Phase transitions in the early Universe

3. Dynamics of first order phase transitions

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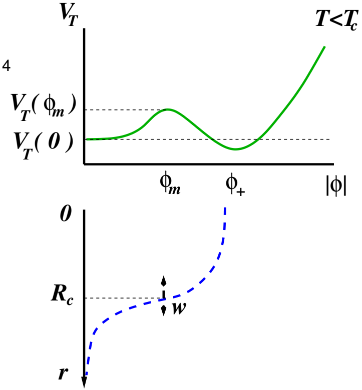
Outline

Dynamics of first order phase transitions

$$\Delta V_T \simeq \frac{D}{2}(T^2 - T_2^2)|\bar{\phi}|^2 - \frac{1}{3}AT|\bar{\phi}|^3 + \frac{1}{4}\lambda|\bar{\phi}|^4$$

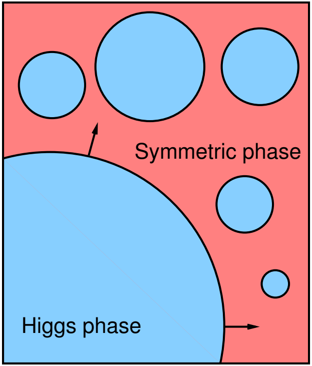
- Below T_c, state $\phi = 0$ metastable
- Separated from equilibrium state by $B = V_T(\phi_m) - V_T(0)$
- Lowest energy path to equilibrium state via critical bubble
- Energy of critical bubble E_c
- Nucleation rate per unit volume (high T): $\Gamma/V \simeq T^4 \exp(-E_c/T)$



Critical bubble field profile

Transition via growth and merger of bubbles

- Thermal fluctuations produce bubbles at rate/volume $\Gamma/V \simeq T^4 \exp(-E_c/T)$
- Bubbles growth speed $v_{\rm w}$ set by interaction with medium
- Bubble merger completes phase transition



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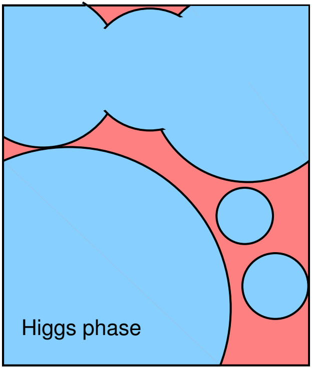

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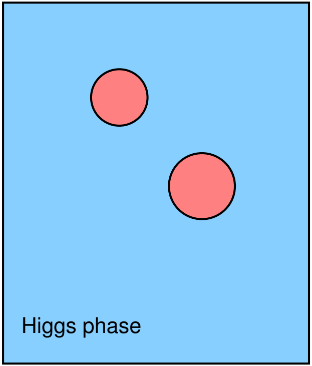


Electroweak phase transition & baryogenesis

Sakharov conditions for baryogenesis:

- **B violation:**
- **C and CP violation:** Antimatter excess violates C and CP
- **non-equilibrium:** $\not{B}$ processes reduce B asymmetry in equilibrium

Electroweak phase transition & baryogenesis

Sakharov conditions for baryogenesis:

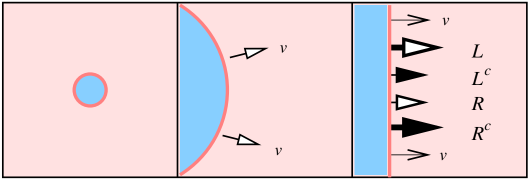
- **B violation:** Electroweak theory has *unstable* topological defects – sphalerons (S)[(1)] Formation and decay of S results in change in $B + L$ of left-handed fermions[(2)]
- **C and CP violation:** C violation automatic in SM. CP violation needs more than CKM at high T
- **non-equilibrium:** Supercooling at 1st order phase transition?

