Phase transitions in the Early Universe

1. Thermodynamics and hydrodynamics in the early Universe

Mark Hindmarsh[1,2]

[1] Department of Physics & Astronomy
University of Sussex

[2] Department of Physics and Helsinki Institute of Physics
Helsinki University

Heraeus School
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Outline

Phase transitions & cosmology

Phase transitions in early Universe:

Thermal Changing $T(t)$ — Vacuum Changing field $\sigma(t)$

- **QCD phase transition**
  - Thermal (Confinement of strong interactions: quarks & gluons $\rightarrow$ hadrons)
- **Electroweak phase transition**
  - Thermal (First order: electroweak baryogenesis[1])
  - Vacuum: cold electroweak baryogenesis[2]
- **Grand Unified Theory & other high-scale phase transitions**
  - Thermal: topological defects[3]
  - Vacuum: hybrid inflation, topological defects, ... [4]

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[1] Kuzmin, Rubakov, Shaposhnikov 1988
[2] Smit and Tranberg 2002-6; Smit, Tranberg & Hindmarsh 2007
[3] Kibble 1976; Zurek 1985, 1996; Hindmarsh & Rajantie 2000
[4] Copeland et al 1994; Kofman, Linde, Starobinsky 1996

Phase transitions and relics

- Relics $\implies$ departure from equilibrium e.g. at phase transition
- e.g. 1st order phase transition – c.f. water boiling
  - Production of gravitational waves
- Topological defects
  - Cosmic Microwave Background fluctuations
  - Cosmic and γ rays
  - Gravitational waves

Event	T	t
QCD transition	100 MeV	10^{-3} s
Electroweak transition	100 GeV	10^{-11} s
GUT/Hybrid inflation	$< 10^{16}$ GeV	$> 10^{-36}$ s

Conventions

- Natural Units: $\hbar = 1$, $c = 1$, $k_B = 1$
- Natural Unit converter:

Quantity	Nat. U.	S.I. Conversion	
Energy:	GeV	1.6022×10^{-10}	Joule
Temperature:	GeV	1.1605×10^{13}	K
Mass:	GeV	1.7827×10^{-27}	kg
Length:	GeV^{-1}	1.9733×10^{-16}	m
Time:	GeV^{-1}	6.5822×10^{-25}	s

- Planck Mass (Energy): $M_P = \sqrt{\hbar c^5/G} = 1.2211 \times 10^{19}$ GeV
- Reduced Planck Mass $m_P = \sqrt{\hbar c^5/8\pi G} = 2.436 \times 10^{18}$ GeV
- $\bar{d}p = \frac{dp}{2\pi}$
- $\bar{\delta}(p) = 2\pi\delta(p)$
- Metric $- + + +$

Thermodynamics of harmonic oscillators 1: bosons

Partition function:

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$$

Leading to:

$$\begin{aligned} \text{free energy} \quad F &= -T \ln Z \\ \text{entropy} \quad S &= -\partial F/\partial T \\ \text{energy} \quad E &= Z^{-1}\,\mathrm{Tr}[\hat{H} e^{-\beta \hat{H}}] = F + TS \end{aligned}$$

Bosonic harmonic oscillator

- $\hat{H} = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)$
- $[\hat{a}, \hat{a}^\dagger] = 1$
- $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$
- $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$,

B.h.o. partition function

$$\begin{aligned} Z_{\text{Bho}} &= \textstyle\sum_{n=0}^{\infty} \langle n|e^{-\beta \hat{H}}|n\rangle \\ &= \textstyle\sum_{n=0}^{\infty} \exp[-\beta\omega(n + \frac{1}{2})] \\ &= e^{-\beta\omega/2}/(1 - e^{-\beta\omega}) \end{aligned}$$

$$\boxed{F_{\text{Bho}} = \frac{1}{2}\omega + T\ln(1 - e^{-\beta\omega})}$$

Free scalar field

Field operator:

$$\hat{\phi}(x) = \int \frac{\bar{d}^3 k}{2\omega_{\mathbf{k}}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x} \right), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 2\omega_{\mathbf{k}}\, \bar{\delta}^3(\mathbf{k} - \mathbf{k}').$$

