Introduction to Classical Gauge Field Theory and to Batalin-Vilkovisky Quantization

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A course by

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Chapter 1

Locality and the horizontal complex

This set of lectures loosely follows [1] by emphasizing Lagrangian field theoretic aspects that are not covered there in detail, and are not usually treated in standard textbooks either. The choice of material is to a large extent idiosyncratic. References to well-known original work underlying the subject, such as [2–5], have generally been omitted. They can be found in the cited reviews.

The goal of the first lecture is to rephrase the basic objects of classical field theory in the language of jet bundles. Although this requires a bit of work in the beginning, it is worthwhile because objects such as actions and functional derivatives can be phrased in a purely algebraic way in this framework.

Original work in this context goes back for instance to [6–8]. We follow here the exposition of [9–11] (see also [12]), where detailed proofs and more references to original work can be found. The first three chapters have been treated in the current form in [13].

1.1 Jet-bundles

In classical mechanics the central object is an **action functional** S, which is the integral of a Lagrangian density L:

$$S[q] = \int_{t_1}^{t_2} dt\, L\left(q^i(t), \dot{q}^i(t), t\right) . \tag{1.1}$$

The equations of motion are obtained by requiring the variation of the action to be an extremum at a classical solution. More precisely, when considering virtual variations at fixed time that vanish at the end points, this amounts to requiring that the **Euler-Lagrange derivatives** of the Lagrangian vanish. If the Lagrangian L is a function of the generalized coordinates q^i and their first order time derivatives $\dot{q}^i$, the latter are defined by

$$\frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}. \tag{1.2}$$

Remark 1. *In order for the Euler-Lagrange operator to make sense, q^i and $\dot{q}^i$ have to be considered as independent variables. The space on which they are coordinates is the (first order) jet-space.*

Remark 2. *When acting on a function that depends on the first order time derivatives, the total time derivative derivative* $\frac{d}{dt}$ *involves second order time derivatives. In other words, it maps functions defined on the first order jet-space to functions defined on the second order jet-space.*

The underlying geometrical structure is described by introducing a new mathematical object, the **jet bundle**. This is a fiber bundle[1] which has as base space M the real line $\mathbb{R}$ in the case of classical mechanics and Minkowski space $\mathbb{R}^{3,1}$ in the case of relativistic field theory, and as fiber V a manifold whose local coordinates are the fields and their derivatives. Coordinates on the base are often called "independent variables", while coordinates on the fiber are dependent ones.

In the case of classical mechanics, restricting to the case of Lagrangians depending only on the coordinates and velocities, the jet bundle will be parametrized by $t, q, \dot{q}$.

For a field theory, let V be locally parametrized by the fields ϕ^i (which we take to be even for simplicity), and let

$$E \equiv M \times V. \tag{1.3}$$

Let us then denote by V^k the **k-th jet space**, i.e. the manifold obtained by extending the set of dependent coordinates to

$$\phi^i, \phi^i_\mu, \ldots, \phi^i_{\mu_1 \ldots \mu_k}, \tag{1.4}$$

where the $\phi^i_{\mu_1 \ldots \mu_k}$ are completely symmetric in their lower indices.

Locally, the **jet bundle** $J^k(E)$ is the direct product

$$J^k(E) = M \times V^k. \tag{1.5}$$

When taking suitable **sections** s, the coordinates of V^k are to be identified with field "histories" and their derivatives up to order k,

$$\begin{aligned} s : M &\longrightarrow J^k(E) \\ x^\mu &\mapsto (x^\mu, \phi^i(x), \frac{\partial \phi^i(x)}{\partial x^\mu}, \ldots). \end{aligned} \tag{1.6}$$

Remark 3. *The components* $g_{\mu\nu}$ *of the metric in General Relativity are not independent local coordinates for the fiber of the jet bundle, since they must satisfy, besides symmetry, the additional condition*

$$\det g_{\mu\nu} \neq 0. \tag{1.7}$$

[1] The reader unfamiliar with the concept of fiber bundles should not be worried: for what concerns our discussion, it suffices to have in mind the conceptual idea that underlies them. As a Riemannian manifold is a smooth deformation of $\mathbb{R}^n$, which locally looks like flat space, a fiber bundle can be thought as a smooth deformation of a direct product of two spaces, that locally looks like the direct product itself.