[(1)] Klinkhamer, Manton (1984)
[(2)] Kuzmin, Rubakov, Shaposhnikov (1985)

(Hot) electroweak baryogenesis

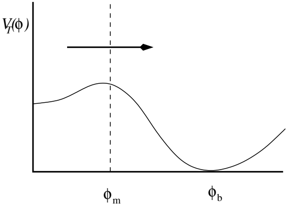
Mechanism:[3]

- CP-violation in bubble wall field profile
- CP-asymmetry in reflection of fermions
- Chiral asymmetry → (Sphalerons) → baryon asymmetry



[3] Cohen, Kaplan, Nelson 1991

Thermal activation: particles in a potential



Simplify: particles in a potential V_T.

- Position ϕ
- momentum π
- Hamiltonian $H = \frac{1}{2}\pi^2 + V_T(\phi)$

What is flux across barrier Γ?

$$\Gamma = \frac{1}{Z}\int d\pi d\phi e^{-\beta H}\delta(\phi - \phi_{\rm m})\pi\theta(\pi) = \frac{1}{Z}\frac{1}{\beta}e^{-\beta V_T(\phi_{\rm m})}$$

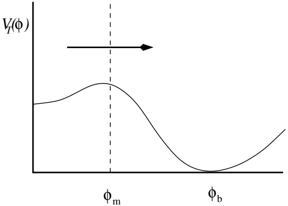
Evaluate Z by steepest descent; assume no particles near $\phi_{\rm b}$

$$Z = \int d\pi d\phi e^{-\beta H} \qquad = \frac{2\pi}{\beta\sqrt{V_T''(0)}}e^{-\beta V_T(0)}$$

$$\boxed{\Gamma = \frac{\sqrt{V_T''(0)}}{2\pi}e^{-\beta\Delta V_T}}$$

$\Delta V_T = V_T(\phi_{\rm m}) - V_T(0)$ – barrier height

Thermal activation: imaginary part of the free energy



- Position ϕ
- momentum π
- Hamiltonian $H = \frac{1}{2}\pi^2 + V_T(\phi)$

What is free energy $F = -\ln Z/\beta$?

Evaluate Z by steepest descent, taking into account particles near ϕ_m

$$Z = \int d\pi d\phi e^{-\beta H} = \frac{2\pi}{\beta}\left(\frac{1}{\sqrt{V_T''(0)}}e^{-\beta V_T(0)} + \frac{1}{2}\frac{1}{\sqrt{V_T''(\phi_\mathrm{m})}}e^{-\beta V_T(\phi_\mathrm{m})}\right)$$

Second term is imaginary: $\mathrm{Im}\, F = \frac{1}{2\beta}\frac{\sqrt{V_T''(0)}}{|V_T''(\phi_\mathrm{m})|}e^{-\beta\Delta V_T}$

Thermal activation rate $\boxed{\Gamma = \frac{\beta\sqrt{V_T''(0)}}{\pi}\mathrm{Im}\, F}$ (steepest descent)

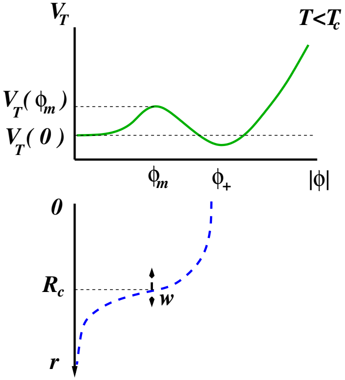
The critical bubble

Evaluate Z by steepest descent: $H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$

$$Z = \int \mathcal{D}\pi\mathcal{D}\phi e^{-\beta H[\pi,\phi]} = \mathcal{N}\mathcal{D}\phi e^{-\beta E[\phi]}$$

$$E[\phi] = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$$

- Critical bubble $\phi_c(\mathbf{x})$ solves $\frac{\delta E[\phi]}{\delta\phi(\mathbf{x})} = 0$
- Spherically symmetric, radius R_c
- Energy E_c
- Activation rate $\Gamma \sim e^{-\beta E_c}$