Field equation:

$$(\Box - m^2)\hat{\phi}(x) = 0 \qquad \Longrightarrow \qquad (k^0)^2 = {\omega_{\mathbf{k}}}^2 = k^2 + m^2$$

Free scalar field is a collection of harmonic oscillators, one for each momentum **k**

Partition function: $Z_B = \prod_{\mathbf{k}} Z_{\mathrm{Bho}}$

Free energy: $F_B = -T \sum_{\mathbf{k}} \ln Z_{\mathrm{Bho}} \implies \boxed{F_B = \sum_{\mathbf{k}} \left(\frac{1}{2}\omega_{\mathbf{k}} + T \ln(1 - e^{-\beta\omega_{\mathbf{k}}}) \right)}$

Quantum statistics of fields: $\sum_{\mathbf{k}} \to V \int \bar{d}^3 k$

Thermodynamics of harmonic oscillators 2: fermions

Partition function:

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$$

Fermionic harmonic oscillator

- $\hat{H} = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)$
- $\{\hat{a}, \hat{a}^\dagger\} = 1$
- $\hat{a}|0\rangle = 0, \hat{a}|1\rangle = |0\rangle$
- $\hat{a}^\dagger|0\rangle = |1\rangle, \hat{a}^\dagger|1\rangle = 0,$

F.h.o. partition function

$$\begin{aligned} Z_{\mathrm{Fho}} &= \textstyle\sum_{n=0}^{1} \langle n | e^{-\beta \hat{H}} | n \rangle \\ &= \textstyle\sum_{n=0}^{1} \exp[-\beta\omega(n + \frac{1}{2})] \\ &= e^{\beta\omega/2}/(1 + e^{-\beta\omega}) \end{aligned}$$

$$\boxed{F_{\mathrm{Fho}} = -\tfrac{1}{2}\omega - T\ln(1 + e^{-\beta\omega})}$$

Free fermionic field

Field operator (Dirac 4-component field):

$$\hat{\psi}(x) = \int \frac{\bar{d}^3 k}{2\omega_{\mathbf{k}}} \left(u_A(\mathbf{k}) \hat{b}^A_{\mathbf{k}} e^{-ik\cdot x} + \bar{v}_A(\mathbf{k}) \hat{d}^{A\dagger}_{\mathbf{k}} e^{ik\cdot x} \right), \quad \begin{array}{ll} \{\hat{b}^A_{\mathbf{k}}, \hat{b}^{B\dagger}_{\mathbf{k}'}\} & = 2\omega_{\mathbf{k}} \delta^{AB}\, \bar{\delta}^3(\mathbf{k}-\mathbf{k}') \\ \{\hat{d}^A_{\mathbf{k}}, \hat{d}^{B\dagger}_{\mathbf{k}'}\} & = 2\omega_{\mathbf{k}} \delta^{AB}\, \bar{\delta}^3(\mathbf{k}-\mathbf{k}') \end{array}$$

Field equation:

$$(i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) = 0 \qquad \Longrightarrow \qquad \begin{array}{l} (k^0)^2 = \omega_{\mathbf{k}}{}^2 = k^2 + m^2 \\ (\not{k} - m) u_A(\mathbf{k}) = 0 \\ (\not{k} + m) \bar{v}_A(\mathbf{k}) = 0 \end{array}$$

Free fermionic field is a collection of harmonic oscillators, 4 for each momentum **k**

Partition function: $Z_F = \prod_{\mathbf{k}} Z_{\rm Fho}$

Free energy: $F_F = -T \sum_{\mathbf{k}} \ln Z_{\rm Fho} \implies \boxed{F = \sum_{\mathbf{k}} \left(-\frac{1}{2}\omega_{\mathbf{k}} - T \ln(1 + e^{-\beta\omega_{\mathbf{k}}}) \right)}$