Local functions

$$f[x,\phi] = f(x,\phi^i,\dots,\phi^i_{\mu_1\dots\mu_k}) \tag{1.8}$$

are defined to be smooth functions on $J^k(E)$, for some (unspecified) k. They are functions that depend on the base space coordinates x^μ, the fields and a finite number of their derivatives. Typical examples of local functions are the Lagrangian (density) L of classical mechanics or of field theory.

A **vector field** on $J^k(E)$ has the form

$$\mathbf{v} = a^\mu \frac{\partial}{\partial x^\mu} + b^i \frac{\partial}{\partial \phi^i} + \sum_{l=1}^{k} \sum_{0\leqslant\mu_1\leqslant\dots\leqslant\mu_k} b^i_{\mu_1\dots\mu_k} \frac{\partial}{\partial \phi^i_{\mu_1\dots\mu_l}}. \tag{1.9}$$

This notation is not very practical, as it does not allow one to use Einstein's summation convention for repeated indices. One thus defines the **symmetrized derivative**

$$\frac{\partial^s}{\partial \phi^i_{\mu_1\dots\mu_l}} = \frac{m_0!\dots m_{n-1}!}{l!} \frac{\partial}{\partial \phi^i_{\mu_1\dots\mu_l}}, \tag{1.10}$$

where m_μ denotes the number of times the index μ appears in $\mu_1\dots\mu_l$, and in terms of which equation (1.9) takes the more compact form

$$\mathbf{v} = a^\mu \frac{\partial}{\partial x^\mu} + b^i \frac{\partial}{\partial \phi^i} + \sum_{l=1}^{k} b^i_{\mu_1\dots\mu_l} \frac{\partial^s}{\partial \phi_{\mu_1\dots\mu_l}}. \tag{1.11}$$

This expression can be simplified further by using a **multi-index notation**:

$$(\mu) \equiv \{\varnothing, \mu_1, \mu_1\mu_2, \dots\}, \quad |\mu| = 0,1,2,\dots, \tag{1.12}$$

with (μ) the multi-index and $|\mu|$ its length, the definition of a vector field (1.9) becomes

$$\mathbf{v} = a^\mu \frac{\partial}{\partial x^\mu} + b^i_{(\mu)} \frac{\partial}{\partial \phi^i_{(\mu)}}, \tag{1.13}$$

in which Einstein's summation convention for multi-indices includes a summation over their lengths.

1.2 Total and Euler-Lagrange derivatives

Total derivatives are a generalization of $\frac{d}{dt}$ in equation (1.2), from classical mechanics to field theory, and to higher order derivatives:

$$\begin{aligned} \partial_\nu &\equiv \frac{\partial}{\partial x^\nu} + \sum_l \phi^i_{\mu_1\dots\mu_l\nu} \frac{\partial^s}{\partial \phi^i_{\mu_1\dots\mu_l}} \\ &= \frac{\partial}{\partial x^\nu} + \phi^i_{((\mu)\nu)} \frac{\partial}{\partial \phi^i_{(\mu)}}. \end{aligned} \tag{1.14}$$

Again, they are not vector fields on some $J^k(E)$ for a fixed k: when acting on smooth functions on $J^k(E)$, they produce smooth functions on $J^{k+1}(E)$.

Total derivatives satisfy two important properties: the first one is that, if a local function is evaluated at a section,

$$f|_s = f\left(x, \phi^i(x), \dots, \frac{\partial \phi^i(x)}{\partial x^{\mu_1} \dots \partial x^{\mu_k}}\right), \tag{1.15}$$

then

$$(\partial_\mu f)\,|_s = \frac{d}{dx^\mu}\left(f|_s\right). \tag{1.16}$$

The second one is that total derivatives commute:

$$[\partial_\mu, \partial_\nu] = 0. \tag{1.17}$$

The **Euler-Lagrange derivative** of a local function f is defined as

$$\begin{aligned} \frac{\delta f}{\delta \phi^i} &= \sum_{l=0}^{k} (-)^l \partial_{\mu_1} \dots \partial_{\mu_l} \frac{\partial^s f}{\partial \phi^i_{\mu_1 \dots \mu_l}} \\ &= (-)^{|\mu|} \partial_{(\mu)} \frac{\partial f}{\partial \phi^i_{(\mu)}}. \end{aligned} \tag{1.18}$$

A first important lemma is the following:

Lemma 1. *The Euler-Lagrange derivative of a local function is zero if and only if it is a total divergence, i.e.,*

$$\frac{\delta f}{\delta \phi^i} = 0 \qquad \Longleftrightarrow \qquad f = \partial_\mu j^\mu, \tag{1.19}$$

for some local functions j^μ.