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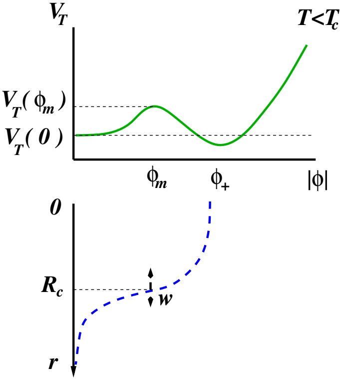
$$E[\phi] = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$$

$$V_T = V_0 + \frac{D}{2}(T^2 - T_2^2)\phi^2 - \frac{1}{3}AT\phi^3 + \frac{1}{4}\lambda\phi^4$$

- Phase boundary surface energy $\sigma \simeq \sqrt{\lambda}\phi_+^3$
- Free energy difference $B = V_T(\phi_+) - V_T(0)$



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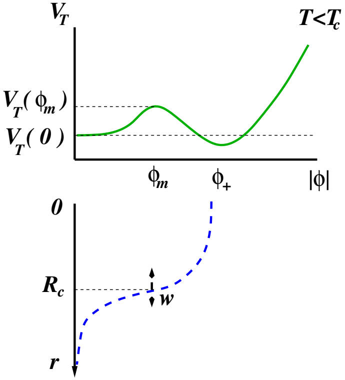


- Estimate (thin wall):

$$E_c \simeq -\frac{4\pi R_c^3}{3}B + 4\pi R_c^2\sigma$$

- Critical bubble radius $R_c \simeq \sigma/B$
- Critical bubble energy $E_c \simeq \sigma^3/B^2$

Nucleation rate formula

Bubble nucleation rate per unit volume[4]

$$\Gamma \sim T \left(\frac{E_c}{2\pi T} \right)^{\frac{3}{2}} \left(V_T''(0) \right)^{\frac{3}{2}} e^{-E_c/T}$$

Notes

- E_c is calculated relative to the energy density of the metastable state
- Bubbles with $R > R_c$ are unstable to growth
- In thin wall approximation $S = E_c/T \gg 1$
- $V_T''(0) = M_h^2$, Higgs mass squared

[4] Langer 1969, Linde 1983

Transition rate parameter

- Will show $S = E_c/T \gg 1$, can write

$$\Gamma(T) = \Gamma_0(T)e^{-S(T)}$$

- Transition rate is exponentially sensitive to the temperature
- Reference time t_r:
  - e.g. t_H tunnelling rate/volume = (Hubble rate)/(Hubble volume)
  - or another, see later

$$\Gamma(t) = \Gamma_r e^{-S'(t_r)(t-t_r)}$$

- Write $\beta = -S'(t_r)$ (this β is not temperature![5])
- β is the transition rate parameter (positive)
- $\beta \simeq -H \dfrac{dS}{d\ln(T)}$
- $\beta \gtrsim H$, otherwise universe stays in metastable state, and inflates forever

[5] Sorry about this terrible notation - it's conventional

Size of transition rate parameter β

What critical bubble energy is needed to get activation rate/volume $\Gamma \sim H^4$ at $T \sim 100$ GeV?

- $\Gamma \sim TM_h(T)^3 e^{-E_c/T} \sim H^4$
- Use Friedmann equation $H^2 \sim T^4/M_P^2$
- Result for $T \sim 10^2$ GeV: $S \equiv E_c/T \simeq \ln(M_P^4/T^4) \simeq 150$

What is the magnitude of the transition rate parameter β?

