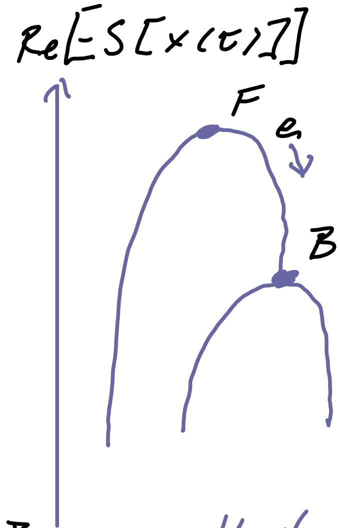
$\longrightarrow \bar{x}(\tau)$ is actually not an extremal point of the action but a saddle point.

Sketch of the situation:

One-dimensional slice of space of the $x(\tau)$.

Choose parameter e that connects F $(x(\tau)=x_+)$ with B $(x(\tau)=\bar{x}(\tau))$ along a gradient of $-S[x(\tau)]$ so that B is approached in the direction of $x^{(0)}(\tau)$. (Picard-Lefschetz theory).





Steepest descent: deform contour at B so that e goes into one of the imaginary directions.

Note: In principle, the contour must be complemented by the backward trip in the imaginary direction and then onward to sweep over all real field configurations.

⟶ In that case:

* Z is real (as its path integral expression is manifestly real)
* Instead of projecting on the false ground state we would project on the true one about the global minimum of the potential.

However, we are practically accounting here only for the fluctuation about $\bar{x}(\tau)$ (and eventually over multi-bounces as well). Therefore:

* We still can assume to project on the false ground state.
* Z_B and eventually also the energy acquire an imaginary part.

In summary, the negative mode can be dealt with by half of a Gaußian integration in the imaginary direction. This imaginary part also emerges from the square root of the negative eigenvalue so that we can incorporate it without ado in the determinant formula.

$\longrightarrow$ Altogether:

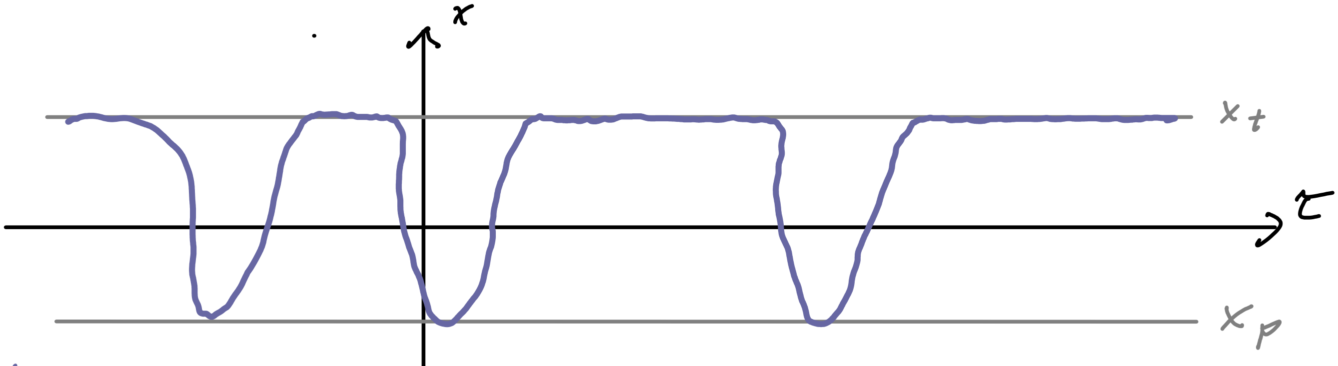
$$Z_B = \mathcal{N} \frac{1}{2} \sqrt{\frac{B}{2\pi}} \, T \, e^{-\overbrace{S_E(\bar{x})}^{=B}} \frac{1}{\sqrt{\det{}' G^{-1}(\bar{x})}}$$

where T is the length of the time interval, to be taken to infinity. The prime on the determinant indicates the omission of the zero eigenvalue.

Normalization:

Divide by the partition function about the false vacuum

$$Z_F = \mathcal{N} \frac{1}{\sqrt{\det G^{-1}(x_f)}}$$



A multi-bounce configuration.

Multi-bounces

Multi-bounces contribute besides the trivial and the single-bounce solutions.

Assuming that these are well separated (and since the bounce tends to x_+ exponentially fast), we can approximately superimpose these.
$\longrightarrow$ "Dilute instanton gas approximation"

Partition function:

$$Z = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int_{-\infty}^{\infty} d\tau_1 \, d\tau_2 \cdots d\tau_\nu \left(\frac{1}{2} \sqrt{\frac{B}{2\pi}} \right)^{\nu} e^{-\nu B} \left(\frac{\det G^{-1}[x_+)}{\det' G^{-1}(\bar{x}(\tau - \tau_2))} \right)^{\frac{\nu}{2}}$$

$$= \exp\left\{ \frac{1}{2} \sqrt{\frac{B}{2\pi}} \, e^{-B} \sqrt{\frac{\det G^{-1}(x_+)}{\det' G^{-1}(\bar{x})}} \, T \right\}$$

$\longrightarrow$ Decay rate $\Gamma = 2 \operatorname{Im} E$

$$\boxed{\Gamma = \sqrt{\frac{B}{2\pi}} \, e^{-B} \sqrt{\frac{\det G^{-1}(x_+)}{\det' G^{-1}(\bar{x})}}}$$

With the conceptual developments for soliton and instanton calculus, we have isolated the hard calculational bit within the functional determinant, which we will go about next.