Saalburg Lectures on Integrable Systems

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Abstract

In these lectures starting from the famous Liouville-Arnold theorem, we cover the basic concepts of integrability concerning classical and quantum factorised scattering and various incarnations of the Bethe Ansatz.

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Preface

This is a course on integrable models given at the 26th Saalburg Summer School for graduate students "Foundations and New Methods in Theoretical Physics". I start with recalling the basic notions of the lagrangian and hamiltonian mechanics, Poisson and symplectic geometry, which provide the necessary background to subsequently formulate the Arnold-Liouville theorem and to outline its proof. I then discuss the Lax representation and the Babelon-Viallet theorem on the special form of the Poisson bracket between the components of a Lax matrix that guarantees involutivity of its spectral invariants.

The main focus of these lectures is on the Factorised Scattering Theory and the Bethe Ansatz. Here I departure from considering scattering in classical integrable models, discuss reflection and transmission representations of the scattering process and show how to compute the classical phase shift (classical S-matrix). I then turn to scattering in quantum integrable models and show that the existence of a large number of conservation laws implies a special form of the asymptotic wave function (the Bethe wave function), renders the scattering process non-diffractive and leads to factorisation of the multi-body scattering matrix into a product of two-body S-matrices. The basic features of the formalism are demonstrated on the examples of the Lieb-Liniger and Calogero-Moser-Sutherland models. Further, the Bethe wave function is considered in a finite one-dimensional volume which amount to imposition of the periodic boundary conditions. This leads to a consistent system of the matrix Bethe-Yang equations on the scattering amplitudes. Resolution of these equations is equivalent to finding a simultaneous eigenstate for a family of commuting operators (matrices) T_j and is conveniently done by means of a procedure known as the nested Bethe Ansatz. I explain the nested Bethe Ansatz construction based on the generalised Bethe hypothesis for the simplest case where the operators T_j act in the permutation module of the symmetric group S_N described by a Young diagram $[N-M, M]$. As soon as the common spectrum of operators T_j is found, one can formulate a system of "scalar" Bethe equations, the latter can be regarded as quantisation conditions for particle momenta.

Finally, in the last part of the course I describe the transfer matrix method which replaces diagonalisation of the commutative family $\{T_j\}$ by diagonalisation of the transfer matrix. This diagonalisation is then performed in two alternative ways – in the framework of the coordinate Bethe Ansatz by Lieb's method and by means of the Algebraic Bethe Ansatz. The course contains a number of exercises (most of them with solutions) which appear in the text in yellow boxes. In the appendix I collected some relevant information about the symmetric group and its representations, as well as some other useful formulae. I also supply a list of papers and books as a suggestion for further reading, all this literature material has been heavily used in preparation of these lectures.

Acknowledgement. I would like to thank the organisers of the School for the very enjoyable atmosphere, perfect organisation and kind hospitality. I am also grateful to the students for their interesting questions and their passion to be guided into the wonderful world of integrable models.

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Lecture 1

Liouville Integrability

1.1 Dynamical systems in classical mechanics

We start with recalling the two ways dynamical systems are described in classical mechanics. The first description is known as the lagrangian formalism and is equivalent to the "principle of stationary action". Consider a point particle with mass m which moves in an N-dimensional space with coordinates $q = (q^1, \dots, q^N)$ and a potential $V(q)$. Newton's equations which describe the particle's trajectory are

$$m\ddot{q}^i = -\frac{\partial V}{\partial q^i}\,. \tag{1.1}$$

These equations can be obtained by extremising the following action functional

$$S[q] = \int_{t_1}^{t_2} \mathrm{d}t\, L(q, \dot{q}, t) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{m\dot{q}^2}{2} - V(q)\right). \tag{1.2}$$

According to the principle of stationary action, the actual trajectories of a dynamical system (particle) are the ones that extremise S.

In general, we consider the *lagrangian* L as an arbitrary function of q, $\dot{q}$ and time t. The equations of motion are obtained by extremising the corresponding action

$$\frac{\delta S}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) = 0$$

and they are called the *Euler-Lagrange equations*. An assumption that L does not involve higher order time derivatives implies that the corresponding dynamical system is fully determined by specifying initial coordinates and velocities. Indeed, for a system with N degrees of freedom there are N Euler-Lagrange equations of second order. Thus, the general solution will depend on $2N$ integration constants, which are determined by specifying *e.g.* the initial coordinates and velocities.

Note that adding to the lagrangian a time derivative of a function which depends on coordinates and time only: $L \to L + \frac{d}{dt}\Lambda(q,t)$ will not influence the Euler-Lagrange equations. Indeed, the variation $\delta S'$ of the new action S' will be

$$\delta S' = \delta S + \int_{t_1}^{t_2} \mathrm{d}t\, \frac{d}{dt}\delta\Lambda(q,t) = \delta S + \frac{\partial \Lambda}{\partial q^i}\delta q^i\Big|_{t=t_1}^{t=t_2},$$

where δS is the variation of the original action S. Since in deriving the equations of motion the variations of coordinates are assumed to vanish at the initial and final moments of motion, we get that $\delta S' = \delta S$ and, as a result, the Euler-Lagrange equations remain unchanged.

If L does not explicitly depend on t, then

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial L}{\partial q^i}\dot{q}^i\,.$$

Substituting here $\frac{\partial L}{\partial q^i}$ from the Euler-Lagrange equations, we get

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big)\dot{q}^i = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\Big)\,.$$

Therefore, we find

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\Big) = 0\,, \tag{1.3}$$

as a consequence of the equations of motion. Thus, the quantity

$$H \equiv \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \tag{1.4}$$

is conserved under the time evolution of our dynamical system. For our particular example,

$$H = m\dot{q}^2 - L = \frac{m\dot{q}^2}{2} + V(q) = T + V \equiv E\,,$$

where T is the kinetic energy, $\dot{q}^2 \equiv \dot{q}^i\dot{q}^i$. Thus, H is nothing else but the energy E of the system; the energy is conserved due to the equations of motion. In general, dynamical quantities which remain unchanged under the time evolution are called *conservation laws* or *integrals of motion*. Conservation of energy is one of the main examples of conservation laws.

Introduce the quantity called the *canonical* momentum

$$p_i = \frac{\partial L}{\partial \dot{q}^i}\,, \qquad p = (p_1, \ldots, p_N)\,.$$

Obviously, for a particle $p_i = m\dot{q}^i$. If $V = 0$, then

$$\dot{p}_i = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) = 0$$

by the Euler-Lagrange equations. Thus, in the case of vanishing potential, the particle momentum is an integral of motion. This is another example of a conservation law.

Let us now we recall the second description of dynamical systems, which exploits the notion of the hamiltonian. The energy of a system expressed via canonical coordinates and momenta is called the *hamiltonian*:

$$H(p,q) = \frac{p^2}{2m} + V(q)\,.$$

where $p^2 \equiv p_i p_i$. Given the hamiltonian, Newton's equations can be rewritten as

$$\dot{q}^j = \frac{\partial H}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial H}{\partial q^j}\,. \tag{1.5}$$

These are equations of motion in the hamiltonian form or Hamilton's equations. These equations can also be obtained by means of the variational principle. The corresponding action has the form, *cf.* (1.2) and (1.4),

$$S[p,q] = \int_{t_1}^{t_2} \big(p_i\dot{q}^i - H(p,q)\big)\mathrm{d}t\,. \tag{1.6}$$

Varying this action with respect to p and q, *considered as independent variables*, one obtains the hamiltonian equations.

Hamilton's equations can be represented in the form of a single equation. Introduce two $2N$-dimensional vectors

$$x = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad \nabla H = \begin{pmatrix} \frac{\partial H}{\partial q^j} \\ \frac{\partial H}{\partial p_j} \end{pmatrix}$$

and $2N \times 2N$ matrix J:

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{1.7}$$

where $\mathbb{1}$ is the $N \times N$ unit matrix. Then (1.5) are concisely written as

$$\dot{x} = -J \cdot \nabla H\,, \qquad \text{or} \qquad J \cdot \dot{x} = \nabla H\,. \tag{1.8}$$

The vector $x = (x^1, \ldots, x^{2N})$ defines a state of a dynamical system in classical mechanics. The set of all states forms the *phase space* $\mathcal{P} = \{x\}$ of the system which in the present case is the $2N$-dimensional space with the euclidean metric $(x, y) = \sum\limits_{i=1}^{2N} x^i y^i$. Solving Hamilton's equations with given initial conditions (p_0, q_0) representing a point in the phase space, we obtain a phase space curve

$$p \equiv p(t; p_0, q_0)\,, \qquad q \equiv q(t; p_0, q_0)$$

passing through this point. As follows from the uniqueness theorem for ordinary differential equations, there is one and only one phase space curve through every phase space point.[1]

Let $F(\mathcal{P})$ be the space of smooth real-valued functions on $\mathcal{P}$. It carries the structure of an algebra with respect to the pointwise multiplication and its elements are called *observables*. Using the matrix J, one can define on $F(\mathcal{P})$ the following *Poisson bracket*

$$\{f, g\}(x) = J^{ij} \partial_i f \partial_j g = \sum_{i=1}^{N} \Big(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \Big)$$

for any $f, g \in F(\mathcal{P})$. The Poisson bracket is a map $F(\mathcal{P}) \times F(\mathcal{P}) \to F(\mathcal{P})$ which has the following properties

1) Linearity $\quad \{f + \alpha h, g\} = \{f, g\} + \alpha\{h, g\}\,;$

2) Skew-symmetry $\quad \{f, g\} = -\{g, f\}\,;$

3) Jacobi identity $\quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0;$

4) Leibniz rule $\quad \{f, gh\} = \{f, g\}h + g\{f, h\}$

for arbitrary functions $f, g, h \in F(\mathcal{P})$ and $\alpha \in \mathbb{R}$. The first three properties imply that the Poisson bracket introduces on $F(\mathcal{P})$ the structure of an infinite-dimensional Lie algebra, while the Leibniz rule expresses the compatibility of the bracket with multiplication in $F(\mathcal{P})$. Due to this rule, the bracket is fully determined by its values on the coordinate functions x^i for which $\{x^i, x^j\} = J^{ij}$ or, explicitly,

$$\{q^i, q^j\} = 0\,, \quad \{p_i, p_j\} = 0\,, \quad \{p_i, q^j\} = \delta_i^j\,. \tag{1.9}$$

[1] The phase curve may consist of a single point. Such a point is called an *equilibrium position*.

Using the Poisson bracket, Hamilton's equations for the coordinate functions can be rephrased in the following concise form

$$\dot{x}^j = \{H, x^j\}\,.$$

As a consequence, evolution of any function f on the phase space is governed by the equation

$$\dot{f} = \{H, f\}\,.$$

Due to the skew-symmetry property of the Poisson bracket, this form of Hamilton's equations makes the conservation law for H obvious.

Poisson and symplectic manifolds. The properties *1) – 4)* provide a general definition of the Poisson bracket for an arbitrary smooth manifold $\mathcal{P}$. Any Poisson bracket is described by a skew-symmetric tensor J on $\mathcal{P}$ satisfying the Jacoby identity. In local coordinates this identity takes the form

$$\sum_{(i,l,m)} J^{ik}\partial_k J^{lm} = 0\,,$$

where the sum is over the cyclic permutation of indices. A manifold endowed with a Poisson bracket is called *Poisson*.

For later we will need the notion of a *Poisson map*. For Poisson manifolds $\mathscr{M}$ and $\mathscr{N}$, a smooth map $\varphi:\ \mathscr{M} \to \mathscr{N}$ is called Poisson, if for any $f, h \in F(\mathscr{N})$

$$\{f, h\}_{\mathscr{N}}(\varphi(x)) = \{\varphi^* f, \varphi^* h\}_{\mathscr{M}}(x)\,, \tag{1.10}$$

where $\varphi^* f(x) = f(\varphi(x))$ and $\varphi^* h(x) = h(\varphi(x))$, $x \in \mathscr{M}$, are pullbacks of f and h. Here $\{\ ,\ \}_{\mathscr{M}}$ and $\{\ ,\ \}_{\mathscr{N}}$ stand for the Poisson brackets on the respective manifolds.

In general, the rank r of the matrix J is less than or equal to the dimension $\dim\mathcal{P}$ of a manifold and it might change from point to point. In the case when $r = \dim\mathcal{P}$ at every point, the matrix J is invertible and the corresponding Poisson bracket is called non-degenerate. This is only possible if $\dim\mathcal{P}$ is even.

Exercise. Show this.

Indeed, since $J^t = -J$, one has

$$\det J = \det(-J) = (-1)^{\dim\mathcal{P}}\det J\,,$$

so that $(-1)^{\dim\mathcal{P}} = 1$ since $\det J \neq 0$.

A manifold $\mathcal{P}$ supplied with a non-degenerate Poisson bracket is called *symplectic*.[2] The inverse of J with entries ω_{ij}, where $J^{ik}\omega_{kj} = \delta^i_j$, defines a skew-symmetric bilinear differential 2-form ω on $\mathcal{P}$

$$\omega = -\tfrac{1}{2}\omega_{ij}(x)\,dx^i \wedge dx^j\,.$$

The Jacobi identity for J implies that this form is closed, *i.e.* $d\omega = 0$. A closed non-degenerate 2-form is called *symplectic*.

[2] As an aside, the term , from the Greek for "intertwined" was symplectic introduced in 1939 by Hermann Weyl in his book *The Classical Groups* as a substitute for the term *complex*.

Exercise. Show that the Jacobi identity for J implies the closeness of ω.
We write the Jacobi identity in the explicit form

$$J^{ik}\partial_k J^{lm} + J^{mk}\partial_k J^{il} + J^{lk}\partial_k J^{mi} = 0\,.$$

Multiply both sides by $\omega_{ij}\omega_{ms}$ and use $J^{ik}\omega_{kj} = \delta^i_j$ to obtain

$$-\partial_j J^{ml}\omega_{ms} - \omega_{ij}\partial_k J^{il} + \omega_{ij}J^{lk}\partial_k J^{mi}\omega_{ms} = 0\,.$$

In the last relation we put derivatives on ω using $\partial_s J^{ik}\omega_{kj} + J^{ik}\partial_s\omega_{kj} = 0$ that gives

$$J^{lm}\partial_j\omega_{ms} - J^{li}\partial_s\omega_{ij} - J^{lm}\partial_m\omega_{js} = 0\,.$$

It remains to multiply the last relation by ω_{kl} and get

$$\partial_i\omega_{jk} + \partial_k\omega_{ij} + \partial_j\omega_{ki} = 0\,,$$

which is equivalent to

$$\partial_{[i}\omega_{jk]} = 0 \quad \longrightarrow \quad d\omega = 0\,.$$

An example of a symplectic manifold is the space $\mathbb{R}^{2N}$ with the bracket (1.9). The corresponding symplectic form is

$$\omega = dp_i \wedge dq^i = d(p_i dq^i)\,.$$

The 1-form $\alpha = p_i dq^i$ is called the *canonical 1-form*.

Given a Poisson manifold, to any function $f \in F(\mathcal{P})$ one can associate a vector field ξ_f defined as

$$\xi_f = \{f,\ \cdot\ \}\,. \tag{1.11}$$

This field is called the *hamiltonian vector field* generated by f, and f is the generating or *hamiltonian function* of ξ_f. In local coordinates x^i we have

$$\xi_f = J^{ij}\partial_i f\partial_j\,. \tag{1.12}$$

If we let $\xi_f = \xi^j_f\partial_j$, then the relation above gives

$$\xi^j_f = J^{ij}\partial_i f\,, \qquad \partial_j f = \omega_{ij}\xi^i_f\,. \tag{1.13}$$

The Jacobi identity for the Poisson bracket implies

$$\xi_{\{f,g\}} = [\xi_f, \xi_g]\,. \tag{1.14}$$

Hence, the map $f \to \xi_f$ is a homomorphism[3] $F(\mathcal{P}) \to \mathfrak{X}(\mathcal{P})$, where $\mathfrak{X}(\mathcal{P})$ is the Lie algebra of vector fields on $\mathcal{P}$.

Exercise. Prove this statement.
We have

$$\begin{aligned}\xi_{\{f,g\}}h = \{\{f,g\},h\} &= -\{\{h,f\},g\} - \{\{g,h\},f\} = -\{g,\{f,h\}\} + \{f,\{g,h\}\}\\ &= -\xi_g\xi_f h + \xi_f\xi_g h = [\xi_f,\xi_g]h\,.\end{aligned}$$

[3]The map is from functions into vector fields, not vice versa, because functions whose difference is a constant lead, in fact, to the one and the same hamiltonian vector field.

If $\mathcal{P}$ is symplectic, the definition (1.11) of the hamiltonian vector field can be formulated with the help of the interior product i_ξ

$$i_{\xi_f}\omega + df = 0\,, \tag{1.15}$$

while the one-to-one correspondence between the Poisson bracket and the symplectic form ω can be expressed as

$$\omega(\xi_f, \xi_h) = \{f, h\} = \xi_f h = -\xi_h f\,. \tag{1.16}$$

A function C is called a *central* or *Casimir function* if it Poisson-commutes with any element of $F(\mathcal{P})$, that is

$$\{C, f\} = 0\,, \quad \forall f \in F(\mathcal{P})\,.$$

Casimir functions form a ring. If C is a Casimir function then it is annihilated by any hamiltonian vector field ξ_f, *i.e.* the latter lies everywhere tangent to the level set of the function C. On the other hand, the hamiltonian vector field ξ_C vanishes as the one-form dC belongs to the kernel of J: $JdC = 0$. Thus, the existence of non-constant Casimir functions means that $r \neq \dim\mathcal{P}$, *i.e.* the Poisson bracket is degenerate.

Let $\{C_i\}$, $i = 1, \ldots, m$, be a complete set of independent Casimir functions. Consider a level set $\mathcal{P}_c = \{x \in \mathcal{P} :\ C_i(x) = c_i\}$, where c_i are constants. Any hamiltonian vector field is tangent to $\mathcal{P}_c$

$$\xi_f C_i = \{f, C_i\} = 0\,, \quad \forall f \in F(\mathcal{P})\,.$$

The same is true for the commutator of any two hamiltonian vector fields. Thus, by the Frobenius theorem, the level set $\mathcal{P}_c$ is an integral submanifold in $\mathcal{P}$. On $\mathcal{P}_c$ one can naturally define a 2-form ω

$$\omega_x(\xi_f, \xi_g) = \{f, g\}(x)\,, \quad x \in \mathcal{P}_c\,, \tag{1.17}$$

where ω_x is the value of ω at x. The differential of ω can be computed with the help of the formula[4]

$$\begin{aligned} 3d\omega(\xi_f, \xi_g, \xi_h) &= \xi_f\omega(\xi_g, \xi_h) + \xi_g\omega(\xi_h, \xi_f) + \xi_h\omega(\xi_f, \xi_g) \\ &- \omega([\xi_f, \xi_g], \xi_h) - \omega([\xi_h, \xi_f], \xi_g) - \omega([\xi_g, \xi_h], \xi_f)\,. \end{aligned}$$

Using (1.14), definition (1.17) and the Jacobi identity, we get $d\omega = 0$. Since the hamiltonian vector of any Casimir function vanishes, the form ω is non-degenerate and, therefore, it is symplectic, *i.e.* $\mathcal{P}_c$ is a symplectic manifold. Thus, the hamiltonian vector fields foliate $\mathcal{P}$ into integral even-dimensional sub-manifolds called *symplectic leaves*, each of which inherits a symplectic form from the original Poisson bracket on $\mathcal{P}$.

Canonical transformations. Consider a smooth coordinate transformation $x \to x' = x'(x)$. In terms of these new coordinates Hamilton's equations (1.8) take the form

$$\frac{dx'^i}{dt} = \frac{\partial x'^i}{\partial x^k}\frac{dx^k}{dt} = -\frac{\partial x'^i}{\partial x^k}J^{km}(x)\nabla^x_m H = -\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}J^{km}(x)\nabla_j H' \equiv -J'^{ij}(x')\nabla_j H'\,,$$

where

$$J'^{ij}(x') = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}J^{km}(x)\,, \tag{1.18}$$

[4]The general formula for the differential of a differential form of order k is

$$(k+1)d\omega(\xi_0, \xi_1, \ldots, \xi_k) = \sum_{i=0}^{k}(-1)^i\xi_i\omega(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_k) + \sum_{0\leqslant i\leqslant j\leqslant k}(-1)^{i+j}\omega([\xi, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots \xi_k)\,.$$

that is under coordinate transformations J transforms as a contravariant anti-symmetric tensor field. Here $H'(x') = H(x(x'))$. Evidently, the equations for x' are of the hamiltonian form with the new hamiltonian $H'(x')$ if and only if

$$\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}J^{km}(x) = J^{ij}(x')\,. \tag{1.19}$$

Diffeomorphisms of the phase space which satisfy this condition are called *canonical*. In other words, canonical transformations do not change the form of the Poisson (tensor) bracket. An infinitesimal diffeomorphism $x'^k = x^k + \xi^k$ is generated by a vector field ξ. Under such a diffeomorphism the form of an arbitrary contravariant tensor J varies according to (1.18),

$$\big(\mathcal{L}_\xi J\big)^{ij} \equiv J^{ij}(x) - J'^{ij}(x) = \xi^k\partial_k J^{ij} - \partial_k\xi^i J^{kj} - \partial_k\xi^j J^{ik}\,. \tag{1.20}$$

Here $\mathcal{L}_\xi$ is the Lie derivative of J along the vector field ξ. It is now obvious that infinitesimal canonical transformations correspond to those ξ for which $\mathcal{L}_\xi J = 0$.

If a manifold $\mathcal{P}$ is symplectic, then canonical transformations preserve the corresponding symplectic form, that is

$$\mathcal{L}_\xi\omega = 0\,. \tag{1.21}$$

For this reason, these transformations are also called *symplectic* or *symplectomorphisms*.

Exercise. Show that hamiltonian vector fields generate canonical transformations.

An important class of canonical transformations is comprised by the hamiltonian vector fields. Consider a diffeomorphism generated by a hamiltonian vector field ξ_f. From the definition (1.20) of the Lie derivative we deduce that

$$\begin{aligned}\big(\mathcal{L}_{\xi_f}J\big)^{ij} &= -J^{km}\partial_m f\partial_k J^{ij} + \partial_k(J^{im}\partial_m f)J^{kj} + \partial_k(J^{jm}\partial_m f)J^{ik}\\ &= \partial_m f\sum_{(i,j,m)} J^{ik}\partial_k J^{jm} = 0\,,\end{aligned}$$

where the sum over the cyclic permutation of indices i, j, k vanishes due to the Jacobi identity. The same result follows immediately from the Cartan formula

$$\mathcal{L}_{\xi_f}\omega = d(i_{\xi_f}\omega) + i_{\xi_f}(d\omega) = -d^2 f = 0\,,$$

since ω is closed. Hence, any hamiltonian vector field generates a canonical transformation. If a Poisson manifold is not symplectic, then hamiltonian vector fields generate symplectomorphisms of the corresponding symplectic leaves.

Generally, a hamiltonian system is characterised by a triple $(\mathcal{P}, \{\ ,\ \}, H)$: a phase space $\mathcal{P}$, a Poisson structure $\{\ ,\ \}$ and a hamiltonian function H. For any function f on the phase space, evolution equation is

$$\frac{df}{dt} = \{H, f\}\,.$$

Since $\{H, H\} = 0$, the hamiltonian is automatically conserved. Therefore, the motion of the system takes place on the submanifold of the phase space defined by the equation $H = E$ where E is a fixed constant.

1.2 Liouville theorem

Among a large variety of physically relevant dynamical systems, those which admit an exact solution turn out to be rather rare. Remarkably, however, for a special class of systems solutions of the corresponding Hamilton's equations can always be found by quadratures, *i.e.* by solving a finite number of algebraic equations and computing a finite number of definite integrals. Dynamical systems falling in this class are generally known as *Liouville integrable systems* because they satisfy the assumptions of the famous Liouville theorem. In essence, for a dynamical system with a $2N$-dimensional phase space $\mathcal{P}$ this theorem states that if there exist N independent functions $f_i \in F(\mathcal{P})$ including the hamiltonian H, which Poisson commute, $\{f_i, f_j\} = 0$, then the corresponding equations of motion can be solved by quadratures. Since $\{H, f_i\} = 0$, the functions f_i do not depend on time, *i.e.* they are integrals of motion. In general, two functions on a phase space that Poisson commute are said to be in involution. Thus, Hamilton's equations of any dynamical system that admits an involutive family of integrals of motion which is equal to half the dimension of its phase space in number can be solved, at least in principle, by means of well-established mathematical operations. For this reason Liouville integrable systems are also called *completely integrable systems*.

Exercise. Show that the number of independent integrals in involution cannot exceed N for a non-degenerate Poisson structure on a manifold on dimension $2N$.

It may happen that a dynamical system defined on a symplectic manifold $\mathcal{P}$ of dimension $2N$ exhibits more than N integrals of motion. In this case the maximal number of independent pairwise commuting integrals f_i can not exceed N. Indeed, the hamiltonian vector fields of f_i, $i = 1, \ldots, k$, span a subspace $V \subset T\mathcal{P}$ of the tangent bundle $T\mathcal{P}$ at any given point $x \in \mathcal{P}$. Assuming f_i in involution, one gets $\omega\big|_V = 0$ and, therefore, $V \subset V^\perp$, where $V^\perp$ is a skew-orthogonal complement of V in $T\mathcal{P}$. The last observation implies that $k = \dim V \leqslant \dim V^\perp$. On the other hand, $\dim V + \dim V^\perp = 2N$, as ω is non-degenerate. Hence, $k \leqslant 2N - k$, so that $k \leqslant N$.

For complete proof we need the following statement. If V is a subspace of a symplectic vector space $W = T\mathcal{P}$, then $\dim V + \dim V^\perp = \dim W$.

This can be proved as follows. There is a linear map $\varphi : \ W \to W^*$ assigning v to the linear transformation $\varphi(v) \ : V \to \mathbb{R}$ that sends $w \to \omega(v, w)$. Since ω is nondegenerate, then φ is injective. Since W and W^* have the same dimension, φ is an isomorphism. The image $\varphi(V^\perp)$ is the space of functions in W^* that vanish on V, which is isomorphic to the space of functions on W/V, *i.e.* $\varphi : \ V^\perp \to (W/V)^*$ is injective, and in fact an isomorphism: any function on W/V is to a function on W vanishing on V. Thus, $\dim V^\perp = \dim(W/V)^* = \dim(W/V) = \mathrm{codim} V$.

To demonstrate the concept of Liouville integrability in a simple setting, consider the example of the one-dimensional harmonic oscillator. The hamiltonian is (mass $m = 1$)

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2$$

and the Poisson bracket is given by $\{p, q\} = 1$. The energy is conserved, therefore, the phase space is fibered into ellipses $H = E$. Perform a change of variables

$$p = \rho\cos(\theta)\,, \qquad q = \frac{\rho}{\omega}\sin(\theta)\,.$$

Then for the Poisson bracket one gets $\{\rho, \theta\} = \frac{\omega}{\rho}$. The hamiltonian becomes $H = \frac{1}{2}\rho^2$, so ρ is an integral of motion. The variable θ evolves according to

$$\dot\theta = \{H, \theta\} = \rho\{\rho, \theta\} = \omega \qquad \to \qquad \theta(t) = \omega t + \theta_0.$$

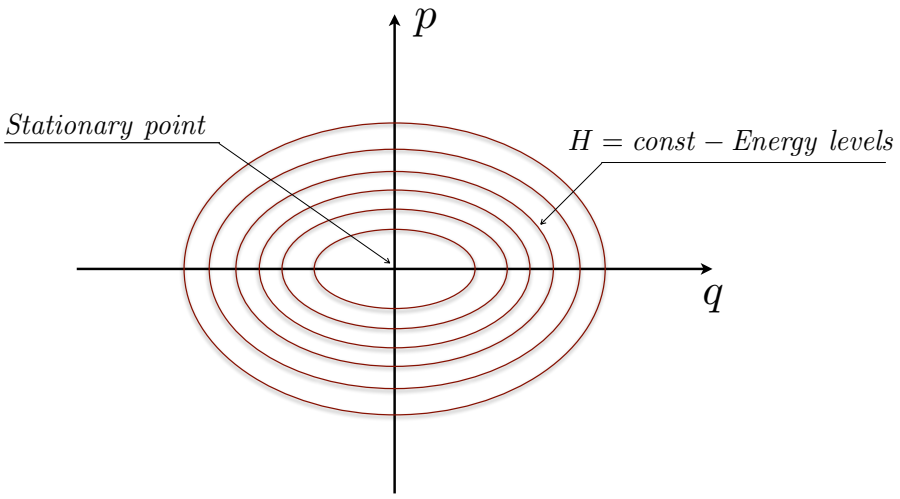


Figure 1.1: Phase space trajectories of the harmonic oscillator.

Thus, the phase space trajectories are ellipses with fixed values of ρ.

The generalisation to the n-dimensional harmonic oscillator is straightforward. The corresponding hamiltonian is[5]

$$H = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + \frac{\omega_i^2}{2} q_i^2 \right) ,$$

while commuting integrals are

$$f_i(p,q) \equiv \frac{1}{2} p_i^2 + \frac{\omega_i^2}{2} q_i^2 , \qquad i = 1, \ldots, N .$$

Define the common level set

$$\mathcal{P}_c = \{x \in \mathcal{P} : \ f_i(p,q) = c_i, \quad i = 1, \ldots, N\},$$

where c_i are constants. This set is a manifold isomorphic to an N-dimensional real torus $\mathbb{T}^N$. These tori foliate the phase space and can be parametrised by N angle variables θ_i that evolve linearly in time with frequencies ω_i.

Consider the equation

$$k_1 \omega_1 + \ldots k_N \omega_N = 0, \tag{1.22}$$

where $k = (k_1, \ldots k_N)$ is a vector with integer components. If (1.22) has at least one non-zero solution solution, the frequency set $(\omega_1, \ldots, \omega_N)$ is called *resonant*, otherwise it is *non-resonant*. For a non-resonant set of frequencies every trajectory is dense on the torus $\mathbb{T}^N$ and the corresponding motion is called *conditionally periodic*. Evidently, if all the frequencies are commensurable (rationally comparable), that is for any ω_i and ω_j there exist integers m and n such that

$$\omega_i \, m = \omega_j \, n \, ,$$

then the motion is periodic.

The multi-dimensional harmonic oscillator is a beautiful example of a Liouville integrable system as any such system exhibits a very similar structure of its phase space flows, the latter are described

[5]To uniformise notations, for the rest of this section we label coordinates by using lower indices.

by the Liouville theorem. The modern version of this theorem and the corresponding proof is due to Arnold.

Arnold-Liouville theorem. Let $\mathcal{P}$ be a $2N$-dimensional symplectic manifold. Suppose there exist N functions $f_i \in F(\mathcal{P})$ that are pairwise in involution with respect to the corresponding Poisson bracket

$$\{f_i, f_j\} = 0\,, \quad \forall i,j = 1,\dots,N\,.$$

Consider a common level set $\mathcal{P}_c$ of these functions,

$$\mathcal{P}_c = \{x \in \mathcal{P}:\ f_i(x) = c_i,\ \ i = 1,\dots,N\}\,, \tag{1.23}$$

where c_i are constants. Assume that functions f_i are independent on $\mathcal{P}_c$, which means that the 1-forms df_i are linearly independent at each point of $\mathcal{P}_c$. Then

1) $\mathcal{P}_c$ is a smooth manifold invariant under the hamiltonian flow with $H = H(f_i)$.

2) If $\mathcal{P}_c$ is compact and connected then it is diffeomorphic to the N-dimensional torus

$$\mathbb{T}^N = \{(\varphi_1,\dots,\varphi_N) \bmod 2\pi\}\,.$$

3) The motion on $\mathcal{P}_c$ under H is conditionally periodic, that is,

$$\frac{d\varphi_i}{dt} = \omega_i(c)\,.$$

4) The equations of motion can be integrated by quadratures.

We sketch the proof of the Arnold-Liouville theorem. Consider the hamiltonian vector fields ξ_i corresponding to the functions f_i. Since $\xi_i f_j = 0$, these vector fields are tangent to $\mathcal{P}_c$ and their linear independence implies that they span the tangent space of $\mathcal{P}_c$ at any point. Taking into account that the vector fields are in involution $[\xi_i, \xi_j] = 0$, we conclude on the base of the Frobenius theorem, that $\mathcal{P}_c$ is a maximal integral submanifold for the distribution spanned by ξ_i. Clearly, the manifold $\mathcal{P}_c$ is invariant under the hamiltonian flow triggered by any $H = H(f_i)$. Varying the constants c_i, we obtain a foliation of almost all[6] $\mathcal{P}$ into invariant submanifolds, see Fig. 1.2.

The main part of the proof consists in showing that whenever $\mathcal{P}_c$ is compact and connected, it is a torus but not, for instance, a sphere. Let $g_i^{t_i}$, $t_i \in \mathbb{R}$, be a one-parametric group of diffeomorphisms of $\mathcal{P}$ corresponding to the hamiltonian vector field ξ_i. The one-parametric groups corresponding to different vector fields commute because the vector fields commute. As a result, one can define the following action of the abelian group $\mathbb{R}^N = \{t_1,\dots,t_N\}$ on $\mathcal{P}_c$:

$$g^t(x) = g_1^{t_1} \cdots g_N^{t_N}(x)\,. \tag{1.24}$$

Since $\mathcal{P}_c$ is an integral manifold for the distribution spanned by ξ_i, this action is transitive and, therefore, $\mathcal{P}_c$ is a homogeneous space. Thus, $\mathcal{P}_c$ is diffeomorphic to the quotient $\mathbb{R}^N/\Gamma$, where Γ is the isotropy subgroup of $\mathbb{R}^N$, *i.e.* a set of all points $t \in \mathbb{R}^N$ for which $g^t(x) = x$. The fact that the fields ξ_i are independent at any point of $\mathcal{P}_c$ implies that the action (1.24) is locally free (none of the group elements has a fixed point) and, therefore, Γ must be a discrete subgroup of $\mathbb{R}^N$. By assumption $\mathcal{P}_c$ is compact and, therefore, Γ should be nothing else[7] but an integral lattice $\mathbb{Z}^N$, so that $\mathcal{P}_c$ is diffeomorphic to $\mathbb{R}^N/\mathbb{Z}^N = \mathbb{T}^N$. By the standard construction of a homogeneous space as a coset, the vector fields ξ_i are mapped by this diffeomorphism to the translation-invariant vector fields on $\mathbb{T}^N$. The angle variables $\{\varphi_i \bmod 2\pi\}$ parametrising the torus provide a coordinate system

[6] There could be values of c_i for which the equations $f_i = c_i$ cease to be independent.

[7] All discrete subgroups of $\mathbb{R}^N$ correspond to integral lattices $\mathbb{Z}^k$, $k \leqslant N$.

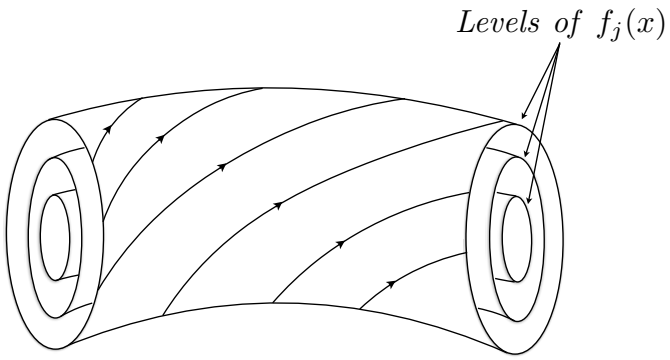


Figure 1.2: Foliation of a phase space by invariant tori. Each torus coincides with a level set $\mathcal{P}_c$. All trajectories on a given torus have the same frequencies $\omega_i(c)$, so one may speak of the "frequency set of a torus".

on $\mathcal{P}_c$ and they can be linearly[8] expressed via $t_1, \ldots, t_N$. The uniform motion on the torus $\mathbb{T}^N$ happens according to the law $\varphi_i = \varphi_i^0 + \omega_i t$ and is conditionally periodic. The numbers $\omega_i = \omega_i(c)$ are called frequencies.

The linear relation between angles and times means that

$$\varphi_i(t) = A_{ji} t_j + \varphi_i^{(0)} \mod 2\pi$$

for some constant matrix A that depends on the level set c. The evolution in the time direction t_j is driven by the vector $\xi_j = \{f_j, \ \}$ and, therefore, the Hamilton's equation for φ_i has the form

$$\frac{d\varphi_i}{dt_j} = \{f_j, \varphi_i\}\big|_{\mathcal{P}_c} = A_{ji} \, .$$

From here we find that the evolution of angle coordinates with respect to $H = H(f)$ will be

$$\frac{d\varphi_i}{dt} = \{H, \varphi_i\}\big|_{\mathcal{P}_c} = \frac{\partial H}{\partial f_j}\{f_j, \varphi_i\}\big|_{\mathcal{P}_c} = \frac{\partial H}{\partial f_j}\Big|_{\mathcal{P}_c} A_{ji} \equiv \omega_i \, .$$

Further, we note that the Arnold-Liouville theorem can be extended to the case when $\mathcal{P}_c$ is not necessarily compact. With an additional assumption that the hamiltonian vector fields ξ_i are complete[9] on $\mathcal{P}_c$, it is possible to show that each connected component of $\mathcal{P}_c$ is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{N-k}$.

Action-angle variables. The variables f_i, φ_j, $i, j = 1, \ldots, N$ featuring in the Arnold-Liouville theorem are not in general canonical coordinates on $\mathcal{P}$. However, such coordinates can be constructed. First we note that in a small neighbourhood of $\mathcal{P}_c$ the symplectic manifold $\mathcal{P}$ is diffeomorphic to the direct product $D \times \mathbb{T}^N$, where D is a small domain in $\mathbb{R}^N$. It turns out that in $D \times \mathbb{T}^N$ there exist coordinates I_i, θ_j, where $I_i \in D$, $\theta_j \in \mathbb{T}^N$ such that in these variables all f_i depend only on I_j and the symplectic structure has the canonical form $\omega = dI_i \wedge d\theta_i$. An explicit construction of the canonical variables I_i, θ_j proceeds as follows.

[8] This follows from $\mathbb{R}^N/\mathbb{Z}^N = \mathbb{T}^N$.

[9] A vector field is complete if any of its flow curves exists for all values of time.

It is clear that in the small neighbourhood of $\mathcal{P}_c$ the non-singular matrix of Poisson brackets takes the form

$$\begin{pmatrix} \{f_i, f_j\} & \{f_i, \varphi_j\} \\ \{\varphi_i, f_j\} & \{\varphi_i, \varphi_j\} \end{pmatrix} = \begin{pmatrix} 0 & A_{ij} \\ -A_{ji} & B_{ij} \end{pmatrix} . \tag{1.25}$$

The matrix A_{ij} is constant on $\mathcal{P}_c$ and therefore $A_{ij} = A_{ij}(f)$. We show that B_{ij} also depends on f_i. Consider the Jacobi identity

$$\{f_m, \{\varphi_i, \varphi_j\}\} + \{\varphi_i, \{\varphi_j, f_m\}\} + \{\varphi_j, \{f_m, \varphi_i\}\} = 0 . \tag{1.26}$$

We have

$$\begin{aligned} \{\varphi_i, \{\varphi_j, f_m\}\} + \{\varphi_j, \{f_m, \varphi_i\}\} = -\{\varphi_i, A_{mj}(f)\} + \{\varphi_j, A_{mi}(f)\} \\ -\frac{\partial A_{mj}}{\partial f_k}\{\varphi_i, f_k\} + \frac{\partial A_{mi}}{\partial f_k}\{\varphi_j, f_k\} = \frac{\partial A_{mj}}{\partial f_k} A_{ki} - \frac{\partial A_{mi}}{\partial f_k} A_{kj} \end{aligned} \tag{1.27}$$

which is independent on φ. Thus, the bracket

$$\{f_m, \{\varphi_i, \varphi_j\}\} = \{f_m, B_{ij}\} = \frac{\partial B_{ij}}{\partial \varphi_k}\{f_m, \varphi_k\} = A_{mk}\frac{\partial B_{ij}}{\partial \varphi_k}$$

is also φ-independent. Since the matrix A is invertible (otherwise the Poisson bracket (1.25) would be degenerate), $\frac{\partial B_{ij}}{\partial \varphi_k}$ also depends only on f which further implies that

$$B_{ij} = c_{ij}^s(f)\varphi_s + g_{ij}(f) .$$

Single-valuedness of the bracket requires that $c_{ij}^s = 0$ (otherwise, the bracket at 0 and at 2π for any of the angles will have different values although it corresponds to the one and the same value on the torus), *i.e.* B_{ij} is a function of f. One of the consequences of this fact is that the Jacobi identity (1.26) reduces to

$$\frac{\partial A_{mj}}{\partial f_k} A_{ki} - \frac{\partial A_{mi}}{\partial f_k} A_{kj} = 0 . \tag{1.28}$$

Now we perform the change of variables $f_i = f_i(I_j)$ such that $\{I_i, \varphi_j\} = \delta_{ij}$. For this we need to solve a system of equations

$$A_{ij} = \{f_i, \varphi_j\} = \frac{\partial f_i}{\partial I_k}\{I_k, \varphi_j\} = \frac{\partial f_i}{\partial I_k}\delta_{kj} = \frac{\partial f_i}{\partial I_j} .$$

The compatibility condition for this system is

$$\frac{\partial A_{ij}}{\partial I_s} = \frac{\partial f_i}{\partial I_s \partial I_j} = \frac{\partial f_i}{\partial I_j \partial I_s} = \frac{\partial A_{is}}{\partial I_j} .$$

Since $\frac{\partial A_{ij}}{\partial I_s} = \frac{\partial f_k}{\partial I_s}\frac{\partial A_{ij}}{\partial f_k} = \frac{\partial A_{ij}}{\partial f_k} A_{ks}$, this condition is equivalent to

$$\frac{\partial A_{ij}}{\partial f_k} A_{ks} = \frac{\partial A_{is}}{\partial f_k} A_{kj} ,$$

which is nothing else but the Jacobi identity (1.28) for functions $f_i, \varphi_j, \varphi_s$.

If the variables φ_i do not commute, then we should pass to new angle coordinates $\theta_i \bmod 2\pi$ by a shift $\varphi_i = \theta_i + h_i(I)$. The functions h_i are determined from the condition

$$B_{ij} = \{h_i, \theta_j\} + \{\theta_i, h_j\} = \frac{\partial h_i}{\partial I_j} - \frac{\partial h_j}{\partial I_i} , \tag{1.29}$$

which solubility condition is equivalent to the system of equations

$$\frac{\partial B_{ij}}{\partial I_k} + \frac{\partial B_{jk}}{\partial I_i} + \frac{\partial B_{ki}}{\partial I_j} = 0$$

which is the same as

$$\frac{\partial B_{ij}}{\partial f_m} A_{mk} + \frac{\partial B_{jk}}{\partial f_m} A_{mi} + \frac{\partial B_{ki}}{\partial f_m} A_{mj} = 0\,. \tag{1.30}$$

Since $\{\{\varphi_i, \varphi_j\}, \varphi_k\} = \{B_{ij}, \varphi_k\} = \frac{\partial B_{ij}}{\partial f_m}\{f_m, \varphi_k\} = \frac{\partial B_{ij}}{\partial f_m} A_{mk}$, one immediately recognises that (1.30) is just the Jacobi identity

$$\{\{\varphi_i, \varphi_j\}, \varphi_k\} + \{\{\varphi_j, \varphi_k\}, \varphi_i\} + \{\{\varphi_k, \varphi_i\}, \varphi_j\} = 0\,.$$

In this way we have constructed the action-angel variables I_i, φ_j realising the canonical structure.

Example of an explicit construction of action-angle variables. Consider a Liouville integrable system with the phase space $\mathbb{R}^{2N}$. According to the Liouville theorem, the motion occurs on a N-dimensional torus $\mathbb{T}^N$ being a common level of N commuting integrals. Let γ_j, $1 \leqslant j \leqslant N$, be the fundamental cycles of this torus depending continuously on the level $\{c_j\}$. Consider a set of equations $f_j(p, q) = c_j$ and solve it for p_j: $p_j = p_j(c, q)$. Introduce the so-called action variables[10]

$$I_j(c) = \frac{1}{2\pi} \oint_{\gamma_j} p_i(q, c) dq_i = \frac{1}{2\pi} \oint_{\gamma_j} \alpha\,, \tag{1.31}$$

where $\alpha = p_i dq_i$ is the canonical 1-form. Since c_j are time-independent as they are values of the integrals of motion, the variables $I_j = I_j(c)$ are also time-independent. Moreover, assuming that I_i are independent functions of c_j, the map $c_j \to I_j(c)$ given by (1.31) has an inverse. The angle variables θ_j are constructed by requiring that the transformation

$$(p_j, q_j) \to (I_j, \theta_j) \tag{1.32}$$

is canonical. To construct this canonical transformation, we will use the following generating function depending on the "old" coordinates q and the "new" momenta I

$$S(I, q) = \int_{q_0}^{q} p_i(\tilde{q}, I) d\tilde{q}_i\,,$$

where an integration path lies on $\mathcal{P}_c$. We have

$$p_j = \frac{\partial S}{\partial q_j} \quad \to \quad p_j = p_j(I, q). \tag{1.33}$$

The angle variables are introduced as

$$\theta_j = \frac{\partial S}{\partial I_j} \quad \to \quad \theta_j = \theta_j(I, q)\,. \tag{1.34}$$

Thus, for the differential of S we then have

$$dS = \frac{\partial S}{\partial q_j} dq_j + \frac{\partial S}{\partial I_j} dI_j = p_j dq_j + \theta_j dI_j\,.$$

[10]The physical dimension of I_j coincide with the dimension of the action that is the same as the dimension of angular momentum.