- Transition rate parameter $\beta \simeq -H\frac{dS}{d\ln(T)} = HS - dE_c/dT$
- First guess: $\beta/H \sim S \sim 100$
- Corresponds to frequency today:

$$f_0 \sim (\beta/H)10^{-5}\ \mathrm{Hz}$$

- First indication of frequency scale of GWs

Fraction of universe in metastable phase $h(t)$

- Once nucleated, bubbles grow with constant speed v_w (see later)
- Volume of bubble nucleated at time t': $V(t,t') = \frac{4\pi}{3} v_\mathrm{w}^3 (t-t')^3$
- Number density of bubbles nucleated in $(t', t'+dt')$ is $dn(t') = \Gamma(t')dt'$
- Fractional volume occupied by bubbles (no overlaps): $dh(t,t') = \frac{4\pi}{3} v_\mathrm{w}^3 (t-t')^3 dn(t')$
- Fractional volume in metastable phase, including overlaps

$$h(t) = \exp\left(- \int^t dt' \frac{4\pi}{3} v_\mathrm{w}^3 (t-t')^3 \Gamma(t') dt' \right)$$

- Reference time t_f such that $h(t_f) = 1/e$, evaluate by steepest descent:

$$h(t) = \exp\left(-e^{\beta(t-t_f)} \right)$$

- Reference time satisfies $\frac{4\pi}{3} v_\mathrm{w}^3 \left(\frac{3!}{\beta^4}\right) \Gamma_0 e^{-S(t_f)} = 1$. Note $t_f \gtrsim t_H$

Bubble density

- Bubbles can nucleate only in metastable phase
- In metastable phase rate/volume is $\Gamma_f = \Gamma_0 e^{-S(t_f)}$ at reference time.
- Nucleation rate averaged over both phases $\dot{n}_{\rm b}(t') = \Gamma(t')h(t')$ so

$$n_{\rm b}(t) = \int^t \Gamma(t')h(t')dt' = \Gamma_f \int^t e^{\beta(t'-t_f)} h(t')dt'$$

- Recall $\frac{4\pi}{3} v_{\rm w}^3 \left(3!/\beta^4\right) \Gamma_f = 1$, and $h(t) = \exp\left(-e^{\beta(t-t_f)}\right)$
- Hence

$$n_{\rm b}(t) = -\frac{\Gamma_f}{\beta} \int^t dt' \frac{dh(t')}{dt'} = \frac{\Gamma_f}{\beta}(1 - h(t))$$

- Final bubble density

$$n_{\rm b} = \frac{\Gamma_f}{\beta} = \frac{\beta^3}{8\pi v_{\rm w}^3}$$

- Mean bubble separation

$$R_* = n_{\rm b}^{-\frac{1}{3}} = (8\pi)^{\frac{1}{3}} v_{\rm w}/\beta$$

Bubble wall speed

- Recall equation for scalar field

$$\Box\phi - V_T'(\phi) \simeq \eta_T(\phi)(U \cdot \partial\phi)$$

- Consider wall frame, where fluid moves with velocity $v^z \simeq -v_\mathrm{w}$

$$\partial_z^2\phi - V_T'(\phi) \simeq -\eta_T(\phi)\gamma_\mathrm{w} v_\mathrm{w} \partial_z\phi$$

- Look for stable time-independent solution $\phi(z)$: constant wall speed
- Multiply both sides by $\partial_z\phi$ and integrate $\int dz$:

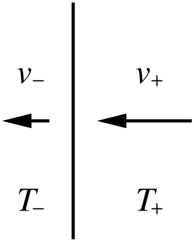
$$\Delta V_T = \gamma_\mathrm{w} v_\mathrm{w} \int dz \eta_T(\phi)(\partial_z\phi)^2$$

- Solve to get v_w (need to calculate $\eta_T(\phi)$ from Boltzmann equation).
- Warning: $\eta_T(\phi)$ may also depend on γ_w. Solutions may not exist ("runaway").[6]