Quantum statistics of fields: $\sum_{\mathbf{k}} \to V \int \bar{d}^3 k$

Free energy (density) of an ideal gas

Free relativistic particles of mass m in equilibrium (zero chemical potential)

$$f = -\eta T \int \bar{d}^3 k \ln(1 + \eta e^{-E/T})$$

where $\eta = \pm 1$ (Fermi-Dirac/Bose-Einstein).

- Entropy density: $s = -\frac{\partial f}{\partial T}$
- Energy density: $e = f + Ts$
- Thermodynamic pressure: $p = Ts - e$ (Note $p = -f$)

To find equilibrium state we minimise free energy

- Dimensions: $f = T^4 \phi(m/T)$ with $\phi(0) = -g_{\rm eff}\pi^2/90$.

Defines effective number of relativistic degrees of freedom $g_{\rm eff}$.

Free energy: exact formulae in high T expansion

Bosons:

$$f_B = -\frac{\pi^2}{90}T^4 + \frac{m^2T^2}{24} - \frac{(m^2)^{\frac{3}{2}}T}{12\pi} - \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_bT^2}\right) - \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell (-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

Fermions:

$$f_F = -\frac{\pi^2}{90}\frac{7}{8}T^4 + \frac{m^2T^2}{48} + \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_fT^2}\right) + \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell (-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}(1-2^{-2\ell-1})\Gamma(\ell+\tfrac{1}{2})\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

$a_b = 16\pi^2\ln(\frac{3}{2} - 2\gamma_E)$, $a_f = a_b/16$, $\gamma_E = 0.5772\ldots$ (Euler's constant)

Effective potential for scalar field with gauge fields and fermions

Let scalar field give masses to

- scalars ($M_S(\bar{\phi})$),
- vectors ($M_V(\bar{\phi})$)
- (Dirac) fermions ($M_F(\bar{\phi})$)

Define effective potential $V_T(\bar{\phi}) = V_0(\bar{\phi}) + f(\bar{\phi}) + g_{\text{eff}}\pi^2 T^4/90$

$$
\begin{aligned}
V_T(\bar{\phi}) &= V_0(\bar{\phi}) + \frac{T^2}{24}\left(\sum_S M_S^2(\bar{\phi}) + 3\sum_V M_V^2(\bar{\phi}) + 2\sum_F M_F^2(\bar{\phi})\right) \\
&- \frac{T}{12\pi}\left(\sum_S (M_S^2(\bar{\phi}))^{\frac{3}{2}} + 3\sum_V (M_V^2(\bar{\phi}))^{\frac{3}{2}}\right) \\
&+ \frac{1}{64\pi^2}\sum_S M_S^4(\bar{\phi}) \ln\left(\frac{M_S^2}{a_b T^2}\right) + \frac{3}{64\pi^2}\sum_V M_V^4(\bar{\phi}) \ln\left(\frac{M_V^2}{a_b T^2}\right) \\
&- \frac{2}{64\pi^2}\sum_F M_F^4(\bar{\phi}) \ln\left(\frac{M_F^2}{a_f T^2}\right) + \cdots
\end{aligned}
$$

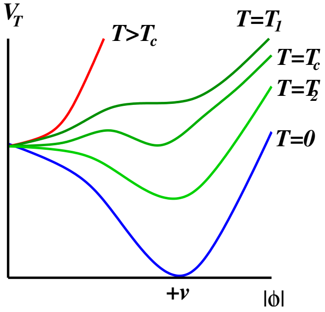
Neglect higher order terms where $M^2(\phi)/T^2 \ll 1$.