Acting on this relation with d and taking into account that $d^2S = 0$, we get

$$\omega = dp_j \wedge dq_j = dI_j \wedge d\theta_j \,,$$

which shows that I_i, θ_j are canonical variables.

A subtle point here concerns a dependence of S on the integration path. Consider a closed path: from q_0 to q and further from q to q_0. If this path is contractable, then by Stokes' theorem

$$\Delta S = \oint_{q_0}^{q_0} \alpha = \int d\alpha = \int \omega = 0 \,.$$

Here the vanishing of the integral of ω is due to the fact that ω vanishes on $\mathcal{P}_c$

$$\omega(\xi_i, \xi_j) = \{f_i, f_j\} = 0 \,.$$

If an integration path encloses a non-trivial cycle γ, the generation function undergoes a shift by an integral of α over this cycle

$$\Delta_\gamma S = \int_\gamma \alpha$$

that depends on I_j only. As a result, going over the cycle the variables θ_j undergo a jump

$$\Delta_\gamma \theta_j = \frac{\partial}{\partial I_j} \int_\gamma p_i(q, I) dq_i \,,$$

i.e. θ_j are multi-valued functions on $\mathcal{P}_c$. In particular, $\Delta_{\gamma_i}\theta_j = 2\pi\delta_{ij}$. This shows that θ_j are independent angle coordinates on the cycles. The same conclusion can be also drawn from the following consideration

$$\oint_{\gamma_j} d\theta_i = \oint_{\gamma_j} d\frac{\partial S}{\partial I_i} = \frac{\partial}{\partial I_i}\Big(\oint_{\gamma_j} dS\Big) = \frac{\partial}{\partial I_i}\Big(\oint_{\gamma_j} \frac{\partial S}{\partial q_k} dq_k\Big) = \frac{\partial}{\partial I_i}\Big(\oint_{\gamma_j} p_k dq_k\Big) = 2\pi\delta_{ij} \,,$$

as on $\gamma_j \in \mathbb{T}^N$ the variables I_j are constants and the function $S(I, q)$ depends on q only.

In the variables I, θ the Hamiltonian is a function of I. Then equations of motion become

$$\dot{I}_j = -\frac{\partial H}{\partial \theta_j} = 0 \,, \qquad \dot{\theta}_j = \frac{\partial H}{\partial I_j} \equiv \omega_j(I)$$

and they are trivially solved, $I_j(t) = I_j^0$, $\theta_j(t) = \theta_j^0 + \omega_j(I^0)t$. On the way of constructing the angle coordinates θ_j, algebraic operations were used to find p_j from $f_j(p, q) = c_j$ and a computation of a definite integral was implicitly done to obtain $S(I, q)$. Finally, the inverse of (1.32) was constructed by solving equations (1.34) for $q_j = q_j(I, \theta)$, which is also an algebraic operation. This way of solving a Liouville integrable system is behind the term "quadrature".

Note that even in the one-dimensional case the action-angle variables are not uniquely defined. The action variable is defined up to an additive constant and the angle variable can be shifted by an arbitrary function h of I: $I \to I + \text{const}$, $\theta \to \theta + h(I)$.

Exercise. Construct the action-angle variables for harmonic oscillator.

We illustrate the construction of the action-angle variables by using the harmonic oscillator as an example. We have

$$E = \frac{1}{2}(p^2 + \omega^2 q^2) \qquad \rightarrow \qquad p(E,q) = \pm\sqrt{2E - \omega^2 q^2}$$

and, therefore,

$$I = \frac{1}{2\pi}\oint_E dq\sqrt{2E - \omega^2 q^2} = \frac{2}{2\pi}\int_{-\frac{\sqrt{2E}}{\omega}}^{\frac{\sqrt{2E}}{\omega}} dq\sqrt{2E - \omega^2 q^2} = \frac{E}{\omega}\,.$$

The generating function of the canonical transformation reads

$$S(I,q) = \int^q dx\sqrt{2I\omega - \omega^2 x^2}\,,$$

while for the angle variable we obtain

$$\theta(I,q) = \frac{\partial S}{\partial I} = \omega\int^q \frac{dx}{\sqrt{2I\omega - \omega^2 x^2}} = \arctan\frac{\omega q}{\sqrt{2I\omega - \omega^2 q^2}}\,. \tag{1.35}$$

The change of θ for the period of motion, which is the same as an integral over the cycle of constant energy, is

$$\frac{1}{2\pi}\oint_E d\theta = \frac{1}{\pi}\omega\int_{-\sqrt{\frac{2I}{\omega}}}^{\sqrt{\frac{2I}{\omega}}} \frac{dx}{\sqrt{2I\omega - \omega^2 x^2}} = 1\,. \tag{1.36}$$

Inverting (1.35) with respect to q, we get

$$q = \sqrt{\frac{2I}{\omega}}\sin\theta\,.$$

We can verify that the transformation to the action-angle variables is indeed canonical

$$\begin{aligned} dp\wedge dq &= \Big(\frac{\omega dI}{\sqrt{2I\omega - \omega^2 q^2}} - \frac{\omega^2 q dq}{\sqrt{2I\omega - \omega^2 q^2}}\Big)\wedge dq \\ &= \frac{\omega}{\sqrt{2I\omega - \omega^2 q^2}} dI\wedge\sqrt{\frac{2I}{\omega}}\,d(\sin\theta) = dI\wedge d\theta\,. \end{aligned}$$

1.3 Some examples of integrable systems

Bose gas with delta-interaction. The so-called delta-interaction model is defined by the hamiltonian

$$H = \frac{1}{2m}\sum_{i=1}^N p_i^2 + \kappa\sum_{i<j}\delta(q_i - q_j)\,, \tag{1.37}$$

where κ is a real coupling constant. For $\kappa > 0$ the interaction is repulsive and for $\kappa < 0$ it is attractive. A solution of the corresponding quantum-mechanical problem for the repulsive case was first obtained in the case of *bosons* by Lieb and Liniger, while the general case of *distinguishable particles* was solved by Yang. Expanded in many directions, this model serves as a prototype example of applications of the Bethe Ansatz techniques.

Calogero-Moser-Sutherland (CMS) models. Inverting the harmonic potential of the one-dimensional oscillator, one obtains a model with the hamiltonian

$$H = \frac{p^2}{2m} + \frac{\gamma^2}{mq^2}\,.$$

Exercise. The corresponding dynamical system can be thought of as describing radial motion of a free particle on a two-dimensional plane with fixed angular momentum L_φ attributed to the coupling constant γ, the latter has the physical dimension of the Planck constant.

The potential gives rise to centrifugal inverse-cube force. As any one-dimensional model with conserved energy, it can be elementary solved by quadratures. It is remarkable, however, that this model admits an integrable generalisation to many degrees of freedom

I. $$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{\gamma^2}{2m}\sum_{i\neq j}^{N}\frac{1}{q_{ij}^2}\,. \tag{1.38}$$

The latter model describes N particles on a line interacting by the inverse-square potential. Here $q_{ij} = q_i - q_j$ is the difference between coordinates of i'th and j'th particle on a line. This mechanical system with n degrees of freedom is historically tied up with names of Calogero and Moser who solved it first in the quantum and classical cases, respectively.

It has been shown by Sutherland that the model (1.38) can be further generalised to account for a periodic boundary conditions. The corresponding potential is a trigonometric generalisation of the one in (1.38) and the hamiltonian is

II. $$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{\gamma^2}{2m}\sum_{i\neq j}^{N}\frac{1}{4\ell^2\sin^2\frac{1}{2\ell}q_{ij}}\,. \tag{1.39}$$

This is the Sutherland model. It can be viewed as an integrable deformation of (1.38) depending on an additional length parameter ℓ. Particles are confined here to a ring of circumference $2\pi\ell$, the decompactification limit $\ell\to\infty$ brings (1.39) back to the rational case (1.38). Sutherland used this model to study thermodynamical properties of quantum fluid based on (1.38). The Sutherland model has an interesting variant where the length ℓ is analytically continued to imaginary values $\ell\to i\ell$, giving rise to the hyperbolic model with the inverse-sinh-squared potential

III. $$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{\gamma^2}{2m}\sum_{i\neq j}^{N}\frac{1}{4\ell^2\sinh^2\frac{1}{2\ell}q_{ij}}\,. \tag{1.40}$$

This time ℓ is naturally interpreted as an interaction length that sets the size of the region where interactions between particles are sizeable. In the limit $\ell\to\infty$ one again recovers the long-range model (1.38).

Evidently, the three models (1.38)-(1.40) are particular instances of the hamiltonian system with a pairwise potential $v(q) = v(-q)$

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \sum_{i<j}^{N} v(q_{ij})\,. \tag{1.41}$$

One can therefore ask a question on the most general function $v(q)$ for which the model defined by the hamiltonian is integrable in the Liouville sense. The answer turns out to be

IV. $$v(q) = \frac{\gamma^2}{m}\wp(q)\,, \tag{1.42}$$

where $\wp(q) \equiv \wp(q|\omega_1, \omega_2)$ is the Weierstrass elliptic function with half-periods ω_1 and ω_2, where we choose $\omega_1, -i\omega_2$ to be any positive numbers, possibly infinite.[11] This potential defines an elliptic model from which the previous models follow as degenerate cases when one or both periods become infinite. Specifically, we have

Rational case: $\omega_1 = \infty, \omega_2 = i\infty$,

$$\wp(q) \to \frac{1}{q^2}\,.$$

Hyperbolic case: $\omega_1 = \infty, \omega_2 = i\pi\ell$,

$$\wp(q) \to \frac{1}{4\ell^2 \sinh^2 \frac{q}{2\ell}} + \frac{1}{12\ell^2}\,.$$

Trigonometric case: $\omega_1 = \pi\ell, \omega_2 = i\infty$,

$$\wp(q) \to \frac{1}{4\ell^2 \sin^2 \frac{q}{2\ell}} - \frac{1}{12\ell^2}\,.$$

The rational, hyperbolic and trigonometric (potentials) models are marked as **I**, **II** and **III** respectively, while the most general elliptic case is refereed to as **IV**. In the following we abbreviate the systems $\mathbf{I} - \mathbf{IV}$ as the CMS (Calogero-Moser-Sutherland) models. These CMS models are related to the root system of the Lie algebra A_{N-1} and can be generalised to other root systems.

1.4 Lax representation and classical r-matrix

Lax representation. Let L and M be two square matrices whose entries are functions on a phase space. Consider the following matrix equation

$$\dot{L} = [M, L]\,, \tag{1.43}$$

where as usual dot stands for the time derivative. If equation (1.43) is identically satisfied as a consequence of hamiltonian equations for a given dynamical system, then this dynamical system is said to admit a *Lax representation* (1.43) with L being the corresponding *Lax matrix*. Such a pair of matrices L and M is often referred to as *Lax pair*.

The importance of the Lax representation is that, once found, it allows for a simple and universal construction of an extended set of conserved quantities as spectral invariants of the corresponding Lax matrix. Indeed, consider

$$I_k = \mathrm{Tr}\, L^k\,.$$

for $k \in \mathbb{Z}$. We have

$$\dot{I}_k = k\mathrm{Tr}(L^{k-1}\dot{L}) = k\mathrm{Tr}(L^{k-1}[M, L]) = \mathrm{Tr}[M, L^k] = 0\,,$$

i.e. the I_k are time-independent as a consequence of the hamiltonian equations implying (1.43). In fact, the matrix equation (1.43) can be readily solved as

$$L(t) = g(t)L(0)g(t)^{-1}\,,$$

where the invertible matrix $g(t)$ is determined from the equation

$$M(t) = \dot{g}g^{-1}\,.$$

[11] The $\wp$-function is homogeneous $\wp(\lambda q|\lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(q|\omega_1, \omega_2)$. With the assumption that ω_1, ω_2 has the physical dimension of length, this property allows one to use in $\wp(q)$ the dimensionful coordinate q.

By Newton's identities, integrals I_k are functions of the eigenvalues of the matrix L and vice versa. Since the eigenvalues of L are preserved in time, evolution of such a dynamical system is called *isospectral*.

It should be emphasised that a Lax pair, if it exists, is not uniquely defined. First, the one and the same dynamical system might admit Lax pairs represented by $n \times n$ matrices of different size n. Second, there is a freedom related to transformations of the type

$$L' = gLg^{-1}\,, \qquad M' = gMg^{-1} + \dot{g}g^{-1}\,, \tag{1.44}$$

where g is an arbitrary invertible matrix possibly depending on dynamical variables. If here L, M is a Lax pair, then L', M' is another one for the same dynamical system. Indeed,

$$\dot{L}' = \dot{g}Lg^{-1} + g[M,L]g^{-1} - gLg^{-1}\dot{g}g^{-1} = [gMg^{-1} + \dot{g}g^{-1}, gLg^{-1}] \equiv [M', L']\,.$$

Note that M undergoes a gauge-type transformation. Lastly, for a fixed L shifting M by any polynomial of L will not influence the Lax equation (1.43).

Exercise. Illustrate the concept of Lax representation on the simple example of a one-dimensional system with the following hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\nu^2}{2q^2}\,. \tag{1.45}$$

This system can be called *Calogero oscillator*, as in the limiting cases $\nu \to 0$ and $\omega \to 0$ it reduces to the usual oscillator and the rational Calogero model, respectively. Show that for this system one can take

$$L = \tfrac{1}{2}\begin{pmatrix} p & \omega q - \frac{\nu}{q} \\ \omega q - \frac{\nu}{q} & -p \end{pmatrix}\,, \qquad M = \tfrac{1}{2}\begin{pmatrix} 0 & -\omega - \frac{\nu}{q^2} \\ \omega + \frac{\nu}{q^2} & 0 \end{pmatrix}\,. \tag{1.46}$$

With this choice for L and M equation (1.43) is satisfied as a consequence of equations of motion $\dot{q} = p$ and $\dot{p} = -\omega^2 q + \frac{\nu^2}{q^3}$, and vice versa, satisfaction of (1.43) implies equations of motion for the Calogero oscillator. Notice that the conserved hamiltonian is expressed as $H = \mathrm{Tr}L^2 + \nu\omega$.

Among further examples we mention the Lax representations for the models CMS models **I** and **III**.

I. Lax representation for the rational CMS model

$$\begin{aligned} L &= \sum_{i=1}^{N} p_i E_{ii} - i\gamma \sum_{i\neq j}^{N} \frac{1}{q_{ij}} E_{ij}\,, \\ M &= i\gamma \sum_{i\neq j}^{N} \frac{1}{q_{ij}^2}(E_{ii} - E_{ij})\,. \end{aligned} \tag{1.47}$$

Using the canonical structure (1.9) and the Hamiltonian[12] (1.38), one can verify the validity of the Lax representation, namely, that

$$\dot{L} = \{H, L\} = [L, M]\,. \tag{1.48}$$

[12]For simplicity we put $m = 1$.

Exercise. Matrix unities.

The space $\mathrm{Mat}(\mathbb{R})$ has a bilinear form

$$\langle A, B\rangle = \mathrm{Tr}(AB) \tag{1.49}$$

and a natural basis of matrix unities. Denote by $E_{ij} \in \mathrm{Mat}(\mathbb{R})$ a matrix unit, *i.e.* a matrix which has only one non-trivial matrix element equal to 1 standing on the intersection of i's row with j's column

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}\,. \tag{1.50}$$

Questions:

1. Show that for the commutator and product one has

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}\,, \quad E_{ij}E_{kl} = \delta_{jk}E_{il}\,.$$

   Note that this basis is not orthogonal with respect to (1.49), rather $\mathrm{Tr}(E_{ij}E_{kl}) = \delta_{il}\delta_{jk}$.

2. Introduce the matrix representation for the so-called *split* Casimir

$$C = \sum_{a,b=1}^{N} E_{ab} \otimes E_{ba} \tag{1.51}$$

   and show the matrix elements of C are

$$C_{ij,kl} = \sum_{a,b=1}^{N} (E_{ab})_{ij} \otimes (E_{ba})_{kl} = \delta_{il}\delta_{jk} = \mathrm{Tr}(E_{ij}E_{kl})\,. \tag{1.52}$$

3. Show that for any matrix A one has

$$A = E_{ij}\mathrm{Tr}(E_{ji}A) \quad \to \quad A_1 = \mathrm{Tr}_2(C_{12}A_2)\,.$$

The first two spectral invariants of L produce the total momentum P and the Hamiltonian $2H \equiv H_2$. Starting from $\mathrm{Tr}L^3$ one produces new integrals of motion that cannot be expressed via P and H. More explicitly,

$$\begin{aligned}
H_1 = \mathrm{Tr}L \quad &= \sum_i p_i \\
H_2 = \mathrm{Tr}L^2 \quad &= \sum_i p_i^2 + \gamma^2 \sum_{i\neq j} \frac{1}{q_{ij}^2} \\
H_3 = \mathrm{Tr}L^3 \quad &= \sum_i p_i^3 + 3\gamma^2 \sum_{i\neq j} \frac{p_i}{q_{ij}^2} \\
H_4 = \mathrm{Tr}L^4 \quad &= \sum_i p_i^4 + 4\gamma^2 \sum_{i\neq j} \frac{p_i^2 + \frac{1}{2}p_ip_j}{q_{ij}^2} + \gamma^4 \sum_{i\neq j} \frac{1}{q_{ij}^4} + 2\gamma^4 \sum_{i\neq j\neq k} \frac{1}{q_{ij}^2 q_{jk}^2} \\
\dots \quad &\quad \dots
\end{aligned} \tag{1.53}$$

As $\gamma \to 0$ the integrals behave as $H_k \sim \sum_j p_j^k$, which gives a reason to call the basis of conserved quantities constituted by these integrals the *power sum basis*.

As we will discuss in the next lecture, the integrals H_k have the distinguished feature of being

local. This means that in the asymptotic limit of large time, these integrals take an additive form with respect to the number of particles

$$H_k \sim \sum_j p_j^k \,,$$

where p_j is an asymptotic momentum of j's particle. Exhibiting a commutative family of operators is not exciting by itself, but the fact that these operators are *local* is truly remarkable. For instance, powers of the hamiltonian, although commuting, are not local operators. Thus, the existence of local commuting operators adds to the notion of (quantum) integrability.

III. Lax representation for the hyperbolic CMS model. It is convenient to introduce $\mathcal{Q}_i = e^{q_i}$. In terms of this variable the Lax pair reads

$$\begin{aligned} L &= \sum_{i=1}^N p_i E_{ii} - i\gamma \sum_{i\neq j}^N \frac{\mathcal{Q}_j}{\mathcal{Q}_{ij}} E_{ij} \,, \\ M &= i\gamma \sum_{i\neq j}^N \frac{\mathcal{Q}_i \mathcal{Q}_j}{\mathcal{Q}_{ij}^2} (E_{ii} - E_{ij}) \,. \end{aligned} \tag{1.54}$$

One can verify that (1.48) holds for the canonical structure (1.9) and the Hamiltonian (1.40), the latter takes in terms of $\mathcal{Q}$-variables the form

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{\gamma^2}{2} \sum_{i\neq j} \frac{\mathcal{Q}_i \mathcal{Q}_j}{\mathcal{Q}_{ij}^2} \,.$$

Computing the spectral invariants of L, we generate the conservation laws for this model

$$\begin{aligned} H_1 = \mathrm{Tr} L \quad &= \sum_i p_i \\ H_2 = \mathrm{Tr} L^2 \quad &= \sum_i p_i^2 + \gamma^2 \sum_{i\neq j} \frac{\mathcal{Q}_i \mathcal{Q}_j}{\mathcal{Q}_{ij}^2} \\ H_3 = \mathrm{Tr} L^3 \quad &= \sum_i p_i^3 + 3\gamma^2 \sum_{i\neq j} \frac{\mathcal{Q}_i \mathcal{Q}_j}{\mathcal{Q}_{ij}^2} p_i \\ H_4 = \mathrm{Tr} L^4 \quad &= \sum_i p_i^4 + 4\gamma^2 \sum_{i\neq j} \frac{\mathcal{Q}_i \mathcal{Q}_j}{\mathcal{Q}_{ij}^2} \Big(p_i^2 + \tfrac{1}{2} p_i p_j \Big) + \gamma^4 \sum_{i\neq j} \frac{\mathcal{Q}_i^2 \mathcal{Q}_j^2}{\mathcal{Q}_{ij}^4} + 2\gamma^4 \sum_{i\neq j\neq k} \frac{\mathcal{Q}_i \mathcal{Q}_j^2 \mathcal{Q}_k}{\mathcal{Q}_{ij}^2 \mathcal{Q}_{jk}^2} \\ \dots \quad &\dots \end{aligned} \tag{1.55}$$

These formula share the features of the rational case.

Babelon-Viallet theorem and dynamical r-matrix. The Lax representation makes no reference to a Poisson structure. Spectral invariants of the Lax matrix are integrals of motion but without specifying this structure it is impossible to conclude anything about their involutive property.

A relation of integrals to the underlying Poisson structure gets established due to the *Babelon-Viallet theorem*. According to this theorem, having the involutive property of the eigenvalues of $L \in \mathrm{Mat}_n(\mathbb{C})$ is equivalent to the existence of a function $\mathbf{r}$ on the phase space with values in $\mathrm{Mat}_n(\mathbb{C})^{\otimes 2}$ such that the Poisson bracket between the entries of L is

$$\{L_1, L_2\} = [\mathbf{r}_{12}, L_1] - [\mathbf{r}_{21}, L_2] \,. \tag{1.56}$$

Here and throughout the book L_1 and L_2 stand for two different embeddings of L in the tensor product $\mathrm{Mat}_n(\mathbb{C})^{\otimes 2}$, namely, $L_1 = L \otimes \mathbb{1}$ and $L_2 = \mathbb{1} \otimes L$, so that $\{L_1, L_2\} = \{L \underset{,}{\otimes} L\}$ represents a

collection of all possible Poisson brackets between the entries of L. Correspondingly, the indices 1 and 2 of $\mathbf{r}_{12}$ refer to the first and second matrix components of $\mathrm{Mat}_n(\mathbb{C})^{\otimes 2}$, respectively. As an explicit matrix, $\mathbf{r}_{12} \equiv (r_{ij,kl})$, where $i,j = 1,\dots,n$ correspond to the first matrix space and $k,l = 1,\dots,n$ to the second one. The matrices on the right hand side of (1.56) are multiplied according to the standard rule of matrix multiplication. Thus, being written in components, formula (1.56) looks like

$$\{L_{ij}, L_{kl}\} = \mathbf{r}_{is,kl} L_{sj} - L_{is}\mathbf{r}_{sj,kl} - \mathbf{r}_{ks,ij} L_{sl} + L_{ks}\mathbf{r}_{sl,ij}\,, \qquad \forall\, i,j,k,l \in 1,\dots,n\,,$$

where we have separated the indices belonging two different matrix spaces of $\mathbf{r}_{12}$ by comma. Clearly, the use of the concise notation as in (1.56) saves a sufficient amount of work and space.

The matrix $\mathbf{r}$ is called *dynamical r-matrix*, which reflects the possibility for this matrix to depend on the phase space variables. Note that the bracket (1.56) is manifestly skew-symmetric. To obtain (1.56), we assume that L is diagonalisable,

$$L = S\Lambda S^{-1}\,,$$

where Λ is a diagonal matrix whose entries Λ_i are prospective integrals of motion. Assuming that the phase space is equipped with a Poisson structure such that $\{\Lambda_i, \Lambda_j\} = 0$ for any i,j, we compute

$$\begin{aligned}
&\{L_1, L_2\} = \{S_1\Lambda_1 S_1^{-1}, S_2\Lambda_2 S_2^{-1}\} = \\
&= \{S_1, S_2\}\Lambda_1 S_1^{-1}\Lambda_2 S_2^{-1} + S_1\{\Lambda_1, S_2\}S_1^{-1}\Lambda_2 S_2^{-1} - S_1\Lambda_1 S_1^{-1}\{S_1, S_2\}S_1^{-1}\Lambda_2 S_2^{-1} \\
&+S_2\{S_1, \Lambda_2\}\Lambda_1 S_1^{-1} S_2^{-1} - S_1\Lambda_1 S_2 S_1^{-1}\{S_1, \Lambda_2\}S_1^{-1} S_2^{-1} - S_2\Lambda_2 S_2^{-1}\{S_1, S_2\}S_2^{-1}\Lambda_1 S_1^{-1} \\
&-S_2\Lambda_2 S_2^{-1} S_1\{\Lambda_1, S_2\}S_2^{-1} S_1^{-1} + S_1\Lambda_1 S_1^{-1} S_2\Lambda_2 S_2^{-1}\{S_1, S_2\}S_1^{-1} S_2^{-1}\,.
\end{aligned}$$

Introducing the notation

$$k_{12} = \{S_1, S_2\}S_1^{-1} S_2^{-1}\,, \qquad q_{12} = S_2\{S_1, \Lambda_2\}S_1^{-1} S_2^{-1}\,, \qquad q_{21} = S_1\{S_2, \Lambda_1\}S_1^{-1} S_2^{-1}\,,$$

we have

$$\begin{aligned}
\{L_1, L_2\} &= k_{12}L_1L_2 + L_1L_2k_{12} - L_1k_{12}L_2 - L_2k_{12}L_1 \\
&- q_{21}L_2 + q_{12}L_1 - L_1q_{12} + L_2q_{21}\,.
\end{aligned}$$

From the explicit form of k_{12} one sees that $k_{21} = -k_{12}$. This allows one to further rearrange the bracket as

$$\begin{aligned}
\{L_1, L_2\} &= [k_{12}L_2 - L_2k_{12}, L_1] + [q_{12}, L_1] - [q_{21}, L_2] \\
&= \tfrac{1}{2}[[k_{12}, L_2], L_1] - \tfrac{1}{2}[[k_{21}, L_1], L_2] + [q_{12}, L_1] - [q_{21}, L_2]\,.
\end{aligned}$$

The last expression has precisely the form (1.56), where the corresponding r-matrix is

$$\mathbf{r}_{12} = q_{12} + \tfrac{1}{2}[k_{12}, L_2]\,.$$

Note that $\mathbf{r}_{12}$ is not assumed to have any specific symmetry properties. Also, it is not uniquely defined: one can readily see that a shift $\mathbf{r}_{12} \to \mathbf{r}_{12} + [\sigma_{12}, L_2]$, where $\sigma_{12} = \sigma_{21}$, does not influence the right hand side of (1.56). Also, the bracket (1.56) does not change its form under symmetry transformations (1.44), although the r-matrix does.

Proceeding with our example of the Calogero oscillator, the r-matrix corresponding to L in (1.46) can be chosen as

$$\mathbf{r} = \frac{\omega q^2 + \nu}{2q(\omega q^2 - \nu)}\Big(E_{12}\otimes E_{21} - E_{21}\otimes E_{12}\Big)\,, \tag{1.57}$$

which can be verified by straightforward calculation. The matrix is dynamical but depends on q only.

Concerning the Jacobi identity for (1.56), it yields the following constraint on the r-matrix

$$[L_1, [\mathbf{r}_{12}, \mathbf{r}_{13}] + [\mathbf{r}_{12}, \mathbf{r}_{23}] + [\mathbf{r}_{32}, \mathbf{r}_{13}] + \{L_2, \mathbf{r}_{13}\} - \{L_3, \mathbf{r}_{12}\}] + \text{cycl. perm} = 0\,.$$

In the case when $\mathbf{r}$ is independent of the dynamical variables, the last equation simplifies to

$$[L_1, [\mathbf{r}_{12}, \mathbf{r}_{13}] + [\mathbf{r}_{12}, \mathbf{r}_{23}] + [\mathbf{r}_{32}, \mathbf{r}_{13}]] + \text{cycl. perm} = 0\,.$$

In particular, the Jacobi identity will be satisfied if $\mathbf{r}$ obeys the following equation

$$[\mathbf{r}_{12}, \mathbf{r}_{13}] + [\mathbf{r}_{12}, \mathbf{r}_{23}] + [\mathbf{r}_{32}, \mathbf{r}_{13}] = 0\,. \tag{1.58}$$

Another important point about the Poisson structure (1.56) is that it yields the Lax representation for evolution equations driven by any of the hamiltonians $H_k = \mathrm{Tr}L^k$, $k \in \mathbb{Z}$. Indeed, from (1.56) one gets

$$\frac{dL}{dt_k} = \{H_k, L\} = [M_k, L]\,, \tag{1.59}$$

where $M_k = -k\mathrm{Tr}_1(\mathbf{r}_{21}L_1^{k-1})$ and t_k is the time evolution parameter along the hamiltonian flow triggered by H_k. One can verify that for the Calogero oscillator with $H = \mathrm{Tr}L^2$, the corresponding matrix M constructed in this way coincides with the one in (1.46).

Lecture 2

Classical scattering and integrability

It is natural to start from the picture of scattering in classical mechanics. Here we deal with well-defined classical trajectories and the simplest situation corresponds to a single non-relativistic particle scattering elastically off a fixed (heavy) target modelled by a potential with a finite interaction range. Asymptotic trajectories describing the motion of a particle far away from an interaction region are straight lines.[1] Restraining to a one-dimensional situation, this means that before the collision, as $t \to -\infty$, the actual orbit asymptotes to a free orbit

$$q(t) \to q_{\rm in}(t) \equiv q^- + v^- t\,, \qquad t \to -\infty\,, \tag{2.1}$$

for some fixed (q^-, v^-). Analogously, after the collision,

$$q(t) \to q_{\rm out}(t) \equiv q^+ + v^+ t\,, \qquad t \to +\infty\,. \tag{2.2}$$

The scattering process is completely characterised by $q_{\rm in}(t)$ and $q_{\rm out}(t)$, the incoming and outgoing asymptotic orbits. The pairs $(q^\pm, v^\pm)$ represent the classical scattering data and a transformation from (q^-, v^-) to (q^+, v^+) defines a classical scattering operator, also known as *classical S-matrix*. Scattering of several particles interacting with each other via an admissible pair-wise potential is considered in a similar manner. Every particle trajectory has well-defined in- and out-asymptotics and scattering results to a transformation of incoming into outgoing scattering data.

To understand what is special about scattering processes in an integrable model, we consider the example of N particles of equal mass $m = 1$ governed by an integrable hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{i<j}^{N} v(q_{ij})\,. \tag{2.3}$$

We assume that the potential v is symmetric: $v(q) = v(-q)$, repulsive and impenetrable, and falls off sufficiently rapidly with the distance between particles, to guarantee the existence of an asymptotic region. Concretely, one can think about the rational or hyperbolic CMS models, which potentials satisfy the above-mentioned conditions. Classical integrability for these models follows from the existence of the Lax representation with the matrix L given by

$$L = \sum_{j=1}^{N} p_j E_{jj} + \sum_{i<j}^{N} u(q_{ij}) E_{ij} \tag{2.4}$$

with an appropriate function $u(q)$.

[1] We exclude bounded orbits which might exist for the case of attractive potentials.

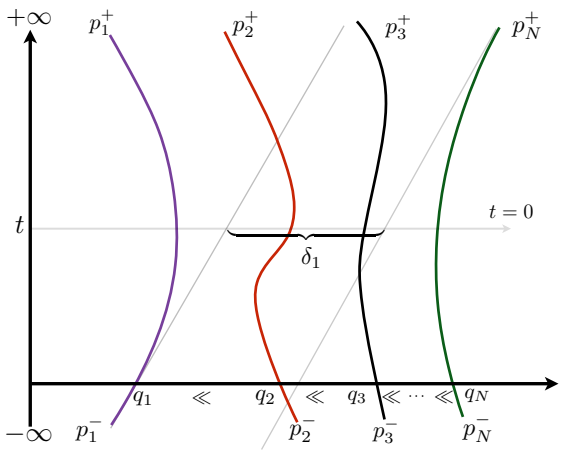


Figure 2.1: The picture of scattering in one dimension. Integrability implies that scattering is non-diffractive.

2.1 Non-diffractive scattering

Since the potential is repulsive and impenetrable, particles cannot overtake each other and one can label them according to the order

$$q_1(t) < q_2(t) < \ldots < q_N(t)\,, \qquad \forall t\,, \tag{2.5}$$

see Fig. 2.1. For $t \to -\infty$ the asymptotic condition has the form

$$q_i(t) = p_i^- t + q_i^- + o(1)\,, \qquad p_i^- \equiv p_i(-\infty)\,, \tag{2.6}$$

and it is described by $2N$ numbers (p_i^-, q_i^-). Analogously, for $t \to +\infty$,

$$q_i(t) = p_i^+ t + q_i^+ + o(1)\,, \qquad p_i^+ \equiv p_i(+\infty)\,. \tag{2.7}$$

From (2.5) that is valid for any t, in particular, for the asymptotic conditions (2.6) and (2.7), we deduce that

$$\begin{aligned} p_1^- > p_2^- > \ldots > p_N^-\,, \\ p_1^+ < p_2^+ < \ldots < p_N^+\,. \end{aligned}$$

Because of the asymptotic conditions, $|q_i - q_j| = \mathcal{O}(t)$ as $t \to \pm\infty$, the Lax matrix has the asymptotic limits $L(\pm\infty)$, where it becomes diagonal; the diagonal supports the set of eigenvalues λ_j of L. These eigenvalues are integrals of motion, and if we set $p_j^- = \lambda_j$ then,

$$\lambda_1 > \lambda_2 > \ldots > \lambda_N\,.$$

Obviously, the same order of eigenvalues must be found at $t = +\infty$, which is only possible if $L(+\infty)$ has the same eigenvalues as $L(-\infty)$ but in the reversed order:

$$L(-\infty) = \begin{pmatrix} p_1^- & & \\ & \ddots & \\ & & p_N^- \end{pmatrix}, \qquad L(+\infty) = \begin{pmatrix} p_N^+ & & \\ & \ddots & \\ & & p_1^+ \end{pmatrix}.$$

This implies in turn a very simple relation between the scattering data:

$$p_{N+1-j}^+ = p_j^-\,. \tag{2.8}$$

Thus, the set of incoming asymptotic momenta $\{p_i^-\}$ coincides with the set of outgoing ones $\{p_i^+\}$. This central result is usually referred to as *conservation of asymptotic momenta* and the corresponding scattering process is described as *non-diffractive*. Note that (2.8) is independent of the value of the coupling constant.

2.2 Classical phase shift

Due to the coincidence of the sets of incoming and outgoing momenta, we can reinterpret the scattering picture in a different way. Namely, we can associate to each particle a unique asymptotic momentum and assume that the order of particles is the same as that of their momenta. In particular, before scattering the fastest particle is the most left one, and after the scattering it reappears on the right of all the others, as if interactions would be completely absent. This is the so-called *transmission* representation of scattering in comparison to the *reflection* representation we started with. We have to say more on these interpretations when it comes to the discussion of quantum scattering.

From the transmission point of view, individual particles always keep their asymptotic momenta, while scattering shows up in the discontinuity δ_j of the asymptotic coordinates

$$\begin{aligned}\delta_j &= q^+_{N+1-j} - q^-_j \\ &= \lim_{t\to+\infty}\big(q_{N+1-j}(t) - q_j(-t) - 2p_j^- t\big)\,.\end{aligned} \tag{2.9}$$

The quantity δ_j, also known as the *classical phase shift*, completely characterises the scattering process: it shows how much the jth particle has advanced in comparison to a freely moving particle with momentum p_j. Our next goal is to understand how to compute δ_j, especially for an integrable model with the hamiltonian (2.3).

We start with the two-body problem. For the two-particle case the equations of motion can be solved in quadratures by making use of the conservation laws of energy and momentum

$$\begin{aligned}\frac{p_1^2}{2} + \frac{p_2^2}{2} + v(q_{12}) &= \frac{(p_1^-)^2}{2} + \frac{(p_2^-)^2}{2}\,,\\ p_1 + p_2 &= p_1^- + p_2^-\,.\end{aligned}$$

Indeed, introducing $k = p_1^- - p_2^- > 0$, from these equations we get

$$p_{1,2} = \frac{1}{2}(p_1^- + p_2^-) \pm \frac{1}{2}\sqrt{k^2 - 4v(q_{12})}\,. \tag{2.10}$$

To correctly associate the particle labels to the signs on the right hand side of the last formula, we recall that for $t \to -\infty$ the potential $v(q_{12})$ vanishes and from equations above we restore the asymptotic conditions for particle momenta, we have chosen

$$\begin{aligned}p_1(t) &= \frac{1}{2}(p_1^- + p_2^-) + \frac{1}{2}\sqrt{k^2 - 4v(q_{12})}\,,\\ p_2(t) &= \frac{1}{2}(p_1^- + p_2^-) - \frac{1}{2}\sqrt{k^2 - 4v(q_{12})}\,.\end{aligned} \tag{2.11}$$

As time goes, p_1 decreases, while p_2 increases and at the value t_0 such that $k^2 = 4v(x_0)$, $x_0 \equiv x(t_0)$, the particle momenta become equal. The difference $x(t) = q_2(t) - q_1(t) > 0$ is governed by the equation

Potential is symmetric $v(q_{12}) = v(q_{21})$

$$\dot{x}(t) = p_2 - p_1 = -\sqrt{k^2 - 4v(x(t))}\,. \tag{2.12}$$

As time grows starting from $-\infty$, the distance between particles diminishes and at $t = t_0$ it reaches its minimum $x = x_0$. By continuity, after passing the value t_0, p_1 and p_2 continue to decrease and increase, respectively, which for $t > t_0$ enforces the identification

$$p_1(t) = \frac{1}{2}(p_1^- + p_2^-) - \frac{1}{2}\sqrt{k^2 - 4v(q_{12})}\,,$$
$$p_2(t) = \frac{1}{2}(p_1^- + p_2^-) + \frac{1}{2}\sqrt{k^2 - 4v(q_{12})}\,.$$

The distance between the particles starts to increase again according to

$$\dot{x}(t) = \sqrt{k^2 - 4v(x(t))}\,. \tag{2.13}$$

This discussion provides a qualitative picture of the dynamics, see Fig. 2.2. In particular, the center–of-mass undergoes free motion in accordance with the prescribed asymptotic behaviour

$$q_1 + q_2 = (p_1^- + p_2^-)t + q_1^- + q_2^- = (p_1^+ + p_2^+)t + q_1^+ + q_2^+\,, \tag{2.14}$$

which, together with (2.8), yields

$$\cancel{(p_1^- + p_2^-)t} + q_1^- + q_2^- = \cancel{(p_1^+ + p_2^+)t} + q_1^+ + q_2^+ \quad \to \quad q_1^+ + q_2^+ = q_1^- + q_2^-$$

that implies for the discontinuities (2.9) that $\delta_1 + \delta_2 = 0$.

Equations (2.12) and (2.13) are solved by quadrature

$$\begin{aligned} t &= -\int^x \frac{\mathrm{d}x}{\sqrt{k^2 - 4v(x)}}\,, \qquad t \leqslant t_0\,, \\ t &= \int^x \frac{\mathrm{d}x}{\sqrt{k^2 - 4v(x)}}\,, \qquad t \geqslant t_0\,. \end{aligned} \tag{2.15}$$

Before we proceed with a general $v(q)$, let us consider the two concrete examples of the rational and hyperbolic CMS models corresponding to the potentials

$$v(q) = \frac{\gamma^2}{q^2} \quad \text{and} \quad v(q) = \frac{\gamma^2}{4\ell^2 \sinh^2 \frac{q}{2\ell}}\,.$$

In the latter case the parameter ℓ controls the interaction range and we set $\ell = 1$ to simplify considerations. For these models, performing integration in the first equation of (2.15) yields

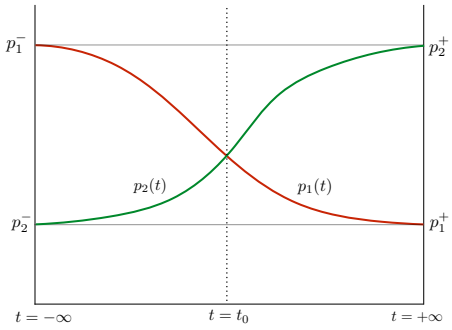


Figure 2.2: Time evolution of particle momenta in the two-body problem.

$$t = -\frac{1}{k}\sqrt{x^2 - \frac{4\gamma^2}{k^2}} + \frac{a}{k} \qquad \text{(rat)}\,,$$
$$t = -\frac{2}{k}\log\left(\cosh\frac{x}{2} + \sinh\frac{x}{2}\sqrt{1 - \frac{\gamma^2}{k^2\sinh^2\frac{x}{2}}}\right) + \frac{a}{k} \quad \text{(hyper)},$$

where in both cases a is an integration constant. This constant is found by matching the above formulae with the asymptotic expansion

$$x(t) = -kt + (q_2^- - q_1^-) + a_1/t + a_2/t^2 + \dots\,,$$

$x(t) = q_2 - q_1$ is always positive.

for $t \to -\infty$. In both cases we find $a = q_2^- - q_1^-$. For $t > t_0$ the solution of the second equation in (2.15) is

$$\begin{aligned} t &= \frac{1}{k}\sqrt{x^2 - \frac{4\gamma^2}{k^2}} - \frac{b}{k} \qquad\qquad (\text{rat})\,, \\ t &= \frac{2}{k}\log\left(\cosh\frac{x}{2} + \sinh\frac{x}{2}\sqrt{1 - \frac{\gamma^2}{k^2\sinh^2\frac{x}{2}}}\right) - \frac{b}{k} \quad (\text{hyper})\,. \end{aligned} \tag{2.16}$$

The integration constant b is found from the condition that at $t = t_0$ corresponding to $x = x_0$, solutions obtained for $t \leqslant t_0$ and $t \geqslant t_0$ must coincide. This gives $b = -a$ for the rational case and $b = -a + 4\log\cosh\frac{x_0}{2}$ for the hyperbolic one. Matching now the asymptotics

$$x(t) = kt + (q_2^+ - q_1^+) + b_1/t + b_2/t^2 + \dots$$

for $t \to \infty$ with that of (2.16), one gets

$$\begin{aligned} q_2^+ - q_1^+ &= b = -a = q_1^- - q_2^- \quad (\text{rat})\,, \\ q_2^+ - q_1^+ &= b = -a + 4\log\cosh\tfrac{x_0}{2} = q_1^- - q_2^- + 4\log\cosh\tfrac{x_0}{2} \quad (\text{hyper})\,. \end{aligned}$$

Combining these formulae with (2.14), yields

$$\begin{aligned} &\delta_1 = 0 = \delta_2 \quad (\text{rat})\,, \\ &\delta_1 = \log\cosh^2\tfrac{x_0}{2} = \log\left(1 + \frac{\gamma^2}{k^2}\right) = -\delta_2 \quad (\text{hyper})\,, \end{aligned} \tag{2.17}$$

where we used the fact that x_0 is defined by $\sinh^2\frac{x_0}{2} = \gamma^2/k^2$. In particular, if we restore the physical dimensions and the dependence on ℓ, the answer for the hyperbolic case $\delta(k) \equiv \delta_1$ reads as

$$\delta(k) = \ell\log\left(1 + \frac{\gamma^2}{k^2\ell^2}\right). \tag{2.18}$$

These examples make clear how to treat the case of a generic potential $v(x)$. A general solution of (2.15) can be written as

$$\begin{aligned} t &= -\int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{k^2 - 4v(x)}} + t_0\,, \qquad t \leqslant t_0\,, \\ t &= \int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{k^2 - 4v(x)}} + t_0\,, \qquad t \geqslant t_0\,, \end{aligned} \tag{2.19}$$

where t_0 is chosen such that for $x_0 = x(t_0)$ one has $4v(x_0) = k^2$. When $x \to \infty$, due to the rapid decrease of the potential at infinity, the first formula in (2.19) admits a well-defined asymptotic expansion

$$\begin{aligned} t &= -\int_{x_0}^{x} \mathrm{d}x\left(\frac{1}{k} + \frac{2}{k^3}v(x) + \frac{6}{k^5}v(x)^2 + \dots\right) + t_0 \\ &= -\left[\frac{1}{k}(x - x_0) + \frac{2}{k^3}V_1(x)\Big|_{x_0}^{x} + \frac{6}{k^5}V_2(x)\Big|_{x_0}^{x} + \dots\right] + t_0\,, \end{aligned}$$

where $V_1'(x) = v(x)$, $V_2' = v(x)^2$ and so on.