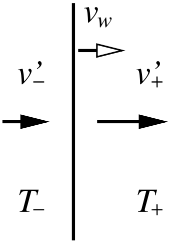
(6) See Bodeker & Moore 2009, 2017

Fluid flow at bubble wall

- Approximate planar symmetry near wall. Wall motion in $+z$ direction



Wall frame

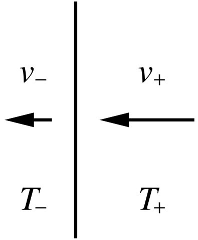


Universe frame

- Fluid motion in $-z$ direction
- Speeds $v_\pm > 0$

- Fluid motion in $+z$ direction
- $v'_\pm = \dfrac{v_\mathrm{w} - v_\pm}{1 - v_\mathrm{w} v_\pm}$

Energy-momentum conservation at bubble wall



Wall frame

- Fluid motion in $-z$ direction
- Speeds $v_\pm > 0$
- $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$

- Energy-Momentum conservation:

$$\partial_t T^{tt} + \partial_z T^{zt} = 0$$
$$\partial_t T^{tz} + \partial_z T^{zz} = 0$$

- Assume steady state and integrate $\int dz$

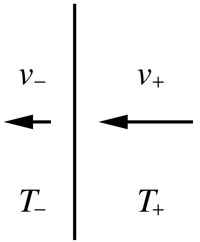
$$T^{zt}_- = T^{zt}_+$$
$$T^{zz}_- = T^{zz}_+$$

- Giving:

$$w_-\gamma_-^2 v_- = w_+\gamma_+^2 v_+$$
$$w_-\gamma_-^2 v_-^2 + p_- = w_+\gamma_+^2 v_+^2 + p_+$$

Bubble wall junction conditions

EM conservation: $w_- \gamma_-^2 v_- = w_+ \gamma_+^2 v_+, \quad w_- \gamma_-^2 v_-^2 + p_- = w_+ \gamma_+^2 v_+^2 + p_+$



Wall frame

- $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$
- Enthalpy $w = e + p$

- Rearrange

$$v_+ v_- = \frac{p_+ - p_-}{e_+ - e_-}, \quad \frac{v_+}{v_-} = \frac{e_- + p_+}{e_+ + p_-}$$

- Define[a] $\epsilon_\pm = \frac{1}{4}(e_\pm - 3p_\pm)$, $\epsilon = \epsilon_+ - \epsilon_-$
- Transition strength $\alpha_+ = \frac{4\epsilon}{3w_+}$
- Define $r = w_+/w_-$

$$v_+ v_- = \frac{1 - (1 - 3\alpha_+)r}{3 - 3(1 + \alpha_+)r},$$

$$\frac{v_+}{v_-} = \frac{3 + (1 - 3\alpha_+)r}{1 + 3(1 + \alpha_+)r}$$

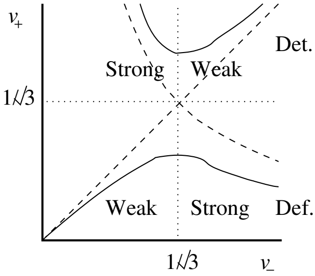
[a] $\frac{1}{4}$ of trace anomaly, or "vacuum" energy

Solution: strong and weak, deflagrations and detonations

$$v_+ v_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r}, \qquad \frac{v_+}{v_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$

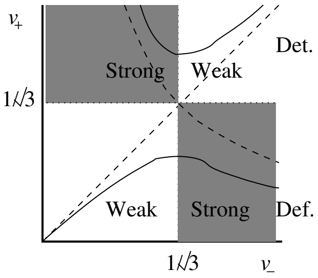
Solve for $v_+ = v_+(v_-, \alpha_+)$ [similar for $v_-(v_+, \alpha_+)$]

$$v_+ = \frac{1}{1+\alpha_+}\left[\frac{v_-}{2} + \frac{1}{6v_-} \pm \sqrt{\left(\frac{v_-}{2} - \frac{1}{6v_-}\right)^2 + \frac{2}{3}\alpha_+ + \alpha_+^2}\right]$$