Phase transition (weakly coupled field theory)

Effective potential: expand in $\bar{\phi}/T$

$$V_T \simeq \frac{D}{2}(T^2 - T_0^2)|\bar{\phi}|^2 - \frac{A}{3}T|\bar{\phi}|^3 + \frac{\lambda_T}{4!}|\bar{\phi}|^4$$

- High temperature: equilibrium at $\bar{\phi} = 0$.
- Second minimum develops at T_1, $\phi_b(T)$.
- Critical temperature T_c: $f(0) = f(\bar{\phi}_b)$.
- System can supercool below T_c (until T_0).
- First order transition (apparently)
- Latent heat $\mathcal{L} = T_c\,\Delta s(T_c)$
- 1st order from cubic term (bosons only)



Degrees of freedom of SM: mostly coloured

		M($T=0$)	g		M($T=0$)	g	
	γ	0	2	g	0	16	
	ν_e	$\lesssim$ 1 eV	2	u	3 MeV	12	
	ν_μ	$\lesssim$ 1 eV	2	d	7MeV	12	
	ν_τ	$\lesssim$ 1 eV	2	s	76 MeV	12	
	e	0.5 MeV	4	c	1.2 GeV	12	
	μ	106 MeV	4	b	4.2 GeV	12	
	τ	1.7 GeV	4	t	174 GeV	12	
	W	80 GeV	6				
	Z	91 GeV	3				
	h	125 GeV	1				
>1 TeV:			$\frac{7}{8}18+8$			$\frac{7}{8}72+16$	72/106.75
40 GeV:			$\frac{7}{8}18+2$			$\frac{7}{8}60+16$	68.5/84.25
0.4 GeV:			$\frac{7}{8}14+2$			$\frac{7}{8}36+16$	47.5/61.75

QCD interactions important, especially around 1GeV

t, W, Z, h contribute most to V_T around 100GeV: largest mass change

Standard Model effective potential in weak coupling approximation

	h	$W^\pm$	Z	t
M/GeV	125	80.4	91.2	174
d.o.f.	1	6	3	$\frac{7}{8}12$
$M(\bar\phi)$	$\sqrt{V_0''(\bar\phi)}$	$\frac{1}{2}g_w\bar\phi$	$\frac{1}{2}\sqrt{g_w^2+g'^2}\bar\phi$	$\sqrt{2}y_t\bar\phi$

Form of effective potential: $V_T \simeq \frac{D}{2}(T^2-T_0^2)|\bar\phi|^2 - \frac{A}{3}T|\bar\phi|^3 + \frac{\lambda_T}{4!}|\bar\phi|^4$

$$D = \frac{1}{12\bar\phi^2}\left(6M_W^2+3M_Z^2+6M_t^2\right) \qquad A = \frac{1}{12\pi\bar\phi^2}\left(6M_W^3+3M_Z^3\right)$$

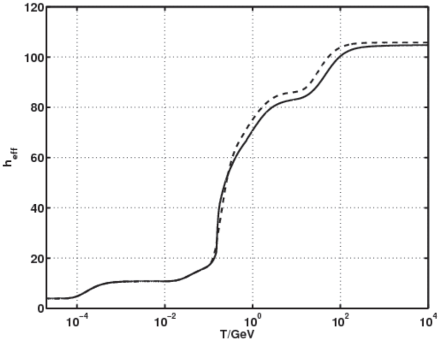
$$\lambda_T = \lambda - \frac{1}{16\pi^2\bar\phi^4}\left(6M_W^4\ln\left(\frac{M_W^2}{a_bT^2}\right)+3M_Z^4\ln\left(\frac{M_Z^2}{a_bT^2}\right)-4M_t^4\ln\left(\frac{M_t^2}{a_fT^2}\right)\right)$$

Predicts: $T_c = 166$ GeV, $T_0 = 165$ GeV

Transition is very weak.