$kt = -(x - x_0) - 2/k^2(V(x) - V(x_0)) + \dots + kt_0$ as $t \to -\infty$.

Assuming $V(x) \to \text{const}$ for $x \to \infty$ and similar behaviour for other terms in the above expansion, one sees that $-x/k$ gives the leading term of the asymptotics $t \to -\infty$, while the constant term equal to $q_2^- - q_1^-$ should be identified with

$$\begin{aligned} q_2^- - q_1^- &= kt_0 + x_0 - k\left[\frac{2}{k^3}\Big(V(\infty) - V(x_0)\Big) + \frac{6}{k^5}\Big(V_2(\infty) - V_2(x_0)\Big)\dots\right] \\ &= kt_0 + x_0 - k\int_{x_0}^{\infty} \mathrm{d}x\Big(\frac{1}{\sqrt{k^2 - 4v(x)}} - \frac{1}{k}\Big)\,. \end{aligned}$$

This expresses the constant t_0 in terms of the asymptotic data and the potential. Analogously, considering the asymptotics of the second formula in (2.19) for $x \to \infty$, we find

$$\begin{aligned} q_2^+ - q_1^+ &= -kt_0 + x_0 - k\int_{x_0}^{\infty} \mathrm{d}x\Big(\frac{1}{\sqrt{k^2 - 4v(x)}} - \frac{1}{k}\Big) \\ &= q_1^- - q_2^- + 2x_0 - 2\int_{x_0}^{\infty} \mathrm{d}x\Big(\frac{k}{\sqrt{k^2 - 4v(x)}} - 1\Big)\,. \end{aligned}$$

This ultimately gives the coordinate discontinuity $\delta_1 \equiv \delta(k)$ for a generic potential subject to the requirements formulated above[2]

$$\delta(k) = x_0(k) - \int_{x_0(k)}^{\infty} \mathrm{d}x\left(\frac{1}{\sqrt{1 - \frac{4v(x)}{k^2}}} - 1\right), \tag{2.20}$$

where $x_0 = x_0(k)$ is found from $4v(x_0) = k^2$. Obviously, the discontinuity depends only on the difference k of asymptotic momenta but not on the asymptotic coordinates q_i^-. From (2.20) we also read that $\delta(k)$ is an even function of k. Now we can turn to the case of many particles.

2.3 Factorisation of the classical S-matrix

First we recall the notion of the scattering matrix in classical theory. Let $f(p,q)$ be an observable defined on a phase space $\mathcal{P} = \mathbb{R}^{2N}$ with the canonical bracket (1.9). Given $f(p,q)$ at some moment of time, say at $t = 0$, we can find its value $f(p,q,t) \equiv f(p(t),q(t))$ at any moment t through the hamiltonian equations. Assuming the time-dependence is continuous, we may expand $f(p,q,t)$ in powers of t

$$f(p,q,t) = f + \frac{t}{1!}\dot{f} + \frac{t^2}{2!}\ddot{f} + \frac{t^3}{3!}\dddot{f} + \dots$$

Using the hamiltonian equations this can be rewritten as

$$f(t) = f + \frac{t}{1!}\{H,f\} + \frac{1}{2!}t^2\{H\{H,f\}\} + \frac{t^3}{3!}\{H\{H\{H,f\}\}\} + \dots$$

Formally, this series can be viewed as an action on f of a certain evolution operator U_t

$$f(p,q,t) = e^{t\{H,\,\cdot\,\}} \circ f(p,q) \equiv (U_t f)(p,q)\,.$$

[2] As a check of the formula (2.20), we substitute $v(x) = \gamma^2/q^2$ for the rational CMS. We then directly find

$$\delta(k) = x_0(k) - \left(q\sqrt{1 - \frac{4\gamma^2}{k^2q^2}} - q\right)\Bigg|_{x_0(k)}^{\infty} = x_0\sqrt{1 - \frac{4\gamma^2}{k^2x_0^2}} = 0\,,$$

where $x_0^2 = 4\gamma^2/k^2$.

The evolution operator U_t defines a one-parametric continuous group of canonical transformations:

$$U_t(\{f,h\}) = \{U_t(f), U_t(h)\}\,.$$

In the context of scattering theory we assume the existence of in- and out-asymptotics, namely,

$$(p_i(t), q_i(t)) \to (p_i^{\pm}, p_i^{\pm} t + q_i^{\pm})\,, \qquad t \to \pm\infty\,, \tag{2.21}$$

where $(p_i^{\pm}, q_i^{\pm})$ are coordinates on the asymptotic phase spaces

$$\begin{aligned}\mathcal{P}^- &= \{(p^-, q^-) \in \mathbb{R}^{2N}\,,\ \ p_1^- > p_2^- > \ldots > p_N^-\}\,,\\ \mathcal{P}^+ &= \{(p^+, q^+) \in \mathbb{R}^{2N}\,,\ \ p_1^+ < p_2^+ < \ldots < p_N^+\}\,.\end{aligned}$$

sectorial regions

Since the time evolution preserves the Poisson brackets between canonical variables, the asymptotic data (p_i^-, q_i^-) and (q_i^+, p_i^+) also form canonical pairs, *i.e.* the asymptotic spaces are symplectic. In terms of evolution maps, eq.(2.21) can be written as

$$e^{t\{H,\,\cdot\,\}} \circ (p_i, q_i) \to e^{t\{H_0^{\pm},\,\cdot\,\}} \circ (p_i^{\pm}, q_i^{\pm}) \qquad t \to \pm\infty\,, \tag{2.22}$$

where $H_0^{\pm}$ are the free hamiltonians constructed from the asymptotic momenta

$$H_0^{\pm} = \frac{1}{2}\sum_{j=1}^{N} (p_j^{\pm})^2\,. \tag{2.23}$$

This allows one to define the classical analogue of the quantum-mechanical Møller operators (the wave maps) $\Omega_{\pm}:\ \mathcal{P}^{\mp} \to \mathcal{P}$, [3]

$$\Omega_{\pm} = \lim_{t \to \mp\infty} e^{-t\{H,\,\cdot\,\}} \circ e^{t\{H_0^{\mp},\,\cdot\,\}} \tag{2.24}$$

The wave maps are canonical and they are used to construct the classical S-matrix

$$S_{\rm class} = \Omega_-^{-1}\Omega_+ :\quad \mathcal{P}^- \to \mathcal{P}^+\,, \tag{2.25}$$

which is also a canonical transformation.

Coming back to the problem of finding the phase shifts δ_j for many-body scattering, we note that this can be done by relying on canonicity of the asymptotic phase spaces and the relations (2.8). Indeed, we have

$$\delta_{ij} = \{p_{N+1-i}^+, q_{N+1-j}^+\} = \{p_i^-, q_j^- + \ \delta^j\} = \delta_{ij} + \frac{\partial\ \delta_j}{\partial q_i^-}\,.$$

This shows that $\partial\ \delta_j/\partial q_i^- = 0$, that is δ_j *depends on* p^- *only* and, therefore, it can be immediately found by arranging the asymptotic data $q_1^-, \ldots, q_N^-$, such that collisions take place pairwise, with asymptotically large times in between of any two subsequent collisions.[4] This shows that the multi-body phase shift can be found by simply summing up the two-body phase shifts arising from collisions of j's particle with the rest[5]

$$\delta_j = q_{N-j+1}^+ - q_j^- = \underbrace{\sum_{k>j} \delta(p_j^- - p_k^-)}_{\text{right particles}} - \underbrace{\sum_{k<j} \delta(p_j^- - p_k^-)}_{\text{left particles}}\,. \tag{2.26}$$

[3] We define $\Omega_{\pm}$ following the tradition of the quantum scattering theory.

[4] Times of collision, like t_0, do depend on q_i^-.

[5] In the S-matrix picture where particles are labelled according to their conserved asymptotic momenta, initially j's particle has $j-1$ particles on its left and $N-j$ on its right, and, after all the collisions, it will have $N-j$ particles on its left and $j-1$ on its right.

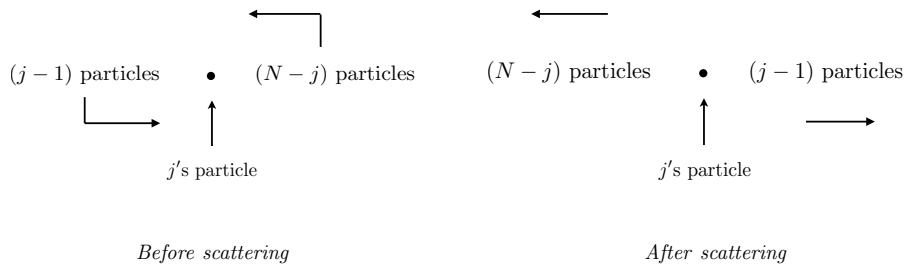


Figure 2.3: Multi-body phase shift

This strikingly simple answer is, of course, a consequence of the existence of a complete set of integrals of motion responsible for (2.8). There are no separate three- and higher-body events and the multi-particle scattering process is completely characterised by the two-body phase shift. A direct argument supporting this statement is based on using evolution equations produced by higher commuting integrals to rearrange the scattering process into a sequence of two-body events.

Finally, to determine the classical S-matrix for the scattering problem at hand, we consider the generating function $\Phi(q^-, p^+)$ of the canonical transformation $(q_i^-, p_i^-) \to (q_i^+, p_i^+)$, namely,

$$p_i^- = \frac{\partial \Phi(q^-, p^+)}{\partial q_i^-} = p_{N+1-i}^+ \,, \qquad q_i^+ = \frac{\partial \Phi(q^-, p^+)}{\partial p_i^+} = q_{N+1-i}^- + \ \delta_{N+1-i} \,, \tag{2.27}$$

where, according to (2.26),

$$\begin{aligned}
\delta_{N+1-i} &= \sum_{k>N+1-i} \delta(p_{N+1-i}^- - p_k^-) - \sum_{k<N+1-i} \delta(p_{N+1-i}^- - p_k^-) \\
&= \sum_{k>N+1-i} \delta(p_i^+ - p_{N+1-k}^+) - \sum_{k<N+1-i} \delta(p_i^+ - p_{N+1-k}^+) \\
&= \sum_{i>N+1-k} \delta(p_i^+ - p_{N+1-k}^+) - \sum_{i<N+1-k} \delta(p_i^+ - p_{N+1-k}^+) \\
&= \sum_{k<i} \delta(p_i^+ - p_k^+) - \sum_{k>i} \delta(p_i^+ - p_k^+) \,.
\end{aligned}$$

Integrating (2.27), we find

$$\Phi(q^-, p^+) = \sum_{i=1}^{N} q_i^- p_{N+1-i}^+ + \sum_{i<j}^{N} \theta(p_i^+ - p_j^+) \,, \tag{2.28}$$

where $\theta(k)$ is an integrated phase shift

$$\theta(k) = k x_0(k) - k \int_{x_0(k)}^{\infty} \mathrm{d}x \left(\sqrt{1 - \frac{4v(x)}{k^2}} - 1 \right) , \qquad \frac{\partial \theta}{\partial k} = \ \delta(k) \,. \tag{2.29}$$

Exercise. Prove (2.28).

We have

$$\begin{aligned}\sum_{i<j}\frac{\partial}{\partial p_k^+}\theta(p_i^+-p_j^+) &= \sum_{i<j}\left[\theta'(p_i^+-p_j^+)\frac{\partial p_i^+}{\partial p_k^+}-\theta'(p_i^+-p_j^+)\frac{\partial p_j^+}{\partial p_k^+}\right]\\ &= \sum_{i<j}\left[\,\delta(p_i^+-p_j^+)\delta_{ik}-\,\delta(p_i^+-p_j^+)\delta_{jk}\right]\\ &= \sum_{k<j}\,\delta(p_k^+-p_j^+)-\sum_{i<k}\,\delta(p_i^+-p_k^+)\\ &= \sum_{k<i}\,\delta(p_i^+-p_k^+)-\sum_{i<k}\,\delta(p_i^+-p_k^+)\end{aligned}$$

where in the first sum we replaced the summation index j for i and use the fact that $\delta(k)$ is an even function.

Obviously, $\theta(k)$ is an odd function of k. Were the theory free, the relation between $\mathcal{P}^+$ and $\mathcal{P}^-$ would reduce to relabelling of particles described by the generating function

$$\Phi_0(q^-,p^+)=\sum_{i=1}^{N}q_i^-p_{N+1-i}^+\,. \tag{2.30}$$

Thus, the non-trivial part of the generating function can be identified with the classical S-matrix that is, therefore, is given by

$$S_{\rm cl}=\sum_{i<j}^{N}\theta(p_i^+-p_j^+)\,. \tag{2.31}$$

This quantity has the physical dimension of the action: $[S_{\rm cl}]=[\hbar]$ and, from the point of view of the correspondence between classical and quantum mechanics, can be thought of as the leading term in the semi-classical expansion of the phase θ of the quantum-mechanical wave function $\Psi=ae^{\frac{i}{\hbar}\theta}$ in powers of $\hbar$.

A fundamental fact about this classical S-matrix is that it has a factorised structure, *i.e.* it is written as the sum of two-body integrated phase shifts. This is a direct consequence of integrability for scattering theory and, as we will see, it will persist in the quantum case as well.

Lecture 3

Bethe wave function and S-matrix

Now we come to multi-body scattering in quantum mechanics. In the coordinate representation a quantum-mechanical system is described by a multi-variable wave function $\Psi(q_1, \ldots, q_N)$. In the time-independent approach the wave function is a solution of the stationary Schrödinger equation ($\hbar = 1$)

$$-\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2}\Psi(q_1,\ldots,q_N) + \sum_{i\neq j} v(q_i - q_j)\Psi(q_1,\ldots,q_N) = E\Psi(q_1,\ldots,q_N)\,. \tag{3.1}$$

The potential v is translation invariant, so that the total momentum P is conserved, $[H, P] = 0$. For the case of two particles, the corresponding wave function is then searched as a common eigenstate of two commuting operators, H and P, and is naturally labeled by the asymptotic momenta p_1 and p_2. In the scattering process asymptotic momenta are conserved, *i.e.* incoming and outgoing plane waves are built on the one and the same set of asymptotic momenta.

For a generic potential there are no conservation laws beyond energy and momentum and, as a result, scattering is diffractive if more than two particles are involved. To make this evident, consider scattering for an initial state $|p_1, p_2, p_3\rangle$ corresponding to three particles with fixed asymptotic momenta p_1, p_2 and p_3. For large separation between the particles, this state is described by an incoming wave $\Psi_{\rm in} \sim e^{i\sum_{i=1}^{3} p_i q_i}$ that solves the free Schrödinger equation. After scattering happens, one expects to find an outgoing wave $\Psi_{\rm out}$, also given by a superposition of plane waves, albeit with all possible asymptotic momenta permitted by the conservation laws of energy and momentum

$$\Psi_{\rm out} \sim \sum_{k_1,k_2,k_3} \mathcal{A}(k_1,k_2,k_3) e^{ik_1q_1+ik_2q_2+ik_3q_3}\delta\Big(\sum_i k_i - P\Big)\delta\Big(\sum_i E(k_i) - E\Big)\,.$$

where $E(k) = k^2/2m$. Obviously, the two conservation laws are not anymore enough to forbid a continuous distribution of momenta among scattering constituents, hence, diffraction and genuine three-body events may occure.

3.1 Conservation laws and Bethe wave function

Let us now assume that our quantum-mechanical model is integrable in the sense that there exists a family of N linearly independent, local in particle momenta, pairwise commuting operators H_m, $m = 1, \ldots, N$, with H and P included in this family. In this case we can search for the wave function as a common solution of N compatible eigenvalue problems

$$H_m\Psi(q_1,\ldots,q_N) = h_m\Psi(q_1,\ldots,q_N)\,, \qquad m = 1,\ldots,N\,. \tag{3.2}$$

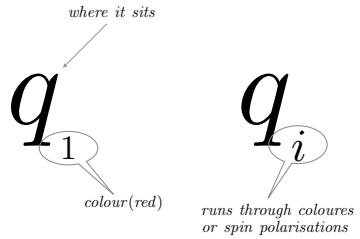


Figure 3.1: The meaning of a label of a distingushable particle. The value of q itself shows where the classical particle is or it gives the value of coordinate of the wave function.

Solutions to this system are thus labelled by the set $\{h_1, \ldots, h_N\}$ constituting the common spectrum of the commutative operator family. Further, we assume that H_m are deformations of the conservation laws $H_m^{(0)}(p_i)$ of free theory, the latter being symmetric functions of particle momenta p_i. For instance, for the hyperbolic CMS model H_m's are given by (1.53) or by (1.55) and $H_m^{(0)}$ is obtained from H_m by putting $\gamma = 0$. Our immediate goal is to show that the system (3.2) implies its scattering solutions to have a peculiar asymptotic form compatible with non-diffractive scattering. To simplify the discussion, we assume hereafter that particles are *distinguishable*, see Fig. 3.1.

Consider a special kinematic domain where particle coordinates are arranged as

$$q_1 < q_2 < \ldots < q_N \,. \tag{3.3}$$

Further, consider a special asymptotic regime in which distances between any two neighbouring particles become very large in comparison to the interaction range set by the potential. In this regime the system (3.2) turns into

$$H_m^{(0)}\Psi(q_1, \ldots, q_N) = h_m \Psi(q_1, \ldots, q_N)\,, \qquad m = 1, \ldots, N\,, \tag{3.4}$$

where $H_m^{(0)}$ are free conservation laws. For the corresponding asymptotic form of wave function we make the following ansatz

$$\Psi \sim e^{ip_1 q_1 + \ldots ip_N q_N}\,, \tag{3.5}$$

with numbers p_i called the asymptotic momenta. Substitution of (3.5) into (3.4) yields a system of equations for the asymptotic momenta

$$H_m^{(0)}(p_i) = h_m\,, \quad m = 1, \ldots, N\,. \tag{3.6}$$

Given a set of h_m, this system imposes very tight restrictions on possible values of the asymptotic momenta. Suppose we found a particular solution of (3.6) for which the individual momenta are enumerated according to the ordering pattern

$\sum_{j=1}^N p_j^m = h_m$.

$$p_1 > p_2 > \ldots > p_N\,. \tag{3.7}$$

Since $H_m^{(0)}$ are assumed to be symmetric functions of the p_i, it is plausible that all the other solution to (3.6) are simply obtained by permutations of the set (3.7). Thus, in the domain (3.3), called the *fundamental sector*, the asymptotic wave function is given by a superposition of plane waves constructed from the set of asymptotic momenta obeying (3.6). Explicitly,

$$\Psi(q_1, \ldots, q_N) = \sum_{\tau \in \mathfrak{S}_N} \mathcal{A}(\tau) e^{iq_1 p_{\tau(1)} + \ldots + iq_N p_{\tau(N)}}\,, \tag{3.8}$$

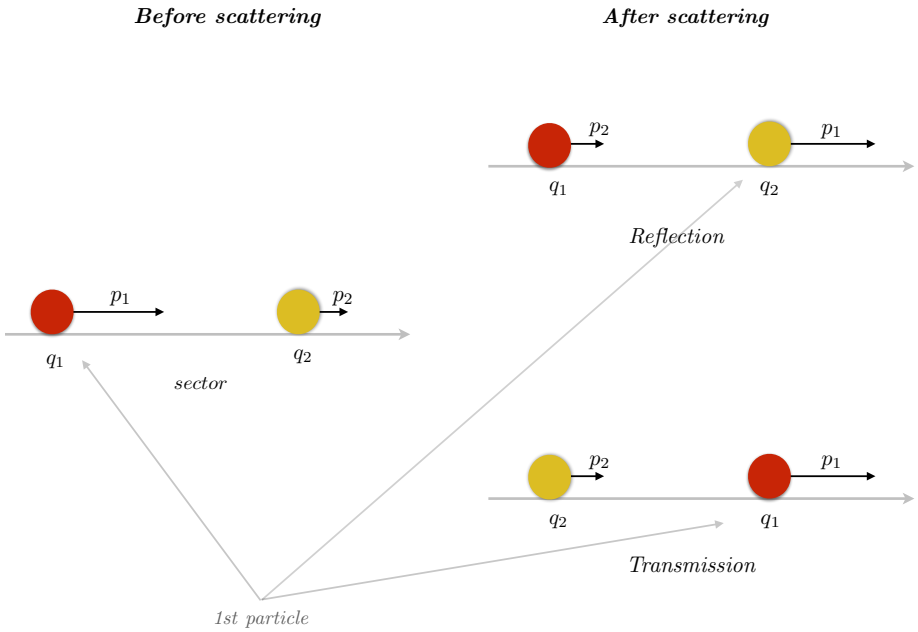


Figure 3.2: Interpretation of quantum-mechanical scattering in the transmission representation. Under scattering particles always keep their momenta. The position of a particle is rigidly tight with the colour (q_1 is red, q_2 is yellow). Under reflection the original order of colours along the line pertains, while it gets interchanged under transmission.

where the sum runs over all permutations τ from the symmetric group $\mathfrak{S}_N$, which act on indices of the asymptotic momenta, the latter form an ordered set according to (3.7).

In general, the configuration space $\mathbb{R}^N$ can be divided into $N!$ disconnected domains, each domain corresponds to a certain ordering of coordinates

$$q_{\sigma(1)} < q_{\sigma(2)} < \ldots < q_{\sigma(N)}\,, \tag{3.9}$$

where the latter are labelled by permutations $\sigma \in \mathfrak{S}_N$. The domain (3.9) will be called *σ-sector*. In particular, $\sigma = e$ corresponds to the fundamental sector (3.3). In each σ-sector we zoom in on the asymptotic region where the difference between any two neighbouring coordinates is very large so that the contribution of the potential terms in (3.2) is negligibly small. Thus, for any σ-sector (3.9) we will have the one and the same asymptotic system (3.4) with the same kind of asymptotic solution (3.8) which we write in the form

$$\Psi(q_1,\ldots,q_N|\sigma) = \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{iq_{\sigma(1)}p_{\tau(1)}+\ldots+iq_{\sigma(N)}p_{\tau(N)}}\,, \tag{3.10}$$

where the complex amplitudes $\mathcal{A}(\sigma|\tau)$ naturally form a $N!\times N!$ matrix depending on the particle momenta.

The expression (3.10) is the celebrated *Bethe wave function* that goes back to the *Bethe hypothesis* on the form of the wave function in the spin-wave problem It was introduced by C. N. Yang in his work on the delta-interaction Bose gas. The variable σ indicates that coordinates are restricted to lie in the domain (3.9). Different domains contribute with different and a priori unrelated amplitudes.

To deal with all sectors at once, it is convenient to introduce the following object

$$\Theta\big(q_{\sigma(1)} < \ldots < q_{\sigma(N)}\big) = \prod_{i=1}^{N-1} \Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big)\,,$$

where $\Theta(x)$ is the Heaviside Θ-function. Multiplying (3.10) with this object to explicitly account to the sector restriction, we then sum over all sectors to obtain

$$\Psi(q_1,\dots,q_N)=\sum_{\sigma\in\mathfrak{S}_N}\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma|\tau)e^{iq_{\sigma(1)}p_{\tau(1)}+\dots+iq_{\sigma(N)}p_{\tau(N)}}\Theta\big(q_{\sigma(1)}<\dots<q_{\sigma(N)}\big)\,. \tag{3.11}$$

Here the coordinates of Ψ are asymptotically unrestricted and since the sectors are not overlapping, the affiliation of a given coordinate configuration to one of them will be automatically detected by the Heaviside function present on the right hand side.

It is important to realise that the Bethe form of the asymptotic wave function is only possible due to integrability. Far away from the boundaries of a given σ-sector the Schrödinger equation becomes free and has a general solution given by a superposition of free waves. The set of asymptotic momenta is, however, the one and the same for each sector and for any type of scattering wave (incoming, outgoing), because this set is uniquely determined from equations (3.6) driven by global spectral invariants h_m. In turn, the fact that any asymptotic wave is determined by the same set of momenta of the incoming wave, up to permutations, means that the scattering process is non-diffractive. Three- and higher-body events that would lead to a continuous redistribution of momenta are prohibited by a sufficiently large number of conservation laws. Further insight into the structure of (3.10) can be thus derived from a relatively simple picture of successive two-body scatterings of classical particles.

Transmission and reflection representations. The fact that we deal with distinguishable particles can be made explicit by indicating the nature of a particle's identity. This can be, for instance, colour, or any other quantum number, like spin or charge. For now we take *colour* as an additional (internal) quantum number, so that all particles have different colours. We assume for simplicity that under collisions no new colours can be created, *i.e.* when they collide particles either keep or interchange their colours.

A collision process in which particles keep the same order of colours along the line before and after the collision is called *reflection* (backward scattering), and a process in which they interchange this order is called *transmission* (forward scattering), see Figs. 3.2 and 3.4. Were the particles to have the same colour, we would not be able to distinguish between reflection and transmission.

Regardless of colour, we have to decide on how to associate initial asymptotic momenta to particles after their two-body collision. Two different assignments are possible and they give rise to the so-called *transmission* and *reflection representations* of scattering. These are two alternative but equivalent ways to describe the scattering theory.

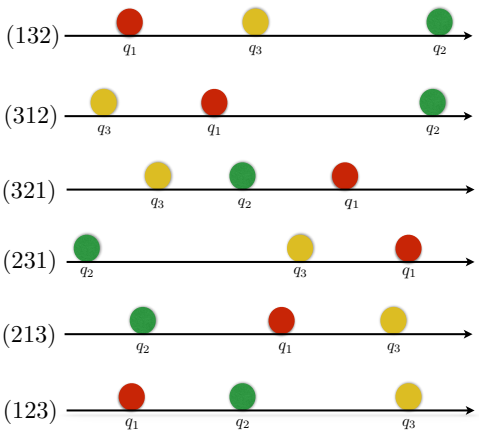


Figure 3.3: Sectors for $N=3$.

In the transmission representation particles are uniquely associated with their momenta, the latter are kept intact through collisions. If the i'th particle in an initial state has momentum p_i, after all possible collisions we have to identify the i'th particle as the one which has the same momentum p_i. In this representation the colour of a particle is rigidly tight to the corresponding coordinate label.[1] Formula (3.10) gives the Bethe wave function in the transmission representation, therefore, implying that we associate to each σ-sector a unique order of colours. Transition from sector to sector happens due to transmission. For instance, in

[1] Say, q_1 is a coordinate of a red particle, q_2 of a yellow one and so on.

Fig. 3.3 we pictured 6 sectors arising for the case of $N = 3$ particles. Each sector is associated to the corresponding permutation $\sigma \in \mathfrak{S}_3$ and it comes with a particular colour ordering. Note that making in (3.11) a change of the summation variable $\tau \to \sigma\tau$, we can write the Bethe wave function in the form

$$\Psi(q_1, \ldots, q_N) = \sum_{\sigma\in\mathfrak{S}_N} \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\sigma\tau) e^{iq_1 p_{\tau(1)} + \ldots + iq_N p_{\tau(N)}} \Theta\big(q_{\sigma(1)} < \ldots < q_{\sigma(N)}\big) \,. \tag{3.12}$$

If interactions are completely absent, the incoming wave $\Psi_{\text{in}} \sim e^{iq_i p_i}$ propagates to all sectors without changing its amplitude, which corresponds to perfect transmission; From the viewpoint of (3.12) perfect transmission means that $\mathcal{A}(\sigma|\sigma\tau) = \delta_{\tau e}$. Note that the conventional definition of the S-matrix relies on the use of the transmission representation. In the simplest two-body situation this *transmission-diagonal representation* follows by changing the order of amplitudes in the column

$$\begin{pmatrix} \mathcal{A}(21|21) \\ \mathcal{A}(12|21) \end{pmatrix} = S \begin{pmatrix} \mathcal{A}(12|12) \\ \mathcal{A}(21|12) \end{pmatrix} . \tag{3.13}$$

Here S is the 2×2 two-body S-matrix made of the reflection A and transmission B coefficients

$$S = \begin{pmatrix} B & A \\ A & B \end{pmatrix} \tag{3.14}$$

The matrix acting on the column of amplitudes becomes diagonal for reflectionless potentials.

In the transmission representation (3.10) each σ-sector has a unique colour ordering, so that the notations of σ- and colour sector are in fact equivalent and one can use them interchangeably. The situation, however, is different for the reflection representation of the scattering process which we now describe.

Type of representation is defined by assigning what happens to particle momenta under collisions

In the reflection representation a collision is an event where particles always interchange their momenta. Because of this interchange, particles cannot overtake each other, *i.e.* in this representation scattering happens within the fundamental sector and the Bethe wave function is understood as

$$\Psi_{\text{refl}}(q_1, \ldots, q_N|\sigma) = \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{iq_1 p_{\tau(1)} + \ldots + iq_N p_{\tau(N)}} \,, \tag{3.15}$$

where q_i satisfy (3.3). As usual, under collisions colours can be kept (reflection) or interchanged (transmission). Permutations σ are now in one-to-one correspondence to colour orderings, which are now unrelated to coordinates q_i, the latter *always* lie within the fundamental sector and cannot be continued outside. In the reflection representation, Fig. 3.3 would look the same as it looks, except now, regardless of its colour, the most left particle on the line will always be labelled as q_1, the second to it as q_2 and so on. Elementary reflections are characterised by passing from one plane wave to another one with interchanged momenta of two particles, but in the same sector which signifies no colour change. Elementary transmissions are the same but they necessarily involve the change of the sector. In this *reflection-diagonal representation* In the simplest two-body case the scattering process would give rise to the following relation between the amplitudes

$$\begin{pmatrix} \mathcal{A}(12|21) \\ \mathcal{A}(21|21) \end{pmatrix} = Y \begin{pmatrix} \mathcal{A}(12|12) \\ \mathcal{A}(21|12) \end{pmatrix} , \tag{3.16}$$

where the scattering matrix Y is the 2×2 matrix

$$Y = \begin{pmatrix} A & B \\ B & A \end{pmatrix} . \tag{3.17}$$

Comparing our discussion here with the treatment of classical scattering in section 2, we note that we started there with scattering in the reflection representation, which is, of course, physically very

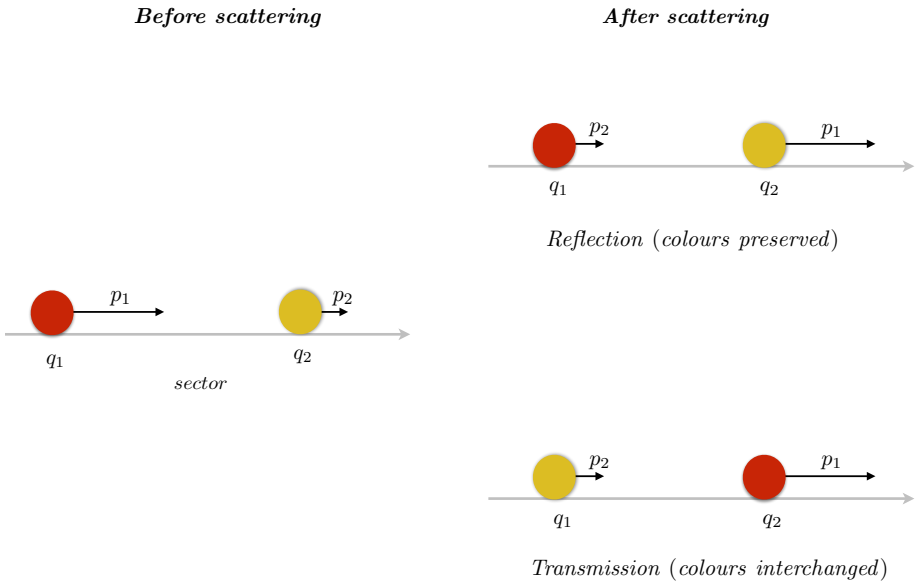


Figure 3.4: Interpretation of quantum-mechanical scattering in reflection representation. Under scattering momenta are always interchanged. Since particles cannot overtake each other, they live in a single sector $q_1 < q_2$. Transmission corresponds to a scattering channel where particles exchange their colours. Under reflection colours are preserved.

appealing if we deal with a repulsive potential. Specifying (2.8) to the two-particle case, we found that $p_1^+ = p_2^-$ and $p_2^+ = p_1^-$, which corresponds to the situation of interchanging the asymptotic momenta under scattering. To compute the classical phase shift, which is the same as the classical S-matrix, we then changed to the transmission representation, see the discussion around (2.9).

Note also that the energy and momentum are

$$\begin{aligned} E &= \frac{1}{2}\sum_{j=1}^{N} p_j^2\,, \\ P &= \sum_{j=1}^{N} p_j\,. \end{aligned} \tag{3.18}$$

In terms of asymptotic momenta the spectrum is additive, which is a characteristic feature of integrable models.

Lieb-Liniger model for distinguishable particles. As an explicit realisation of the Bethe wave function construction above, we consider the two-body problem for the Lieb-Liniger model described by the Hamiltonian (1.37). The Bethe wave function (3.11) for the two-particle case reads as

$$\begin{aligned} \Psi(q_1,q_2) = \Theta(q_1<q_2)\Big\{\mathcal{A}(12|12)e^{i(p_1q_1+p_2q_2)} + \mathcal{A}(12|21)e^{i(p_2q_1+p_1q_2)}\Big\} \\ + \Theta(q_2<q_1)\Big\{\mathcal{A}(21|12)e^{i(p_1q_2+p_2q_1)} + \mathcal{A}(21|21)e^{i(p_2q_2+p_1q_1)}\Big\}\,. \end{aligned} \tag{3.19}$$

Thus, we have four amplitudes involved. Separating the center of mass

$$q = q_1 - q_2, \qquad Q = q_1 + q_2\,,$$

so that

$$q_1 = \frac{Q+q}{2}\,, \quad q_2 = \frac{Q-q}{2}\,,$$

we get

$$\Psi(q_1,q_2)=e^{\frac{i}{2}(p_1+p_2)Q}\psi(q)\,, \tag{3.20}$$

where

$$\begin{aligned}\psi(q) &= \theta(-q)\left\{\mathcal{A}(12|12)e^{\frac{i}{2}(p_1-p_2)q}+\mathcal{A}(12|21)e^{-\frac{i}{2}(p_1-p_2)q}\right\}\\ &+ \theta(q)\left\{\mathcal{A}(21|12)e^{-\frac{i}{2}(p_1-p_2)q}+\mathcal{A}(21|21)e^{\frac{i}{2}(p_1-p_2)q}\right\}.\end{aligned}$$

Continuity of $\psi(q)$ at 0 requires

$$\mathcal{A}(12|12)+\mathcal{A}(12|21)=\mathcal{A}(21|21)+\mathcal{A}(21|21)\,. \tag{3.21}$$

The left and right derivatives at zero are

$$\begin{aligned}\psi'(+0) &= -\frac{i}{2}(p_1-p_2)\mathcal{A}(21|12)+\frac{i}{2}(p_1-p_2)\mathcal{A}(21|21)\\ \psi'(-0) &= \frac{i}{2}(p_1-p_2)\mathcal{A}(12|12)-\frac{i}{2}(p_1-p_2)\mathcal{A}(12|21)\end{aligned}$$

The wave function for the relative motion is the subject of the following Schrödinger equation

$$-\psi''(x)+\kappa\delta(x)\psi(x)=\frac{k^2}{4}\psi(x)\,,\quad k=p_1-p_2\,. \tag{3.22}$$

We then integrate both sides of equation (3.22) over a small segment $[-\epsilon,\epsilon]$ and then send $\epsilon\to 0$. Due to continuity of the wave function, we get an equation for the discontinuity of its derivative at the origin

$$\psi'(+0)-\psi'(-0)=\kappa\psi(0)\,. \tag{3.23}$$

Substituting here the derivatives found above, we get

$$\frac{i}{2}(p_1-p_2)\Big\{-\mathcal{A}(21|12)+\mathcal{A}(21|21)-\mathcal{A}(12|12)+\mathcal{A}(12|21)\Big\}=\kappa\Big\{\mathcal{A}(12|12)+\mathcal{A}(12|21)\Big\}\,. \tag{3.24}$$

By using (3.21) we first remove from this equation $\mathcal{A}(21|21)$

$$\begin{aligned}\frac{i}{2}(p_1-p_2)\Big\{-\mathcal{A}(21|12)+\cancel{\mathcal{A}(12|12)}+\mathcal{A}(12|21)-\mathcal{A}(21|12)-\cancel{\mathcal{A}(12|12)}+\mathcal{A}(12|21)\Big\}&\\ =\kappa\Big\{\mathcal{A}(12|12)+\mathcal{A}(12|21)\Big\}\,,&\end{aligned}$$

so that

$$\mathcal{A}(12|21)-\mathcal{A}(21|12)=-x_{12}\Big\{\mathcal{A}(12|12)+\mathcal{A}(12|21)\mathcal{A}\Big\}\,, \tag{3.25}$$

where we have introduced

$$x_{12}=\frac{i\kappa}{p_1-p_2}\,. \tag{3.26}$$

The last equation can be solved for $\mathcal{A}(12|21)$

$$\mathcal{A}(12|21)=-\frac{x_{12}}{1+x_{12}}\mathcal{A}(12|12)+\frac{1}{1+x_{12}}\mathcal{A}(21|12)\,. \tag{3.27}$$

Now we look at (3.24) again and remove this time $\mathcal{A}(12|21)$ by using (3.21)

$$\frac{i}{2}(p_1-p_2)\Big\{-\cancel{\mathcal{A}(21|12)}+\mathcal{A}(21|21)-\mathcal{A}(12|12)+\cancel{\mathcal{A}(21|12)}+\mathcal{A}(21|21)-\mathcal{A}(12|12)\Big\}$$
$$=\kappa\Big\{\mathcal{A}(21|21)+\mathcal{A}(21|21)\Big\}\,.$$

This gives

$$\mathcal{A}(21|21)=-\frac{x_{12}}{1+x_{12}}\mathcal{A}(21|12)+\frac{1}{1+x_{12}}\mathcal{A}(12|12)\,. \tag{3.28}$$

Equations (3.27) and (3.28) can be compactly written as

$$\Phi(21)=Y_{12}\Phi(12)\,,\quad Y_{12}=-\frac{x_{12}}{1+x_{12}}\mathbb{1}+\frac{1}{1+x_{12}}\pi_{12}\,, \tag{3.29}$$

where $\Phi(\tau)$ are columns of $\mathcal{A}(\sigma|\tau)$, namely,

$$\Phi(12)=\begin{pmatrix}\mathcal{A}(12|12)\\ \mathcal{A}(21|12)\end{pmatrix}\,,\qquad \Phi(21)=\begin{pmatrix}\mathcal{A}(12|21)\\ \mathcal{A}(21|21)\end{pmatrix}\,. \tag{3.30}$$

Comparing (3.29) to the general form (3.17), we read off the reflectiona and transmission coefficients for the Lieb-Liniger model

$$A=-\frac{i\kappa}{p_1-p_2+i\kappa}\,,\qquad B=\frac{p_1-p_2}{p_1-p_2+i\kappa}\,. \tag{3.31}$$

The vector $\Phi(21)$ is fully determined by $\Phi(12)$, on the other hand $\Phi(12)$ remains arbitrary, *i.e.* amplitudes in different sectors remain unrelated. This is unsurprising because we did not impose on the wave function any symmetry requirements.

3.2 S-matrix

The S-matrix of the problem is an $N!\times N!$ matrix, the elements of which encode how one of the $N!$ initial configurations of particles on a line couples to each of the final $N!$ configurations. To obtain the whole S-matrix it is enough to consider one distinguished configuration as an initial state. For instance, if we set up an incoming wave in the fundamental sector

$$e^{ip_1q_1+ip_2q_2+\ldots+ip_Nq_N}\,, \tag{3.32}$$

where momenta satisfy (3.7) to guarantee that scattering happens, then in the σ-sector we register an outgoing wave

$$e^{ip_Nq_{\sigma(1)}+ip_{N-1}q_{\sigma(2)}+\ldots+ip_1q_{\sigma(N)}} \tag{3.33}$$

with the amplitude given by the S-matrix element $S(\sigma|\varpi)$, where $\tau=\varpi$ is the reversed permutation

$$\varpi\equiv\begin{pmatrix}1 & 2 & \cdots & N\\ N & N-1 & & 1\end{pmatrix}\,. \tag{3.34}$$

In particular, (3.33) for $\sigma=e$ is a reflected wave in the fundamental sector. The remaining elements of the $N!\times N!$ matrix are obtained by permutations of particles in the initial state.

Thus, finding the S-matrix requires an extrapolation of the wave function from one asymptotic sector to another through sectorial boundaries where particle interactions are essential and cannot be

neglected. Solving the multi-body interacting problem is, in general, very complicated. However, one could notice the following: consider two sectors which differ only by a permutation of two neighbouring particles with coordinates q_i and q_j, so that in the first sector $q_i < q_j$ and $q_i > q_j$ in the second. Geometrically, these sectors are neighbours and have the hyperplane $q_i = q_j$ as a common boundary. Extrapolation of the wave function through this boundary can always be done in the asymptotic regime where all the other coordinates are kept far away from $q_i \approx q_j$ and from each other. Physically, this extrapolation corresponds to a two-body scattering event. Starting from any sector, one can obviously reach any other by passing through the adjacent sectorial boundaries, albeit not in a unique way. The sectors are thus connected by simple transpositions α_j, $j = 1, \dots, N-1$, the latter generate the symmetric group $\mathfrak{S}_N$.

Scattering operators in the reflection representation. The scattering process is described most elementary in the reflection representation. If we have two neighbouring particles at q_j and q_{j+1}, then under collision they interchange their momenta $p_{\tau(j)}$ and $p_{\tau(j+1)}$. If this collision is a pure reflection, then colours are preserved, if this is a pure transmission then also the colour sector changes as $\sigma \to \alpha_j \sigma$. This picture suggests that for these two pure processes the amplitudes must be related as

$$\begin{aligned} \mathcal{A}(\sigma|\alpha_j\tau) &= A(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)\,, \\ \mathcal{A}(\alpha_j\sigma|\alpha_j\tau) &= B(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)\,, \end{aligned} \tag{3.35}$$

where A and B are reflection and transmission coefficients, respectively. They depend on the momenta of scattered particles. The second formula in (3.35) is equivalent to

$$\mathcal{A}(\sigma|\alpha_j\tau) = B(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\alpha_j^{-1}\sigma|\tau) = B(p_{\tau(j)}, p_{\tau(j+1)})(\pi(\alpha_j)\mathcal{A})(\sigma|\tau)\,,$$

where we have written the final result via the action of the left regular representation π of $\mathfrak{S}_N$. To combine two scattering processes into one formula, we introduce a column vector $\Phi(\tau)$ which comprises the amplitudes in all the sectors corresponding to the same permutation τ of momenta

$$\Phi(\tau) \equiv \{\mathcal{A}(\sigma|\tau),\ \sigma \in \mathfrak{S}_N\}\,. \tag{3.36}$$

Then (3.35) can be combined into a single formula

$$\Phi(\alpha_j\tau) = Y_j(p_{\tau(j)}, p_{\tau(j+1)})\Phi(\tau)\,, \tag{3.37}$$

where we introduced Yang's scattering operators Y_j

$$Y_j(p_1, p_2) = A(p_1, p_2)\,\mathbb{1} + B(p_1, p_2)\,\pi(\alpha_j)\,, \tag{3.38}$$

where $j = 1, \dots, N-1$. Each Y_j is a $N! \times N!$ matrix which acts on the vector $\Phi(\tau)$. This matrix can be naturally viewed as a momentum-dependent connection on the symmetric group that defines the transport of the vector $\Phi(\tau)$ by a "discrete" amount α_j. As such, it must satisfy certain compatibility conditions that render the system of $(N-1)(N!)^2$ equations (3.37) for $(N!)^2$ unknowns $\mathcal{A}(\sigma|\tau)$ consistent. Indeed, we have

$$\Phi(\alpha_j^2\tau) = Y_j\big(p_{(\alpha_j\tau)(j)}, p_{(\alpha_j\tau)(j+1)}\big)\Phi(\alpha_j\tau) = Y_j(p_{\tau(j+1)}, p_{\tau(j)})Y_j(p_{\tau(j)}, p_{\tau(j+1)})\Phi(\tau)\,,$$

where we have taken into account that according to our rules $(\alpha_j\tau)(j) = \tau(\alpha_j(j)) = \tau(j+1)$ and $(\alpha_j\tau)(j+1) = \tau(\alpha_j(j+1)) = \tau(j)$. The defining relation $\alpha_j^2 = e$ then demands the fulfilment of the following relation

$$Y_j(p_1, p_2)Y_j(p_2, p_1) = \mathbb{1}\,. \tag{3.39}$$

Analogously, the second relation in (5.74) implies that

$$\Phi(\alpha_j\alpha_{j+1}\alpha_j) = \Phi(\alpha_{j+1}\alpha_j\alpha_{j+1})\,,$$

so that Y_j must satisfy

$$Y_j(p_2,p_3)Y_{j+1}(p_1,p_3)Y_j(p_1,p_2) = Y_{j+1}(p_1,p_2)Y_j(p_1,p_3)Y_{j+1}(p_2,p_3)\,. \tag{3.40}$$

Finally, the third relation in (5.74) leads to

$$Y_iY_j = Y_jY_i \tag{3.41}$$

for $|i-j| \geqslant 2$.