- Strong: v_+ and v_- on opposite sides of $\frac{1}{\sqrt{3}}$
- Weak: v_+ and v_- on same side of $\frac{1}{\sqrt{3}}$
- Detonation: $v_+ > \frac{1}{\sqrt{3}}$
- Deflagration: $v_+ < \frac{1}{\sqrt{3}}$

Deflagrations and detonations: general remarks



- Recall

$$\alpha_+ = \frac{4(\epsilon_+ - \epsilon_-)}{3w_+}, \qquad r = \frac{w_+}{w_-}$$

- No strong deflagrations or detonations
- No deflagrations for $\alpha_+ > 1/3$
- Turning points at $v_- > \frac{1}{\sqrt{3}}$
- In bulk fluid $\alpha_+ = 0$, shocks obey

$$v_+ v_- = \frac{1}{3}, \qquad \frac{v_+}{v_-} = \frac{3+r}{1+3r}$$

Similarity solution: equations for v and T

- Recall: $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$
- Recall: EM conservation (away from wall): $\partial_\mu T^{\mu\nu} = 0$
- Project onto $U^\mu = \gamma(1, \mathbf{v})$ and $\bar{U}^\mu = \gamma(v, \hat{\mathbf{v}})$ ($\bar{U}^2 = +1$, $\bar{U} \cdot U = 0$)

$$U_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu (wU^\mu) + U \cdot \partial p = 0$$
$$\bar{U}_\nu \partial_\mu T^{\mu\nu} = w\bar{U}^\nu U \cdot \partial U_\nu + \bar{U} \cdot \partial p = 0$$

- Bubbles spherical, radius $R = v_{\rm w} t$ (take nucleation time $t' = 0$)
- Fluid velocity $\mathbf{v} = v(r,t)\hat{\mathbf{r}} \to v(\xi)\hat{\mathbf{r}}$, with $\xi = r/t$
- Speed of sound $c_{\rm s}^2 = \frac{\partial p}{\partial T} / \frac{\partial e}{\partial T}$

$$\frac{dv}{d\xi} = 2\frac{v}{\xi}\frac{1}{\gamma^2(1-\xi v)(\mu^2/c_{\rm s}^2 - 1)}$$
$$\frac{dw}{d\xi} = w\left(1 + \frac{1}{c_{\rm s}^2}\right)\gamma^2 \mu \frac{dv}{d\xi}$$

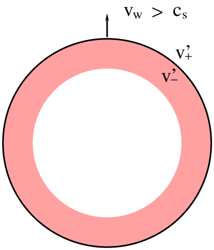
- $\mu = \frac{\xi - v}{1 - \xi v}$, fluid speed in expanding frames

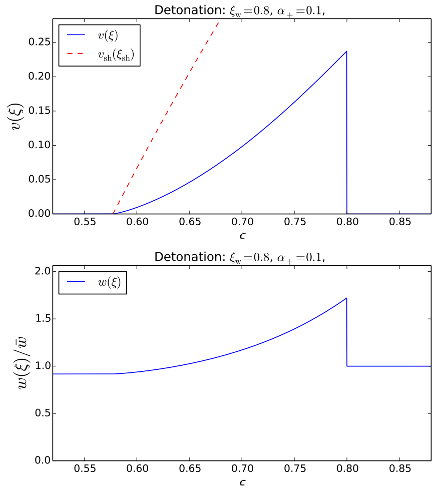
Similarity solution: invariant profiles (solutions for $v(\xi)$, $T(\xi)$)

- Boundary conditions:
  - $v \to 0$ for $\xi \to 0, \infty$
  - $v \to v'_\pm$ for $\xi \to \xi_\mathrm{w}\pm$[7]
- $v'_\pm$ are fluid speeds just ahead/behind of bubble wall in universe frame
  - e.g. $v'_+ = \mu(\xi_\mathrm{w}, v_+) = \frac{\xi_\mathrm{w} - v_+(\xi_\mathrm{w}, \alpha_+)}{1 - \xi_\mathrm{w} v_+(\xi_\mathrm{w}, \alpha_+)}$
- Intricate reasoning leads to three classes of solution:
  - Detonations
  - Deflagrations
  - Supersonic deflagrations ("hybrids")