Standard Model effective degrees of freedom

Ideal gas, model QCD transition[5] (dashed)
With interactions, lattice QCD[6] (solid)



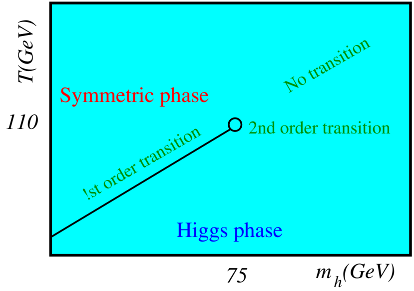
Temp.	Event
100 GeV	t non-relativistic
1 GeV	b non-relativistic
500 GeV	c, τ non-relativistic
200 MeV	QCD phase transition
30 MeV	μ non-relativistic
2 MeV	ν freeze-out
0.2 MeV	e non-relativistic
1 eV	matter = radiation
0.1 eV	photon decoupling

(5) Olive 1981
(6) Hindmarsh & Philipsen 2005, Laine & Schroder 2006, Borsanyi et al 2016

Electroweak phase transition in the Standard Model

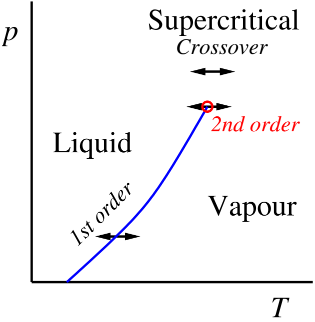
Interactions are important!

Standard Model phase diagram



Kajantie et al 1996

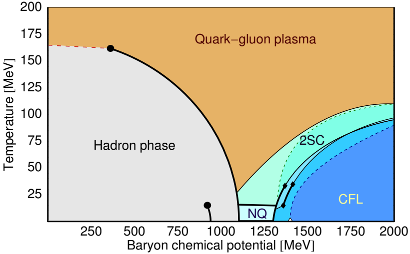
SM is cross-over



Water phase diagram (sketch)

QCD phase diagram

- $\eta_B = n_B/n_\gamma = (6.10 \pm 0.04) \times 10^{-10}$ (Planck)[7]
- Low $\eta_B \implies$ low chemical potential

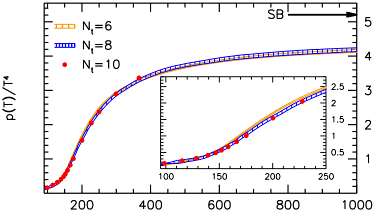


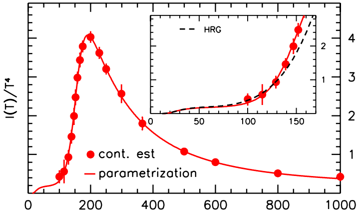
Ruester et al hep-ph/0503184

(7) Ade et al 2015

QCD equation of state

- Budapest-Marseille-Wuppertal lattice (physical quark masses)[8]
- Shown: pressure and trace anomaly $I(T) = \rho(T) - 3p(T)$ (with fit)





- Can model with hadronic resonance gas at low T

[8] Borsányi et al. (2010)

1st order phase transitions in SM extensions

- 2HDM (2 Higgs doublet model)
  - Extra scalars (A^0, H^0, $H^\pm$) increase strength of cubic term.
  - Strong phase transition when $m_{A^0} \gtrsim 400$ GeV[9]
- Extra singlet scalars
  - Tree level first order phase transition
  - Strong phase transition with SM-like phenomenology allowed[10]
- Effective field theory with h^6 operator[11]
  - e.g. by integrating out singlet[12]
  - $V_T(\phi) \simeq c_0 + c_1(T)h^2 + c_2 h^4 + c_3 h^6 + \cdots$
  - $c_2 < 0$ gives 1st order transition at tree level.
- etc. etc. etc.

(9) Dorsch, Huber, No (2015)
(10) Ashoorioon, Konstandin (2009)
(11) Grojean, Servant, Wells (2005)
(12) Huber et al (2006)