Provided the matrices Y_j satisfy the conditions above, the system (3.37) is consistent. Since α_j generate the whole $\mathfrak{S}_N$, the connection (3.37) transports the value of Φ at one point, for instance, at the identity e, to any other point of the group. The value $\Phi(e)$ is thus an initial condition for (3.37), it depends on $N!$ arbitrary parameters which are nothing else but the amplitudes $\mathcal{A}(\sigma|e)$ of purely incoming waves. Note that if the transmission is absent, *i.e.* $B = 0$, then each Y_j is the identity matrix times the reflection coefficient A, justifying the name *reflection* for the corresponding representation of the Bethe wave function. Thus, Y_j can be interpreted as the two-body scattering matrices in the reflection representation. In particular, for $N = 2$ there is only one transposition α represented by the matrix

$$\pi(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{3.42}$$

and, therefore, Y coincides with S-matrix (3.17).

Now we are ready to construct the full scattering matrix. The momenta of the incoming wave (3.7) are related to those of the outgoing wave by means of permutation (3.34) that acts as $\varpi(j) = N-j+1$. Writing ϖ in two-line notation makes it obvious that it can be represented as the following product of transpositions

$$\varpi = (1\,|N)(2\,|N-1)(3\,|N-2)\dots\,. \tag{3.43}$$

In turn, each of the transpositions entering this expression can be represented as a product of simple transpositions

$$\begin{aligned}
(1|N) &= (1|2)(2|3)(3|4)\dots(N-2|N-1)(N-1|N)(N-2|N-1)\dots(3|4)(2|3)(1|2)\,,\\
(2|N-1) &= (2|3)(3|4)\dots(N-3|N-2)(N-2|N-1)(N-3|N-2)\dots(3|4)(2|3)\,,\\
(3|N-2) &= (3|4)\dots(N-4|N-3)(N-3|N-2)(N-4|N-3)\dots(3|4)
\end{aligned}$$

and so on. Successively multiplying these expressions and using the defining relations of $\mathfrak{S}_N$, one finds that ϖ reduces to

$$\varpi = (1|2)\cdot(2|3)(1|2)\cdot(3|4)(2|3)(1|2)\cdot\ \dots\ \cdot(N-1|N)(N-2|N-1)\dots(1|2)\,, \tag{3.44}$$

where to make the structure of ϖ more visible, we separated the groups of simple transpositions by an explicit multiplication sign. Taking this structure of ϖ into account, formula (3.37) yields

$$\begin{aligned}
\Phi(\varpi) &= Y_1(p_{N-1},p_N)\\
&\times\ Y_2(p_{N-2},p_N)Y_1(p_{N-2},p_{N-1})\\
&\times\ Y_3(p_{N-3},p_N)Y_2(p_{N-3},p_{N-1})Y_1(p_{N-3},p_{N-2})\\
&\times\ \dots\\
&\times\ Y_N(p_1,p_N)Y_{N-1}(p_1,p_{N-1})\dots Y_1(p_1,p_2)\ \Phi(e)\,.
\end{aligned} \tag{3.45}$$

Here the order of Y matrices follows the pattern of simple transpositions in (3.44). The arguments of Y's were determined according to the scattering history built in (3.44).

To illustrate the last point, consider an example of $N = 4$. As the first step of using the connection formula (3.37), we have

$$\Phi((1|2)\cdot(2|3)(1|2)\cdot(3|4)(2|3)(1|2)) = Y_1(p_{\varpi^{(1)}(1)}, p_{\varpi^{(1)}(2)})\Phi((2|3)(1|2)\cdot(3|4)(2|3)(1|2))\,.$$

Here $\varpi^{(1)}$ is the permutation

$$\varpi^{(1)} = (2|3)(1|2)\cdot(3|4)(2|3)(1|2) = (3421)\,,$$

so that $\varpi^{(1)}(1) = 3$ and $\varpi^{(1)}(2) = 4$. Therefore,

$$\begin{aligned}\Phi((1|2)\cdot(2|3)(1|2)\cdot(3|4)(2|3)(1|2)) &= \\ &= Y_1(p_3,p_4)Y_2(p_{\varpi^{(2)}(2)}, p_{\varpi^{(2)}(3)})\Phi((1|2)\cdot(3|4)(2|3)(1|2))\,,\end{aligned}$$

where $\varpi^{(2)} = (1|2)\cdot(3|4)(2|3)(1|2) = (3241)$ yielding $\varpi^{(2)}(2) = 2$ and $\varpi^{(2)}(3) = 4$. Thus, we have

$$\Phi((1|2)\cdot(2|3)(1|2)\cdot(3|4)(2|3)(1|2)) = Y_1(p_3,p_4)Y_2(p_2,p_4)\Phi((1|2)\cdot(3|4)(2|3)(1|2))\,.$$

Continuing along the same lines, we will arrive at the final expression

$$\begin{aligned}\Phi((1|2)\cdot(2|3)(1|2)\cdot(3|4)(2|3)(1|2)) &= \\ &= Y_1(p_3,p_4)Y_2(p_2,p_4)Y_1(p_2,p_3)Y_3(p_1,p_4)Y_2(p_1,p_3)Y_1(p_1,p_2)\Phi(e)\,,\end{aligned}$$

which is a specification of (3.45) to the four-particle case.

S-matrix in the transmission representation. More generally, we can associate the scattering operator of Yang with an arbitrary transposition α_{ij}, namely,

$$Y_{ij}(p_1,p_2) = A(p_1,p_2)\,\mathbb{1} + B(p_1,p_2)\,\pi(\alpha_{ij})\,,$$

so that Y_j's introduced in (3.38) are $Y_j \equiv Y_{jj+1}$. In turn, by using Y_{ij}, we define the following matrix

$$S_{ij}(p_1,p_2) \equiv \pi(\alpha_{ij})Y_{ij} = B(p_1,p_2)\mathbb{1} + A(p_1,p_2)\pi(\alpha_{ij})\,. \tag{3.46}$$

Due to (5.73), we observe the following "braiding" property

$$S_{kj}\pi(\alpha_{ik}) = \pi(\alpha_{ik})S_{ij}\,. \tag{3.47}$$

Now we rewrite the main formula (3.45) via S_{ij} and, using (3.47), bring the answer to the following form

$$\begin{aligned}\Phi(\varpi) \;=\;& \pi(\varpi)\cdot S_{N-1\,N}(p_{N-1},p_N) \\ \times\;& S_{N-2\,N}(p_{N-2},p_N)S_{N-2\,N-1}(p_{N-2},p_{N-1}) \\ \times\;& S_{N-3\,N}(p_{N-3},p_N)S_{N-3\,N-1}(p_{N-3},p_{N-1})S_{N-3\,N-2}(p_{N-3},p_{N-2}) \\ \times\;& \dots \\ \times\;& S_{1N}(p_1,p_N)S_{1\,N-1}(p_1,p_{N-1})\dots S_{12}(p_1,p_2)\;\Phi(e)\,,\end{aligned} \tag{3.48}$$

where ϖ is the permutation (3.43). A welcome feature of this formula is that the index of each S-matrix perfectly matches with the index of the momenta on which this S-matrix depends. Thus, in the future we may not indicate the momentum dependence of S, as the latter can be unambiguously restored from the S-matrix subscript. Note that this is not true for the Y-representation (3.45). If we introduce

$$S \equiv S_{N-1\,N}\cdot S_{N-2\,N}S_{N-2\,N-1}\cdot\;\dots\;\cdot S_{1N}S_{1\,N-1}\dots S_{12}\,, \tag{3.49}$$

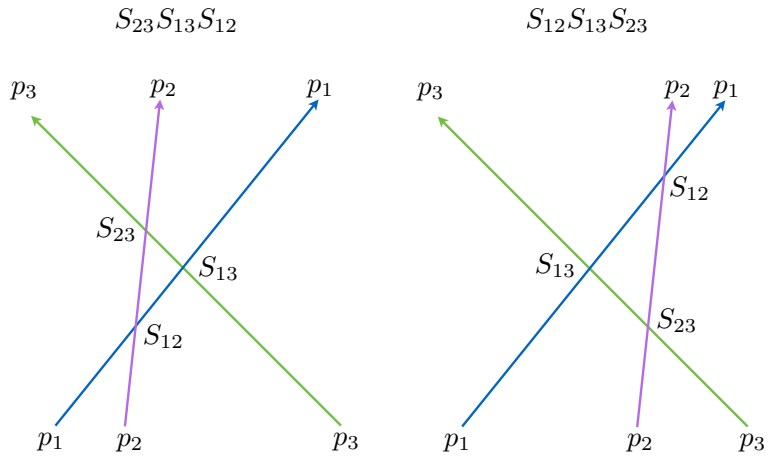


Figure 3.5: Two topologically different three-body space-time diagrams and factorisation of the three-body S-matrix. The result of the three-body scattering process does not depend on the order in which two-body scattering events take place.

then, keeping in mind that $\varpi^2 = e$, we obtain

$$\pi(\varpi)\Phi(\varpi) = S\,\Phi(e)\,. \tag{3.50}$$

The expression on the left hand side is $\pi(\varpi)\Phi(\varpi) = \mathcal{A}(\varpi\sigma|\varpi)$. This is the amplitude of the outgoing wave in the $\varpi\sigma$-sector where

$$\varpi\sigma = \begin{pmatrix} 1 & 2 & \cdots & N \\ N & N-1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & N \\ \sigma(1) & \sigma(2) & & \sigma(N) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & N \\ \sigma(N) & \sigma(N-1) & & \sigma(1) \end{pmatrix}.$$

It is clear that in this sector p_i couples to $q_{\sigma(i)}$, *i.e.* precisely in the same way as in the incoming wave in the σ-sector. The incoming wave in the σ-sector

$$e^{ip_1q_{\sigma(1)}+ip_2q_{\sigma(2)}+\ldots+ip_Nq_{\sigma(N)}}$$

with an amplitude $\mathcal{A}(\sigma|e)$ is transmitted to the $\varpi\sigma$-sector

$$e^{ip_Nq_{(\varpi\sigma)(1)}+ip_{N-1}q_{(\varpi\sigma)(2)}+\ldots+ip_1q_{(\varpi\sigma)(N)}} = e^{ip_1q_{\sigma(1)}+ip_2q_{\sigma(2)}+\ldots+ip_Nq_{\sigma(N)}}$$

with the amplitude $\mathcal{A}(\varpi\sigma|\varpi)$. Thus, S in (3.50) is nothing else but the S-matrix in the transmission representation. In particular, if the reflection coefficient $A = 0$ the S-matrix is diagonal, as can be seen from (3.46).

The structure of the scattering matrix encoded in (3.48) and (3.49) has also a very clear physical meaning: in the transmission picture the fastest particle with momentum p_1, which is the most left before scattering should undergo collisions with the remaining particles with momenta p_2, p_3, …, p_N to appear the most right after scattering. Every time it transfers through the i'th particle, its amplitude undergoes a change (a phase shift) by the corresponding two-body S-matrix $S_{1i}(p_1, p_i)$. After all these collisions the accumulated change of the amplitude is

$$S_{1N}S_{1\,N-1}\ldots S_{12}\,\Phi(e).$$

Then the p_2-particle, which is now the most left, goes to cross p_3, p_4, …, p_N and take its position in between p_N and p_1. This leads to further accumulation of successive amplitude changes and we get

$$S_{2N}S_{2\,N-1}\ldots S_{23}\cdot S_{1N}S_{1\,N-1}\ldots S_{12}\,\Phi(e).$$

Continuing in the same fashion, we find that after all particles crossed and reached the final order p_N, p_{N-1}, ..., p_1, the initial amplitude turns into $S\Phi(e)$, where S is exactly the S-matrix (3.48). Obviously, the number of two-body collisions that happens before the final configuration is reached is

$$N-1+N-2+N-3+\ldots+1=\frac{N(N-1)}{2}\,.$$

Hence, the N-body S-matrix factorises into the product of $N(N-1)/2$ two-body S-matrices. This factorised structure of the S-matrix is a consequence of a large number of conservation laws that prohibit diffraction and render the wave function in each asymptotic domain to be a superposition of a finite number of waves.

Another important observation about the factorised structure of the S-matrix is that the order in which $N(N-1)/2$ two-body collisions occur *does not matter*. This statement is a consequence of the consistency conditions obeyed by the two-body scattering matrix. These conditions can be immediately derived from those satisfied by the corresponding S-matrix in the reflection representation. Indeed, using the relation (3.46), the formulae (3.39) and (3.40) yield

$$S_{12}(p_1,p_2)S_{21}(p_2,p_1)=\mathbb{1} \tag{3.51}$$

and

$$S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3)=S_{23}(p_2,p_3)S_{13}(p_1,p_3)S_{12}(p_1,p_2)\,, \tag{3.52}$$

respectively. In addition, it follows from (3.46) that $S_{ij}S_{kl}=S_{kl}S_{ij}$, if among the indices i,j,k and l there are no two coincident ones.

Relation (3.52) is the *Yang-Baxter equation* for the two-body S-matrix. Physically, it expresses the equivalence of two different ways to factorise a three-body S-matrix S_{123} into a product of two-body S-matrices, see Fig. 3.5. Thus, integrability implies consistent factorisation of scattering process and the corresponding S-matrix in a sequence of two-body events and S-matrices, giving rise to the notion of *Factorised Scattering Theory*. This result is of fundamental nature, it reduces the problem of calculating the multi-body S-matrix in an integrable model to the one just for the two-body S-matrix, making the latter the main object of study, at least in the context of scattering theory.

Among further properties of the S-matrix, we point out that it is unitary and symmetric. Unitarity of (3.49) follows immediately if we require the two-body S-matrix to be unitary. Taking into account the unitarity of the representation π, the latter requirement reduces to the familiar conditions: $|A|^2+|B|^2=1$ and $\bar{A}B+\bar{B}A=0$. As to the symmetric property required by the time-reversal invariance of the interaction, using the fact that the two-body S-matrix is symmetric,[2] we have

$$S^t=S_{12}\ldots S_{1\,N-1}S_{1N}\cdot\ldots\cdot S_{N-2\,N-1}S_{N-2\,N}\cdot S_{N-1\,N}.$$

Now by successive use of (3.52) the right hand side of the last formula can be brought to the original form (3.49) that proves the relation $S^t=S$.

3.3 Bethe wave function and statistics

First we recall the standard treatment of a quantum-mechanical system of many particles with internal degrees of freedom. Suppose that the hamiltonian of such a system does not involve terms

[2]Representation π is orthogonal and, since for permutation $\pi(\alpha_{ij})^2=e$, one gets $\pi(\alpha_{ij})=\pi(\alpha_{ij})^t$.

acting on internal degrees of freedom. In this situation the wave function $\mathbf{\Psi}$ can be searched in a factorised form

$$\mathbf{\Psi}(x_1, \ldots, x_N) = \Psi(q_1, \ldots, q_N)\chi(s_1, \ldots, s_N)\,, \tag{3.53}$$

where Ψ depends on the particle coordinates q_i, while χ – on variables s_i describing internal degrees of freedom (spin). The variables x_i stand for the pairs (q_i, s_i). The Schrödinger equation determines the coordinate wave function Ψ only, leaving χ arbitrary. Indistinguishable particles are either bosons or fermions. Accordingly, the wave function $\mathbf{\Psi}$ is symmetric or anti-symmetric, which can only be possible if there is a rigid correlation between symmetry properties of Ψ and χ with respect to simultaneous permutations of their respective

Assume that the hamiltonian is invariant under permutations of particles, and, therefore, commutes with all operators representing permutations. Permutations, however, do not all commute with each other, and, as a consequence, can not be simultaneously brought to a diagonal form. This means that the spectrum of the hamiltonian is degenerate, and, in general, there will be several solutions Ψ of the Schrödinger equation with the same energy transforming into each other under the action of the symmetric group. In other words, these solutions can be combined into an irreducible multiplet λ of $\mathfrak{S}_N$. Correspondingly, the wave function Ψ_λ is labelled by an index λ and is said to be of the symmetry type λ. Similarly, the action of $\mathfrak{S}_N$ on χ can also be decomposed into the sum of irreducible components.

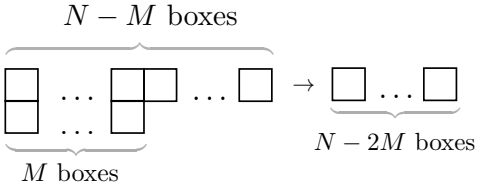


Figure 3.6: Young diagram $\bar{\lambda} = [N-M, M]$ represents an irrep of $\mathfrak{S}_N$ under which the spin wave function of electrons with the total spin $S = 1/2(N-2M)$ transforms.

Now, it is well known how to choose χ such that it can be combined with the coordinate wave function of a given symmetry type λ to produce symmetric or anti-symmetric $\mathbf{\Psi}$. Namely, if particles are bosons, then the symmetry of Ψ and χ must be defined by the same Young diagram, and the full symmetric wave function is expressed via certain bilinear combinations of those. If particles are spin-$\frac{1}{2}$ fermions, then the full wave function is anti-symmetric and the Young diagrams of the coordinate and the spin wave functions must be conjugate, *i.e.* one is obtained from the other by replacing rows for columns and vice versa. This follows from the fact that if $\pi_\lambda(\sigma)$ is the representation of $\mathfrak{S}_N$ corresponding to the partition λ, then the representation corresponding to the conjugate (the same as "transposed") partition $\bar{\lambda}$ is $\pi_{\bar{\lambda}}(\sigma) = \mathrm{sign}(\sigma)\pi_\lambda(\sigma)$, $\sigma \in \mathfrak{S}_N$. Hence,

$$\mathbf{\Psi} = \sum_{\sigma \in \mathfrak{S}_N} \Psi_\lambda(q_{\sigma(1)}, \ldots, q_{\sigma(N)})\chi_{\bar{\lambda}}(s_{\sigma(1)}, \ldots, s_{\sigma(N)})\,. \tag{3.54}$$

For the purpose of our present discussion, the most convenient way to describe representations λ (or $\bar{\lambda}$) of $\mathfrak{S}_N$ in the space of functions of N-variables is to use *Hund's method* [**?**]. According to this method, a function $\Psi_\lambda(q_1, \ldots q_N)$ has a definite symmetry type λ, *i.e.* it is *one of the basis functions* of a representation $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$ of $\mathfrak{S}_N$, if

1) it is anti-symmetric in a set of λ_1 arguments, anti-symmetric in another set of λ_2 arguments and so on,

2) it satisfies the Fock symmetry conditions

$$\Big[\mathbb{1} - \sum_{k \in \lambda_i} \alpha_{km}\Big]\Psi = 0\,, \tag{3.55}$$

where α_{km} is a transposition such that m is in λ_j for all choices of λ_i and λ_j with $\lambda_i \geqslant \lambda_j$.

One usually writes a function Ψ satisfying the first condition as

$$\Psi\big([q_1,\dots,q_{\lambda_1}][q_{\lambda_1+1},\dots,q_{\lambda_1+\lambda_2}]\dots[q_{N-\lambda_l-\lambda_{l-1}+1},\dots,q_{N-\lambda_l}][q_{N-\lambda_l+1},\dots,q_N]\big)\,,$$

meaning that it is separately anti-symmetric in the λ_1 variables $q_1,\dots,q_{\lambda_1}$, in the λ_2 variables $q_{\lambda_1+1},\dots q_{\lambda_1+\lambda_2}$, and so forth.

Later, by using a different realisation of irreducible representations of $\mathfrak{S}_N$, we show that for electrons the spin wave functions are associated to the two-row Young diagrams $\bar\lambda$ depicted in Fig. (3.6). Any such diagram with N boxes has $N-M$ boxes in the first row and M boxes in the second row, where M takes values from 0 to the integer part of $N/2$. For a fixed M the $\mathfrak{sl}(2)$-representation associated with this diagram has spin $S=1/2(N-2M)$. Therefore, for electrons the corresponding coordinate wave function must transform in the representation λ depicted in Fig 3.7 and, according to Hund's method, be of the type

$$\Psi([q_1,\dots,q_{N-M}][q_{N-M+1},\dots,q_N])\,, \tag{3.56}$$

and satisfy the Fock condition

$$\left[\mathbb{1}-\sum_{i=1}^{N-M}\alpha_{ij}\right]\Psi=0\,,\qquad j>N-M\,. \tag{3.57}$$

This completes our recollection of the standard treatment of multi-particle systems in quantum mechanics.

Coming back to the Bethe wave function, we would like to understand the conditions it must satisfy in order to be of a definite symmetry type. To this end, we consider this function in the transmission representation (3.10) and determine how it transforms under permutations of coordinates. For any particle configuration from the σ-sector the wave function is

$$\Psi(q_1,\dots,q_N)=\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma|\tau)e^{iq_{\sigma(i)}p_{\tau(i)}}\,. \tag{3.58}$$

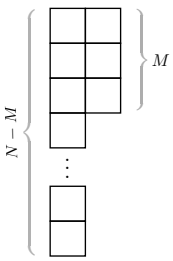


Figure 3.7: Young diagram $\lambda=[2^M,1^{N-2M}]$ for a coordinate function of electrons.

Consider $\Psi(q_{\varsigma(1)},\dots q_{\varsigma(N)})$, where ς is any permutation. Denote $q'_i\equiv q_{\varsigma(i)}$, so that $q_i=q'_{\varsigma^{-1}(i)}$. Replacing in the last equality the index i with $\sigma(i)$, we get $q_{\sigma(i)}=q'_{\varsigma^{-1}(\sigma(i))}=q'_{\sigma\varsigma^{-1}(i)}$. This means that the configuration of q'_i belongs to the $\sigma\varsigma^{-1}$-sector and we have

$$\Psi(q_{\varsigma(1)},\dots q_{\varsigma(N)})=\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma\varsigma^{-1}|\tau)e^{iq'_{\sigma\varsigma^{-1}(i)}p_{\tau(i)}}=\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma\varsigma^{-1}|\tau)e^{iq_{\sigma(i)}p_{\tau(i)}} \tag{3.59}$$

Comparison of (3.58) with (3.59) shows that the action of the symmetric group on the wave function by permuting its arguments induces the following action on the amplitudes

$$\mathcal{A}(\sigma|\tau)\to\mathcal{A}(\sigma\varsigma^{-1}|\tau)\,, \tag{3.60}$$

or for the vector $\Phi(\tau)$

$$\Phi(\tau)\to\pi'(\varsigma^{-1})\Phi(\tau)\,, \tag{3.61}$$

where this time π' is the *right* regular representation of $\mathfrak{S}_N$. It is now clear that if we want the Bethe wave function to be of the symmetry type λ, the vector $\Phi(\tau)$ for any τ must obey the following two conditions:

1) The anti-symmetry requirement

$$\pi'(\alpha_i)\Phi(\tau) = -\Phi(\tau)\,, \tag{3.62}$$

where $i \in \{1, \dots \lambda_1 - 1\} \cup \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2 - 1\} \cup \dots \cup \{N - \lambda_l + 1, \dots, N - 1\}$.

2) The Fock condition

$$\left[\mathbb{1} - \sum_{k \in \lambda_i} \pi'(\alpha_{km})\right]\Phi(\tau) = 0\,, \tag{3.63}$$

where m is in λ_j for all choices of λ_i and λ_j with $\lambda_i \geqslant \lambda_j$.

It is now time to recall that Φ transforms linearly under the left regular representation π of $\mathfrak{S}_N$. Because π' and π commute, if Φ satisfies the constraints (3.62) and (5.106), the vector $\pi(\sigma)\Phi$ will also satisfy them for any σ. In fact, these constraints project out in decomposition (5.77) all components but one that coincides with λ. Thus, we arrive at the important conclusion that the requirement for the Bethe wave function to have the symmetry type λ is equivalent for the amplitude vector Φ to transform in the irreducible representation λ of the symmetric group [**?**].

Example. To illustrate how the constraints (3.62) and (5.106) single out an irreducible component of π, we look at a simple example of $N = 3$ particles. The group $\mathfrak{S}_3$ and its representations π and π' were discussed in 5.5. Consider the diagram $\lambda = [2, 1]$ and fill it as

1	3
2	

.

This gives one of two possible standard tableaux. There is one anti-symmetry condition on the six-dimensional vector Φ, namely,

$$\pi'(\sigma_2)\Phi = -\Phi\,, \tag{3.64}$$

where $\sigma_2 = \alpha_{12}$ is the second element from the list (5.79). The Fock condition is[3]

$$\Big[\mathbb{1} - \pi'(\sigma_4) - \pi'(\sigma_6)\Big]\Phi = 0\,, \tag{3.65}$$

where σ_4 and σ_6 are the corresponding permutations from the same list. A solution of the first equation leaves 3 parameters undetermined, while a subsequent imposition of the second equation leaves a two-dimensional vector space

$$\Phi^t = (u_1 - u_2, -u_1 + u_2, -u_1, -u_2, u_2, u_1)\,, \qquad u_1, u_2 \in \mathbb{C}\,.$$

It is not hard to see that this is a two-dimensional invariant subspace of the representation π. On this subspace π acts irreducibly and coincides, in fact, with one of the irreducible components $\pi_{[2,1]}$ in the decomposition (5.81).

Analogously, we can consider the second standard tableau

1	2
3	

.

[3] One cannot anisymmetrise the 3d particle with the first and the second.

The conditions on Φ are

$$\pi'(\sigma_4)\Phi = -\Phi\,,$$
$$\Big[\mathbb{1} - \pi'(\sigma_2) - \pi'(\sigma_6)\Big]\Phi = 0\,.$$

Together they single out another two-dimensional invariant subspace

$$\Phi^t = (-v_2, -v_1 - v_2, v_1 + v_2, v_2, -v_1, v_1)\,, \quad v_1, v_2 \in \mathbb{C}\,,$$

on which π acts irreducibly and coincides with another component $\pi_{[2,1]}$ in (5.81).

Further, there is an anti-symmetric representation

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\quad .$$

It is singled out by the conditions

$$\pi'(\sigma_2)\Phi = -\Phi\,, \quad \pi'(\sigma_6)\Phi = -\Phi\,,$$

that have the solution $\Phi^t = w(1, -1, 1, -1, 1, -1)$. Finally, the trivial representation [3] completes the list of irreducible components appearing in (5.81) for the regular representation of $\mathfrak{S}_3$.

Compatibility of scattering with statistics. Recall that the action of scattering operators on amplitudes is realised through the *left* regular representation π. Taking into account that π and π' commute, we conclude that imposition of symmetry conditions on the wave function should also be compatible with scattering. More precisely, by construction the S-matrix is an element of the group algebra of $\mathfrak{S}_N$ evaluated in the left regular representation π. This representation is reducible and its decomposition into a sum of irreducibles is given by (5.77). Projecting the Bethe wave function on an irreducible component λ, we obtain the wave function with a type of symmetry described by the Young diagram λ. The corresponding S-matrix is still given by (3.49), where the two-body S-matrices are substituted with

$$S_{ij}(p_1, p_2) = B_\lambda(p_1, p_2)\mathbb{1} + A_\lambda(p_1, p_2)\pi_\lambda(\alpha_{ij})\,,$$

where the subscript λ of A and B is used to emphasise that these scattering coefficients can be, in fact, different for different representations. If $\lambda = [1^N]$ is the anti-symmetric representation, then $\pi_\lambda(\alpha_{ij}) = -1$ and $S = B - A$. If $\lambda = [N]$ is the symmetric representation, then $\pi_\lambda(\alpha_{ij}) = 1$ and $S = A + B$. In both cases the S-matrices are scalar.

Example. Consider the delta-interaction model (1.37) for the case of fermions. The delta-function potential produce no effect and the two-body S-matrix is trivial, $S = B - A = 1$. Since we deal with fermions, the Bethe wave function must transform in the anti-symmetric representation for which $\pi_\lambda(\alpha_{ij}) = -1$. Therefore, from (3.37) we obtain the following equation

$$\Phi(\alpha_j\tau) = (A - B)\Phi(\tau) = -(B - A)\Phi(\tau) = -\Phi(\tau).$$

It follows from here that for an arbitrary ς the vector Φ satisfies $\Phi(\varsigma\tau) = \mathrm{sign}(\varsigma)\Phi(\tau)$, the latter equation is obviously solved as $\Phi(\tau) = \mathrm{sign}(\tau)\Phi(e)$. Further, the requirement (3.62) of anti-symmetry of the wave function allows one to completely determine the amplitudes $\mathcal{A}(\sigma|\tau) = \mathrm{sign}(\sigma^{-1}\tau)\mathcal{A}(e|e)$. Up to an overall normalisation factor $\mathcal{A}(e|e)$, the Bethe wave function (3.10) is then given by the Slater determinant

$$\begin{aligned}\Psi(q_1, \ldots, q_N) &= \sum_{\tau\in\mathfrak{S}_N} \mathrm{sign}(\sigma^{-1}\tau)e^{iq_{\sigma(i)}p_{\tau(i)}} \\ &= \sum_{\tau\in\mathfrak{S}_N} \mathrm{sign}(\tau)e^{iq_ip_{\tau(i)}} = \det\nolimits_{N\times N}\big(e^{iq_ip_j}\big)\,. \end{aligned} \tag{3.66}$$

This completes our discussion of the fermionic delta-interaction model.

Lecture 4

Coordinate Bethe Ansatz

4.1 Periodicity condition for the Bethe wave function

Consider a system of N interacting particles confined in a one-dimensional box. The full description of this system requires an imposition of certain boundary conditions. In the following we choose *periodic boundary conditions*. These conditions relate the quantum-mechanical amplitude for finding a particle with given colour and momentum at one end of the box to the amplitude for finding a particle with the same colour and momentum at the other end. To be able to use the Bethe function formalism developed in the previous section, we assume that the size L of the box is large enough to fit in a kinematic configuration corresponding to asymptotically free particles.

Under these assumptions, let us recall the Bethe wave function Ψ in the transmission representation (3.11)

$$\Psi(q_1,\dots,q_N)=\sum_{\sigma\in\mathfrak{S}_N}\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma|\tau)e^{i\sum\limits_{i=1}^{N}q_{\sigma(i)}p_{\tau(i)}}\Theta\big(q_{\sigma(1)}<\ldots<q_{\sigma(N)}\big)\,. \tag{4.1}$$

Coordinates q_i can take any values but now within a segment of length L, that is $0\leqslant q_i\leqslant L$ for $\forall i$. The periodic boundary conditions for Ψ mean that

$$\Psi(q_1,\dots,q_j=0,\dots,q_N)=\Psi(q_1,\dots,q_j=L,\dots,q_N)\,,\quad\forall j\,. \tag{4.2}$$

For the left hand side of this equation, denoted by LHS, we get

$$\mathrm{LHS}=\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \sigma(1)=j}}\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma|\tau)e^{i\sum\limits_{k=2}^{N}q_{\sigma(k)}p_{\tau(k)}}\Theta\big(q_{\sigma(2)}<\ldots<q_{\sigma(N)}\big)\,, \tag{4.3}$$

while for the right hand side called RHS,

$$\mathrm{RHS}=\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \sigma(N)=j}}\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\sigma|\tau)e^{iLp_{\tau(N)}}e^{i\sum\limits_{k=1}^{N}q_{\sigma(k)}p_{\tau(k)}}\Theta\big(q_{\sigma(1)}<\ldots<q_{\sigma(N-1)}\big)\,, \tag{4.4}$$

where in the exponent we have now $q_{\sigma(N)}=q_j=0$.

To compare (4.4) with (4.3), we introduce the following element $\xi\in\mathfrak{S}_N$ (cyclic permutation)

$$\xi=\alpha_{N-1}\ldots\alpha_1=\begin{pmatrix}1&2&\cdots&N-1&N\\2&3&&N&1\end{pmatrix} \tag{4.5}$$

and make a replacement of the summation variable σ as $\sigma \to \xi\sigma$. Since σ obeys $\sigma(1) = j$, then $\xi\sigma(N) = \sigma(\xi(N)) = \sigma(1) = j$, and we get

$$\begin{aligned}\text{RHS} &= \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \xi\sigma(N)=j}} \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\xi\sigma|\tau) e^{iLp_{\tau(N)}} e^{i\sum\limits_{k=1}^{N} q_{\xi\sigma(k)}p_{\tau(k)}} \Theta\big(q_{\xi\sigma(1)} < \ldots < q_{\xi\sigma(N-1)}\big) \\ &= \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \sigma(1)=j}} \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\xi\sigma|\tau) e^{iLp_{\tau(N)}} e^{i\sum\limits_{k=1}^{N} q_{\sigma(k)}p_{\xi^{-1}\tau(k)}} \Theta\big(q_{\sigma(2)} < \ldots < q_{\sigma(N)}\big)\,.\end{aligned}$$

Here the argument of the theta-function and the condition $\sigma(1) = j$ are precisely the same as in the expression LHS. Further, making a change of variable $\tau \to \xi\tau$, we arrive at

$$\text{RHS} = \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \sigma(1)=j}} \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\xi\sigma|\xi\tau) e^{iLp_{\xi\tau(N)}} e^{i\sum\limits_{k=2}^{N} q_{\sigma(k)}p_{\tau(k)}} \Theta\big(q_{\sigma(2)} < \ldots < q_{\sigma(N)}\big)\,, \tag{4.6}$$

where we have taken into account that $q_{\sigma(1)} = q_j = 0$. Note also that $p_{\xi\tau(N)} = p_{\tau(\xi(N))} = p_{\tau(1)}$. Finally, comparison of (4.6) with LHS yields the following equations

$$\mathcal{A}(\sigma|\tau) = \mathcal{A}(\xi\sigma|\xi\tau) e^{iLp_{\tau(1)}}\,. \tag{4.7}$$

This is a requirement on the coefficients of the asymptotic wave function in order for the latter to satisfy periodic boundary conditions. Using the left regular representation π of $\mathfrak{S}_N$, equations (4.7) can be written as conditions for the vector $\Phi(\tau)$ defined in (3.36):

$$\pi(\xi)\Phi(\tau) = e^{iLp_{\tau(1)}}\Phi(\xi\tau)\,. \tag{4.8}$$

To make further progress, we note that (4.8) must be satisfied for any momentum ordering τ, which means that, in order to proceed, we can make for τ a convenient choice. We take

$$\tau = \alpha_1 \ldots \alpha_{j-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & j & j+1 & \cdots & N \\ j & 1 & 2 & \cdots & j-1 & j+1 & \cdots & N \end{pmatrix}\,,$$

so that $\xi\tau = \alpha_{N-1} \ldots \alpha_j$. With this choice $\tau(1) = j$ and (4.8) boils down to

$$\pi(\alpha_{N-1} \ldots \alpha_1)\Phi(\alpha_1 \ldots \alpha_{j-1}) = e^{iLp_j}\,\Phi(\alpha_{N-1} \ldots \alpha_j)\,. \tag{4.9}$$

Next, we evaluate both sides of the last expression with the help of connection formula (3.37) and get

$$\begin{aligned}&\pi(\alpha_{j-1}) \ldots \pi(\alpha_1)\, Y_1(p_1, p_j) \ldots Y_{j-1}(p_{j-1}, p_j)\Phi(e) \\ &\qquad = e^{iLp_j}\,\pi(\alpha_j) \ldots \pi(\alpha_{N-1}) Y_{N-1}(p_j, p_N) \ldots Y_j(p_j, p_{j+1})\Phi(e)\,.\end{aligned}$$

Using the definition (3.46) of the two-body S-matrix, the last expression can be rewritten in the following elegant form

$$S_{j+1\,j} S_{j+2\,j} \ldots S_{Nj} \cdot S_{1j} \ldots S_{j-1\,j}\,\Phi(e) = e^{iLp_j}\,\Phi(e)\,. \tag{4.10}$$

If we introduce the following matrix operators

$$T_j = S_{j+1\,j} S_{j+2\,j} \ldots S_{Nj} \cdot S_{1j} \ldots S_{j-1\,j}\,, \tag{4.11}$$

then (4.10) tells that $|\Phi\rangle \equiv \Phi(e)$ is a common eigenvector for N matrix operators T_j

$$T_j\,|\Phi\rangle = \Lambda_j\,|\Phi\rangle\,, \tag{4.12}$$

where $j = 1, \ldots, N$. In the following we refer to (4.12) as the *(matrix) Bethe-Yang equations*. Once a common eigenvalue, which is a function of momenta, is found, one is left to solve a system of (scalar) Bethe equations

$$\Lambda_j = e^{iLp_j}\,, \tag{4.13}$$

to determine the momenta p_j.

Compatibility of the system (4.12) requires that matrices T_j for various j pair-wise commute. The important fact that they do so is a consequence of the condition (3.51) and the Yang-Baxter equation (3.52). In the case of scalar S-matrices, where Φ is one-dimensional, the diagonalisation problem of T_j does not arise.

As was argued in subsection 3.3, demanding the Bethe wave function to be of the symmetry type λ implies that the vector $|\Phi\rangle$ transforms in the same representation. Correspondingly, the scattering operators and the operators T_j are also restricted to λ. The problem now is to solve the system (4.12) for a given irreducible representation of $\mathfrak{S}_N$ and, subsequently, use this solution to reconstruct the corresponding Bethe wave function.

Lieb-Liniger model for two particles. The matrix Bethe-Yang equations are

$$\begin{aligned} S_{21}\Phi(e) &= e^{iLp_1}\Phi(e)\,,\\ S_{12}\Phi(e) &= e^{iLp_2}\Phi(e)\,, \end{aligned} \tag{4.14}$$

where

$$S_{12} \;=\; \frac{p_1-p_2}{p_1-p_2+i\kappa}\,\mathbb{1} - \frac{i\kappa}{p_1-p_2+i\kappa}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\,, \tag{4.15}$$

It is trivial to diagonalise S_{12} by hand. This S-matrix has two different eigenvalues

$$\lambda_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = 1\,, \qquad \lambda_{\square\square} = \frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa}\,, \tag{4.16}$$

corresponding to the anti-symmetric representation, where particles are speenless fermions, and the symmetric one, where particles are indistinguishable bosons. Correspondingly, we have for $\Phi(e)$ the following expressions

$$\Phi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e) = \begin{pmatrix} 1\\ -1\end{pmatrix}\,, \qquad \Phi_{\square\square}(e) = \begin{pmatrix} 1\\ 1\end{pmatrix}\,, \tag{4.17}$$

where we recall that

$$\Phi(e) = \begin{pmatrix} \mathcal{A}(12|12)\\ \mathcal{A}(21|12)\end{pmatrix}\,. \tag{4.18}$$

Starting from three particles, in addition to the anti-symmetric and symmetric representations

there appears also a hook corresponding to spin-$\frac{1}{2}$ fermions

4.2 Incarnations of the Lieb-Liniger model

To shed more light on the Bethe-Yang equations (4.12), we consider a few concrete examples. The first example is provided by the delta-interaction model for three cases different by the nature of interacting particles.

1) Interacting particles with internal degrees of freedom, the representation λ for the wave function remains unspecified.[1] Using the expression (3.31) for the reflection and transmission coefficients, we can write down for the Yang operator and the two-body S-matrix the following expressions

$$Y_j(p_1,p_2) = -\frac{i\kappa}{p_1-p_2+i\kappa}\,\mathbb{1} + \frac{p_1-p_2}{p_1-p_2+i\kappa}\pi_\lambda(\alpha_j)\,, \tag{4.19}$$

$$S_{ij}(p_i,p_j) = \frac{p_i-p_j}{p_i-p_j+i\kappa}\,\mathbb{1} - \frac{i\kappa}{p_i-p_j+i\kappa}\pi_\lambda(\alpha_{ij})\,, \tag{4.20}$$

where $\pi_\lambda(\alpha_{ij})$ is the transposition α_{ij} evaluated in the representation λ. In this case T_j are matrices, and the non-trivial problem of their diagonalisation will be discussed later.

2) The case of spinless fermions corresponds to picking up the anti-symmetric representation, $\lambda = [1^N]$, so that the wave function is anti-symmetric. Since for this case for any α_{ij} we have $\pi_\lambda(\alpha_{ij}) = -1$, formula (4.19) yields $S = 1$. As a result, equations (4.12) become the familiar quantisation condition $e^{ip_jL} = 1$ for momenta of free fermions put on a circle of length L.

To find the wave function, we notice that $Y_j = -1$ for this case. Therefore, from (3.37) we obtain the following equation

$$\Phi(\alpha_j\tau) = Y_j\Phi(\tau) = -\Phi(\tau).$$

It follows from here that for an arbitrary ς the vector Φ satisfies $\Phi(\varsigma\tau) = \mathrm{sign}(\varsigma)\Phi(\tau)$, the latter equation is obviously solved as $\Phi(\tau) = \mathrm{sign}(\tau)\Phi(e)$. Since $\Phi(e)$ belongs to the anti-symmetric irrep $\lambda = [1^N]$, we have $\Phi(e) = \mathcal{A}(\sigma|e) = \mathrm{sign}(\sigma^{-1})\mathcal{A}(e|e)$. Up to an overall normalisation factor $\mathcal{A}(e|e)$, the Bethe wave function (3.10) is then given by the Slater determinant

$$\begin{aligned}\Psi_{\{p_j\}}(q_1,\dots,q_N) &= \sum_{\tau\in\mathfrak{S}_N}\mathrm{sign}(\sigma^{-1}\tau)e^{iq_{\sigma(i)}p_{\tau(i)}} \\ &= \sum_{\tau\in\mathfrak{S}_N}\mathrm{sign}(\tau)e^{iq_ip_{\tau(i)}} = \det\bigl(e^{iq_ip_j}\bigr)\,.\end{aligned} \tag{4.21}$$

3) The Lieb-Liniger model describes Bose gas with repulsive delta-function interaction (1.37). The corresponding wave function transforms in the symmetric representation, $\lambda = [N]$. The two-body S-matrix is scalar and reads as

$$S(p_1,p_2) = \frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa}\,, \tag{4.22}$$

where $\kappa > 0$ is the coupling constant. Equations (4.12) reduce to

$$e^{ip_jL} = \prod_{\substack{k=1\\k\neq j}}^{N} S_{kj}(p_k,p_j) = \prod_{\substack{k=1\\k\neq j}}^{N}\frac{p_j-p_k+i\kappa}{p_j-p_k-i\kappa} = -\prod_{k=1}^{N}\frac{p_j-p_k+i\kappa}{p_j-p_k-i\kappa}\,. \tag{4.23}$$

[1] Later, considering the case of spin-1/2 fermions, we specify λ as the permutation module $M^{[N-M,M]}$.

This set of N equations determine the allowed values of the particle momenta in this model.The vector Φ in (4.8) is one-dimensional, *i.e.* the amplitude $\mathcal{A}(\sigma|\tau)$ does not depend on σ. From (3.37) we then have

$$\mathcal{A}(\alpha_j\tau) = \frac{p_{\tau(j)} - p_{\tau(j+1)} - i\kappa}{p_{\tau(j)} - p_{\tau(j+1)} + i\kappa}\mathcal{A}(\tau)\,.$$

The last equation has a unique, up to an overall normalisation, solution

$$\mathcal{A}(\tau) = a\prod_{i<j}\frac{p_{\tau(i)} - p_{\tau(j)} + i\kappa}{p_{\tau(i)} - p_{\tau(j)}}\,, \tag{4.24}$$

where a is τ-independent constant that might be a function of p. To preclude singularities at coincident momenta, we choose it as

$$a = \prod_{i<j}(p_i - p_j)\,.$$

Then due to the identity[2]

$$\prod_{i<j}\big(p_{\tau(i)} - p_{\tau(j)})\big) = \operatorname{sign}\tau\prod_{i<j}(p_i - p_j)\,, \tag{4.25}$$

we will have

$$\mathcal{A}(\tau) = \operatorname{sign}\tau\prod_{i<j}\big(p_{\tau(i)} - p_{\tau(j)} + i\kappa\big)\,.$$

Substituting this expression into the Bethe wave function (3.11), we find

$$\begin{aligned}\Psi(q_1,\dots,q_N) &= \sum_{\sigma\in\mathfrak{S}_N}\sum_{\tau\in\mathfrak{S}_N} e^{\,i\sum\limits_{k=1}^{N} q_{\sigma(k)}p_{\tau(k)}} \\ &\times \operatorname{sign}\tau\prod_{i<j}\big(p_{\tau(i)} - p_{\tau(j)} + i\kappa\big)\prod_{i=1}^{N-1}\Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big)\,.\end{aligned}$$

Using the invariance of the scalar product and making the shift $\tau \to \sigma\tau$, we write

$$\begin{aligned}\Psi(q_1,\dots,q_N) &= \sum_{\tau\in\mathfrak{S}_N} e^{\,i\sum\limits_{k=1}^{N} q_k p_{\tau(k)}}\operatorname{sign}\tau \\ &\times \sum_{\sigma\in\mathfrak{S}_N}\operatorname{sign}\sigma\prod_{i<j}\big(p_{\tau(\sigma(i))} - p_{\tau(\sigma(j))} + i\kappa\big)\prod_{i=1}^{N-1}\Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big)\,.\end{aligned}$$

The sum over σ can be taken explicitly with the following result[3]

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_N}\operatorname{sign}\sigma\prod_{i<j}\big(p_{\tau(\sigma(i))} - p_{\tau(\sigma(j))} + i\kappa\big)\prod_{i=1}^{N-1}\Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big) \\ = \prod_{i<j}\big(p_{\tau(i)} - p_{\tau(j)} - i\kappa\,\epsilon(q_i - q_j)\big)\,.\end{aligned} \tag{4.26}$$

[2]We recall that $\prod_{i<j}^{N}(x_i - x_j) = \det x_i^{N-j}$ is the Vandermond determinant.
[3]See appendix 5.8 for the sketch of the proof.