[7] Write ξ_w for wall speed to avoid too many vs

Detonations



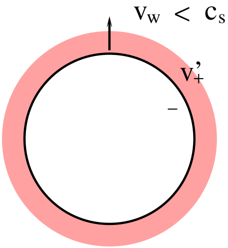


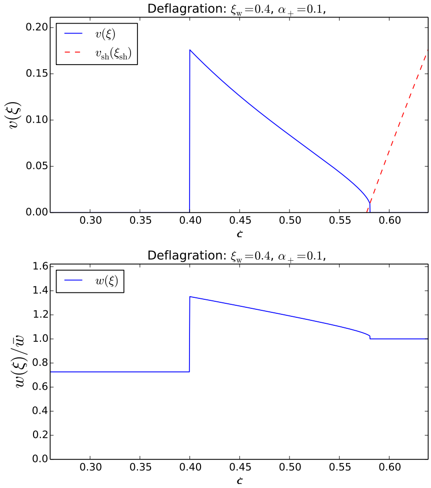
- Fluid at rest in front of wall $v'_+ = 0$
- Fluid dragged out behind (rarefaction wave):

$$v'_- = \frac{\xi_w - v_-(\xi_w, \alpha_+)}{1 - \xi_w v_-(\xi_w, \alpha_+)}$$

- $v \to 0$ at $\xi = c_s$

Deflagrations



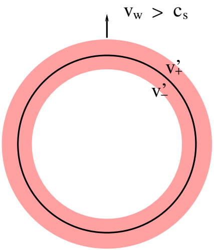


- Fluid at rest in behind wall $v'_- = 0$
- Fluid pushed out in front (compression wave):

$$v'_+ = \frac{\xi_w - v_+(\xi_w, \alpha_+)}{1 - \xi_w v_+(\xi_w, \alpha_+)}$$

- $v \to 0$ at $\xi = \xi_{sh}$
- For ξ_{sh} and $v(\xi_{sh})$, use $v_- = \frac{1}{3}\xi_{sh}$

Supersonic deflagrations (hybrids)

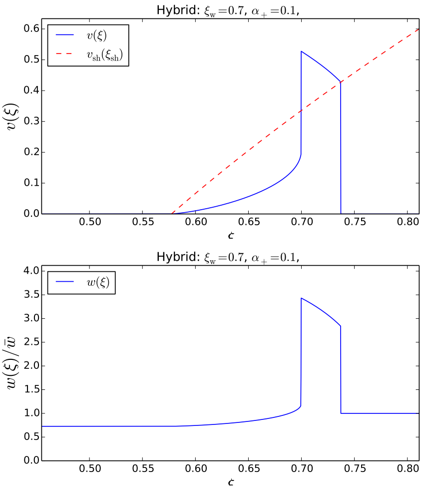


- Both compression and rarefaction

$$v'_+ = \frac{\xi_w - v_+(\xi_w, \alpha_+)}{1 - \xi_w v_+(\xi_w, \alpha_+)}$$

$$v'_- = \frac{\xi_w - c_s}{1 - \xi_w c_s}$$

- Behind wall $v_- = c_s$ (wall frame)



Conversion efficiency: vacuum energy to kinetic energy

- Kinetic energy of fluid $\int d^3x T^i_i$.
- Take ratio of kinetic energy of bubble to energy of bubble

$$K = \frac{\int d^3x\, w\gamma^2 v^2}{\frac{4\pi}{3} R^3 \bar{e}}$$

- Define a mean-square velocity for the fluid:

$$\overline{U}_{\rm f}^2 = \frac{3}{v_{\rm w}^3 \bar{w}} \int d\xi \xi^2 w\gamma^2 v^2$$