Finally, using again (4.25), we find for the wave function the following expression

$$\Psi_{\{p_j\}}(q_1,\dots,q_N)=\prod_{i<j}^{N}(p_i-p_j)\sum_{\tau\in\mathfrak{S}_N}e^{\,i\sum\limits_{k=1}^{N}q_kp_{\tau(k)}}\prod_{i<j}^{N}\left[1-\frac{i\kappa\,\epsilon(q_i-q_j)}{p_{\tau(i)}-p_{\tau(j)}}\right].$$

The wave function is parametrised by a set of N momenta $\{p_j\}$ which was reflected in its notation above. The function is symmetric under permutations of coordinates and anti-symmetric under permutations of momenta, so that it vanishes if any two momenta coincide. The coordinate symmetry of the wave function is compatible with the boson statistics. In the momentum space the particles behave rather like fermions: each value of momentum can be occupied by at most one particle. To emphasise this behaviour, in the expression for Ψ we singled out the overall anti-symmetric factor $\prod_{i<j}(p_j-p_i)$, without which the wave function would be symmetric under permutations of momenta. Dropping this factor is not allowed, however, because this would lead to an ill-defined wave function that would not be bounded on the whole $\mathbb{R}^N$ when two momenta coincide.

4.3 Low-dimensional eigenvalues

In this section we would like to describe one approach to solve the eigenvalue problem

$$T_j\,|\Phi\rangle=\Lambda_j\,|\Phi\rangle\,, \tag{4.27}$$

for the commuting operators

$$T_j=S_{j+1\,j}S_{j+2\,j}\dots S_{Nj}S_{1j}\dots S_{j-1\,j} \tag{4.28}$$

acting in a given irreducible representation of the symmetric group $\mathfrak{S}_N$. This approach, known as the *Nested Bethe Ansatz*, is an extension of the original treatment of Yang of the Lieb-Liniger model for spin-$\frac{1}{2}$ electrons and it is based on the *generalised Bethe hypothesis*. It is therefore natural to first demonstrate how the latter model was solved by Yang.

Due to the simple relationship between representations λ and $\bar\lambda$, for the S-matrix in the representation λ we can write

$$S_{ij}=\frac{(p_i-p_j)\mathbb{1}-i\kappa\,\pi_\lambda(\alpha_{ij})}{p_i-p_j+i\kappa}=\frac{(p_i-p_j)\mathbb{1}+i\kappa\,\pi_{\bar\lambda}(\alpha_{ij})}{p_i-p_j+i\kappa}\,. \tag{4.29}$$

and think about the S-matrix as acting in the representation $\bar\lambda$. Recall that for spin-$\frac{1}{2}$ electrons $\lambda=[2^M,1^{N-2M}]$ and, therefore, the original eigenvalue problem (4.27) with $|\Phi\rangle$ transforming in this irrep is fully equivalent to the one where the operators (4.28) act in the conjugate irrep $\bar\lambda=[N-M,M]$ and $|\Phi\rangle$ belongs to the corresponding representation space. Recall that $\bar\lambda=[N-M,M]$ is the representation in which the spin wave function of electrons transforms. Thus, we want to find the eigenbasis for the commuting operators (4.28) acting in the space of $[N-M,M]$.

The direct evaluation of eigenvalues of commuting operators (4.28) becomes straightforward if the representation for the module S^λ for $\lambda=[N-M,M]$ is explicitly known. The experience with representations of $\mathfrak{S}_N$ shows, however, that representation matrices for an irreducible Specht module are rather intricate as typically they contain $+1$ and -1 as their matrix elements, the position of those depend on a basis chosen. Moreover, when the dimension of an irrep S^λ becomes sufficiently high the straightforward diagonalisation of T_j becomes technically problematic. On the other hand, permutation modules have rather simple representation matrices having only $+1$ as their matrix elements, the latter are distributed in a controllable way. In addition, any permutation module M^λ contains the Specht module, $S^\lambda\subset M^\lambda$, as a sub-representation with multiplicity one. Thus, instead

of diagoinalising T_j on $S^{[N-M,M]}$, one can diagonalise them on $M^{[N-M,M]}$ and then single out those eigenvalues which correspond to the sub-representation $S^{[N-M,M]}$. Before we present the general Yang's construction, we worked out the eigenvalues of T_j on various modules we studied above. This gives us further intuition of how these eigenvalues present themselves.

In the following we will often use the more concise notation

$$S_{ij} = a_{ij} + b_{ij}\,\pi_{ij}\,, \tag{4.30}$$

where

$$a_{ij} = \frac{p_i - p_j}{p_i - p_j + i\kappa}\,, \qquad b_{ij} = \frac{i\kappa}{p_i - p_j + i\kappa}\,, \tag{4.31}$$

and depending on the context transposition π_{ij} is realised as an operator acting either in the representation space $M^{[N-M,M]}$ or in $S^{[N-M,M]}$. Note the relation $a_{ij} + b_{ij} = 1$ which we always use in the evaluation of T_j.

It is enough to look at eigenvalues of T_1 and compute them numerically by picking up a random value of momenta and κ. Throughout our analysis we use the following values

$$p_1 = \frac{1}{5}\,,\quad p_2 = \frac{11}{13},\quad p_3 = \frac{17}{37}\,,\quad p_4 = \frac{19}{81}\,,\quad p_5 = \frac{81}{377}\,,\quad \kappa = \frac{1}{15}\,. \tag{4.32}$$

The group $\mathfrak{S}_3$. We have 3 commuting operators

$$\begin{aligned} T_1 &= S_{21}S_{31} \\ T_2 &= S_{32}S_{12} \\ T_3 &= S_{13}S_{23} \end{aligned}$$

In the following it is enough to look at eigenvalues of one matrix, for instance, T_1. Specifying the relevant representation and computing the eigenvalues μ_1 of T_1 we find

Permutation module $\begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$:

$$\mu_1 = \Big\{\underline{0.8403 - 0.5421i},\, \underline{0.9908 - 0.1353i},\, 1\Big\}$$

Specht module $\begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$:

$$\mu_1 = \Big\{0.8403 - 0.5421i,\, 0.9908 - 0.1353i\Big\}$$

In comparison to the Specht module, the permutation module contains one more eigenvalue equal to 1 which corresponds to the trivial sub-representation.

The group $\mathfrak{S}_4$. We have 4 commuting operators

$$\begin{aligned} T_1 &= S_{21}S_{31}S_{41} \\ T_2 &= S_{32}S_{42}S_{12} \\ T_3 &= S_{43}S_{13}S_{23} \\ T_4 &= S_{14}S_{24}S_{34} \end{aligned}$$

Permutation module $\begin{array}{|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$:

$$\mu_1 = \Big\{\underline{-0.7245 - 0.6892i},\, \underline{0.9270 - 0.3751i},\, \underline{0.9918 - 0.1275i}, 1\Big\}$$

Specht module $\begin{array}{|c|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}\\\cline{1-1}\end{array}$:

$$\mu_1 = \Big\{ -0.7245 - 0.6892i,\, 0.9270 - 0.3751i,\, 0.9918 - 0.1275i \Big\}$$

Once again, the permutation module contains one more eigenvalue equal to 1 which corresponds to the trivial sub-representation.

Permutation module $\begin{array}{|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\end{array}$:

$$\begin{aligned}\mu_1 \;\; = \;\; \Big\{ & \underline{\underline{-0.7245 - 0.6892i}},\; \underline{-0.7167 - 0.6974i}\,, \underline{0.86660 - 0.5001i},\\ & \underline{\underline{0.9270 - 0.3751i}},\; \underline{\underline{0.9918 - 0.1275i}}\,, 1 \Big\}\end{aligned}$$

Specht module $\begin{array}{|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\end{array}$:

$$\mu_1 = \{ -0.7167 - 0.6974i,\, 0.8660 - 0.5001i \}$$

In addition to two eigenvalues of the Specht module (uderlined with one line), the permutation module gives rise to three eigenvalues which coincide with those in the Specht module $[3,1]$. In addition, there is again the eigenvalue 1 that corresponds to the trivial representation. This result is in accord with the decomposition of the permutation module into the following sum of Specht modules

$$M^{[2,2]} = [4] + [3,1] + [2,2]\,.$$

The group $\mathfrak{S}_5$. We have 5 commuting operators

$$\begin{aligned} T_1 &= S_{21} S_{31} S_{41} S_{51} \\ T_2 &= S_{32} S_{42} S_{52} S_{12} \\ T_3 &= S_{43} S_{53} S_{13} S_{23} \\ T_4 &= S_{45} S_{14} S_{24} S_{34} \\ T_5 &= S_{15} S_{25} S_{35} S_{45} \end{aligned}$$

Permutation module $\begin{array}{|c|c|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}\\\cline{1-1}\end{array}$:

$$\mu_1 = \Big\{ \underline{-0.8899 + 0.4561i},\; \underline{0.0498 - 0.9988i},\; \underline{0.9436 - 0.3312i},\; \underline{0.9924 - 0.1227i},\; 1 \Big\}$$

Specht module $\begin{array}{|c|c|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}\\\cline{1-1}\end{array}$:

$$\mu_1 = \Big\{ -0.8899 + 0.4561i,\; 0.0498 - 0.9988i,\; 0.9436 - 0.3312i,\; 0.9924 - 0.1227i \Big\}$$

Once again, the permutation module contains one more eigenvalue equal to 1 which corresponds to the trivial sub-representation.

Permutation module $\begin{array}{|c|c|c|}\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\hline\multicolumn{1}{|c|}{}&\multicolumn{1}{|c|}{}\\\cline{1-2}\end{array}$:

$$\begin{aligned}\mu_1 \;\; = \;\; \Big\{ & \underline{\underline{-0.8899 + 0.4561i}},\; \underline{-0.8845 + 0.4666i},\; \underline{-0.8580 + 0.5136i},\; \underline{-0.2534 - 0.9674i},\end{aligned}$$

$$\underline{-0.0248-0.9997i},\ \underline{\underline{0.0498-0.9988i}},\ \underline{0.8949-0.4463i},$$
$$\underline{\underline{0.9436-0.3312i}},\ \underline{\underline{0.9924-0.1227i}},\ 1\Big\}$$

Specht module $\begin{array}{|c|c|c|}\hline \multicolumn{1}{|c|}{} & \multicolumn{1}{|c|}{} & \multicolumn{1}{|c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{|c|}{} \\ \cline{1-2}\end{array}$:

$$\mu_1 \quad = \quad \Big\{ -0.8845+0.4666i,\ -0.8580+5136i,\ -0.2534-0.9674i,$$
$$-0.0248-0.9997i,\ 0.8949-0.4463i\Big\}$$

The structure of the eigenvalues reflects the following decomposition of the permutation module over the Specht modules

$$M^{[3,2]} = [5]+[4,1]+[3,2]\,.$$

4.4 Spin chain representation of the permutation module

It is possible to realise permutation modules $M^{[N-M,M]}$ of $\mathfrak{S}_N$ in a more convenient and physically appealing way by embedding them in a larger vector space of dimension 2^N. This is a spin chain representation of the permutation modules which is constructed as follows.

Consider a discrete circle which is a collection of ordered points labelled by the index n with the identification $n \equiv n+N$ reflecting periodic boundary conditions. Here N is a positive integer which plays the role of the length (volume) of the space. The numbers $n = 1,\ldots,N$ form the fundamental domain. To each integer n along the chain we associate a two-dimensional vector space $V = \mathbb{C}^2$ with a basis

$$|\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}\,, \qquad |\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}\,. \tag{4.33}$$

As this notation suggests, the basis elements are identified as "spin up" and "spin down", see Fig. 4.1. The Hilbert space $\mathscr{H}$ of the spin chain is the tensor product of N copies of V, $\mathscr{H} = V^{\otimes N}$, and has dimension 2^N.

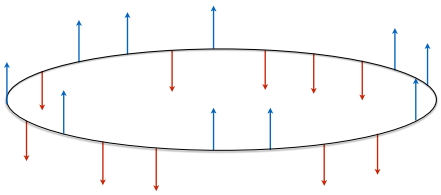


Figure 4.1: Spin chain. A state of the spin chain can be represented as $|\psi\rangle = |\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\uparrow\downarrow\rangle$.

The symmetric group $\mathfrak{S}_N$ acts in $\mathcal{H}$ in the following way. Each transposition α_{ij} when acting on an individual basis element permutes the spins standing in the i's and j's position. With the help of the standard 2×2 matrix unities, its action on spin chain states can be written as

$$P_{ij} = \sum_{a,b=1}^{2} \mathbb{1}\otimes\ldots\otimes \overset{i}{\underset{\downarrow}{E_{ab}}} \otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes \overset{j}{\underset{\downarrow}{E_{ba}}} \otimes\ldots\otimes\mathbb{1}\,, \tag{4.34}$$

where the matrices E_{ab} and E_{ba} occur in the tensor product at positions i and j, respectively. From transpositions the action is extended on arbitrary elements by using the group property. Obviously, the action of $\mathfrak{S}_N$ preserves a subspace of $\mathcal{H}$ with M down spins. Within each subspace with M down spins, the latter has dimension C_N^M, the action of $\mathfrak{S}_N$ coincides precisely with the one on the permutation module $M^{[N-M,M]}$. This can be seen from identifying the basis of the M-particle permutation module

$$|n_1,\ldots,n_M\rangle, \quad 1\leqslant n_1<n_2<\ldots<n_M\leqslant N\,, \tag{4.35}$$

with states of the spin chain with M down spins occurring in the positions $n_1, n_2, \ldots, n_M$ along the chain. Thus, with respect to the action of $\mathfrak{S}_N$ the Hilbert space of the spin chain decomposes as

$$\mathcal{H} = \bigoplus_{M=0}^{N} M^{[N-M,M]}$$

which is, of course, compatible with counting dimensions $2^N = \sum_{M=0}^{N} C_N^M$. The wave function (4.38) is then naturally interpreted as an element of $\mathcal{H}$ lying in the subspace spanned by states with M down spins.

4.5 Generalised Bethe hypothesis

We consider the eigenvalue problem

$$T_j|\Phi\rangle = \Lambda_j|\Phi\rangle\,, \tag{4.36}$$

for the commuting operators

$$T_j = S_{j+1\,j}S_{j+2\,j}\ldots S_{Nj}S_{1j}\ldots S_{j-2\,j}S_{j-1\,j}\,, \tag{4.37}$$

where $j = 1, \ldots, N$. The matrix $S_{ij} = a_{ij}\mathbb{1} + b_{ij}P_{ij}$.

The generalised Bethe hypothesis gives an ansatz for a vector $|\Phi\rangle \in M^{[N-M,M]}$ which diagonalises the commuting operators T_j on the sub-representation corresponding to the Specht module $S^{[N-M,M]}$. The hypothesis also provides an expression for the corresponding eigenvalues Λ_j. Namely, the following expression for $|\Phi\rangle$ is postulated

$$|\Phi\rangle = \sum_{1\leqslant n_1<n_2<\ldots<n_M\leqslant N} c_{n_1\ldots n_M}|n_1,\ldots,n_M\rangle\,, \tag{4.38}$$

where the coefficients are

$$c_{n_1\ldots n_M} = \sum_{\tau\in\mathfrak{S}_M} \mathcal{A}(\tau)F(v_{\tau(1)},n_1)\cdots F(v_{\tau(M)},n_M)\,. \tag{4.39}$$

Here n_i with $i = 1, \ldots, M$ are distinct integers $1 \leqslant n_i \leqslant N$ and $|n_1, \ldots, n_M\rangle$ are C_N^M states (4.35) which form a basis of the permutation module $M^{[N-M,M]}$. In the spin chain picture N is identified with the number of spin chain sites and M with the number of down spins. To solve the eigenvalue problem for T_j, the function $F(v,n)$ must be chosen as

$$F(v,n) = \frac{i\kappa}{p_n - v - \frac{i\kappa}{2}}\prod_{j=1}^{n-1}\frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}}\,, \tag{4.40}$$

while a set of unequal numbers $v_1, \ldots, v_M$ must solve the following set of *Bethe-Yang equations*

$$\prod_{j=1}^{N}\frac{p_j - v_k + \frac{i\kappa}{2}}{p_j - v_k - \frac{i\kappa}{2}} = \prod_{l\neq k}^{M}\frac{v_k - v_l - i\kappa}{v_k - v_l + i\kappa}\,. \tag{4.41}$$

The eigenvalues of T_j on the irreducible Specht module $[N-M, M]$ are then given by

$$\Lambda_j = \prod_{k=1}^{M}\frac{p_j - v_k + \frac{i\kappa}{2}}{p_j - v_k - \frac{i\kappa}{2}}\,. \tag{4.42}$$

Finally, the coefficients $\mathcal{A}(\tau)$ are given by (up an overall normalisation)

$$\mathcal{A}(\tau) = \prod_{1\leqslant k<l\leqslant M} \frac{v_{\tau(k)} - v_{\tau(l)} - i\kappa}{v_{\tau(k)} - v_{\tau(l)}}\,. \tag{4.43}$$

Formula (4.38) is similar to the one which was suggested by Bethe for describing the wave function of the Heisenberg spin chain, the role of plane waves[4]

$$e^{ipn} = \left(\frac{v - \frac{i}{2}\kappa}{v + \frac{i}{2}\kappa}\right)^n \tag{4.44}$$

in the original Bethe ansatz is now played by the factors $F(v,n)$. The formula (4.44) follows from $-(v-\frac{i\kappa}{2})F(v,n)$ in the homogeneous limit $p_j \to 0$. This is the reason why the ansatz (4.38) is referred to as the generalised Bethe hypothesis. It described the wave function of the inhomogeneous XXX model. In the following we will prove this hypothesis for $M = 1, 2$.

One-particle case. We start out consideration with the simplest case of one overturned spin

$$|\Phi\rangle = \sum_{n=1}^{N} c_n |n\rangle\,, \tag{4.45}$$

where $|n\rangle$ denotes a state of the spin chain with all spins up except one, which is the down spin at position n and c_n is the corresponding amplitude. Consider the operator $S_{j-1\,j}$. If $n \neq j$ and $n \neq j-1$, then this operator leaves $|n\rangle$ invariant, $S_{j-1\,j}|n\rangle = |n\rangle$, because $a_{j-1\,j} + b_{j-1\,j} = 1$. On the other hand,

$$\begin{aligned} &S_{j-1\,j}\Big[\,\ldots + c_{j-1}|j-1\rangle + c_j|j\rangle + \ldots\,\Big] \\ &\qquad = \Big[\,\ldots + (a_{j-1\,j}c_{j-1} + b_{j-1\,j}c_j)|j-1\rangle + (a_{j-1\,j}c_j + b_{j-1\,j}c_{j-1})|j\rangle + \ldots\,\Big]\,. \end{aligned}$$

Now we note that none of the remaining operators in T_j will act on the overturned spin at position $j-1$ and, therefore, the corresponding amplitude must obey

$$\Lambda_j c_{j-1} = a_{j-1\,j}c_{j-1} + b_{j-1\,j}c_j\,,$$

so that

$$\Lambda_j = a_{j-1\,j} + b_{j-1\,j}\frac{c_j}{c_{j-1}}\,.$$

Next, we denote $c_j^{(1)} = a_{j-1\,j}c_j + b_{j-1\,j}c_{j-1}$ and consider the action of $S_{j-2\,j}$ on the previous result

$$\begin{aligned} &S_{j-2\,j}\Big[\,\ldots + c_{j-2}|j-2\rangle + c_j^{(1)}|j\rangle + \ldots\,\Big] \\ &\qquad = \Big[\,\ldots + (a_{j-2\,j}c_{j-2} + b_{j-2\,j}c_j^{(1)})|j-2\rangle + (a_{j-2\,j}c_j^{(1)} + b_{j-2\,j}c_{j-2})|j\rangle + \ldots\,\Big]\,. \end{aligned}$$

The remaining operators in T_j will not act on $|j-2\rangle$ and, therefore, we should have

$$\Lambda_j = a_{j-2\,j} + b_{j-2\,j}\frac{c_j^{(1)}}{c_{j-2}}\,.$$

Analogously, we introduce

$$c_j^{(2)} = a_{j-2\,j}c_j^{(1)} + b_{j-2\,j}c_{j-2}\,.$$

[4]From the formula below the rapidity v is related to the particle momentum p as $v = -\frac{\kappa}{2}\cot\frac{p}{2}$.

Proceeding in the same manner, we formulate a recurrent scheme

$$\begin{aligned}\Lambda_j &= a_{j-k\,j} + b_{j-k\,j}\frac{c_j^{(k-1)}}{c_{j-k}}\,,\\ c_j^{(k)} &= a_{j-k\,j}c_j^{(k-1)} + b_{j-k\,j}c_{j-k}\,.\end{aligned} \tag{4.46}$$

Since the number of the S-matrices in T_j is $N-1$, here $k=1,\dots,N-1$ with understanding that $a_{-m\,j}\equiv a_{N-m\,j}$, $b_{-m\,j}\equiv b_{N-m\,j}$ for any $m=0,\dots,N-j-1$. The first of these equations implies that Λ_j depends on neither k nor $j-k$. We use this equation to solve for $c_j^{(k-1)}$

$$\begin{aligned}c_j^{(k-1)} &= \frac{\Lambda_j - a_{j-k\,j}}{b_{j-k\,j}}c_{j-k} = \frac{\Lambda_j(a_{j-k\,j}+b_{j-k\,j}) - a_{j-k\,j}}{b_{j-k\,j}}c_{j-k}\\ &= \Big[\Lambda_j + (\Lambda_j-1)\frac{a_{j-k\,j}}{b_{j-k\,j}}\Big]c_{j-k}\,.\end{aligned}$$

Changing here $k\to k+1$ gives

$$c_j^{(k)} = \Big[\Lambda_j + (\Lambda_j-1)\frac{a_{j-k-1\,j}}{b_{j-k-1\,j}}\Big]c_{j-k-1}\,. \tag{4.47}$$

We substitute these formulae in the second equation of (4.46) and get

$$\begin{aligned}&\Big[\Lambda_j + (\Lambda_j-1)\frac{a_{j-k-1\,j}}{b_{j-k-1\,j}}\Big]c_{j-k-1} = a_{j-k\,j}\Big[\Lambda_j + (\Lambda_j-1)\frac{a_{j-k\,j}}{b_{j-k\,j}}\Big]c_{j-k} + b_{j-k\,j}c_{j-k}\\ &= \Big[a_{j-k\,j}\Lambda_j + b_{j-k\,j} + (\Lambda_j-1)\frac{a_{j-k\,j}^2}{b_{j-k\,j}}\Big]c_{j-k} = \Big[1 + (\Lambda_j-1)\frac{a_{j-k\,j}}{b_{j-k\,j}}\Big]c_{j-k}\,,\end{aligned}$$

from where upon substituting the explicit expressions for a and b in terms of particle momenta, we obtain

$$\frac{c_{j-k}}{c_{j-k-1}} = \frac{\Lambda_j + (\Lambda_j-1)\frac{a_{j-k-1\,j}}{b_{j-k-1\,j}}}{1 + (\Lambda_j-1)\frac{a_{j-k\,j}}{b_{j-k\,j}}} = \frac{\frac{i\kappa\Lambda_j}{\Lambda_j-1} + p_{j-k-1} - p_j}{\frac{i\kappa}{\Lambda_j-1} + p_{j-k} - p_j}\,. \tag{4.48}$$

Now, setting $j-k=n$, for the ratio of two successive coefficients we find

$$\frac{c_n}{c_{n-1}} = \frac{\frac{i\kappa}{\Lambda_j-1} + p_{n-1} - p_j + i\kappa}{\frac{i\kappa}{\Lambda_j-1} + p_n - p_j}\,. \tag{4.49}$$

Here the right hand side has an apparent dependence on the index j. On the other hand, the left hand side must depend on the index n only to account for the fact that $|\Phi\rangle$ is a simultaneous eigenvector for all T_j.[5] Thus, the right hand side, should not depend on j as well, which is possible only if $\frac{i\kappa}{\Lambda_j-1} - p_j$ is some constant, which we choose for further convenience as

$$\frac{i\kappa}{\Lambda_j-1} - p_j = -v - \frac{i\kappa}{2}\,. \tag{4.50}$$

Hence,

$$\frac{c_n}{c_{n-1}} = \frac{p_{n-1} - v + \frac{i\kappa}{2}}{p_n - v - \frac{i\kappa}{2}}\,.$$

[5]This is precisely the place where we use the information that we diagonalise the commuting family $\{T_j\}$ rather than one matrix.

Iterating this equation, we find

$$c_n = c_1 \prod_{j=1}^{n-1} \frac{p_j - v + \frac{i\kappa}{2}}{p_{j+1} - v - \frac{i\kappa}{2}} = c_1 \frac{p_1 - v - \frac{i\kappa}{2}}{p_n - v - \frac{i\kappa}{2}} \prod_{j=1}^{n-1} \frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} \,. \tag{4.51}$$

The coefficient c_1 remains arbitrary and we choose it[6]

$$c_1 = \frac{i\kappa}{p_1 - v - \frac{i\kappa}{2}} \,, \tag{4.52}$$

so that

$$c_n = \frac{i\kappa}{p_n - v - \frac{i\kappa}{2}} \prod_{j=1}^{n-1} \frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} \,. \tag{4.53}$$

From equation (4.50) we obtain the eigenvalue Λ_j

$$\Lambda_j = \frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} \,. \tag{4.54}$$

Equation (4.47) yields

$$c_j^{(k)} = \frac{p_{j-k-1} - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} c_{j-k-1} \,. \tag{4.55}$$

In these formulae v plays the role of integration constant. In the following it is convenient to introduce the following variables

$$\underline{x}_j = \frac{i\kappa}{p_j - v - \frac{i\kappa}{2}} \,, \qquad \underline{y}_j = 1 + \underline{x}_j = \frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} \,. \tag{4.56}$$

In terms of this variables formula (4.53) reads as

$$c_n = \underline{y}_1 \cdots \underline{y}_{n-1} \underline{x}_n \,. \tag{4.57}$$

We then find that

$$\begin{aligned} c_j^{(k)} &= \frac{p_{j-k-1} - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} \frac{i\kappa}{p_{j-k-1} - v - \frac{i\kappa}{2}} \underline{y}_1 \cdots \underline{y}_{j-k-2} = \frac{i\kappa}{p_j - v - \frac{i\kappa}{2}} \underline{y}_1 \cdots \underline{y}_{j-k-1} \\ &= x'_j \underline{y}_1 \cdots \underline{y}_{j-k-1} \frac{\underline{y}_{j-k} \cdots \underline{y}_{j-1}}{\underline{y}_{j-k} \cdots \underline{y}_{j-1}} = \frac{c_j}{\underline{y}_{j-k} \cdots \underline{y}_{j-1}} \,. \end{aligned}$$

and the formulae (4.46) can be now regarded as identities satisfied by functions c_j, namely,

$$\begin{aligned} a_{j-k\,j} \frac{c_j}{\underline{y}_{j-k+1} \cdots \underline{y}_{j-1}} + b_{j-k\,j} c_{j-k} &= \frac{c_j}{\underline{y}_{j-k} \cdots \underline{y}_{j-1}} \,, \\ a_{j-k\,j} c_{j-k} + b_{j-k\,j} \frac{c_j}{\underline{y}_{j-k+1} \cdots \underline{y}_{j-1}} &= \underline{y}_j c_{j-k} \,. \end{aligned} \tag{4.58}$$

We have not yet solved the eigenvalue problem completely. Indeed, choose $j = N$. Then, after all $k = N - 1$ applications of the S-matrices, we will be left with

$$T_N |\Phi\rangle = S_{1N} \ldots S_{N-1\,N} |\Phi\rangle = \Lambda_N \Big[c_1 |1\rangle + \ldots + c_{N-1} |N-1\rangle \Big] + c_N^{(N-1)} |N\rangle \,. \tag{4.59}$$

[6]The choice $c_1 = 1$ was implemented in the original work [**?**].

Thus, to satisfy the eigenstate equation $T_N|\Phi\rangle = \Lambda_N|\Phi\rangle$, we have to require that

$$c_N^{(N-1)} = \Lambda_N c_N \,. \tag{4.60}$$

Plugging here the value of $c_N^{(N-1)}$ and $\Lambda_N = \underline{y}_N$, we get

$$\frac{c_N}{\underline{y}_1 \cdots \underline{y}_{N-1}} = \underline{y}_N c_N \,.$$

Thus, satisfaction of the last relation requires that

$$\prod_{j=1}^{N} \underline{y}_j = 1 \,. \tag{4.61}$$

Explicitly, it reads as

$$\prod_{j=1}^{N} \frac{p_j - v + \frac{i\kappa}{2}}{p_j - v - \frac{i\kappa}{2}} = 1 \,. \tag{4.62}$$

This is the Bethe equation to determine v. It implies, in particular, that $c_1 = c_{N+1}$, as follows from (4.53) provided $p_{N+1} = p_1$. As a result, identifying $|N+1\rangle = |1\rangle$ we render $|\Phi\rangle$ as a state of the *periodic* spin chain that is also an eigenstate of the commuting operators T_j.

In this way we have proved the validity of the generalised Bethe Ansatz and the expression for the eigenvalues in the one-particle sector. The Bethe equation (4.62) is a polynomial equation for v of degree $N-1$ and, therefore, for generic momenta p_j it has $N-1$ roots. Thus, formula (4.54) for Λ_j does not give all the eigenvalues of T_j on the permutation module, but only $N-1$ eigenvalues on the Specht submodule $[N-1,1]$. One extra eigenvalue of T_j is equal to 1 and it corresponds to the trivial submodule.

We conclude our consideration of the one-particle case with numerical results. Using the same set of momenta and the coupling constant (4.32), we solve equation (4.62) for different values $N = 3, 4, 5$ and find for each case the corresponding solutions which we enumerate as $v^{(i)}$, $i = 1, \ldots, N-1$. Once the roots are found, we evaluate the eigenvalue $\Lambda_1^{(i)}$ for each of them. The results are summarised below

$N = 3$,  :

$$v^{(1)} = 0.3332 \quad \Lambda_1^{(1)} = 0.8403 - 0.5421i \,; \qquad v^{(2)} = 0.6906 \quad \Lambda_1^{(2)} = 0.9908 - 0.1353i \,;$$

$N = 4$,  :

$$v^{(1)} = 0.2133 \quad \Lambda_1^{(1)} = -0.7245 - 0.6892i \,; \quad v^{(2)} = 0.3713 \quad \Lambda_1^{(2)} = 0.9270 - 0.3751i \,; \quad v^{(3)} = 0.7206 \quad \Lambda_1^{(3)} = 0.9918 - 0.1275i \,;$$

$N = 5$,  :

$$v^{(1)} = 0.1920 \quad \Lambda_1^{(1)} = -0.8899 + 0.4561i \,; \quad v^{(2)} = 0.2350 \quad \Lambda_1^{(2)} = 0.0498 - 0.9988i \,; \quad v^{(3)} = 0.3956 \quad \Lambda_1^{(3)} = 0.9436 - 0.3312i \,;$$
$$v^{(4)} = 0.7414 \quad \Lambda_1^{(4)} = 0.9924 - 0.1227i \,.$$

One can see that the eigenvalues found here coincide precisely with those obtained earlier through the direct diagonalisation of T_1.

Two-particle case. To consider multi-particle excitations, it is convenient to reintroduce a set of functions

$$F_j \equiv F(v,j) = \frac{i\kappa}{p_j - v - \frac{i\kappa}{2}} \prod_{k=1}^{j-1} \frac{p_k - v + \frac{i\kappa}{2}}{p_k - v - \frac{i\kappa}{2}} = \underline{y}_1 \cdots \underline{y}_{j-1} \underline{x}_j \,, \tag{4.63}$$

which coincide with c_j from the one-particle problem, but now we attach to them an independent meaning as to the set of basis functions satisfying the identities

$$\begin{aligned} a_{j-k\,j}\frac{F_j}{\underline{y}_{j-k+1}\cdots\underline{y}_{j-1}} + b_{j-k\,j}F_{j-k} &= \frac{F_j}{\underline{y}_{j-k}\cdots\underline{y}_{j-1}}\,, \\ a_{j-k\,j}F_{j-k} + b_{j-k\,j}\frac{F_j}{\underline{y}_{j-k+1}\cdots\underline{y}_{j-1}} &= \underline{y}_j F_{j-k}\,. \end{aligned} \tag{4.64}$$

These identities can be verified in a direct manner using the definitions (4.56) and (4.64). We also reintroduce

$$F_j^{(k)} = \frac{F_j}{\underline{y}_{j-k}\cdots\underline{y}_{j-1}} \tag{4.65}$$

and regard this function as as a result, of k-th iteration of F_j.

For the two particle case we then try the following ansatz

$$c_{mn} = \mathcal{A}F(v,m)F(v',n) + \mathcal{A}'F(v',m)F(v,n)\,. \tag{4.66}$$

For conciseness in the following we denote $F(v,m) \equiv F_m$ and $F(v',m) \equiv F'_m$, on other words,

$$c_{mn} = \mathcal{A}F_mF'_n + \mathcal{A}'F'_mF_n \equiv \mathcal{A}F_mF'_n + \text{conj}\,. \tag{4.67}$$

The ansatz requires two terms with coefficients $\mathcal{A}$ and $\mathcal{A}'$ to be present, as well as two different integration constants v and v'. Operating on the coefficients c_{ij} with T_j, we need to consider only the fate of the first term in (4.67), as the second one is obtained from it by replacing $\mathcal{A} \leftrightarrow \mathcal{A}'$ and $v \leftrightarrow v'$,[7] and so we will not write the second term and its transformations explicitly. For simplicity we assume that we act on the two-particle state

$$|\Phi\rangle = \sum_{m<n} c_{mn}|mn\rangle\,.$$

with the operator T_j with $j = N$, so there are no overturned spins on the right of j. Under the action of $S_{j-1\,j}$ all amplitudes remain invariant unless there is a down spin either at $j-1$ or j. We have[8]

$$\begin{aligned} &S_{j-1,j}\Big[\ldots + \sum_{i\leqslant j-2}\mathcal{A}F_iF'_{j-1}|i,j-1\rangle + \sum_{i\leqslant j-1}\mathcal{A}F_iF'_j|i,j\rangle\Big] \\ &\quad= S_{j-1,j}\Big[\ldots + \sum_{i\leqslant j-2}\mathcal{A}F_i\Big[F'_{j-1}|i,j-1\rangle + F'_j|i,j\rangle\Big] + \mathcal{A}F_{j-1}F'_j|j-1,j\rangle\Big] \\ &\quad= \ldots + \sum_{i\leqslant j-2}\mathcal{A}F_i\Big[\underbrace{\underline{y}'_jF'_{j-1}|i,j-1\rangle}_{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim} + F'^{(1)}_j|i,j\rangle\Big] + \underbrace{\mathcal{A}F_{j-1}F'_j|j-1,j\rangle}_{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}\,, \end{aligned}$$

where we have taken into account that the state $|j-1,j\rangle$ with two neighbouring spins down is invariant under $S_{j-1\,j}$ and use identities (4.64). To be able to use recurrent formulae (4.64) after the second step, we will try to impose on $\mathcal{A}$ and $\mathcal{A}'$ the requirement that

$$\mathcal{A}F_{j-1}F'_j + \mathcal{A}'F'_{j-1}F_j = \underline{y}'_j\mathcal{A}F^{(1)}_jF'_{j-1} + \underline{y}_j\mathcal{A}'F'^{(1)}_jF_{j-1}\,. \tag{4.68}$$

[7] In the present context we call this operation of replacement of primed for unprimed quantities *conjugation*.

[8] For more visibility we separated the positions of spins i and j in $|i,j\rangle$ by comma.

With this requirement the result above reads as

$$(\mathbf{I}) \quad \ldots + \underline{y}'_j \mathcal{A} F'_{j-1} \Big[\sum_{i \leqslant j-2} F_i |i, j-1\rangle + F_j^{(1)} |j-1, j\rangle \Big] + \mathcal{A} \sum_{i \leqslant j-2} F_i F_j'^{(1)} |i, j\rangle \,.$$

Now we are prepared to act on (**I**) with $S_{j-2\,j}$. In the application of $S_{j-2\,j}$ we have activated (in red) a set of terms in $|\Phi\rangle$ which will now undergo a non-trivial action

$$\begin{aligned}
&S_{j-2,j} \Big[\ldots + \sum_{i \leqslant j-3} \mathcal{A} F_i F'_{j-2} |i, j-2\rangle + \underline{y}'_j \mathcal{A} F'_{j-1} \Big[\sum_{i \leqslant j-2} F_i |i, j-1\rangle + F_j^{(1)} |j-1, j\rangle \Big] + \mathcal{A} \sum_{i \leqslant j-2} F_i F_j'^{(1)} |i, j\rangle \Big] \\
&= \ldots + \underline{y}'_j \mathcal{A} F'_{j-1} \sum_{i \leqslant j-3} F_i |i, j-1\rangle + \underline{y}'_j \mathcal{A} F'_{j-1} S_{j-2,j} \Big[F_{j-2} |j-2, j-1\rangle + F_j^{(1)} |j-1, j\rangle \Big] \\
&\quad + \mathcal{A} S_{j-2,j} \Big[\sum_{i \leqslant j-2} F_i F_j'^{(1)} |i, j\rangle + \sum_{i \leqslant j-3} \mathcal{A} F_i F'_{j-2} |i, j-2\rangle \Big] \\
&= \ldots + \underline{y}'_j \mathcal{A} F'_{j-1} \Big[\sum_{i \leqslant j-3} F_i |i, j-1\rangle + F_j^{(2)} |j-1, j\rangle \Big] + \underline{y}_j \underline{y}'_j \mathcal{A} F_{j-2} F'_{j-1} |j-2, j-1\rangle \\
&+ \mathcal{A} \sum_{i \leqslant j-3} F_i \Big[(a_{j-2\,j} F_j'^{(1)} + b_{j-2\,j} F'_{j-2}) |i, j\rangle + (b_{j-2\,j} F_j'^{(1)} + a_{j-2\,j} F'_{j-2}) |i, j-2\rangle \Big] + \mathcal{A} F_j'^{(1)} F_{j-2} |j-2, j\rangle \\
&= \ldots + \underline{y}'_j \mathcal{A} F'_{j-1} \Big[\sum_{i \leqslant j-3} F_i |i, j-1\rangle + F_j^{(2)} |j-1, j\rangle \Big] + \underline{y}_j \underline{y}'_j \mathcal{A} F_{j-2} F'_{j-1} |j-2, j-1\rangle \\
&\quad + \sum_{i \leqslant j-3} \mathcal{A} F_i \Big[\underline{y}'_j F'_{j-2} |i, j-2\rangle + F_j'^{(2)} |i, j\rangle \Big] + \mathcal{A} F_j'^{(1)} F_{j-2} |j-2, j\rangle \,.
\end{aligned}$$

Here the underwaved term will stay unchanged under action of any further S-matrices, and, therefore, from the eigenvalue equation, we find

$$\Lambda_j = \underline{y}_j \underline{y}'_j \,. \tag{4.69}$$

To proceed and bring the result to the form similar to the one obtained after the first application and, therefore, to prepare it to the third application, it would be nice to require that

$$\mathcal{A} F_{j-2} F_j'^{(1)} + \mathcal{A}' F'_{j-2} F_j^{(1)} = \underline{y}'_j \mathcal{A} F'_{j-2} F_j^{(2)} + \underline{y}_j \mathcal{A}' F_{j-2} F_j'^{(2)} \,, \tag{4.70}$$

although it is not at all clear if a simultaneous solution of (4.68) and (4.70) does exist. Nevertheless, accepting (4.70), after two applications we will have

$$\begin{aligned}
(\mathbf{II}) \quad \ldots \quad + \quad & \mathcal{A} \underline{y}'_j F'_{j-1} \Big[\sum_{i \leqslant j-3} F_i |i, j-1\rangle + F_j^{(2)} |j-1, j\rangle \Big] \\
& \qquad + \underline{y}_j \underline{y}'_j \mathcal{A} F_{j-2} F'_{j-1} |j-2, j-1\rangle \\
+ \quad & \mathcal{A} \underline{y}'_j F'_{j-2} \Big[\sum_{i \leqslant j-3} F_i |i, j-2\rangle + F_j^{(2)} |j-2, j\rangle \Big] + \sum_{i \leqslant j-3} \mathcal{A} F_i F_j'^{(2)} |i, j\rangle \,,
\end{aligned}$$

where we have omitted the underwaved unchangeable term. This result is similar to (**I**) and is ready for the next application of $S_{j-3\,j}$. We get

$$\begin{aligned}
(\mathbf{III}) \quad \ldots \quad + \quad & \mathcal{A} \underline{y}'_j F'_{j-1} \Big[\sum_{i \leqslant j-4} F_i |i, j-1\rangle + \underline{F_j^{(3)} |j-1, j\rangle} \Big] \\
& \qquad + \underline{y}_j \underline{y}'_j \mathcal{A} F_{j-3} F'_{j-1} |j-3, j-1\rangle + \underline{y}_j \underline{y}'_j \mathcal{A} F_{j-2} F'_{j-1} |j-2, j-1\rangle \\
+ \quad & \mathcal{A} \underline{y}'_j F'_{j-2} \Big[\sum_{i \leqslant j-4} F_i |i, j-2\rangle + \underline{F_j^{(3)} |j-2, j\rangle} \Big]
\end{aligned} \tag{4.71}$$

$$+\,\underline{y}_j\underline{y}'_j\mathcal{A}F_{j-3}F'_{j-2}|j-3,j-2\rangle$$

$$+\quad \mathcal{A}\underline{y}'_jF'_{j-3}\left[\sum_{i\leqslant j-4}F_i|i,j-3\rangle+\underline{F_j^{(3)}|j-3,j\rangle}\right]+\sum_{i\leqslant j-4}\mathcal{A}F_iF_j'^{(3)}|i,j\rangle\,.$$

We see that more and more applications of S-matrices "work out" more and more terms into their eigenstate form with the unique eigenvalue $\Lambda_j=\underline{y}_j\underline{y}'_j$. Proceeding in the same manner, after k steps we will have the result

$$\begin{aligned}\ldots\quad+\quad&\mathcal{A}\underline{y}'_jF'_{j-1}\left[\sum_{i\leqslant j-k-1}F_i|i,j-1\rangle+\underline{F_j^{(k)}|j-1,j\rangle}\right]\\&+\,\underline{y}_j\underline{y}'_j\left[\mathcal{A}F_{j-k+1}F'_{j-1}|j-k+1,j-1\rangle+\ldots+\mathcal{A}F_{j-2}F'_{j-1}|j-2,j-1\rangle\right]\\+\quad&\cdots\\&+\underline{y}_j\underline{y}'_j\mathcal{A}F_{j-k}F'_{j-k+1}|j-k,j-k+1\rangle\\+\quad&\mathcal{A}\underline{y}'_jF'_{j-k}\left[\sum_{i\leqslant j-k-1}F_i|i,j-k\rangle+\underline{F_j^{(k)}|j-k,j\rangle}\right]+\sum_{i\leqslant j-k-1}\mathcal{A}F_iF_j'^{(k)}|i,j\rangle\,.\end{aligned}\tag{4.72}$$

All spin states $|i,m\rangle$, where $i<m$, with

$$\begin{aligned}j-k\leqslant i\leqslant j-2\,,\\ j-k+1\leqslant m\leqslant j-1\end{aligned}$$

have been brought to the eigenstate form. To be able to write down this formula, before performing the k-th application, we require that

$$\mathcal{A}F_{j-k}F_j'^{(k-1)}+\mathcal{A}'F'_{j-k}F_j^{(k-1)}=\underline{y}'_j\mathcal{A}F'_{j-k}F_j^{(k)}+\underline{y}_j\mathcal{A}'F_{j-k}F_j'^{(k)}\,,\tag{4.73}$$

which is

$$\mathcal{A}\frac{F_{j-k}F'_j}{\underline{y}'_{j-k+1}\cdots\underline{y}'_{j-1}}+\mathcal{A}'\frac{F'_{j-k}F_j}{\underline{y}_{j-k+1}\cdots\underline{y}_{j-1}}=\mathcal{A}\frac{\underline{y}'_jF'_{j-k}F_j}{\underline{y}_{j-k}\cdots\underline{y}_{j-1}}+\mathcal{A}'\frac{\underline{y}_jF_{j-k}F'_j}{\underline{y}'_{j-k}\cdots\underline{y}'_{j-1}}\,.$$

From here we have an equation

$$\begin{aligned}\frac{\mathcal{A}}{\mathcal{A}'}&=\frac{\frac{F_{j-k}F'_j\underline{y}_j}{\underline{y}'_{j-k}\cdots\underline{y}'_{j-1}}-\frac{F'_{j-k}F_j}{\underline{y}_{j-k+1}\cdots\underline{y}_{j-1}}}{\frac{F_{j-k}F'_j}{\underline{y}'_{j-k+1}\cdots\underline{y}'_{j-1}}-\frac{F'_{j-k}F_j\underline{y}'_j}{\underline{y}_{j-k}\cdots\underline{y}_{j-1}}}\\&=\frac{\underline{y}_{j-k}}{\underline{y}'_{j-k}}\frac{F_{j-k}F'_j\;\underline{y}_{j-k+1}\cdots\underline{y}_j-F'_{j-k}F_j\;\underline{y}'_j\cdots\underline{y}'_{j-1}}{F_{j-k}F'_j\;\underline{y}_{j-k}\cdots\underline{y}_{j-1}-F'_{j-k}F_j\;\underline{y}'_{j-k+1}\cdots\underline{y}'_j}\,.\end{aligned}$$

which must be satisfied independent of k. Substituting here the expressions for F-functions in terms of $\underline{y}$, see (4.63), and cancelling the common factor $\underline{y}_1\ldots\underline{y}_{j-1}\underline{y}'_1\ldots\underline{y}'_{j-1}$, we arrive at

$$\frac{\mathcal{A}}{\mathcal{A}'}=\frac{\frac{p_1-v-\frac{i\kappa}{2}}{p_{j-k}-v-\frac{i\kappa}{2}}\frac{p_1-v'-\frac{i\kappa}{2}}{p_j-v'-\frac{i\kappa}{2}}\underline{y}_j-\frac{p_1-v-\frac{i\kappa}{2}}{p_j-v-\frac{i\kappa}{2}}\frac{p_1-v'-\frac{i\kappa}{2}}{p_{j-k}-v'-\frac{i\kappa}{2}}\underline{y}_{j-k}}{\frac{p_1-v-\frac{i\kappa}{2}}{p_{j-k}-v-\frac{i\kappa}{2}}\frac{p_1-v'-\frac{i\kappa}{2}}{p_j-v'-\frac{i\kappa}{2}}\underline{y}'_{j-k}-\frac{p_1-v-\frac{i\kappa}{2}}{p_j-v-\frac{i\kappa}{2}}\frac{p_1-v'-\frac{i\kappa}{2}}{p_{j-k}-v'-\frac{i\kappa}{2}}\underline{y}'_j}\,.$$

Substituting here the expressions for $\underline{y}$, we observe remarkable cancellations leaving us the result

$$\frac{\mathcal{A}}{\mathcal{A}'} = \frac{v-v'-i\kappa}{v-v'+i\kappa}\,, \tag{4.74}$$

which is independent of k, as desired.

We recall that for the operator T_N we have $j=N$, and we can perform applications of S-matrices down to $k=N-1$. At the end we will be left with all the terms in the eigenstate form except a certain number of them (these are the underlined terms in (4.71) and in (4.72), where in (4.72) only the first and the last of k such terms are shown explicitly). Namely, after the full application of T_N, we will be left over with

$$\sum_{k=1}^{N-1}\Big[\mathcal{A}\underline{y}'_N F'_{N-k}F_N^{(N-1)} + (\mathcal{A}\leftrightarrow\mathcal{A}', v\leftrightarrow v')\Big]|N-k,N\rangle$$
$$= \underline{y}_N\underline{y}'_N\sum_{k=1}^{N-1}\Big[\mathcal{A}F_{N-k}F'_N + (\mathcal{A}\leftrightarrow\mathcal{A}', v\leftrightarrow v')\Big]|N-k,N\rangle$$

which we have equated above to the right hand side of the eigenstate equation. This equation can be satisfied by requiring the "cross-cancelation", namely,

$$\mathcal{A}\underline{y}'_N F'_{N-k}F_N^{(N-1)} = \underline{y}_N\underline{y}'_N\mathcal{A}'F'_{N-k}F_N\,, \qquad \mathcal{A}'\underline{y}_N F_{N-k}F_N'^{(N-1)} = \underline{y}'_N\underline{y}_N\mathcal{A}F_{N-k}F'_N$$

that is

$$\begin{aligned}\frac{\mathcal{A}}{\mathcal{A}'} &= \frac{\underline{y}_N F_N}{F_N^{(N-1)}} = \prod_{j=1}^{N}\underline{y}_j\,,\\ \frac{\mathcal{A}'}{\mathcal{A}} &= \frac{\underline{y}'_N F'_N}{F_N'^{(N-1)}} = \prod_{j=1}^{N}\underline{y}'_j\,.\end{aligned} \tag{4.75}$$

These are *Bethe-Yang equations* which have the following explicit form

$$\begin{aligned}\frac{v_1-v_2-i\kappa}{v_1-v_2+i\kappa} &= \prod_{j=1}^{N}\frac{p_j-v_1+\frac{i\kappa}{2}}{p_j-v_1-\frac{i\kappa}{2}} = \frac{F(v_1,N+1)}{F(v_1,1)}\,,\\ \frac{v_1-v_2+i\kappa}{v_1-v_2-i\kappa} &= \prod_{j=1}^{N}\frac{p_j-v_2+\frac{i\kappa}{2}}{p_j-v_2-\frac{i\kappa}{2}} = \frac{F(v_2,N+1)}{F(v_2,1)}\,,\end{aligned} \tag{4.76}$$

where we have written the product of $\underline{y}$'s via F-functions by assuming the periodic identification $p_{N+1}=p_1$ and set $v=v_1$ and $v'=v_2$.

Finally, let us give some numerical results which will allow to understand further restrictions on relevant values Bethe roots. Using the same set of momenta and the coupling constant (4.32), we solve equation (4.76) for $N=4$ and get the following ordered pairs[9] (v_1,v_2):

$$(0.2165, 0.6536);\ (0.4350-0.0659i, 0.4350+0.0659i);$$
$$(0.8525, 0.8525);\ (0.4656, 0.4656);\ (0.2483, 0.2483);\ (0.1738, 0.1738)\,.$$

We got 6 pairs of roots but only the first two pairs, where v_1 and v_2 are not equal, give rise to physical eigenvalues

$$\Lambda_1^{(1)} = -0.7167 - 0.6974i\,, \qquad \Lambda_1^{(2)} = 0.8660 - 0.5001i\,,$$

[9]Solutions which differ by permutations of roots lead to the same eigenvalue Λ_1.

which agree with the direct generalisation of T_1 on the Specht module $[2,2]$. Similar results hold for higher N. We thus conclude that the physically relevant solutions of the Bethe equations (4.41) are those for which Bethe roots are pairwise different. If Bethe roots coincide, $v_1 = v_2$, then $\mathcal{A} = -\mathcal{A}'$ and the wave function (4.67) vanishes.

The results obtained for the one- and two-particle cases are generalised to multi-particle ones according to (4.38), (4.41) and (4.42). In the next section we provide an independent and general derivation of the Bethe-Yang equations (4.41) as the cyclicity condition for the coefficients $c_{n_1 \ldots n_M}$ of the wave function (4.38).

Lecture 5

Transfer Matrix Method

Here we explain another method of solving the inhomogeneous spin-1/2 chain by using the concept of the transfer matrix. We therefore introduce the transfer matrix for this chain and show that the problem of its diagonalisation is equivalent to finding the eigenbasis for the commuting family $\{T_j\}$. We then diagonalise the transfer matrix by Lieb's method as explained in Gaudin's book.

5.1 Transfer matrix

Now, to treat all T_j at once and, most importantly, to prove their commutativity, we use the spin chain representation to introduce the concept of transfer matrix. Consider the following object

$$\mathcal{M}(p) = S_{1a}(p_1, p)S_{2a}(p_2, p)\dots S_{Na}(p_N, p)\,. \tag{5.1}$$

Here "a" stands for an extra copy of V, called "auxiliary space" and p is an associated auxiliary momentum variable. The quantity $\mathcal{M}(p)$ is called *monodromy matrix* or simply *monodromy*. The monodromy acts on the space $V^{\otimes N} \otimes V_a$.

Taking the trace of $\mathcal{M}(p)$ with respect to the auxiliary space, we obtain an operator

$$T(p) = \mathrm{Tr}_a\, \mathcal{M}(p) = \mathrm{Tr}_a\Big[S_{1a}(p_1, p)S_{2a}(p_2, p)\dots S_{Na}(p_N, p)\Big]\,, \tag{5.2}$$

called *transfer matrix*. The transfer matrix is an operator on the configuration space of the spin chain

$$T(p):\ V^{\otimes N} \to V^{\otimes N}\,.$$

The fundamental property of the transfer matrix is that it is a generating function for the commuting operators T_j. According to (3.46), the two-body S-matrix for coincident momenta degenerates into a permutation $S_{ij}(p, p) = P_{ij}$, which is, of course, compatible with the condition (3.51). We then evaluate the transfer matrix at $p = p_j$

$$\begin{aligned} T(p_j) &= \mathrm{Tr}_a\Big[S_{1a}(p_1, p_j)S_{2a}(p_2, p_j)\dots S_{ja}(p_j, p_j)\dots S_{Na}(p_N, p_j)\Big] \\ &= \mathrm{Tr}_a\Big[S_{1a}(p_1, p_j)S_{2a}(p_2, p_j)\dots P_{ja}\dots S_{Na}(p_N, p_j)\Big]\,. \end{aligned}$$

Then, using the braiding property (3.47), we pull P_{ja} to the left

$$T(p_j) = \mathrm{Tr}_a\Big[P_{ja}S_{1j}(p_1, p_j)\dots S_{j-1j}(p_{j-1}, p_j)\cdot S_{j+1a}(p_{j+1}, p_j)\dots S_{Na}(p_N, p_j)\Big]\,.$$

Here we separated by $\cdot$ two strings of S-matrices. Since the indices of any S-matrix from one string are different from those for any S-matrix from other string, these two strings commute and we can interchange their position under the trace

$$T(p_j) \quad = \quad \mathrm{Tr}_a\Big[P_{ja}S_{j+1a}(p_{j+1},p_j)\dots S_{Na}(p_N,p_j)\cdot S_{1j}(p_1,p_j)\dots S_{j-1j}(p_{j-1},p_j)\Big]\,.$$

Next, we move P_{ja} into the position between two strings to get

$$T(p_j) \quad = \quad S_{j+1j}(p_{j+1},p_j)\dots S_{Nj}(p_N,p_j)\cdot \mathrm{Tr}_a(P_{ja})\cdot S_{1j}(p_1,p_j)\dots S_{j-1j}(p_{j-1},p_j)\,.$$

It remains to note that due to $\mathrm{Tr}_a(P_{ja})=\mathbb{1}$, we get $T(p_j)=T_j$, where T_j is given by (4.11).

Working with the spin chain representation and the transfer matrix has an advantage that it allows for an easy proof of commutativity of T_j. It follows from the commutation relation between two monodromy matrices, $\mathcal{M}_a(p)$ and $\mathcal{M}_b(q)$, each of which is defined with the help of its own independent auxiliary spaces, V_a and V_b, respectively. This commutation relation is derived as follows. Let us consider

$$S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p)\,, \tag{5.3}$$

where S_{ab} is the S-matrix which acts on auxiliary spaces only. Using the definition of the monodromies, we write

$$\begin{aligned} S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p) &= S_{ab}(p,q)S_{1b}(p_1,q)\dots S_{Nb}(p_N,q)S_{1a}(p_1,p)\dots S_{Na}(p_N,p)\\ &= S_{ab}(p,q)S_{1b}(p_1,q)S_{1a}(p_1,p)\Big(S_{2b}(p_1,q)\dots S_{Nb}(p_N,q)S_{2a}(p_1,p)\dots S_{Na}(p_N,p)\Big)\,, \end{aligned}$$

Here we freely moved the matrix S_{1a} next to S_{1b} because it commutes with all the matrices on its way until it meets S_{1b}. Now we can use the Yang-Baxter equation

$$S_{ab}(p,q)S_{1b}(p_1,q)S_{1a}(p_1,p) = S_{1a}(p_1,p)S_{1b}(p_1,q)S_{ab}(p,q)$$

that yields at this stage the following answer

$$\begin{aligned} &S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p) =\\ &\qquad S_{1a}(p_1,p)S_{1b}(p_1,q)S_{ab}(p,q)\Big(S_{2b}(p_1,q)\dots S_{Nb}(p_N,q)S_{2a}(p_1,p)\dots S_{Na}(p_N,p)\Big)\,. \end{aligned}$$

Clearly, the matrices S_{1a} and S_{1b} interchanged their initial order and S_{ab} stands again in front of monodromies, the latter being reduced by the elements S_{1a} and S_{1b}. Clearly, we can now repeat the same manipulation for S_{2b} and S_{2a}, and so on until we commute with the help of repeated application of the Yang-Baxter equation the matrix S_{ab} to the right of all the matrices. As a result of these manipulations, we obtain the the following commutation relation between the components of the monodromy matrix

$$S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p) = \mathcal{M}_a(p)\mathcal{M}_b(q)S_{ab}(p,q)\,. \tag{5.4}$$

This relation is of fundamental importance, it provides a starting point for the algebraic Bethe Ansatz approach, which will be discussed later. Here we note that (5.4) immediately implies commutativity of T_j. Indeed, we rewrite it as

$$\mathcal{M}_a(p)\mathcal{M}_b(q) = S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p)S^{-1}_{ab}(p,q)$$

and then take the trace with respect to each of the two auxiliary spaces. This gives

$$T(p)T(q) = \mathrm{Tr}_{a,b}\Big(S_{ab}(p,q)\mathcal{M}_b(q)\mathcal{M}_a(p)S^{-1}_{ab}(p,q)\Big) = T(q)T(p)\,.$$

Thus, the values of the transfer matrix at different values of momenta commute

$$T(p)T(q) = T(q)T(p)\,. \tag{5.5}$$

Taking $p = p_j$ and $q = p_k$ completes the argument.

Considering the system (4.12) in the context of the spin chain representation, we can replace the problem of diagonalising the set $\{T_j\}$ by an equivalent problem of diagonalising the transfer matrix[1]

$$T = \mathrm{Tr}_0 S_{10}(p_1, p)\dots S_{N0}(p_N, p) \tag{5.6}$$

for all values of p. If we denote by $\Lambda(p)$ an eigenvalue of the transfer matrix, that is,

$$T|\Phi\rangle = \Lambda|\Phi\rangle\,, \tag{5.7}$$

then (4.12) results into a set of *Bethe equation*

$$e^{ip_j L} = \Lambda(p_j|\{p_k\})\,, \quad j = 1, \dots, N\,, \tag{5.8}$$

where by $\{p_k\}$ we have indicated an implicit dependence of the eigenvalue on all the other momenta than p_j. Equations (5.8) are implications of the periodicity condition for the real space wave function and they can be thought of as the quantisation conditions for asymptotic momenta. The range of their applicability is the same as of the asymptotic wave function.

5.2 Diagonalisation of the transfer matrix by Lieb's method

Here we explain how to diagonalise the transfer matrix. First we introduce a local spin algebra generated by the spin operators S_n^α, where $\alpha = 1, 2, 3$, with commutation relations

$$[S_m^\alpha, S_n^\beta] = i\epsilon^{\alpha\beta\gamma} S_n^\gamma \delta_{mn}\,. \tag{5.9}$$

The spin operators have the following realisation in terms of the Pauli matrices: $S_n^\alpha = \frac{1}{2}\sigma^\alpha$. Spin variables are subject to the periodic boundary condition $S_n^\alpha \equiv S_{n+N}^\alpha$. We then introduce the raising and lowering operators $S_n^\pm = S_n^1 \pm iS_n^2$. They are realised as

$$S_n^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad S_n^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,.$$

The spin operators $S_n^\pm$, S_n^3 acts non-trivially only on the n'th site of the chain, where this action reads as

$$\begin{aligned} &S_n^+|\uparrow_n\rangle = 0\,, && S_n^+|\downarrow_n\rangle = |\uparrow_n\rangle\,, && S_n^3|\uparrow_n\rangle = \tfrac{1}{2}|\uparrow_n\rangle\,, \\ &S_n^-|\downarrow_n\rangle = 0\,, && S_n^-|\uparrow_n\rangle = |\downarrow_n\rangle\,, && S_n^3|\downarrow_n\rangle = -\tfrac{1}{2}|\downarrow_n\rangle\,. \end{aligned}$$

The Hilbert space of the spin chain carries a tensor product representation of the Lie algebra $\mathfrak{sl}(2)$; the corresponding generators are realised as

$$S^\alpha = \sum_{n=1}^{N} \mathbb{1}\otimes\dots\otimes S_n^\alpha \otimes\dots\otimes \mathbb{1}\,. \tag{5.10}$$

In particular, $S^\pm = S^1 \pm iS^2$ are the raising and lowering operators. The transfer matrix has the property that it commutes with S^α, $[T, S^\alpha] = 0$. In particular, $[T, S^3] = 0$. The later property implies that the matrix elements

$$\langle m_1, \dots, m_M|\, T\,|n_1, \dots, n_R\rangle \tag{5.11}$$

[1] Here we denote the auxiliary space by 0.

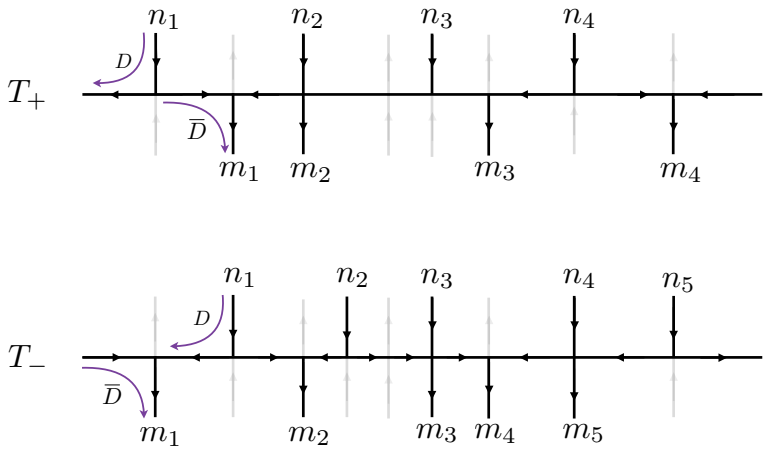


Figure 5.1: Matrix elements of the transfer matrix T.

are non-zero if and only if $M = R$, because T preserves the number of the overturned spins. The operator S_{n0} entering the monodromy is the following 2×2-matrix in the auxiliary space "0"

$$S_{n0} = \begin{pmatrix} a_n\left(\frac{1}{2} + S_n^3\right) + b_n\left(\frac{1}{2} - S_n^3\right) & c_n S_n^- \\ c_n S_n^+ & a_n\left(\frac{1}{2} - S_n^3\right) + b_n\left(\frac{1}{2} + S_n^3\right). \end{pmatrix},$$

where

$$a_n = 1\,, \quad b_n = \frac{1}{y_n}\,, \quad c_n = \frac{x_n}{y_n}\,, \tag{5.12}$$

and we have introduced

$$x_n = \frac{i\kappa}{p_n - p}\,, \quad y_n = 1 + x_n\,. \tag{5.13}$$

The matrix S_{n0} has operator-valued entries that in the local spin basis give rise to the following matrix elements

$$\langle\uparrow_n |S_{n0}| \uparrow_n\rangle = \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}, \qquad \langle\downarrow_n |S_{n0}| \downarrow_n\rangle = \begin{pmatrix} b_n & 0 \\ 0 & a_n \end{pmatrix},$$
$$\langle\uparrow_n |S_{n0}| \downarrow_n\rangle = \begin{pmatrix} 0 & 0 \\ c_n & 0 \end{pmatrix}, \qquad \langle\downarrow_n |S_{n0}| \uparrow_n\rangle = \begin{pmatrix} 0 & c_n \\ 0 & 0 \end{pmatrix},$$

The next important observation is that S_{n0} defines the *six-vertex model* of statistical mechanics. The vertices of this model satisfy the *ice rule* which allow for the transfer matrix to have in the spin basis non-zero matrix elements of only two types symbolically depicted in the Fig. 5.1.

Construction of the transfer matrix. Computing the matrix elements of the monodromy $\mathcal{M}$ for the spin spin configuration on the upper picture of Fig. 5.1, we find

$$\langle m|\mathcal{M}|n\rangle\Big|_{\text{1st conf}} = \begin{pmatrix} a_1 \dots a_{n_1-1} & 0 \\ 0 & b_1 \dots b_{n_1-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c_{n_1} & 0 \end{pmatrix}$$

$$
\begin{aligned}
&\times \begin{pmatrix} a_{n_1+1}\dots a_{m_1-1} & 0 \\ 0 & b_{n_1+1}\dots b_{m_1-1} \end{pmatrix}\begin{pmatrix} 0 & c_{m_1} \\ 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} a_{m_1+1}\dots a_{n_2-1} & 0 \\ 0 & b_{m_1+1}\dots b_{n_2-1} \end{pmatrix}\begin{pmatrix} b_{n_2} & 0 \\ 0 & a_{n_2} \end{pmatrix} \\
&\times \begin{pmatrix} a_{n_2+1}\dots a_{n_3-1} & 0 \\ 0 & b_{n_2+1}\dots b_{n_3-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c_{n_3} & 0 \end{pmatrix}\times\dots \\
\dots\ &\times \begin{pmatrix} a_{m_M+1}\dots a_N & 0 \\ 0 & b_{m_M+1}\dots b_N \end{pmatrix}.
\end{aligned}
$$

Using the projector nature of upper and lower triangular matrices, we can first replace the redundant matrix elements by zeros

$$
\begin{aligned}
\langle m|\mathcal{M}|n\rangle\Big|_{\text{1st conf}} &= \begin{pmatrix} 0 & 0 \\ 0 & b_1\dots b_{n_1-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c_{n_1} & 0 \end{pmatrix} \\
&\times \begin{pmatrix} a_{n_1+1}\dots a_{m_1-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & c_{m_1} \\ 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 \\ 0 & b_{m_1+1}\dots b_{n_2-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a_{n_2} \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 \\ 0 & b_{n_2+1}\dots b_{n_3-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c_{n_3} & 0 \end{pmatrix}\times\dots\times\begin{pmatrix} 0 & 0 \\ 0 & b_{m_M+1}\dots b_N \end{pmatrix}.
\end{aligned}
$$

Then we multiply the matrices pairwise to get

$$
\begin{aligned}
\langle m|\mathcal{M}|n\rangle\Big|_{\text{1st conf}} &= \begin{pmatrix} 0 & 0 \\ b_1\dots b_{n_1-1}c_{n_1} & 0 \end{pmatrix}\times\begin{pmatrix} 0 & a_{n_1+1}\dots a_{m_1-1}c_{m_1} \\ 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 \\ b_{m_1+1}\dots b_{n_2-1}c_{n_2} & 0 \end{pmatrix}\times\begin{pmatrix} 0 & \frac{a_{n_2}}{c_{n_2}} \\ 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 \\ b_{n_2+1}\dots b_{n_3-1}c_{n_3} & 0 \end{pmatrix}\times\dots\times\begin{pmatrix} 0 & 0 \\ 0 & b_{m_M+1}\dots b_N \end{pmatrix},
\end{aligned}
$$

where we factored out

$$
\begin{pmatrix} 0 & 0 \\ 0 & b_{m_1+1}\dots b_{n_2-1}a_{n_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_{m_1+1}\dots b_{n_2-1}c_{n_2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{a_{n_2}}{c_{n_2}} \\ 0 & 0 \end{pmatrix}
$$

Finally, we can take away the common factor $y = y_1, \dots y_N$, obtaining

$$
\begin{aligned}
\langle m|\mathcal{M}|n\rangle\Big|_{\text{1st conf}} = \frac{1}{y}&\begin{pmatrix} 0 & 0 \\ x_{n_1} & 0 \end{pmatrix}\times\begin{pmatrix} 0 & y_{n_1+1}\dots y_{m_1-1}x_{m_1} \\ 0 & 0 \end{pmatrix} \\
&\times\begin{pmatrix} 0 & 0 \\ x_{n_2} & 0 \end{pmatrix}\times\begin{pmatrix} 0 & \frac{y_{n_2}}{x_{n_2}} \\ 0 & 0 \end{pmatrix} \\
&\times\begin{pmatrix} 0 & 0 \\ x_{n_3} & 0 \end{pmatrix}\times\dots\times\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Thus, multiplying all the matrices, we will end up with the following answer

$$
\langle m|\mathcal{M}|n\rangle\Big|_{\text{1st conf}} = \begin{pmatrix} 0 & 0 \\ 0 & \langle m|T_+|n\rangle \end{pmatrix}. \tag{5.14}
$$

Here

$$
y\langle m|T_+|n\rangle = D(0n_1)\overline{D}(n_1m_1)D(m_1n_2)\overline{D}(n_2m_2)\dots\overline{D}(n_Mm_M)\,, \tag{5.15}
$$

where

$$1 \leqslant n_1 \leqslant m_1 \leqslant \ldots \leqslant n_M \leqslant m_M \leqslant N \tag{5.16}$$

and we have introduced the set of basis functions

$$\begin{aligned} D(mn) &= x_n\,, \qquad m < n \\ D(nn) &= \frac{1}{x_n}\,, \qquad m = n \\ \overline{D}(nm) &= y_{n+1}y_{n+1}\ldots y_{m-1}x_m\,, \quad n < m \\ \overline{D}(nn) &= \frac{y_n}{x_n}\,, \quad n = m\,. \end{aligned} \tag{5.17}$$

In the example depicted in Fig. 5.1 the term with $D(nn)$ does not arise, but it will arise when, for instance, $n_2 = m_1$ and we will get the structure

$$y\langle m|T_+|n\rangle = D(0n_1)\overline{D}(n_1m_1)D(m_1n_2)\overline{D}(n_2m_2)\ldots\,, \tag{5.18}$$

where the highlighted term has coincident arguments $n_2 = m_1$.

Proceeding in a similar way, we compute the matrix elements of $\mathcal{M}$ for the spin spin configuration on the lower picture of Fig. 5.1

$$\begin{aligned} \langle m|\mathcal{M}|n\rangle\Big|_{\text{2nd conf}} &= \begin{pmatrix} a_1\ldots a_{m_1-1} & 0 \\ 0 & b_1\ldots b_{m_1-1} \end{pmatrix} \times \begin{pmatrix} 0 & c_{m_1} \\ 0 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} a_{m_1+1}\ldots a_{n_1-1} & 0 \\ 0 & b_{m_1+1}\ldots b_{n_1-1} \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ c_{n_1} & 0 \end{pmatrix} \times \ldots \\ \ldots &\times \begin{pmatrix} a_{n_M+1}\ldots a_N & 0 \\ 0 & b_{n_M+1}\ldots b_N \end{pmatrix}. \end{aligned}$$

Further simplifying this result, we find

$$\begin{aligned} \langle m|\mathcal{M}|n\rangle\Big|_{\text{2nd conf}} &= \frac{1}{y}\begin{pmatrix} 0 & 0 \\ 0 & y_1\ldots y_{m_1-1}x_{m_1} \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ x_{n_1} & 0 \end{pmatrix} \times \ldots \\ \ldots &\times \begin{pmatrix} y_{n_M+1}\ldots y_N & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, multiplying the matrices, we will have

$$\langle m|\mathcal{M}|n\rangle\Big|_{\text{2nd conf}} = \begin{pmatrix} \langle m|T_-|n\rangle & 0 \\ 0 & 0 \end{pmatrix}.$$

Here

$$y\langle m|T_-|n\rangle = \overline{D}(0m_1)D(m_1n_1)\overline{D}(n_1m_2)\ldots D(m_Mn_M)\,y_{n_M+1}\ldots y_N\,, \tag{5.19}$$

where

$$1 \leqslant n_1 \leqslant m_1 \leqslant \ldots \leqslant m_N \leqslant N\,. \tag{5.20}$$

In this way we have computed the matrix elements of the monodromy for arbitrary configuration of admissible spin states with the following result

$$\langle m|\mathcal{M}|n\rangle = \begin{pmatrix} \langle m|T_-|n\rangle & 0 \\ 0 & \langle m|T_+|n\rangle \end{pmatrix}.$$

Taking the trace in the auxiliary space, we finally find the matrix elements of the transfer matrix

$$\langle m|T|n\rangle = \langle m|T_+|n\rangle + \langle m|T_-|n\rangle\,, \tag{5.21}$$

where the quantities $\langle m|T_\pm|n\rangle$ are given by (5.15) and (5.19), with restrictions (5.16) and (5.20), respectively.

Eigenvalue equation. An equation for the amplitudes $c_{n_1\ldots n_M}$ which diagonalise the transfer matrix reads as

$$\sum_{n_1<\ldots<n_M} \Big\{\langle m|T_+|n\rangle + \langle m|T_-|n\rangle\Big\} c_{n_1\ldots n_M} = \Lambda\, c_{m_1\ldots m_M}\,. \tag{5.22}$$

Although we already know the coefficients $c_{n_1\ldots n_M}$ that satisfy this equation from Yang's tratment, it is instructive to derive them again by using a different method, the latter admits a generalisation to the problem of arbitrary spin.

Note that (5.15) can be written as

$$y\,\langle m|T_+|n\rangle = \prod_{j=1}^{M} D(m_{j-1}n_j)\overline{D}(n_j m_j)\,, \tag{5.23}$$

where we by definition set $m_0 = m_M - N \leqslant 0$.

Assuming the periodicity conditions $x_{k+N} = x_k$ and so that $y_{k+N} = y_k$, we can write

$$\begin{aligned} y_{n_M+1}\ldots y_N\,\overline{D}(0m_1) &= y_{n_M+1}\ldots y_N\cdot y_1\ldots y_{m_1-1}x_{m_1} \\ &= y_{n_M+1}\ldots y_N\cdot y_{N+1}\ldots y_{N+m_1-1}x_{m_1+N} = \overline{D}(n_M m_{M+1})\,, \end{aligned}$$

where we identify $m_{M+1} = m_1 + N$. This allows us to obtain the more concise expression analogous to (5.23)

$$\begin{aligned} y\,\langle m|T_-|n\rangle &= D(m_1n_1)\overline{D}(n_1m_2)\ldots D(m_Mn_M)\overline{D}(n_Mm_{M+1}) \\ &= \prod_{j=1}^{M} D(m_jn_j)\overline{D}(n_jm_{j+1})\,. \end{aligned} \tag{5.24}$$

To perform the summation in (5.22), we assume that the *coefficients $c_{n_1\ldots n_M}$ can be extended outside the interval $1 \leqslant n_i \leqslant N$ with the condition of their cyclic invariance*, that is

$$c_{n_1n_2\ldots n_M} = c_{n_0n_1\ldots n_{M-1}} = c_{n_2n_3\ldots n_{M+1}}\,, \tag{5.25}$$

where $n_0 = n_M - N$ and $n_{M+1} = n_1 + N$. Then we have

$$\sum_{1\leqslant n_1<\ldots<n_M\leqslant N} \langle m|T_+|n\rangle c_{n_1\ldots n_M} = \frac{1}{y}\sum_{1\leqslant n_1<\ldots<n_M\leqslant N}\prod_{j=1}^{M} D(m_{j-1}n_j)\overline{D}(n_jm_j)c_{n_1\ldots n_M}\,.$$

Here we make the change of the summation variables

$$n_1 \to n_0, \quad n_2 \to n_1, \quad \ldots \quad n_M \to n_{M-1}$$

so that inequalities (5.16) becomes

$$1 \leqslant n_0 \leqslant m_1 \leqslant n_1 \leqslant m_2 \leqslant \ldots \leqslant n_{M-1} \leqslant m_M \leqslant N \tag{5.26}$$

and the sum takes the form

$$\frac{1}{y}\sum_{1\leqslant n_0<\ldots<n_{M-1}\leqslant N}\prod_{j=1}^{M}D(m_{j-1}n_{j-1})\overline{D}(n_{j-1}m_j)c_{n_0n_1\ldots n_{M-1}}$$
$$=\frac{1}{y}\sum_{1<n_1\ldots<n_{M-1}\leqslant N<n_M}\prod_{j=1}^{M}D(m_{j-1}n_{j-1})\overline{D}(n_{j-1}m_j)c_{n_1\ldots n_{M-1}n_M}\,,$$

where we used the cyclicity of the coefficients and set $n_M\equiv n_0+N$, where

$$N<n_M<n_1+N\leqslant m_1+N\equiv m_{M+1}\,,$$

see equation (5.26). Taking into account that $m_0=m_M-N<1<n_0$, we notice that

$$D(m_0n_0)\overline{D}(n_0m_1)=x_{n_0}y_{n_0+1}\ldots y_{m_1-1}x_{m_1}=x_{n_0+N}y_{n_0+N+1}\ldots y_{m_1+N-1}x_{m_1+N}$$
$$=x_{n_M}y_{n_M}\ldots y_{m_{M+1}-1}x_{m_{M+1}}=D(m_Mn_M)\overline{D}(n_Mm_{M+1})\,.$$

Using this property, the sum can be finally written as

$$\sum_{1\leqslant n_1<\ldots<n_M\leqslant N}\langle m|T_+|n\rangle c_{n_1\ldots n_M}=\frac{1}{y}\sum_{\{n\}}\prod_{j=1}^{M}D(m_jn_j)\overline{D}(n_jm_{j+1})c_{n_1\ldots n_M}\,,\tag{5.27}$$

where summation extends over all n_j satisfying the conditions

$$m_j\leqslant n_j\leqslant m_{j+1}\,,\quad j=1,\ldots,M\,,\tag{5.28}$$

as well as

$$1<n_1\ldots<n_{M-1}\leqslant N<n_M<n_{M+1}\,,\qquad n_{M+1}=n_1+N\,.\tag{5.29}$$

Next, for T_- the sum is given by the same formula (5.27), except in addition to (5.28) the summation variables have to satisfy the condition

$$1\leqslant n_1<n_2\ldots<n_M\leqslant N\,.\tag{5.30}$$

Since domains (5.29) and (5.30) do not intersect, we can unite them in one domain

$$n_1<n_2\ldots<n_M<n_{M+1}=n_1+N\tag{5.31}$$

and write the eigenvalue equation (5.22) in the form

$$\sum_{\{n\}}\prod_{j=1}^{M}D(m_jn_j)\overline{D}(n_jm_{j+1})c_{n_1\ldots,n_M}=y\Lambda c_{m_1\ldots m_M}\,,\tag{5.32}$$

where for a given set of m_i the summation on the left hand side runs over all n_i obeying (5.28) and (5.31). Also, for $n_M>N$, n_1 is not allowed to take the value 1, see (5.29).

Example. Before we proceed with (5.32), we look at an explicit example of the permutation module $M^{[N-2,2]}$ which corresponds to the have of 5 momentum-carrying particles and two overturned spins. We have

$$\sum_{m_1\leqslant n_1\leqslant m_2}\sum_{m_2\leqslant n_2\leqslant m_1+N}\begin{aligned}&c_{n_1n_2}D(m_1n_1)\overline{D}(n_1m_2)D(m_2n_2)\overline{D}(n_2m_1+N)\\&-c_{m_2m_2}D(m_1m_2)\overline{D}(m_2m_2)D(m_2m_2)\overline{D}(m_2m_1+N)\\&-c_{m_1m_1+N}D(m_1m_1)\overline{D}(m_1m_2)D(m_2m_1+N)\overline{D}(m_1+N\,m_1+N)\\&=y\Lambda\,c_{m_1m_2}\,,\end{aligned}\tag{5.33}$$

where $m_1 < m_2$. Here we set up to sum over all values of $m_1 \leqslant n_1 \leqslant m_2$ and $m_2 \leqslant n_2 \leqslant m_3 = m_1+N$ but then explicitly subtract tho terms – the first one represents the overlapping case $n_1 = m_2 = n_2$, and the second one corresponds to $n_1 = m_1$, $n_2 = m_1 + N$ and it must be excluded because for $n_1 = m_1$ we should have $n_2 < n_3 = n_1 + N = m_1 + N$.

Lieb's method. To solve (5.32), for the coefficients $c_{n_1\ldots,n_M}$ we adopt the same ansatz (4.39). Then, substituting (4.39) into (5.32), we note that the expression under the sum would factorise if we would not impose the condition that all n_i must be distinct. To use this welcome factorisation property, we can first disregard that n_i are distinct and extend the coefficients $c_{n_1\ldots,n_M}$ to the domain $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_M$. However, once the summation is performed, we have to subtract the (unwanted) terms which originate from coincident $n_j = n_{j+1}$ on the walls separating the regions $n_j < n_{j+1}$ for all $j = 1, \ldots, M$. This approach constitutes *Lieb's method* of solving (5.32) and this is precisely what is implemented in example (5.33).

Thus, according to Lieb's method, we disregard the conditions

$$n_1 < n_2 < \ldots < n_M$$

factorise the sums over n_j and subtract the unwanted terms

$$\sum_{\tau\in\mathfrak{S}_M} \mathcal{A}(\tau) \sum_{n_j=m_j}^{m_{j+1}} \prod_{j=1}^{M} D(m_j n_j)\overline{D}(n_j m_{j+1})F(v_{\tau(1)}, n_j) - \Big\{\text{unwanted terms}\Big\} = y\Lambda\, c_{m_1\ldots m_M}\,.$$

Summation over n in each of these sums can be performed in an explicit manner. We have

$$\begin{aligned}\sum_{n=m_j}^{m_{j+1}} D(m_j n)\overline{D}(n m_{j+1})F(v,n) = \\ = \frac{1}{x_{m_j}} y_{m_j+1}\cdots y_{m_{j+1}-1}x_{m_{j+1}}\underline{F(v,m_j)} + y_{m_{j+1}}\underline{F(v,m_{j+1})} \\ + \sum_{n=m_j+1}^{m_{j+1}-1} x_n y_{n+1}\ldots y_{m_{j+1}-1}x_{m_{j+1}}F(v,n)\,.\end{aligned}$$

Substituting in the last sum an explicit expression for $F(v,n)$, we then need to evaluate the sum

$$\mathcal{S} = \sum_{n=m_j+1}^{m_{j+1}-1} \underline{y}_1\underline{y}_2\ldots\underline{y}_{n-1}x_n\underline{x}_n y_{n+1}\ldots y_{m_{j+1}-1}\,.$$

The key formula to perform this sum is

$$x_n\underline{x}_n = (x_n - \underline{x}_n)\underline{x}_0 = (y_n - \underline{y}_n)\underline{x}_0\,, \qquad \underline{x}_0 = \frac{i\kappa}{p - v - \kappa}\,. \tag{5.34}$$

It immediately follows from definitions (5.13) and (4.56). Here $\underline{x}_0$ is similar to $\underline{x}_n$ in (4.56) except p_n is replaced with $p_0 = p$. Then we have

$$\begin{aligned}\mathcal{S} &= \underline{x}_0 \sum_{n=m_j+1}^{m_{j+1}-1} \underline{y}_1\ldots\underline{y}_{n-1}y_n\ldots y_{m_{j+1}-1} - \underline{y}_1\ldots\underline{y}_n y_{n+1}\ldots y_{m_{j+1}-1} \\ &= \underline{x}_0\,\underline{y}_1\ldots\underline{y}_{m_j}y_{m_j+1}\ldots y_{m_{j+1}-1} - \underline{x}_0\,\underline{y}_1\ldots\underline{y}_{m_{j+1}-1}\,.\end{aligned}$$

Therefore,

$$\sum_{n=m_j}^{m_{j+1}} D(m_j n)\overline{D}(n m_{j+1})F(v,n) =$$

$$= \frac{1}{x_{m_j}} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} \underline{F(v,m_j)} + y_{m_{j+1}} \underline{F(v,m_{j+1})}$$
$$+ \underline{x}_0 \, \underline{y}_1 \cdots \underline{y}_{m_j} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} - \underline{x}_0 \, \underline{y}_1 \cdots \underline{y}_{m_{j+1}-1} x_{m_{j+1}} \,.$$

We can further rewrite the last two terms in terms of the corresponding functions F, namely,

$$\sum_{n=m_j}^{m_{j+1}} D(m_j n)\overline{D}(n m_{j+1}) F(v,n) =$$
$$= \frac{1}{x_{m_j}} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} \underline{F(v,m_j)} + y_{m_{j+1}} \underline{F(v,m_{j+1})}$$
$$+ \frac{\underline{x}_0 \underline{y}_{m_j}}{\underline{x}_{m_j}} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} F(v,m_j) - \frac{\underline{x}_0}{\underline{x}_{m_{j+1}}} x_{m_{j+1}} F(v,m_{j+1})$$
$$= \Big(1 + \frac{\underline{x}_0 \underline{y}_{m_j} x_{m_j}}{\underline{x}_{m_j}}\Big) \frac{1}{x_{m_j}} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} F(v,m_j) + \Big(y_{m_{j+1}} - \underline{x}_0 \frac{x_{m_{j+1}}}{\underline{x}_{m_{j+1}}}\Big) F(v,m_{j+1}) \,.$$

Further, we have

$$1 + \underline{x}_0 \frac{\underline{y}_{m_j} x_{m_j}}{\underline{x}_{m_j}} = 1 + \underline{x}_0 \frac{(1+\underline{x}_{m_j}) x_{m_j}}{\underline{x}_{m_j}} = 1 + \underline{x}_0 x_{m_j} + \underline{x}_0 \frac{x_{m_j}}{\underline{x}_{m_j}} = 1 + \underline{x}_0 (y_{m_j} - 1) + \underline{x}_0 \frac{x_{m_j}}{\underline{x}_{m_j}}$$
$$= \underline{x}_0 y_{m_j} + 1 - \underline{x}_0 + \underline{x}_0 \frac{x_{m_j}}{\underline{x}_{m_j}} = \underline{x}_0 y_{m_j} + \frac{\underline{x}_{m_j} + \underline{x}_0 (x_{m_j} - \underline{x}_{m_j})}{x_{m_j}} = \underline{x}_0 y_{m_j} + \frac{\underline{x}_{m_j} + x_{m_j} \underline{x}_{m_j}}{\underline{x}_{m_j}}$$
$$= (1 + \underline{x}_0) y_{m_j} \,. \tag{5.35}$$

Analogously,

$$y_{m_{j+1}} - \underline{x}_0 \frac{x_{m_{j+1}}}{\underline{x}_{m_{j+1}}} = 1 + x_{m_{j+1}} - \underline{x}_0 \frac{x_{m_{j+1}}}{\underline{x}_{m_{j+1}}} = 1 + \frac{x_{m_{j+1}} \underline{x}_{m_{j+1}} - \underline{x}_0 x_{m_{j+1}}}{\underline{x}_{m_{j+1}}}$$
$$= 1 + \frac{\underline{x}_0 (x_{m_{j+1}} - \underline{x}_{m_{j+1}}) - \underline{x}_0 x_{m_{j+1}}}{\underline{x}_{m_{j+1}}} = 1 - \underline{x}_0 \,. \tag{5.36}$$

Thus, we have evalueated the necessary sum with the following result

$$\sum_{n=m_j}^{m_{j+1}} D(m_j n)\overline{D}(n m_{j+1}) F(v,n)$$
$$= (1 + \underline{x}_0) \frac{1}{x_{m_j}} y_{m_j} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} F(v,m_j) + (1 - \underline{x}_0) F(v,m_{j+1}) \,.$$

Note that this formula remains valid also for the limiting case when $m_{j+1} = m_j + 1$. Introducing the concise notation

$$\begin{aligned} X_j &= (1 + \underline{x}_0) \frac{1}{x_{m_j}} y_{m_j} y_{m_j+1} \cdots y_{m_{j+1}-1} x_{m_{j+1}} F(v,m_j) \,, \\ Y_{j+1} &= (1 - \underline{x}_0) F(v,m_{j+1}) \,. \end{aligned} \tag{5.37}$$

We can write the final sum together with unwanted terms subtracted in the schematic form

$$\sum_{\tau \in \mathfrak{S}_M} \mathcal{A}(\tau) \, \overbrace{\underbrace{(X_1 + Y_2)}_{v_{\tau(1)}} \underbrace{(X_2 + Y_3)}_{v_{\tau(2)}} \underbrace{(X_3 + Y_4)}_{v_{\tau(3)}} \ldots \underbrace{(X_M + Y_{M+1})}_{v_{\tau(M)}}} \,. \tag{5.38}$$

Here contractions connect the terms which product upon opening the brackets will contain two functions, $F(v_{\tau(j)}, m_{j+1}) F(v_{\tau(j+1)}, m_{j+1})$, with the same argument m_{j+1}. The idea now is to try

to choose the coefficient $\mathcal{A}(\tau)$ in such away as to cancel all F-functions with coincident arguments including those which come from unwanted terms, that is,

$$\mathcal{A}(\tau)\Big(Y_{j+1}X_{j+1}-\text{unwanted}\Big)\Big|_{\tau}+\mathcal{A}(\alpha_j\tau)\Big(Y_{j+1}X_{j+1}-\text{unwanted}\Big)\Big|_{\alpha_j\tau}=0\,. \tag{5.39}$$

Submitting here the expressions (5.37) and cancelling out the common multiplier

$$\frac{1}{x_{m_{j+1}}}y_{m_{j+1}}y_{m_{j+1}+1}\cdots y_{m_{j+2}-1}x_{m_{j+2}}\,,$$

we obtain the condition

$$\begin{aligned}\mathcal{A}(\tau)\Big[\big(1-\underline{x}_0(v_{\tau(j)})\big)\big(1+\underline{x}_0(v_{\tau(j+1)})\big)-1\Big]\\ +\mathcal{A}(\alpha_j\tau)\Big[\big(1-\underline{x}_0(v_{\tau(j+1)})\big)\big(1+\underline{x}_0(v_{\tau(j)})\big)-1\Big]=0\,,\end{aligned} \tag{5.40}$$

where we blued the subtracted unwanted terms. This yields an equation

$$\frac{\mathcal{A}(\alpha_j\tau)}{\mathcal{A}(\tau)}=\frac{v_{\tau(j)}-v_{\tau(j+1)}+i\kappa}{v_{\tau(j)}-v_{\tau(j+1)}-i\kappa}\,, \tag{5.41}$$

which has the unique up to an overall constant solution

$$\mathcal{A}(\tau)=\prod_{1\leqslant i<j\leqslant M}\frac{v_{\tau(i)}-v_{\tau(j)}-i\kappa}{v_{\tau(i)}-v_{\tau(j)}}\,. \tag{5.42}$$

Formula (5.41) gives a simple example of the *connection formulae* that we will meet in full generality in the next section when discussion the nested Bethe Ansatz construction.