- Kinetic energy fraction $K = \Gamma \overline{U}_{\rm f}^2$, where adiabatic index $\Gamma = barw/\bar{e}$
- Strength parameter $\alpha = 4\epsilon/3\bar{w}$, where $\epsilon = |\epsilon_+ - \epsilon_-|$, $\epsilon_\pm = (e_\pm - 3p_\pm)/4$
- $K = K(\alpha, v_{\rm w})$
- Radial perturbation of radial velocity - compression/rarefaction $\to$ sound

Sound waves

Consider EM tensor for perturbations with z dependence only

$$T^{tt} = w\gamma^2 - p, \qquad T^{tz} = w\gamma^2 v^z, \qquad T^{zz} = w\gamma^2 (v^z)^2 + p$$

Perturbations: $\delta e = e - \bar{e}$, $\delta p = p - \bar{p}$, v^z all $\ll 1$

$$\partial_t T^{tt} + \partial_z T^{zt} = 0 \implies \partial_t(\delta e) + \bar{w}\partial_z v^z = 0 \tag{1}$$

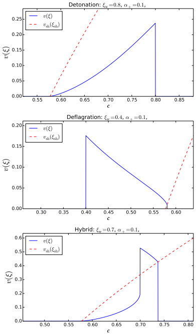
$$\partial_t T^{tz} + \partial_z T^{zz} = 0 \implies \bar{w}\partial_t v^z + \partial_z(\delta p) = 0 \tag{2}$$

Note that δp and δe both depends temperature T: $\delta p = \left(\frac{\partial p}{\partial T} / \frac{\partial e}{\partial T}\right) \delta e = c_\mathrm{s}^2 \delta e$
Hence equations (1) and (2) can be combined

$$\left(\partial_t^2 - c_\mathrm{s}^2 \partial_z^2\right) v^z = 0, \qquad \left(\partial_t^2 - c_\mathrm{s}^2 \partial_z\right) \delta p = 0$$

Sound wave is a collective mode of fluid velocity v^i and temperature T.
It is longitudinal: v^i is in direction of travel of wave.

Summary



- Bubble nucleation rate/volume

$$\Gamma(T) = \Gamma_0(T) e^{-S(T)}$$

- Transition rate parameter β

$$\Gamma(t) = \Gamma_f e^{\beta(t-t_f)}$$

with $\frac{4\pi}{3} v_{\rm w}^3 \left(3!/\beta^4\right) \Gamma_f = 1$

- Wall speed $v_{\rm w}$
- Transition strength $\alpha_+ = \frac{4\epsilon}{3w_+}$
  - Junction conditions ($r = w_+/w_-$)

$$v_+ v_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r}, \qquad \frac{v_+}{v_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$

  - Similarity solution for bubble growth: detonation, deflagration, hybrid
  - Kinetic energy fraction K

Reading

Relativistic hydrodynamics

- *Relativistic Hydrodynamics*, L. Rezzolla and O. Zanotti (OUP, 2013)
- *Fluid Mechanics*, L. Landau and Lifshitz ()

Bubble nucleation and growth

- *Decay of the false vacuum at finite temperature*, A. Linde (1983)
- *From Boltzmann equations to steady wall velocities*, T. Konstantin, G. Nardini, I. Rues [arXiv:1407.3132]
- *Energy Budget of Cosmological First-order Phase Transitions*, J.R. Espinosa, T. Konstantin, J.M. No, G. Servant [arXiv:1004.4187]
- *Bubble growth and droplet decay in cosmological phase transitions*, H. Kurki-Suonio, M. Laine [1996]
- *Nucleation and bubble growth in cosmological electroweak phase transitions*, K. Enqvist, J. Ignatius, K. Kajantie, K. Rummukainen [1992]
- *Growth of bubbles in cosmological phase transitions*, J. Ignatius, K. Kajantie, H. Kurki-Suonio, M. Laine [arXiv:astro-ph/9309059]