Meanwhile, implementing (5.42), all contacted terms in (5.38) cancel[2] and the eigenvalue equation takes the form

$$\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)(X_1\dots X_M+Y_2\dots Y_{M+1})=y\Lambda c_{m_1\dots m_M}\,. \tag{5.43}$$

We find

$$\begin{aligned}X_1\dots X_M &= \prod_{i=1}^{M}\big(1+\underline{x}_0(v_{\tau(i)})\big)F(v_{\tau(i)},m_i)\\ &\times \frac{1}{x_{m_1}}y_{m_1}\dots y_{m_2-1}x_{m_2}\times\frac{1}{x_{m_2}}y_{m_2}\dots y_{m_3-1}x_{m_3}\times\\ \dots &\times \frac{1}{x_{m_M}}y_{m_M}y_{m_M+1}\dots y_N\underbrace{y_{N+1}\dots y_{m_1+N-1}x_{m_1+N}}_{y_1\dots y_{m_1-1}x_{m_1}}\\ &= \underbrace{y_1\dots y_N}_{y}\prod_{i=1}^{M}\big(1+\underline{x}_0(v_i)\big)\prod_{i=1}^{M}F(v_{\tau(i)},m_i)\,,\end{aligned}$$

where the the product of factors $1+x_0$ does not actually depend on permutation τ. Hence, the eigenvalue equation becomes

$$y\prod_{i=1}^{M}\big(1+\underline{x}_0(v_i)\big)\Bigg[\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\prod_{i=1}^{M}F(v_{\tau(i)},m_i)\Bigg]$$

[2]For cancellation of $X_1Y_{M+1}-\{\text{unwanted}\}$ one needs to use the Bethe-Yang equations, as will be discussed in a moment.

$$+\prod_{i=1}^{M}\big(1-\underline{x}_0(v_i)\big)\underbrace{\Big[\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\prod_{i=2}^{M+1}F(v_{\tau(i)},m_i)\Big]}_{c_{m_2,\dots,m_{M+1}}}=y\Lambda c_{m_1\dots m_M}\ .$$

Due to the cyclic property of the coefficients c, we observe that the eigenstate equation is satisfied and we read off the eigenvalue

$$\Lambda=\prod_{i=1}^{M}\big(1+\underline{x}_0(v_i)\big)+\frac{1}{y_1\dots y_N}\prod_{i=1}^{M}\big(1-\underline{x}_0(v_i)\big)\,. \tag{5.44}$$

Substituting here the functions $\underline{x}_0$ and y_j, we finally get

$$\Lambda(p)=\prod_{i=1}^{M}\frac{p-v_i+\kappa}{p-v_i-\kappa}+\prod_{j=1}^{N}\frac{p-p_j}{p-p_j-i\kappa}\prod_{k=1}^{M}\frac{p-v_i-\frac{3i\kappa}{2}}{p-v_i-\kappa}\,. \tag{5.45}$$

This is an eigenvalue of the transfer matrix $T(p)$ for arbitrary p. From this expression we immediately see that $\Lambda(p_j)=\Lambda_j$.

Cyclicity condition and Bethe-Yang equations. To carry out the diagonalisation procedure, we heavily used the condition of cyclic invariance (5.25) of the coefficients (4.39). Let us now show that this condition leads to the Bethe-Yang equations (4.41). According to (4.39) we should have

$$\begin{aligned}c_{n_2\dots n_{M+1}}&=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)F(v_{\tau(1)},n_2)\dots F(v_{\tau(M-1)},n_M)F(v_{\tau(M)},n_{M+1})\\&=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\xi\tau)F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)F(v_{\tau(1)},n_{M+1})\,.\end{aligned}$$

Here we made a change of summation variable $\tau\to\xi\tau$, where

$$\xi=\alpha_{M-1}\dots\alpha_1$$

is the cyclic permutation of the set $(12\dots M)$. The cyclic invariance requires the coefficient above to be equal to

$$c_{n_1\dots n_M}=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)F(v_{\tau(1)},n_1)F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)\,,$$

which is only possible if

$$\frac{\mathcal{A}(\tau)}{\mathcal{A}(\xi\tau)}=\frac{F(v_{\tau(1)},n_{M+1})}{F(v_{\tau(1)},n_1)}\,. \tag{5.46}$$

Using the definition (4.63) and $n_{M+1}=n_1+N$, and assuming momenta p_j to be periodic, $p_{j+N}=p_j$, we obtain that

$$\frac{F(v_{\tau(1)},n_{M+1})}{F(v_{\tau(1)},n_1)}=\prod_{j=1}^{N}\underline{y}_j(v_{\tau(1)})$$

and this expression is independent on n_1. Finally, from the explicit solution (5.42), we find

$$\frac{\mathcal{A}(\tau)}{\mathcal{A}(\xi\tau)}=\prod_{l\neq1}^{M}\frac{v_{\tau(1)}-v_{\tau(l)}-i\kappa}{v_{\tau(1)}-v_{\tau(l)}+i\kappa}\,.$$

It remains to choose $\tau=\alpha_{1k}$, so that $v_{\tau(1)}=v_k$ and equations (5.46) reduce to

$$\prod_{l\neq k}^{M}\frac{v_k-v_l-i\kappa}{v_k-v_l+i\kappa}=\prod_{\beta=1}^{N}\underline{y}_j(v_k)\,, \tag{5.47}$$

5.3 Fock condition

For any τ the vector $\Phi(\tau)$ must be an element of the the permutation module $M^{[N-M,M]}$ and it expanded over the basis $|n_1,\dots,n_M\rangle$ of C_N^M elements, which in terms of the Young tabloids are

(5.48)

To show that $|\Phi\rangle$ has an irreducible symmetry type $[N-M,M]$, we have to show that we cannot symmetrise in more than $N-M$ variables, which amounts to the following *Fock condition*

$$\mathrm{Sym}_{\bullet\bullet}|\Phi\rangle = 0\,, \tag{5.49}$$

i.e. picking, for instance, a spin down visualised as red particle at position n_1, we symmetrise its position with positions of all black particles (spins up). This leaves us with the following coefficient in front of the corresponding symmetrised term

$$\mathcal{H} = \sum_{n=1}^{n_2-1} c_{nn_2\dots n_M} + \sum_{n=n_2+1}^{n_3-1} c_{n_2nn_3\dots n_M} + \sum_{n=n_3+1}^{n_4-1} c_{n_2n_3nn_4\dots n_M} + \dots + \sum_{n=n_M+1}^{N} c_{n_2n_3\dots n_Mn}\,,$$

where $c_{n_1\dots n_M}$ is given by (4.39).

To evaluate this sum, we first compute

$$\begin{aligned}\sum_a^b F(v,n) &= \underline{y}_1\dots\underline{y}_{a-1}(\underline{x}_a + \underline{y}_a\underline{x}_{a+1} + \dots + \underline{y}_a\dots\underline{y}_{b-1}\underline{x}_b)\\ &= \underline{y}_1\dots\underline{y}_{a-1}(\underline{y}_a\dots\underline{y}_{b-1}\underline{y}_b - 1) = \frac{1}{\underline{x}_{b+1}}F(v,b+1) - \frac{\underline{y}_{a-1}}{\underline{x}_{a-1}}F(v,a-1)\,.\end{aligned}$$

With this formula at hand we then find[3]

$$\begin{aligned}\mathcal{H} &= \sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\left(\frac{F(v_{\tau(1)},n_2)}{\underline{x}_{n_2}(v_{\tau(1)})} - \frac{F(v_{\tau(1)},1)}{\underline{x}_1(v_{\tau(1)})}\right)F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)\\ &+ \sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)F(v_{\tau(1)},n_2)\left(\frac{F(v_{\tau(2)},n_3)}{\underline{x}_{n_3}(v_{\tau(2)})} - \frac{\underline{y}_{n_2}(v_{\tau(2)})F(v_{\tau(2)},n_2)}{\underline{x}_{n_2}(v_{\tau(2)})}\right)F(v_{\tau(3)},n_3)\dots F(v_{\tau(M)},n_M)\\ &+ \sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)F(v_{\tau(1)},n_2)F(v_{\tau(2)},n_3)\left(\frac{F(v_{\tau(3)},n_4)}{\underline{x}_{n_4}(v_{\tau(3)})} - \frac{\underline{y}_{n_3}(v_{\tau(3)})F(v_{\tau(3)},n_3)}{\underline{x}_{n_3}(v_{\tau(3)})}\right)\dots F(v_{\tau(M)},n_M)\\ &+ \dots\\ &+ \sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\left(\frac{F(v_{\tau(M)},N+1)}{\underline{x}_{N+1}(v_{\tau(M)})} - \frac{\underline{y}_{n_M}(v_{\tau(M)})F(v_{\tau(M)},n_M)}{\underline{x}_{n_M}(v_{\tau(M)})}\right)F(v_{\tau(1)},n_2)\dots F(v_{\tau(M-1)},n_M)\end{aligned}$$

To proceed, we recombine the terms according to the underlined patterns and get

$$\begin{aligned}\mathcal{H} &= \sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_2}(v_{\tau(2)}) - \underline{x}_{n_2}(v_{\tau(1)}) - \underline{x}_{n_2}(v_{\tau(1)})\underline{x}_{n_2}(v_{\tau(2)})\Big)\mathcal{A}(\tau)\\ &\qquad\times\frac{F(v_{\tau(1)},n_2)F(v_{\tau(2)},n_2)}{\underline{x}_{n_2}(v_{\tau(1)})\underline{x}_{n_2}(v_{\tau(2)})}\dots F(v_{\tau(M)},n_M)\\ &+ \sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_3}(v_{\tau(3)}) - \underline{x}_{n_3}(v_{\tau(2)}) - \underline{x}_{n_3}(v_{\tau(2)})\underline{x}_{n_3}(v_{\tau(3)})\Big)\mathcal{A}(\tau)\end{aligned}$$

[3]Here the term $\frac{F(v_{\tau(1)},1)}{\underline{x}_1(v_{\tau(1)})} = \frac{\underline{x}_1(v_{\tau(1)})}{\underline{x}_1(v_{\tau(1)})} = 1$.

$$\times F(v_{\tau(1)},n_2)\frac{F(v_{\tau(2)},n_3)F(v_{\tau(3)},n_3)}{\underline{x}_{n_3}(v_{\tau(2)})\underline{x}_{n_3}(v_{\tau(3)})}\dots F(v_{\tau(M)},n_M)$$
$$\dots$$
$$+\quad\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\frac{F(v_{\tau(M)},N+1)}{\underline{x}_{N+1}(v_{\tau(M)})}F(v_{\tau(1)},n_2)\dots F(v_{\tau(M-1)},n_M)$$
$$-\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\frac{F(v_{\tau(1)},1)}{\underline{x}_1(v_{\tau(1)})}F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)\,.$$

First we consider the term

$$W=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\frac{F(v_{\tau(M)},N+1)}{\underline{x}_{N+1}(v_{\tau(M)})}F(v_{\tau(1)},n_2)\dots F(v_{\tau(M-1)},n_M)$$

and perform a change of variables $\tau\to\xi\tau$ obtaining thereby

$$W=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\xi\tau)\frac{F(v_{\tau(1)},N+1)}{\underline{x}_1(v_{\tau(1)})}F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)\,.$$

Then, the Bethe-Yang equations

$$\frac{\mathcal{A}(\tau)}{\mathcal{A}(\xi\tau)}=\frac{F(v_{\tau(1)},N+1)}{F(v_{\tau(1)},1)}$$

allow one to replace $F(v_{\tau(1)},N+1)$ in favour of $F(v_{\tau(1)},1)$, so that

$$W=\sum_{\tau\in\mathfrak{S}_M}\mathcal{A}(\tau)\frac{F(v_{\tau(1)},1)}{\underline{x}_1(v_{\tau(1)})}F(v_{\tau(2)},n_2)\dots F(v_{\tau(M)},n_M)\,.$$

As the result the last two terms in $\mathcal{H}$ cancel and we are left with

$$\begin{aligned}\mathcal{H}\quad=\quad&\sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_2}(v_{\tau(2)})-\underline{x}_{n_2}(v_{\tau(1)})-\underline{x}_{n_2}(v_{\tau(1)})\underline{x}_{n_2}(v_{\tau(2)})\Big)\mathcal{A}(\tau)\\&\qquad\times\frac{F(v_{\tau(1)},n_2)F(v_{\tau(2)},n_2)}{\underline{x}_{n_2}(v_{\tau(1)})\underline{x}_{n_2}(v_{\tau(2)})}\dots F(v_{\tau(M)},n_M)\\+\quad&\sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_3}(v_{\tau(3)})-\underline{x}_{n_3}(v_{\tau(2)})-\underline{x}_{n_3}(v_{\tau(2)})\underline{x}_{n_3}(v_{\tau(3)})\Big)\mathcal{A}(\tau)\\&\qquad\times F(v_{\tau(1)},n_2)\frac{F(v_{\tau(2)},n_3)F(v_{\tau(3)},n_3)}{\underline{x}_{n_3}(v_{\tau(2)})\underline{x}_{n_3}(v_{\tau(3)})}\dots F(v_{\tau(M)},n_M)\\&\qquad\dots\\+\quad&\sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_M}(v_{\tau(M)})-\underline{x}_{n_M}(v_{\tau(M-1)})-\underline{x}_{n_M}(v_{\tau(M-1)})\underline{x}_{n_M}(v_{\tau(M)})\Big)\mathcal{A}(\tau)\\&\qquad\times F(v_{\tau(1)},n_2)\dots F(v_{\tau(M-2)},n_{M-1})\ \frac{F(v_{\tau(M-1)},n_M)F(v_{\tau(M)},n_M)}{\underline{x}_{n_M}(v_{\tau(M-1)})\underline{x}_{n_M}(v_{\tau(M)})}\,.\end{aligned}$$

At j's linr we will have the contribution of the form

$$\sum_{\tau\in\mathfrak{S}_M}\Big(\underline{x}_{n_{j+1}}(v_{\tau(j+1)})-\underline{x}_{n_{j+1}}(v_{\tau(j)})-\underline{x}_{n_{j+1}}(v_{\tau(j)})\underline{x}_{n_{j+1}}(v_{\tau(j+1)})\Big)\mathcal{A}(\tau)$$
$$\times F(v_{\tau(1)},n_2)\dots\frac{F(v_{\tau(j)},n_{j+1})F(v_{\tau(j+1)},n_{j+1})}{\underline{x}_{n_{j+1}}(v_{\tau(j)})\underline{x}_{n_{j+1}}(v_{\tau(j+1)})}\dots F(v_{\tau(M)},n_M)\,.$$

Since every permutation in the sum is accompanied by a permutation $\alpha_j\tau$, we will have the contribution in the form

$$\Big(\underline{x}_{n_{j+1}}(v_{\tau(j+1)})-\underline{x}_{n_{j+1}}(v_{\tau(j)})-\underline{x}_{n_{j+1}}(v_{\tau(j)})\underline{x}_{n_{j+1}}(v_{\tau(j+1)})\Big)\mathcal{A}(\tau)$$

$$+\Big(\underline{x}_{n_{j+1}}(v_{\tau(j)})-\underline{x}_{n_{j+1}}(v_{\tau(j+1)})-\underline{x}_{n_{j+1}}(v_{\tau(j)})\underline{x}_{n_{j+1}}(v_{\tau(j+1)})\Big)\mathcal{A}(\alpha_j\tau)\,,$$

which vanishes because it is nothing else but (5.40) where p is replaced with p_{n_j+1}. Thus, $\mathcal{H}=0$.

5.4 Algebraic Bethe Ansatz

After having understood the combinatorial and group-theoretical properties of the spin chain representation, we are facing the problem of diagonalising the transfer matrix (5.2) in this representation. This will be done with a special technique known under the name *Algebraic Bethe Ansatz*[4] which represents a far-reaching embodiment of the general Bethe Ansatz idea. Introducing the so-called R-matrix

$$R_{ab}(p_a,p_b)=S_{ab}^{-1}(p_a,p_b)=\frac{(p_a-p_b)\mathbb{1}-i\kappa\,\pi_{ab}}{p_a-p_b-i\kappa}\,, \tag{5.50}$$

we rewrite the commutation relations (5.4) between the entries of the monodromy matrix in the form

$$R_{ab}(p_a,p_b)T_a(p_a)T_b(p_b)=T_b(p_b)T_a(p_a)R_{ab}(p_a,p_b)\,, \tag{5.51}$$

where we recall that the indices a and b stand for two auxiliary spaces. The relation (5.51) is an essential starting point in the algebraic Bethe Ansatz approach to diagonalisation of the transfer matrix. The formulae (5.51) comprise the so-called *fundamental commutation relations*.

To proceed, we first point out that the monodromy (5.1) and the corresponding transfer matrix (5.2) are *inhomogeneous*, with $p_1,\dots,p_N$ playing the role of inhomogeneities. Second, using the explicit form (4.29) of the S-matrix, we obtain the following expression for the monodromy

$$T_a(p)=\Omega(p)\prod_{j=1}^{N}\Big((p-p_j)\mathbb{1}-i\kappa\,\pi_{ja}\Big)\,, \tag{5.52}$$

where

$$\Omega(p)=\prod_{j=1}^{N}\frac{1}{p-p_j-i\kappa} \tag{5.53}$$

is a scalar prefactor which was singled out for later convenience. Every term in the product (5.52) acts as a 2×2 matrix in the auxiliary space, which we can write down explicitly with the help of the local spin operators S_j^3 and $S_j^\pm$ as

$$L_{ja}(p)\equiv(p-p_j)\mathbb{1}-i\kappa\,\pi_{ja}=\begin{pmatrix} p-p_j-\frac{i\kappa}{2}-i\kappa S_j^3 & -i\kappa S_j^- \\ -i\kappa S_j^+ & p-p_j-\frac{i\kappa}{2}+i\kappa S_j^3\end{pmatrix}. \tag{5.54}$$

In the context of the Algebraic Bethe Ansatz, $L_{ja}(p)$ is called Lax operator. The definition of the Lax operator involves the local "quantum" space $V_j\simeq\mathbb{C}^2$. The Lax operator L_{ja} acts in $V_j\otimes V_a$:

$$L_{ja}(p):\qquad V_j\otimes V_a\to V_j\otimes V_a\,. \tag{5.55}$$

Taking the ordered product (5.52) over all sites of the chain, we obtain a realisation of the monodromy as the 2×2 matrix acting in the auxiliary space and we parametrise its entries as

$$T(p)=\begin{pmatrix} A(p) & B(p)\\ C(p) & D(p)\end{pmatrix}. \tag{5.56}$$

[4] See the review [7] in the annotated literature.

Here $A(p),\dots,D(p)$ are operators that act on the Hilbert space of the spin chain; they implicitly depend on the inhomogeneities $p_1,\dots,p_N$. The relations between these operators follow from the fundamental commutation relations (5.51) and those which we need here are

$$\begin{aligned} B(p)B(q) &= B(q)B(p)\,, \\ A(p)B(q) &= \frac{p-q+i\kappa}{p-q}B(q)A(p) - \frac{i\kappa}{p-q}B(p)A(q)\,, \\ D(p)B(q) &= \frac{p-q-i\kappa}{p-q}B(q)D(p) + \frac{i\kappa}{p-q}B(p)D(q)\,. \end{aligned} \tag{5.57}$$

The transfer matrix which we aim to diagonalise is given by the following operator

$$\tau(p) = \mathrm{Tr}_a(T_a) = A(p) + D(p)\,. \tag{5.58}$$

The main idea of the algebraic Bethe Ansatz relies on the existence of a reference state $|0\rangle$, also called pseudo-vacuum, such that $C(p)|0\rangle = 0$ for any p and the eigenvectors of $\tau(p)$ with M spins down have the form

$$|\lambda_1,\lambda_2,\cdots,\lambda_M\rangle = B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle\,, \tag{5.59}$$

where $\{\lambda_i\}$ are unequal numbers called *Bethe roots*. In our present case the pseudo-vacuum can be naturally identified with the state

$$|0\rangle = \bigotimes_{n=1}^{N} |\uparrow_n\rangle\,, \tag{5.60}$$

that is the unique state with all spins up. Indeed, since (5.54) acts on this state as

$$L_{ja}(p)|\uparrow_n\rangle = \begin{pmatrix} (p-p_j-i\kappa)|\uparrow_n\rangle & i\kappa|\downarrow_n\rangle \\ 0 & (p-p_j)|\uparrow_n\rangle \end{pmatrix},$$

we find that

$$T(p)|0\rangle = \Omega(p)\begin{pmatrix} \prod_{j=1}^{N}(p-p_j-i\kappa)|0\rangle & \bigstar \\ 0 & \prod_{j=1}^{N}(p-p_j)|0\rangle \end{pmatrix},$$

where $\bigstar$ stands for terms whose explicit form is irrelevant for our further treatment. Taking into account (5.53), we thus have

$$C(p)|0\rangle = 0\,, \qquad A(p)|0\rangle = |0\rangle\,, \qquad D(p)|0\rangle = \prod_{j=1}^{N}\frac{p-p_j}{p-p_j-i\kappa}|0\rangle\,, \tag{5.61}$$

where the first relation confirms the status of $|0\rangle$ as the pseudo-vacuum. The second two relations show that $|0\rangle$ is an eigenstate of the transfer matrix

$$\tau(p)|0\rangle = \left[1 + \prod_{j=1}^{N}\frac{p-p_j}{p-p_j-i\kappa}\right]|0\rangle\,.$$

Excited states are then obtained by multiple application of the "raising operator" B to the vacuum, in accordance with the formula (5.59). These states will form the eigenstates of the transfer matrix, provided the Bethe roots $\{\lambda_i\}$ satisfy certain restrictions, which we are going to determine.

An explicit computation done for small M indicates that the result of acting with $A(p)$ on the state (5.59) should have the following structure

$$A(p)B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle = \left(\prod_{n=1}^{M}\frac{p-\lambda_n+i\kappa}{p-\lambda_n}\right)B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle + \sum_{n=1}^{M} W_n^A(p,\{\lambda_i\})B(p)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle\,.$$

Here the coefficients $W_n^A(p,\{\lambda_i\})$ depend on p and the set of Bethe roots $\{\lambda_i\}_{i=1}^M$. To determine these coefficients we note that since the operators $B(\lambda)$ commute with each other at different values of λ, we can write

$$|\lambda_1,\lambda_2,\cdots,\lambda_M\rangle = B(\lambda_n)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle\,.$$

Thus,

$$\begin{aligned} A(p)|\lambda_1,\lambda_2,\cdots,\lambda_M\rangle &= \frac{p-\lambda_n+i\kappa}{p-\lambda_n}B(\lambda_n)A(p)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle \\ &\quad - \frac{i\kappa}{p-\lambda_n}B(p)A(\lambda_n)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle\,. \end{aligned}$$

It is clear from this equation that only the second term on its right hand side will contribute to W_n^A since this term does not contain $B(\lambda_n)$. On the other hand, moving in this term $A(\lambda_n)$ past the string of $B(\lambda_j)$, we see that the only way to avoid the appearance of $B(\lambda_n)$ is to restrict an application of the commutation relation (5.57) to the first term on its right hand side. With this restricted application, we pull the operator $A(\lambda_n)$ through all $B(\lambda_j)$ close to the pseudo-vacuum and, taking into account the second equation in (5.61), find the following contribution

$$-\frac{i\kappa}{p-\lambda_n}\prod_{\substack{i=1\\ i\neq n}}^{M}\frac{\lambda_n-\lambda_i+i\kappa}{\lambda_n-\lambda_i}B(p)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle\,,$$

from which we read off the coefficient W_n^A

$$W_n^A(p,\{\lambda_i\}) = -\frac{i\kappa}{p-\lambda_n}\prod_{\substack{j=1\\ j\neq n}}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\,.$$

We should point out that this expression for W_n^A is a non-trivial result that comes from cancelling many individual terms arising upon the use of the full commutation relation (5.57). In the same way we obtain

$$\begin{aligned} &D(p)B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle = \\ &\quad = \left(\prod_{j=1}^{N}\frac{p-p_j}{p-p_j-i\kappa}\right)\left(\prod_{n=1}^{M}\frac{p-\lambda_n-i\kappa}{p-\lambda_n}\right)B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle \\ &\qquad + \sum_{n=1}^{M} W_n^D(p,\{\lambda_i\})B(p)\prod_{\substack{j=1\\ j\neq n}}^{M} B(\lambda_j)|0\rangle\,, \end{aligned}$$

where

$$W_n^D(p,\{\lambda_i\}) = \frac{i\kappa}{p-\lambda_n}\prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}\prod_{\substack{j=1\\ j\neq n}}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\,.$$

Thus, we will solve the eigenvalue problem

$$\tau(p)|\lambda_1,\cdots,\lambda_M\rangle = \Lambda(p,\{\lambda_n\})|\lambda_1,\cdots,\lambda_M\rangle$$

with

$$\Lambda(p,\{\lambda_n\}) = \prod_{n=1}^{M}\frac{p-\lambda_n+i\kappa}{p-\lambda_n} + \prod_{j=1}^{N}\frac{p-p_j}{p-p_j-i\kappa}\prod_{n=1}^{M}\frac{p-\lambda_n-i\kappa}{p-\lambda_n}\,,$$

provided $W_n^A + W_n^D = 0$ for all n, which means that

$$\prod_{\substack{j=1\\ j\neq n}}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j} = \prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}\prod_{\substack{j=1\\ j\neq n}}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\,. \tag{5.62}$$

Making a uniform shift of all Bethe roots $\lambda_n \to \lambda_n + \frac{i\kappa}{2}$, we rewrite the above equations as

$$\prod_{j=1}^{N}\frac{p_j-\lambda_n+\frac{i\kappa}{2}}{p_j-\lambda_n-\frac{i\kappa}{2}} = \prod_{j\neq n}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j+i\kappa}\,. \tag{5.63}$$

These are the *Bethe equations*. Their solutions for the set $\{\lambda_j\}_{j=1}^M$ enumerate the eigenstates of the transfer matrix.

Solution of Yang's spin-$\frac{1}{2}$ problem. Thus, Yang's fermion spin-$\frac{1}{2}$ problem reduces to the following set of equations

$$e^{ip_jL} = \prod_{n=1}^{M}\frac{p_j-\lambda_n+\frac{i\kappa}{2}}{p_j-\lambda_n-\frac{i\kappa}{2}}\,, \qquad j=1,\ldots,N\,, \tag{5.64}$$

$$\prod_{j=1}^{N}\frac{p_j-\lambda_n+\frac{i\kappa}{2}}{p_j-\lambda_n-\frac{i\kappa}{2}} = \prod_{j\neq n}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j+i\kappa}\,. \qquad n=1,\ldots,M\,. \tag{5.65}$$

Here (5.64) are the equations (5.8) for the eigenvalues of the transfer matrix and they express the periodicity condition for the coordinate Bethe wave function. In the present context the variables p_j are called *momentum carrying roots*, while the variables λ_n are usually referred to as *auxiliary roots*. We see that equations (5.65) for auxiliary roots are algebraic and they involve momentum carrying roots as parameters. Once the momenta p_j are found the energy of the state described by the corresponding Bethe wave function is determined as

$$E = \frac{1}{2}\sum_{j=1}^{N}p_j^2\,. \tag{5.66}$$

The system of equations (5.64), (5.8) is an example of the so-called *nested Bethe Ansatz*, with one level of nesting given by equations for auxiliary roots. The term *nesting* originates from the hierarchical way of applying the Bethe Ansatz technique for diagonalising an auxiliary spin chain with spins transforming in an arbitrary irreducible representation of the symmetric group $\mathfrak{S}_N$.

To make a connection of our findings with the representation theory of $\mathfrak{sl}(2)\times\mathfrak{S}_N$, let us rewrite the fundamental commutation relations (5.51) in the form

$$R_{ab}(\lambda-\mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda-\mu)\,, \tag{5.67}$$

where the momentum variables p_a, p_b were replaced by λ and μ which play the role of *spectral parameters*. We study the behaviour of (5.67) in the limit $\mu \to \infty$. From (5.54) we obtain that in this limit the monodromy expands as

$$T_b(\mu) = \mathbb{1} + \frac{i\kappa}{\mu}\left(\frac{N}{2} - \sum_{j=1}^{N} S_j^\alpha \otimes \sigma_b^\alpha\right) + \dots\,, \tag{5.68}$$

where we recognised in the sum $\sum_{j=1}^{N} S_j^\alpha$ the generator (5.10) of the global $\mathfrak{sl}(2)$ algebra and σ_b^α denotes the corresponding Pauli matrix acting in the auxiliary space. Then the relation (5.67) expands as

$$\begin{aligned}&\Big((\lambda-\mu)\mathbb{1} - i\kappa\,\pi_{ab}\Big)T_a(\lambda)\left(\mathbb{1} + \frac{i\kappa}{\mu}\Big(\tfrac{N}{2} - S^\alpha\otimes\sigma^\alpha\Big) + \dots\right) = \\ &= \left(\mathbb{1} + \frac{i\kappa}{\mu}\Big(\frac{N}{2} - S^\alpha\otimes\sigma^\alpha\Big) + \dots\right)T_a(\lambda)\Big((\lambda-\mu)\mathbb{1} - i\kappa\,\pi_{ab}\Big)\,.\end{aligned}$$

Here the leading term in the large μ expansion cancels out and the subleading contribution yields the relation

$$[\pi_{ab}, T_a(\lambda)] + [S^\alpha, T_a(\lambda)]\otimes\sigma^\alpha = 0\,. \tag{5.69}$$

Writing π_{ab} via Pauli matrices, we conclude that (5.69) implies the fulfilment of the following relation

$$[S^\alpha, T_a(\lambda)] = [T_a(\lambda), \tfrac{1}{2}\sigma_a^\alpha]\,. \tag{5.70}$$

The spin operator S^α acts on the Hilbert space $\mathscr{H}$ of the spin chain. On the left hand side of (5.70) one has the commutator of this operator with each entry of the 2×2 monodromy matrix (5.56), the latter being also operators on $\mathscr{H}$. On the right hand side, one finds a matrix commutator of the monodromy matrix with the corresponding Pauli matrix in the auxiliary space. Thus, equation (5.70) is equivalent to three distinct equations

$$\begin{aligned}[S^3, T_a(\lambda)] &= \frac{1}{2}[T_a(\lambda), \sigma_a^3] = \begin{pmatrix} 0 & -B(\lambda) \\ C(\lambda) & 0 \end{pmatrix}, \\ [S^+, T_a(\lambda)] &= [T_a(\lambda), \sigma_a^+] = \begin{pmatrix} -C(\lambda) & A(\lambda)-D(\lambda) \\ 0 & C(\lambda) \end{pmatrix}\end{aligned}$$

and

$$[S^-, T_a(\lambda)] = [T_a(\lambda), \sigma_a^-] = \begin{pmatrix} B(\lambda) & 0 \\ D(\lambda)-A(\lambda) & -B(\lambda) \end{pmatrix}.$$

Essentially, we need the following commutation relations

$$[S^3, B] = -B\,, \qquad [S^+, B] = A - D\,. \tag{5.71}$$

The action of the symmetry generators on the pseudo-vacuum is

$$S^+|0\rangle = 0\,, \qquad S^3|0\rangle = \frac{N}{2}|0\rangle\,.$$

Therefore, the pseudo-vacuum is the highest weight state of the $\mathfrak{sl}(2)$ algebra. With the help of (5.71) we then compute

$$S^3|\lambda_1, \cdots, \lambda_M\rangle = \Big(\frac{N}{2} - M\Big)|\lambda_1, \cdots, \lambda_M\rangle$$

and

$$S^+|\lambda_1,\cdots,\lambda_M\rangle = \sum_j B(\lambda_1)\dots B(\lambda_{j-1})(A(\lambda_j)-D(\lambda_j))B(\lambda_{j+1})\dots B(\lambda_M)|0\rangle\,.$$

An attentive look at the last expression reveals that it can be re-expanded as

$$S^+|\lambda_1,\cdots,\lambda_M\rangle = \sum_{n=1}^{M} O_n B(\lambda_1)\dots B(\lambda_{n-1})\cancel{B(\lambda_n)}B(\lambda_{n+1})\dots B(\lambda_M)|0\rangle\,,$$

where the crossed out term does not appear in the sum. The coefficients O_n are unknown but they can be calculated by invoking the arguments similar to those used for computing W_n^A and W_n^D. The only contributions to O_n will come from

$$B(\lambda_1)\dots B(\lambda_{k-1})(A(\lambda_k)-D(\lambda_k))B(\lambda_{k+1})\dots B(\lambda_M)|0\rangle \qquad \text{with} \quad k\leqslant n.$$

If $k=n$ this contribution will be

$$\prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}-\prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}$$

and if $k<n$ the contribution will be

$$W_n^A(\lambda_k,\{\lambda\}_{k+1}^M)-W_n^D(\lambda_k,\{\lambda\}_{k+1}^M)\,,$$

where it is convenient to represent W_n^A and W_n^D in the following split form

$$\begin{aligned}
W_n^A(\lambda_k,\{\lambda\}_{k+1}^M) &= \frac{i\kappa}{\lambda_n-\lambda_k}\prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\prod_{j=k+1}^{n-1}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\,,\\
W_n^D(\lambda_k,\{\lambda\}_{k+1}^M) &= -\frac{i\kappa}{\lambda_n-\lambda_k}\prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\prod_{j=k+1}^{n-1}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}\,.
\end{aligned}$$

Thus, adding up, we obtain

$$\begin{aligned}
O_n &= \prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}+\sum_{k=1}^{n-1}W_n^A(\lambda_k,\{\lambda\}_{k+1}^M)\\
&- \prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}-\sum_{k=1}^{n-1}W_n^D(\lambda_k,\{\lambda\}_{k+1}^M)=\\
&= \prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\left(1+\sum_{k=1}^{n-1}\frac{i\kappa}{\lambda_n-\lambda_k}\prod_{j=k+1}^{n-1}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\right)\\
&- \prod_{j=n+1}^{M}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\prod_{j=1}^{N}\frac{\lambda_n-p_j}{\lambda_n-p_j-i\kappa}\left(1-\sum_{k=1}^{n-1}\frac{i\kappa}{\lambda_n-\lambda_j}\prod_{j=k+1}^{n-1}\frac{\lambda_n-\lambda_j-i\kappa}{\lambda_n-\lambda_j}\right).
\end{aligned}$$

To proceed, we note the following useful identity

$$t_m\equiv 1+\sum_{k=m}^{n-1}\frac{i\kappa}{\lambda_n-\lambda_k}\prod_{j=k+1}^{n-1}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}=\prod_{j=m}^{n-1}\frac{\lambda_n-\lambda_j+i\kappa}{\lambda_n-\lambda_j}\,. \tag{5.72}$$

We will prove this identity by induction over m. For $m=n-1$ and $m=n-2$ we have

$$t_{n-1} = 1+\frac{i\kappa}{\lambda_n-\lambda_{n-1}}=\frac{\lambda_n-\lambda_{n-1}+i\kappa}{\lambda_n-\lambda_{n-1}}\,,$$

$$t_{n-2} \quad = \quad 1 + \frac{i\kappa}{\lambda_n - \lambda_{n-1}} + \frac{i\kappa}{\lambda_n - \lambda_{n-2}} \frac{\lambda_n - \lambda_{n-1} + i\kappa}{\lambda_n - \lambda_{n-1}} = \frac{\lambda_n - \lambda_{n-1} + i\kappa}{\lambda_n - \lambda_{n-1}} \frac{\lambda_n - \lambda_{n-2} + i\kappa}{\lambda_n - \lambda_{n-2}} \,.$$

Now we suppose that the formula holds for $m = l$, then we have

$$t_{l-1} = t_l + \frac{i\kappa}{\lambda_n - \lambda_{l-1}} \prod_{j=l}^{n-1} \frac{\lambda_n - \lambda_j + i\kappa}{\lambda_n - \lambda_j} = \prod_{j=l-1}^{n-1} \frac{\lambda_n - \lambda_j + i\kappa}{\lambda_n - \lambda_j} \,,$$

which proves the identity. With formula (5.72) at hand we get

$$1 + \sum_{k=1}^{n-1} \frac{i\kappa}{\lambda_n - \lambda_k} \prod_{j=k+1}^{n-1} \frac{\lambda_n - \lambda_j + i\kappa}{\lambda_n - \lambda_j} = \prod_{j=1}^{n-1} \frac{\lambda_n - \lambda_j + i\kappa}{\lambda_n - \lambda_j} \,.$$

In the same way one can show that

$$1 - \sum_{k=1}^{n-1} \frac{i\kappa}{\lambda_n - \lambda_j} \prod_{j=k+1}^{n-1} \frac{\lambda_n - \lambda_j - i\kappa}{\lambda_n - \lambda_j} = \prod_{j=1}^{n-1} \frac{\lambda_n - \lambda_j - i\kappa}{\lambda_n - \lambda_j} \,.$$

This, we found for O_n the following answer

$$O_n = \prod_{\substack{j=1 \\ j\neq n}}^{M} \frac{\lambda_n - \lambda_j + i\kappa}{\lambda_n - \lambda_j} - \prod_{\substack{j=1 \\ j\neq n}}^{M} \frac{\lambda_n - \lambda_j - i\kappa}{\lambda_n - \lambda_j} \prod_{j=1}^{N} \frac{\lambda_n - p_j}{\lambda_n - p_j - i\kappa} \,.$$

It is quite remarkable that this expression is nothing else but the Bethe equation (5.62) for the root λ_n, and, therefore, if the Bethe equations are satisfied all the coefficients O_n vanish. This proves that the eigenstates of the transfer matrix are annihilated by the raising operator S^+, *i.e.* they are the highest weight vectors of the spin algebra and, for this reason, belong to the representation $[N-M, M]$ of $\mathfrak{S}_N$. For a given M, the number of distinct solutions[5] $\{\lambda_i\}$ of (5.62) with no two λ coincident is equal to $\dim[N-M, M]$. Showing this is, however, non-trivial and represents a variant of the so-called *completeness problem* for the Bethe Ansatz.

[5] That is the solutions which are not related to each other by permutations of some Bethe roots.

Appendix

5.5 Symmetric group $\mathfrak{S}_N$

The symmetric group $\mathfrak{S}_N$ is defined as the set of all one-to-one mappings of the set of numbers $\{1, \ldots, N\}$ to itself. Every element $\sigma \in \mathfrak{S}_N$, called permutation, can be written in two-line notation as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & N \\ \sigma(1) & \sigma(2) & & \sigma(N) \end{pmatrix},$$

meaning that σ maps 1 to $\sigma(1)$, 2 to $\sigma(2)$ and so on. Since in two-line notation the top line is fixed, one can drop it obtaining a one-line notation. The product $\sigma\tau$ of two permutations σ and τ is constructed as follows. In the τ-string one takes an integer standing in the position $\sigma(j)$ and moves it in the position j of the product $\sigma\tau$. For instance, given two permutations $\sigma = (132)$ and $\tau = (213)$ in $\mathfrak{S}_3$, written in one-line notation, one has for their product[6]

$$(132)(213) = (231)\,.$$

In the two-line notation the same product looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

As the flow of indices shows, this result means that $(\sigma\tau)(j) = \tau(\sigma(j))$, that is it corresponds to the application to the index j of σ followed by τ.

Next, we need a transposition α_{ij} that interchanges the positions of i and j and leaves all the other elements unchanged

$$\alpha_{ij} \equiv (i|j) = \begin{pmatrix} 1 & \cdots & i & \cdots & j & \cdots & N \\ 1 & & j & & i & & N \end{pmatrix}.$$

Multiplication of σ by α_{ij} from the right exchanges the positions of $\sigma(i)$ and $\sigma(j)$. Thus,

$$\alpha_{ij}\sigma = (\sigma(1), \ldots, \sigma(j), \ldots, \sigma(i), \ldots, \sigma(N))\,.$$

Transpositions satisfy $\alpha_{ij}^2 = e$, where e is the identity, and the relations

$$\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{ij}\,, \quad i \neq j\,. \tag{5.73}$$

[6]To display a permutation σ in the one-line notation, we confine the corresponding sequence of numbers from the set $\{1, 2, \ldots, N\}$ within the brackets (...). This notation should not be confused with the one used to represent σ via its *cycles*. In this book we never use cycle notation.

Every permutation can be expressed as a product of transpositions, albeit not in a unique way. For a given permutation $\sigma \in \mathfrak{S}_N$ the number of transpositions in its decomposition is always either even or odd. This allows to define the *signature function*, also called *parity*, on $\mathfrak{S}_N$: $\mathrm{sign}(\sigma) = 1$, if σ is given by an even number of transpositions, and $\mathrm{sign}(\sigma) = -1$, if the corresponding number is odd. The sign function has the properties

$$\mathrm{sign}(\sigma\tau) = \mathrm{sign}(\sigma)\mathrm{sign}(\tau)\,, \qquad \mathrm{sign}(\sigma^{-1}) = \mathrm{sign}(\sigma)\,.$$

Importantly, the group $\mathfrak{S}_N$ is generated by simple transpositions $\alpha_j \equiv \alpha_{jj+1}$, $i = 1, \ldots, N-1$, subject to the Coxeter relations

$$\begin{aligned} &\alpha_j^2 = e\,, && 1 \leqslant j \leqslant N-1\,, \\ &\alpha_j\alpha_{j+1}\alpha_j = \alpha_{j+1}\alpha_j\alpha_{j+1}\,, && 1 \leqslant j \leqslant N-2\,, \\ &\alpha_i\alpha_j = \alpha_j\alpha_i\,, && 1 \leqslant i,j \leqslant N-1 \ \text{ and } \ |i-j| \geqslant 2\,. \end{aligned} \tag{5.74}$$

It is useful to have in mind that the invariance of the euclidean scalar product under the action of $\sigma, \tau \in \mathfrak{S}_N$ implies that

$$\sum_j q_{\sigma(j)} p_{\tau(j)} = \sum_j q_j p_{(\sigma^{-1}\tau)(j)} = \sum_j q_{(\tau^{-1}\sigma)(j)} p_j\,.$$

Let $\mathrm{Fun}(\mathfrak{S}_N)$ be the algebra of functions on $\mathfrak{S}_N$. The left π and the right π' regular representations of $\mathfrak{S}_N$ are defined as

$$\pi(\sigma_0)\mathcal{A}(\sigma) = \mathcal{A}(\sigma_0^{-1}\sigma)\,, \tag{5.75}$$

$$\pi'(\sigma_0)\mathcal{A}(\sigma) = \mathcal{A}(\sigma\sigma_0)\,, \tag{5.76}$$

for $\mathcal{A} \in \mathrm{Fun}(\mathfrak{S}_N)$. The left and right regular representations are equivalent: $\pi'\mathfrak{I} = \mathfrak{I}\pi$, where an intertwining operator $\mathfrak{I}$ acts as $(\mathfrak{I}\mathcal{A})(\sigma) = \mathcal{A}(\sigma^{-1})$.

The left (right) regular representation is decomposed into a sum of irreducible representations according to

$$\pi \simeq \bigoplus_\lambda \dim \pi_\lambda \cdot \pi_\lambda\,, \tag{5.77}$$

where π_λ is the irreducible representation of $\mathfrak{S}_N$ corresponding to a partition λ of N and $\dim \pi_\lambda$ is the dimension of this representation. The sum runs over all partitions of N. According to (5.77), the multiplicity with which a representation π_λ appears in the decomposition of the regular representation is equal to $\dim \pi_\lambda$. This dimension is given by the determinant formula

$$\dim \pi_\lambda = N!\, \det_{l\times l} \frac{1}{(\lambda_i - i + j)!}\,, \tag{5.78}$$

where $i, j = 1, \ldots, l$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$ is the associated partition or Young diagram.

Example. As an illustrative example for later use, we consider the decomposition (5.77) for $\mathfrak{S}_3$. There are 6 permutations which we enumerate as

$$\sigma_1 = (123) = e\,,\ \sigma_2 = (213)\,,\ \sigma_3 = (231)\,,\ \sigma_4 = (321)\,,\ \sigma_5 = (312)\,,\ \sigma_6 = (132)\,. \tag{5.79}$$

The corresponding multiplication table $\sigma_i\sigma_j$, where i and j enumerate its rows and columns, respec-

tively, looks as

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_2	σ_2	σ_1	σ_4	σ_3	σ_6	σ_5
σ_3	σ_3	σ_6	σ_5	σ_2	σ_1	σ_4
σ_4	σ_4	σ_5	σ_6	σ_1	σ_2	σ_3
σ_5	σ_5	σ_4	σ_1	σ_6	σ_3	σ_2
σ_6	σ_6	σ_3	σ_2	σ_5	σ_4	σ_1

In the basis $\mathbf{g}_i \equiv \mathcal{A}(\sigma_i)$ the representation π is realised by the following 6×6 real orthogonal matrices

$$\pi(\sigma_1) = \mathbb{1}\,, \qquad \pi(\sigma_2) = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \quad \pi(\sigma_3) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix},$$

$$\pi(\sigma_4) = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \quad \pi(\sigma_5) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \quad \pi(\sigma_6) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0 \end{pmatrix},$$

and, hence, this representation is unitary. Transpositions are σ_2, σ_4 and σ_6. They are realised by symmetric matrices. In particular, the simple transpositions are $\alpha_1 = \sigma_2$ and $\alpha_2 = \sigma_6$. The operator $\mathfrak{I}$ that intertwines π' and π, $\mathfrak{I}\,\pi' = \pi\,\mathfrak{I}$, is

$$\mathfrak{I} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad \mathfrak{I}^2 = \mathbb{1}\,. \tag{5.80}$$

The matrix realisation of π' on the same basis is

$$\pi'(\sigma_1) = \mathbb{1}\,, \qquad \pi'(\sigma_2) = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \quad \pi'(\sigma_3) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix},$$

$$\pi'(\sigma_4) = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \quad \pi'(\sigma_5) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \quad \pi'(\sigma_6) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix},$$

The representation π has three irreducible components: $\lambda = [3]$, $\lambda = [2,1]$ and $\lambda = [1,1,1]$. The first is a trivial (symmetric) representation, the last is a one-dimensional anti-symmetric one. The representation $[2,1]$ is the two-dimensional standard (defining) representation and it occurs with multiplicity 2. The decomposition (5.77) is obtained by performing a similarity transformation

$\pi \to T\pi T^{-1}$, where T is the following unitary matrix

$$T = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ -1 & 1 & -e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & -e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & -1 & e^{\frac{2\pi i}{3}} & -e^{-\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & -e^{\frac{2\pi i}{3}} \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Under the action of T the basis $\mathbf{g} = \{\mathbf{g}_i\}$ of π transforms into

$$T\mathbf{g} = \frac{1}{\sqrt{6}} \begin{pmatrix} \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3 + \mathbf{g}_4 + \mathbf{g}_5 + \mathbf{g}_6 \\ \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}^{-\frac{2\pi i}{3}}(\mathbf{g}_3 + \mathbf{g}_6) + e^{\frac{2\pi i}{3}}(\mathbf{g}_4 + \mathbf{g}_5) \\ \mathbf{g}_1 + \mathbf{g}_2 + e^{-\frac{2\pi i}{3}}(\mathbf{g}_4 + \mathbf{g}_5) + e^{\frac{2\pi i}{3}}(\mathbf{g}_3 + \mathbf{g}_6) \\ -\mathbf{g}_1 + \mathbf{g}_2 - e^{-\frac{2\pi i}{3}}(\mathbf{g}_3 - \mathbf{g}_6) + e^{\frac{2\pi i}{3}}(\mathbf{g}_4 - \mathbf{g}_5) \\ \mathbf{g}_1 - \mathbf{g}_2 - e^{-\frac{2\pi i}{3}}(\mathbf{g}_4 - \mathbf{g}_5) + e^{\frac{2\pi i}{3}}(\mathbf{g}_3 - \mathbf{g}_6) \\ \mathbf{g}_1 - \mathbf{g}_2 + \mathbf{g}_3 - \mathbf{g}_4 + \mathbf{g}_5 - \mathbf{g}_6 \end{pmatrix} \equiv \frac{1}{\sqrt{6}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}.$$

In this basis the matrices of $T\pi T^{-1}$ take a block-diagonal form which corresponds to the decomposition

$$T\pi T^{-1} = \pi_{[3]} \oplus \pi_{[2,1]} \oplus \pi_{[2,1]} \oplus \pi_{[1,1,1]}\,. \tag{5.81}$$

In particular, v_1 is a projection on the invariant subspace of the trivial representation of $\pi_{[3]}$ and v_6 plays a similar role for the anti-symmetric one-dimensional representation $\pi_{[1,1,1]}$. Analogously, (v_2, v_3) and (v_4, v_5) are invariant subspaces for the two-dimensional representations $\pi_{[2,1]}$. This example shows a clear pattern of how the tensor product decomposition of the regular representation looks like. In particular, for the general case of $\mathfrak{S}_N$, a decomposition of π will always contain trivial and anti-symmetric representations.

5.6 Some facts on representations of $\mathfrak{S}_N$

Let $\lambda = [\lambda_1, \ldots, \lambda_l]$ be a partition on N. A *Young tableau of shape* λ is an array obtained by replacing boxes of the Young diagram with the numbers $1, 2, \ldots, N$ bijectively. A *Young tabloid of shape* λ is defined as an equivalence class of row-equivalent tableaux. A tabloid is denoted in the following way

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \tag{5.82}$$

This notation means that we do not distinguish between tableaux which differ from each other by permutations of $1, 2, 3$ and $4, 5$. The number of different tableaux is a given equivalence glass corresponding to a single tabloid is $\lambda_1! \ldots \lambda_l!$ and, therefore, the number of λ-tabloids is $N!/(\lambda_1! \ldots \lambda_l!)$.

Permutation modules M^λ. The symmetric group acts on λ-tabloids in well-defined way, which for any λ defines an associated representation of dimension

$$\dim M^\lambda = \frac{N!}{\lambda_1! \ldots \lambda_l!}\,. \tag{5.83}$$

Elements of $\mathfrak{S}_N$ act on λ-tabloids by permuting their elements. For instance, if $\pi \in \mathfrak{S}_6$, then its action on the tabloid (5.82) gives another tabloid

$$\begin{array}{|c|c|c|}\hline \pi(1) & \pi(2) & \pi(3) \\ \hline \pi(4) & \pi(5) \\ \cline{1-2} \pi(6) \\ \cline{1-1}\end{array}$$

The M^λ is called the permutation module corresponding to λ. The importance of M^λ is that it contains a unique irreducible module S^λ called the Specht module. There are three basic permutation modules. The first one corresponds to the diagram $\lambda = [N]$

$$M^{[n]} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & N \\ \hline\end{array}$$

with the trivial action of $\mathfrak{S}_N$. Next, consider $[1^N]$. Each equivalence class consists of a single tableau, each tableau can be identified with a permutation itself, and therefore, the module is isomorphic to $\mathfrak{S}_N$ itself: $M^{[1^N]} \simeq \mathbb{C}\mathfrak{S}_N$, Thus, this module realises the regular representation of $\mathfrak{S}_N$. Finally, if $\lambda = [N-1, 1]$, each tabloid is identified by the element standing in the second row, so that the basis of $M^{[N-1,1]}$ contains precisely N-elements. This is the so-called *defining representation* of $\mathfrak{S}_N$ that is realised in a vector space of $\dim V = N$. In the following we will be interested in representations corresponding to the Young diagrams $[N-M, M]$. The dimension of the corresponding permutation module $M^{[N-M,M]}$ is

$$\dim M^{[N-M,M]} = \frac{N!}{(N-M)!M!} = C_N^M \,. \tag{5.84}$$

The states of this permutation module can be realised as spin configurations with M spins down of a closed spin-$\frac{1}{2}$ chain. Later on, when discussion the construction of the algebraic Bethe Ansatz, we will use all advantages of this physical interpretation of the permutation module $M^{[N-M,M]}$.

Specht modules S^λ. Specht modules provide an explicit realisation of all irreducible representations of the symmetric group, the latter are in one-to-one correspondence with Young diagrams. Suppose that the tableau t has rows $R_1, \dots, R_t$ and columns $C_1, \dots, C_k$. Define

$$R_t = \mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \dots \times \mathfrak{S}_{R_t} \tag{5.85}$$

and

$$C_t = \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \dots \times \mathfrak{S}_{C_k} \,, \tag{5.86}$$

which are the row-stabiliser and column-stabiliser of t, respectively. Note that the equivalence classes which at the same time are tabloids can be written as

$$\{t\} = R_t t = \sum_{\sigma \in R_t} \sigma t$$

Define

$$\kappa_t = \sum_{\pi \in C_t} \mathrm{sign}(\pi)\pi \,. \tag{5.87}$$

Note that κ_t factorises as

$$\kappa_t = \kappa_{C_1} \kappa_{C_2} \cdots \kappa_{C_k} \,.$$

If t is a tableau, then we construct an associated *polytabloid* as

$$e_t = \kappa_t\{t\} \;=\; \sum_{\substack{\sigma\in R_t\\ \pi\in C_t}} \mathrm{sign}(\pi)\pi\sigma t. \tag{5.88}$$

Here the element

$$Y = \sum_{\substack{\sigma\in R_t\\ \pi\in C_t}} \mathrm{sign}(\pi)\pi\sigma \tag{5.89}$$

of the group algebra of $\mathfrak{S}_N$ is called *Young symmetriser*.

For any partition λ, the corresponding *Specht module* S^λ is a submodule of M^λ spanned by the polytabloids e_t, where t is of shape λ. The dimension of the Specht modules is found, for instance, from the hook formula and its is also equal to the number of the standard Young tableaux[7].

For the Specht module S^λ with $\lambda = [N-M, M]$ the hook formula yields

$$\dim S^\lambda = \frac{N!(N-2M+1)}{(N-M+1)!M!}\,. \tag{5.90}$$

Another direct and useful construction of the Specht modules is as follows. The Specht module S^λ can be identified the subspace of the polynomial ring $\mathbb{C}[x_1,\ldots,x_N]$ spanned by all polynomials p_t, where $p_t = \prod(x_i - x_j)$, the product over all pairs $i<j$ which occur in the same column of a tableau t of shape λ.

Examples. To provide a large set of explicit examples of Yang's approach, below we work out the representation matrices for permutation modules $M^{[N-1,1]}$ and $M^{[N-2,2]}$ corresponding to the symmetric groups for $N = 3, 4, 5$.

The group $\mathfrak{S}_3$. Consider a Young diagram $[2,1]$ corresponding to the defining representation of $\mathfrak{S}_3$ and specify the following basis of tabloids

$$e_1 = \begin{array}{|c|c|}\hline 2 & 3\\ \hline 1\\ \cline{1-1}\end{array}\,, \qquad e_2 = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2\\ \cline{1-1}\end{array}\,, \qquad e_3 = \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3\\ \cline{1-1}\end{array}\,. \tag{5.91}$$

We have the following table of actions of transpositions[8] on the tabloids

$$\begin{array}{lll} \sigma_{12}e_1 = e_2 & \sigma_{13}e_1 = e_3 & \sigma_{23}e_1 = e_1\\ \sigma_{12}e_2 = e_1 & \sigma_{13}e_2 = e_2 & \sigma_{23}e_2 = e_3\\ \sigma_{12}e_3 = e_3 & \sigma_{13}e_3 = e_1 & \sigma_{23}e_3 = e_2 \end{array}$$

In this basis transpositions are realised by 3×3 permutation matrices

$$\sigma_{12} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \sigma_{13} = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \sigma_{23} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}. \tag{5.92}$$

Let us now construct the corresponding Specht module $S^{[2,1]}$ which has dimension 2. Let is take as the cyclic vector of the basis

$$v_1 \;=\; \begin{array}{|c|c|}\hline 2 & 3\\ \hline 1\\ \cline{1-1}\end{array} \;-\; \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2\\ \cline{1-1}\end{array} = e_1 - e_2\,.$$

[7] A tableau t is *standard* if the rows and and columns of t are increasing sequences.

[8] Since the symmetric group is generated by transpositions, it is enough to build up a representation on transpositions and extend it to the whole group by multiplication.

As the vector v_2 we take $v_2 = \sigma_{13} v_1 = e_3 - e_2$. We then have the following table of actions

$$\begin{array}{lll} \sigma_{12} v_1 = -v_1 & \sigma_{13} v_1 = v_2 & \sigma_{23} v_1 = v_1 - v_2 \\ \sigma_{12} v_2 = v_2 - v_1 & \sigma_{13} v_2 = v_1 & \sigma_{23} v_2 = -v_2 \end{array}$$

As the result, in this basis (v_1, v_2) transpositions are realised by 2×2 matrices

$$\sigma_{12} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{23} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{5.93}$$

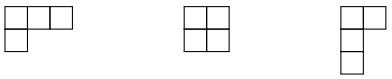
Finally, we note that since the Young diagram $[2,1]$ is self-conjugate, the conjugate representation must be equivalent to the original one. Indeed, taking

$$h = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix},$$

we observe that

$$h \sigma_{12} h^{-1} = -\sigma_{12}\,, \quad h \sigma_{13} h^{-1} = -\sigma_{13}\,, \qquad h \sigma_{23} h^{-1} = -\sigma_{23}\,.$$

The group $\mathfrak{S}_4$. For this group we have the following non-trivial Young diagrams

 (5.94)

The first two diagrams are the most relevant for our purposes and we construct the corresponding permutation and Specht modules explicitly.

We start with the defining representation $M^{[3,1]}$ for which we have

$$\dim M^{[3,1]} = \frac{4!}{3! \cdot 1!} = 4$$

and choose a basis in the representation space as

$$e_1 = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1} \end{array}, \qquad e_2 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}, \qquad e_3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad e_4 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

In this bases the transposition act as

$$\begin{array}{lll} \sigma_{12} e_1 = e_2 & \sigma_{13} e_1 = e_3 & \sigma_{14} e_1 = e_4 \\ \sigma_{12} e_2 = e_1 & \sigma_{13} e_2 = e_2 & \sigma_{14} e_2 = e_2 \\ \sigma_{12} e_2 = e_3 & \sigma_{13} e_3 = e_1 & \sigma_{14} e_3 = e_4 \\ \sigma_{12} e_4 = e_4 & \sigma_{13} e_4 = e_4 & \sigma_{14} e_4 = e_3 \end{array}$$

$$\begin{array}{lll} \sigma_{23} e_1 = e_1 & \sigma_{24} e_1 = e_1 & \sigma_{34} e_1 = e_1 \\ \sigma_{23} e_2 = e_3 & \sigma_{24} e_2 = e_4 & \sigma_{34} e_2 = e_2 \\ \sigma_{23} e_2 = e_2 & \sigma_{24} e_3 = e_3 & \sigma_{34} e_3 = e_4 \\ \sigma_{23} e_4 = e_4 & \sigma_{24} e_4 = e_2 & \sigma_{34} e_4 = e_3 \end{array}$$

and are realised by the following permutation matrices

$$\sigma_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\sigma_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.95}$$

The Specht module $S^{[3,1]}$ has $\dim S^{[3,1]} = 3$. We can pick up a basis spanned by

$$\begin{aligned} v_1 &= e_1 - e_2 \\ v_2 &= e_3 - e_2 \\ v_3 &= e_3 - e_4 \end{aligned}$$

Computing the action of transpositions in this basis we find

$$\sigma_{12} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{13} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_{14} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$
$$\sigma_{23} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{24} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \sigma_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}. \tag{5.96}$$

Note that the determinant of any of these matrices is equal to -1. This shows, in particular, that the conjugate representation $[2,1,1]$ corresponding to the utmost right diagram in (5.94) for which any transposition is realised as $-\sigma$ with σ in irrep $[3,1]$ cannot be equivalent to irrep $[3,1]$, because $-\sigma$ would then have the unit determinant. It is still a useful exercise to construct the representation $[2,1,1]$ and show that it is equivalent to the one where the matrices (5.96) are taken with minus sign in front. To proceed with solving this exercise, we use the method of polynomials and pick up a subring of $\mathbb{C}[x_1, x_2, x_3, x_4]$ with the following basis

$$\begin{aligned} p_1 &= (x_1 - x_3)(x_3 - x_4)(x_1 - x_4)\,, \\ p_2 &= (x_2 - x_3)(x_3 - x_4)(x_2 - x_4)\,, \\ p_3 &= (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)\,. \end{aligned}$$

The symmetric group acts on x_i by $x_i \to x_{\sigma(i)}$. If we identify

$$p_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.97}$$

then the action of $\mathfrak{S}_3$ on these polynomials becomes equivalent to the following matrix action

$$t_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad t_{13} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad t_{14} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$t_{23} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad t_{24} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t_{34} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.98}$$

Introduce a matrix

$$w = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

One can now check that $wt_{ij}w^{-1} = -\sigma_{ij}$ for all $i < j$, where σ_{ij} are given by (5.96).

Let us now consider the permutation module $M^{[2,2]}$ for the middle diagram in (5.94). The dimension of this module is

$$\dim M^{[2,2]} = \frac{4!}{2! \cdot 2!} = 6$$

and we choose the following basis

$$e_1 = \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}\,, \qquad e_2 = \begin{array}{|cc|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\,, \qquad e_3 = \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array}\,,$$

$$e_4 = \begin{array}{|cc|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}\,, \qquad e_5 = \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,, \qquad e_6 = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\,.$$

In this bases the transposition act as

$$\begin{array}{lll}
\sigma_{12}e_1 = e_1 & \sigma_{13}e_1 = e_4 & \sigma_{14}e_1 = e_5 \\
\sigma_{12}e_2 = e_4 & \sigma_{13}e_2 = e_2 & \sigma_{14}e_2 = e_6 \\
\sigma_{12}e_3 = e_5 & \sigma_{13}e_3 = e_6 & \sigma_{14}e_3 = e_3 \\
\sigma_{12}e_4 = e_2 & \sigma_{13}e_4 = e_1 & \sigma_{14}e_4 = e_4 \\
\sigma_{12}e_5 = e_3 & \sigma_{13}e_5 = e_5 & \sigma_{14}e_5 = e_1 \\
\sigma_{12}e_6 = e_6 & \sigma_{13}e_6 = e_3 & \sigma_{14}e_6 = e_2
\end{array}$$

$$\begin{array}{lll}
\sigma_{23}e_1 = e_2 & \sigma_{24}e_1 = e_3 & \sigma_{34}e_1 = e_1 \\
\sigma_{23}e_2 = e_1 & \sigma_{24}e_2 = e_2 & \sigma_{34}e_2 = e_3 \\
\sigma_{23}e_3 = e_3 & \sigma_{24}e_3 = e_1 & \sigma_{34}e_3 = e_2 \\
\sigma_{23}e_4 = e_4 & \sigma_{24}e_4 = e_6 & \sigma_{34}e_4 = e_5 \\
\sigma_{23}e_5 = e_6 & \sigma_{24}e_5 = e_5 & \sigma_{34}e_5 = e_4 \\
\sigma_{23}e_6 = e_5 & \sigma_{24}e_6 = e_4 & \sigma_{34}e_6 = e_6
\end{array}$$

and the corresponding matrix realisation reads

$$\begin{array}{lll}
\sigma_{12} = \begin{pmatrix} 1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1 \end{pmatrix}, & \sigma_{13} = \begin{pmatrix} 0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&1&0&0&0 \end{pmatrix}, & \sigma_{14} = \begin{pmatrix} 0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0 \end{pmatrix}, \\
\sigma_{23} = \begin{pmatrix} 0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0 \end{pmatrix}, & \sigma_{24} = \begin{pmatrix} 0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0 \end{pmatrix}, & \sigma_{34} = \begin{pmatrix} 1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&1 \end{pmatrix}.
\end{array} \tag{5.99}$$

The Specht module $S^{[2,2]}$ has dimension 2. We can start with the following tabloid

$$\{t\} = e_1 = \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}\,,$$

which has

$$\kappa_t = (\epsilon - (13))(\epsilon - (24)) = \epsilon - (13) - (24) + (13)(24)\,.$$

Applying κ_t to $\{t\}$, we obtain the first basis vector of our Specht module

$$v_1 = \frac{\begin{array}{|cc|}\hline 3 & 4 \\ \hline\end{array}}{\begin{array}{|cc|}\hline 1 & 2 \\ \hline\end{array}} - \frac{\begin{array}{|cc|}\hline 1 & 4 \\ \hline\end{array}}{\begin{array}{|cc|}\hline 2 & 3 \\ \hline\end{array}} - \frac{\begin{array}{|cc|}\hline 2 & 3 \\ \hline\end{array}}{\begin{array}{|cc|}\hline 1 & 4 \\ \hline\end{array}} + \frac{\begin{array}{|cc|}\hline 1 & 2 \\ \hline\end{array}}{\begin{array}{|cc|}\hline 3 & 4 \\ \hline\end{array}}.$$

Obviously,

$$v_1 = e_1 - e_3 - e_4 + e_6\,.$$

We supplement this vector with another one

$$v_2 = e_1 - e_2 - e_5 + e_6\,.$$

Together v_1 and v_2 form a basis of the 2dim irrep $S^{[2,2]}$. Computing the action of transposition on these two basis vectors, we find the following matrix representation

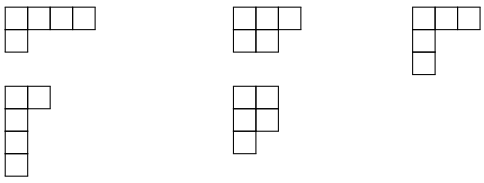
$$\sigma_{12} = \sigma_{34} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{13} = \sigma_{24} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{14} = \sigma_{23} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{5.100}$$

Introducing a matrix

$$k = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix},$$

we verify that $k\sigma_{ij}k^{-1} = -\sigma_{ij}$ for $i < j$, $i, j = 1, \ldots, 4$, which shows that the conjugate irrep is equivalent to the original one. This is, of course, is a consequence of the fact that the diagram $[2, 2]$ is self-conjugate.

The group $\mathfrak{S}_5$. This is the last example we work out in an explicit manner. For $\mathfrak{S}_5$ we have the following non-trivial Young diagrams



(5.101)

We are primarily interested in the first two diagrams in the first row. The defining representation $M^{[4,1]}$ has the dimension 5 and to describe it, we pick up the following basis

$$e_1 = \frac{\begin{array}{|cccc|}\hline 2 & 3 & 4 & 5 \\ \hline\end{array}}{\begin{array}{|c|}\hline 1 \\ \hline\end{array}}, \qquad e_2 = \frac{\begin{array}{|cccc|}\hline 1 & 3 & 4 & 5 \\ \hline\end{array}}{\begin{array}{|c|}\hline 2 \\ \hline\end{array}}, \qquad e_3 = \frac{\begin{array}{|cccc|}\hline 1 & 2 & 4 & 5 \\ \hline\end{array}}{\begin{array}{|c|}\hline 3 \\ \hline\end{array}},$$

$$e_4 = \frac{\begin{array}{|cccc|}\hline 1 & 2 & 3 & 5 \\ \hline\end{array}}{\begin{array}{|c|}\hline 4 \\ \hline\end{array}}, \qquad e_5 = \frac{\begin{array}{|cccc|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}}{\begin{array}{|c|}\hline 5 \\ \hline\end{array}},$$

The corresponding representation of $\mathfrak{S}_4$ will again be given by the standard 5×5 permutation matrices. The Specht module $S^{[4,1]}$ has dimension 4. To single it out, we can use the following basis

$$\begin{aligned} v_1 &= e_1 - e_2\,, \\ v_2 &= e_3 - e_2\,, \\ v_3 &= e_4 - e_2\,, \\ v_4 &= e_5 - e_2\,, \end{aligned} \tag{5.102}$$

In this basis the corresponding representation matrices are

$$
\begin{aligned}
\sigma_{12} &= \begin{pmatrix} \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{13} = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{14} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \\
\sigma_{15} &= \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{23} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{24} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \\
\sigma_{25} &= \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \end{pmatrix}, \quad \sigma_{34} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{35} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}, \\
\sigma_{45} &= \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{pmatrix}.
\end{aligned} \tag{5.103}
$$

All these matrices have the determinant equal to -1.

The permutation module $M^{[3,2]}$ has dimension 10 and to construct an explicit matrix representation we pick up the following basis

$$
e_1 = \begin{array}{|ccc|} \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}, \quad e_2 = \begin{array}{|ccc|} \hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}, \quad e_3 = \begin{array}{|ccc|} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array}, \quad e_4 = \begin{array}{|ccc|} \hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2} \end{array}, \quad e_5 = \begin{array}{|ccc|} \hline 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array},
$$

$$
e_6 = \begin{array}{|ccc|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \quad e_7 = \begin{array}{|ccc|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}, \quad e_8 = \begin{array}{|ccc|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \quad e_9 = \begin{array}{|ccc|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \quad e_{10} = \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}.
$$

In this bases the transposition act as

$$
\begin{array}{lllll}
\sigma_{12}e_1 = e_1 & \sigma_{13}e_1 = e_5 & \sigma_{14}e_1 = e_6 & \sigma_{15}e_1 = e_7 & \sigma_{23}e_1 = e_2 \\
\sigma_{12}e_2 = e_5 & \sigma_{13}e_2 = e_2 & \sigma_{14}e_2 = e_8 & \sigma_{15}e_2 = e_9 & \sigma_{23}e_2 = e_1 \\
\sigma_{12}e_3 = e_6 & \sigma_{13}e_3 = e_8 & \sigma_{14}e_3 = e_3 & \sigma_{15}e_3 = e_{10} & \sigma_{23}e_3 = e_3 \\
\sigma_{12}e_4 = e_7 & \sigma_{13}e_4 = e_9 & \sigma_{14}e_4 = e_{10} & \sigma_{15}e_4 = e_4 & \sigma_{23}e_4 = e_4 \\
\sigma_{12}e_5 = e_2 & \sigma_{13}e_5 = e_1 & \sigma_{14}e_5 = e_5 & \sigma_{15}e_5 = e_5 & \sigma_{23}e_5 = e_5 \\
\sigma_{12}e_6 = e_3 & \sigma_{13}e_6 = e_6 & \sigma_{14}e_6 = e_1 & \sigma_{15}e_6 = e_6 & \sigma_{23}e_6 = e_8 \\
\sigma_{12}e_7 = e_4 & \sigma_{13}e_7 = e_7 & \sigma_{14}e_7 = e_7 & \sigma_{15}e_7 = e_1 & \sigma_{23}e_7 = e_9 \\
\sigma_{12}e_8 = e_8 & \sigma_{13}e_8 = e_3 & \sigma_{14}e_8 = e_2 & \sigma_{15}e_8 = e_8 & \sigma_{23}e_8 = e_6 \\
\sigma_{12}e_9 = e_9 & \sigma_{13}e_9 = e_4 & \sigma_{14}e_9 = e_9 & \sigma_{15}e_9 = e_2 & \sigma_{23}e_9 = e_7 \\
\sigma_{12}e_{10} = e_{10} & \sigma_{13}e_{10} = e_{10} & \sigma_{14}e_{10} = e_4 & \sigma_{15}e_{10} = e_3 & \sigma_{23}e_{10} = e_{10}
\end{array}
$$

$$
\begin{array}{lllll}
\sigma_{24}e_1 = e_3 & \sigma_{25}e_1 = e_4 & \sigma_{34}e_1 = e_1 & \sigma_{35}e_1 = e_1 & \sigma_{45}e_1 = e_1 \\
\sigma_{24}e_2 = e_2 & \sigma_{25}e_2 = e_2 & \sigma_{34}e_2 = e_3 & \sigma_{35}e_2 = e_4 & \sigma_{45}e_2 = e_2 \\
\sigma_{24}e_3 = e_1 & \sigma_{25}e_3 = e_3 & \sigma_{34}e_3 = e_2 & \sigma_{35}e_3 = e_3 & \sigma_{45}e_3 = e_4 \\
\sigma_{24}e_4 = e_4 & \sigma_{25}e_4 = e_1 & \sigma_{34}e_4 = e_4 & \sigma_{35}e_4 = e_2 & \sigma_{45}e_4 = e_3 \\
\sigma_{24}e_5 = e_8 & \sigma_{25}e_5 = e_9 & \sigma_{34}e_5 = e_6 & \sigma_{35}e_5 = e_7 & \sigma_{45}e_5 = e_5 \\
\sigma_{24}e_6 = e_6 & \sigma_{25}e_6 = e_{10} & \sigma_{34}e_6 = e_5 & \sigma_{35}e_6 = e_6 & \sigma_{45}e_6 = e_7 \\
\sigma_{24}e_7 = e_{10} & \sigma_{25}e_7 = e_7 & \sigma_{34}e_7 = e_7 & \sigma_{35}e_7 = e_5 & \sigma_{45}e_7 = e_6 \\
\sigma_{24}e_8 = e_5 & \sigma_{25}e_8 = e_8 & \sigma_{34}e_8 = e_8 & \sigma_{35}e_8 = e_{10} & \sigma_{45}e_8 = e_9 \\
\sigma_{24}e_9 = e_9 & \sigma_{25}e_9 = e_5 & \sigma_{34}e_9 = e_{10} & \sigma_{35}e_9 = e_9 & \sigma_{45}e_9 = e_8 \\
\sigma_{24}e_{10} = e_7 & \sigma_{25}e_{10} = e_6 & \sigma_{34}e_{10} = e_9 & \sigma_{35}e_{10} = e_8 & \sigma_{45}e_{10} = e_{10}
\end{array}
$$

This action gives rise to the following matrix representation

$$\sigma_{12} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix},$$

$$\sigma_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_{23} = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \quad \sigma_{24} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix},$$

$$\sigma_{25} = \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_{34} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix},$$

$$\sigma_{35} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix}, \quad \sigma_{45} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

All these matrices has the determinant equal to -1.

The Specht module $S^{[3,2]}$ has dimension 5. It can be singled out from the permutation module $M^{[3,2]}$ constructed above by picking up a basis

$$\begin{aligned} v_1 &= e_1 - e_4 - e_6 + e_{10}\,, \\ v_2 &= e_1 - e_3 - e_7 + e_{10}\,, \\ v_3 &= e_5 - e_6 - e_9 + e_{10}\,, \\ v_4 &= e_2 - e_3 - e_9 + e_{10}\,, \\ v_5 &= e_5 - e_7 - e_8 + e_{10}\,. \end{aligned}$$

then in the basis where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.104}$$

irreducible action of $\mathfrak{S}_5$ is realised by the following 5×5 matrices

$$\sigma_{12} = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \qquad \sigma_{13} = \begin{pmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \end{pmatrix},$$

$$\sigma_{14} = \begin{pmatrix} \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{-1} \\ 1 & 0 & 0 & \mathbf{-1} & 0 \end{pmatrix}, \qquad \sigma_{15} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{-1} & 0 & 0 & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{-1} & \mathbf{-1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix},$$

$$\sigma_{23} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{-1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{-1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \end{pmatrix}, \qquad \sigma_{24} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{-1} & 0 & \mathbf{-1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{-1} & 0 & \mathbf{-1} \end{pmatrix}, \tag{5.105}$$

$$\sigma_{25} = \begin{pmatrix} \mathbf{-1} & \mathbf{-1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \qquad \sigma_{34} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{-1} & 0 & \mathbf{-1} & 0 & \mathbf{-1} \\ 0 & \mathbf{-1} & 0 & \mathbf{-1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix},$$

$$\sigma_{35} = \begin{pmatrix} 0 & 0 & 0 & \mathbf{-1} & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{-1} & 0 & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \end{pmatrix}, \qquad \sigma_{45} = \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 \\ 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}.$$

The determinant of any of these matrices is 1.

5.7 More on the Fock condition

According to the group theory, the symmetry of the coordinate wave function of the multi-electron system should be defined by a Young diagram on the Fig. 5.2.

A wave function of the required symmetry can be obtained by from an arbitrary wave function by first symmetrising with respect to the pair of variables that stand in each row, *i.e.* with respect to $(1, M+1)$, $(2, M+2), \ldots (M, 2M)$, and then by antisymmetrsing with respect to the columns, *e.g.* with respect to the variables $(1, 2, \ldots M)$ and $(M+1, M+2, \ldots, N)$. This procedure is equivalent to an application to the corresponding Young tableau the Young operator (5.89). As is known, this gives an irreducible representation representation of $\mathfrak{S}_N$, which is the Specht module $[2^M, 1^{N-2M}]$. Obviously, the sequence of variables in the Young tableaux may be different yielding different Young operators. The number of linearly independent operators coincides with the dimension of an irreducible representation of the symmetric group corresponding to this Young diagram.

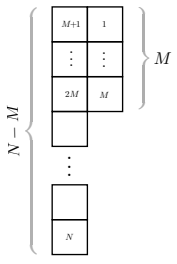


Figure 5.2: Young diagram $\lambda = [2^M, 1^{N-2M}]$ for a coordinate function of electrons.

In Fock's method the coordinate wave function must obey the following two conditions:

1) The anti-symmetry requirement: it must be anti-symmetric in the variables $(1, 2, \dots M)$ and $(M+1, M+2, \dots, N)$

2) The Fock cyclic symmetry condition: the operator

$$F = \mathbb{1} - P_{M\,M+1} - P_{M,M+2} - \dots - P_{M\,N}\,, \tag{5.106}$$

must annihilate the coordinate wave function.

A function which is obtained by application of the Young operator does satisfy *1)*. We now show that such a function also obeys the Fock condition, *i.e.* $F(Yt) = 0$, the latter is equivalent to the following product of operators $YF = 0$, see the footnote below.

The Young operator can be written in terms of the antisymmetrizer and symmetriser as[9]

$$Y = \sum_{j=1}^{M} S(j, j+M) A(1, 2, \dots, M) A(M+1, M+2, \dots, N) \equiv S A_1 A_2\,.$$

We show that all the terms in SA_1A_2F cancel in pairs. Consider one of the $M!(N-M)!$ terms in the product A_1A_2. It can be written as a permutation in the two-line notation

$$\pm \left(\begin{array}{ccc|ccc} 1 & \cdots & M & M+1 & \cdots & N \\ \sigma(1) & \cdots & \sigma(M) & \tau(M+1) & \cdots & \tau(N) \end{array} \right),$$

where the sign is determined by the parity of σ. Let us now apply to this term one of the transpositions in F.

$$\begin{aligned} J &= \left(\begin{array}{ccccc|ccccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & d & \cdots & N \\ \sigma(1) & \cdots & M & \cdots & \sigma(M) & \tau(M+1) & \cdots & c & \cdots & a & \cdots & \tau(N) \end{array} \right) \left(\begin{array}{cc} M & a \\ a & M \end{array} \right) \\ &= \left(\begin{array}{ccccc|ccccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & d & \cdots & N \\ \sigma(1) & \cdots & a & \cdots & \sigma(M) & \tau(M+1) & \cdots & c & \cdots & M & \cdots & \tau(N) \end{array} \right), \end{aligned}$$

where $M+1 \leqslant a \leqslant N$, $M+1 \leqslant c \leqslant N$, $M+1 \leqslant d \leqslant N$ and $1 \leqslant b \leqslant M$. This expression is the same as

$$J = \left(\begin{array}{cc} b & b+M \\ b+M & b \end{array} \right) \left(\begin{array}{ccccc|ccccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & d & \cdots & N \\ \sigma(1) & \cdots & M & \cdots & \sigma(M) & \tau(M+1) & \cdots & a & \cdots & c & \cdots & \tau(N) \end{array} \right) \left(\begin{array}{cc} M & c \\ c & M \end{array} \right).$$

We now see that both permutations

$$\left(\begin{array}{ccccc|ccccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & d & \cdots & N \\ \sigma(1) & \cdots & M & \cdots & \sigma(M) & \tau(M+1) & \cdots & c & \cdots & a & \cdots & \tau(N) \end{array} \right),$$

$$\left(\begin{array}{ccccc|ccccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & d & \cdots & N \\ \sigma(1) & \cdots & M & \cdots & \sigma(M) & \tau(M+1) & \cdots & a & \cdots & c & \cdots & \tau(N) \end{array} \right)$$

are in A_1A_2 and they differ by a transposition which introduces an additional "$-$" sign, so in A_1A_2 these two permutations will appear with opposite signs. The operator

$$P_{b,b+M} = \left(\begin{array}{cc} b & b+M \\ b+M & b \end{array} \right)$$

does not change the operator S. Thus, the permutations above cancel under the action of the Fock operator F.

[9]The opposite order is related to our convention of multiplying permutations in the two-line notation. When applying to the Young diagram, symmetriser acts first and then anti-symmetriser.

It remains to consider a particular case when $a = c$ and $b + M = d$. In this case the following identity is valid

$$\begin{aligned} J &= \left(\begin{array}{ccccc|ccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & N \\ \sigma(1) & & M & & \sigma(M) & \tau(M+1) & & a & & \tau(N) \end{array}\right)\begin{pmatrix} M & a \\ a & M \end{pmatrix} \\ &= \begin{pmatrix} b & b+M \\ b+M & b \end{pmatrix}\left(\begin{array}{ccccc|ccccc} 1 & \cdots & b & \cdots & M & M+1 & \cdots & b+M & \cdots & N \\ \sigma(1) & & M & & \sigma(M) & \tau(M+1) & & a & & \tau(N) \end{array}\right). \end{aligned}$$

This term in the product A_1A_2 will cancel against the same term but under the identity in the operator F. Tuis, it is proved that any function that is obtained by applying the Young operator satisfies Fock's condition.

The reverse relation is more involved, because any function which satisfies *1)* and *2)* corresponds in general to a set of Young tableaux that differ from each other by permutations of indices $M + 1, M + 2, \ldots, N$. Any linear combination of the corresponding Young operators, when acting on an arbitrary function, gives a function which satisfies *1)* and *2)*.

It remains to note that application of the Young operator is convenient when we want to construct a wave function from some given non-symmetric function. On the other hand, if the function is already known, then it is easier to check if it obeys Fock's condition than to see how it behaves under the action of Young operators.

5.8 Miscellenia

Here we prove the formula

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_N} \operatorname{sign}\sigma \prod_{i<j}\big(p_{\tau(\sigma(i))} - p_{\tau(\sigma(j))} + i\kappa\big) \prod_{i=1}^{N-1} \Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big) \\ = \prod_{i<j}\big(p_{\tau(i)} - p_{\tau(j)} - i\kappa\,\epsilon(q_i - q_j)\big)\,. \end{aligned}$$

First, it is clear that the product of successive Θ-functions defining for a fixed σ the corresponding coordinate sector can be replaced as

$$\prod_{i=1}^{N-1} \Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big) = \prod_{i<j} \Theta\big(q_{\sigma(j)} - q_{\sigma(i)}\big)\,, \tag{5.107}$$

as the additional Θ-functions yield trivial contribution. Making this replacement, we then obtain the sum

$$\sum_{\sigma\in\mathfrak{S}_N} \operatorname{sign}\sigma \prod_{i<j}\big(p_{\tau(\sigma(i))} - p_{\tau(\sigma(j))} + i\kappa\big)\Theta\big(q_{\sigma(j)} - q_{\sigma(i)}\big)$$

Let us concentrate on two particular terms in the product with some fixed indices $\underline{i}$ and $\underline{j}$, in other words, we consider

$$\operatorname{sign}\sigma\,\big(p_{\tau(k)} - p_{\tau(l)} + i\kappa\big)\Theta\big(q_l - q_k\big)\,,$$

where we set $\sigma(\underline{i}) = k$ and $\sigma(\underline{j}) = l$. For every σ, there will be an accompanying permutation of the form $\alpha_{\underline{ij}}\sigma$ which acts non-trivially only on the indices $\underline{i}$ and $\underline{j}$, namely, $(\alpha_{\underline{ij}}\sigma)(\underline{i}) = \sigma(\underline{j}) = l$ and $(\alpha_{\underline{ij}}\sigma)(\underline{j}) = \sigma(\underline{i}) = k$. Thus, summing in pairs σ and $\alpha_{\underline{ij}}\sigma$, we will have

$$\big(p_{\tau(k)} - p_{\tau(l)} + i\kappa\big)\Theta\big(q_l - q_k\big) + \operatorname{sign}\alpha_{\underline{ij}}\big(p_{\tau(l)} - p_{\tau(k)} + i\kappa\big)\Theta\big(q_k - q_l\big)\,,$$

Since $\operatorname{sign}\alpha_{\underline{ij}} = -1$, we have

$$\big(p_{\tau(k)} - p_{\tau(l)} + i\kappa\big)\Theta\big(q_l - q_k\big) + \big(p_{\tau(k)} - p_{\tau(l)} - i\kappa\big)\Theta\big(q_k - q_l\big) = p_{\tau(k)} - p_{\tau(l)} - i\kappa\,\epsilon(q_k - q_l)\,,$$

where we used the properties of the Heaviside Θ-function

$$\Theta(x) + \Theta(-x) = 1\,, \qquad \Theta(x) - \Theta(-x) = \epsilon(x)\,.$$

Summing up in this way, we obtain the desired result.

As an example, of more explicit evaluation of the Bethe wave function, we consider the two-body function (3.19) in the symmetric representation.

$$\begin{aligned}\Psi(q_1,q_2) &= \Theta(q_1<q_2)\left\{\mathcal{A}(12)e^{i(p_1q_1+p_2q_2)} + \frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa}\mathcal{A}(12)e^{i(p_2q_1+p_1q_2)}\right\}\\ &+ \Theta(q_2<q_1)\left\{\mathcal{A}(12)e^{i(p_1q_2+p_2q_1)} + \frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa}\mathcal{A}(12)e^{i(p_2q_2+p_1q_1)}\right\}.\end{aligned}$$

$$\Theta(q_1<q_2) + \Theta(q_2<q_1)\frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa} = \frac{(p_1-p_2+i\kappa)\Theta(q_2-q_1) + (p_1-p_2-i\kappa)\Theta(q_1-q_2)}{p_1-p_2+i\kappa}\,.$$

Next, we take into account that

$$\Theta(x) + \Theta(-x) = 1 \quad \& \quad \Theta(x) - \Theta(-x) = \operatorname{sign}x$$

from where we find that

$$\Theta(x) = \frac{1+\operatorname{sign}x}{2}\,.$$

Thus,

$$\Theta(q_1<q_2) + \Theta(q_2<q_1)\frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa} = \frac{p_1-p_2-i\kappa\operatorname{sign}(q_1-q_2)}{p_1-p_2+i\kappa}$$

Analogously,

$$\Theta(q_2<q_1) + \Theta(q_1<q_2)\frac{p_1-p_2-i\kappa}{p_1-p_2+i\kappa} = \frac{p_1-p_2+i\kappa\operatorname{sign}(q_1-q_2)}{p_1-p_2+i\kappa}$$

Finally, choosing $\mathcal{A}(12) = p_1 - p_2 + i\kappa$, the wave function becomes

$$\Psi(q_1,q_2) = (p_1-p_2)\left[e^{i(p_1q_1+p_2q_2)}\left(1 - \frac{i\kappa\operatorname{sign}(q_1-q_2)}{p_1-p_2}\right) + e^{i(p_1q_2+p_2q_1)}\left(1 - \frac{i\kappa\operatorname{sign}(q_1-q_2)}{p_2-p_1}\right)\right].$$

Recommended literature

[1] V. Arnold, *Mathematical methods of classical mechanics*, vol. 60, Springer, 1989.

[2] O. Babelon, D. Bernard, and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, 2003.

[3] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-body Problems*, World Scientific, 381 p., 2004.

[4] C.N. Yang, Some exact results for the many body problems in one dimension with repulsive delta function interaction, Phys. Rev. Lett., 19:1312–1314, 1967.

[5] A.B. Zamolodchikov and Al.B. Zamolodchikov, Factorized S Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. Annals Phys., 120:253–291, 1979.

[6] H.B. Thacker, Exact integrability in quantum field theory and statistical systems, Rev. Mod. Phys., 53:253–285, 1981.

[7] L. D. Faddeev, How Algebraic Bethe Ansatz works for integrable model, Les-Houches lectures, arXiv:hep-th/9605187.

[8] M. Gaudin, *The Bethe Wavefunction*, Translated from French original "La fonction d'onde de Bethe' (1983) by J.-S. Caux, Cambridge University Press, 2014.

[9] G. Arutyunov, *Elements of classical and quantum integrable systems*, Springer, 2019.