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1.2.1 Exercise 1

Prove Lemma 1.

Proof. Let us first prove that

$$f = \partial_\mu j^\mu \qquad \Longrightarrow \qquad \frac{\delta f}{\delta \phi^i} = 0. \tag{1.20}$$

In order to show this, we need to use the commutation relation

$$\left[\frac{\partial^s}{\partial \phi_{\mu_1 \dots \mu_k}}, \partial_\nu\right] = \delta^{\mu_1}_{(\nu} \delta^{\mu_2}_{\lambda_1} \dots \delta^{\mu_k}_{\lambda_{k-1})} \frac{\partial^s}{\partial \phi^i_{(\lambda_1 \dots \lambda_{k-1})}} \equiv \delta^{(\mu)}_{(\nu(\lambda))} \frac{\partial}{\partial \phi^i_{(\lambda)}}, \tag{1.21}$$

which follows directly from the various definitions. In multi-index notation, it becomes

$$\left[\frac{\partial}{\partial \phi^i_{(\mu)}}, \partial_\nu\right] = \delta^{(\mu)}_{((\lambda)\nu)} \frac{\partial}{\partial \phi^i_{(\lambda)}}. \tag{1.22}$$

With this, we are ready to show (1.20):

$$\begin{aligned}
\frac{\delta}{\delta\phi^i}(\partial_\mu j^\mu) &= (-)^{|\nu|}\partial_{(\nu)}\left(\frac{\partial}{\partial\phi^i_{(\nu)}}(\partial_\mu j^\mu)\right)\\
&= (-)^{|\nu|}\partial_{(\nu)}\left(\left[\frac{\partial}{\partial\phi^i_{(\nu)}},\partial_\mu\right]j^\mu\right) + (-)^{|\nu|}\partial_{(\nu)}\partial_\mu\left(\frac{\partial j^\mu}{\partial\phi^i_{(\nu)}}\right)\\
&= (-)^{|\nu|}\delta^{(\nu)}_{(\mu(\lambda))}\partial_{(\nu)}\frac{\partial j^\mu}{\partial\phi^i_{(\lambda)}} + (-)^{|\nu|}\partial_{(\nu)}\partial_\mu\frac{\partial j^\mu}{\partial\phi^i_{(\nu)}}\\
&= (-)^{|\lambda|+1}\partial_{(\mu(\lambda))}\frac{\partial j^\mu}{\partial\phi^i_{(\lambda)}} + (-)^{|\lambda|}\partial_{(\lambda)\mu}\frac{\partial j^\mu}{\partial\phi^i_{(\lambda)}} = 0.
\end{aligned} \tag{1.23}$$

We now turn to the proof of the other implication, namely

$$f = \partial_\mu j^\mu \qquad \Longleftarrow \qquad \frac{\delta f}{\delta\phi^i} = 0. \tag{1.24}$$

We can write

$$\begin{aligned}
f[x,\phi] - f[x,0] &= \int_0^1 d\lambda\frac{d}{d\lambda}f(x,\lambda\phi)\\
&= \int_0^1 \frac{d\lambda}{\lambda}(\phi^i_{(\nu)}\frac{\partial}{\partial\phi^i_{(\nu)}}f)[x,\lambda\phi)].
\end{aligned} \tag{1.25}$$

By repeated "integrations by parts"[2], we have on the one hand that

$$\phi^i_{(\nu)}\frac{\partial}{\partial\phi^i_{(\nu)}}f = \phi^i\frac{\delta f}{\delta\phi^i} + \partial_\mu\tilde{k}^\mu[x,\phi], \tag{1.26}$$

with $\tilde{k}^\mu[x,0] = 0$. On the other hand, $f[x,0] = \frac{\partial}{\partial x^\mu}\xi^\mu(x)$ for some $\xi^\mu(x)$ on account of the ordinary Poincaré lemma for $\mathbb{R}^n$ (of which we will provide a proof at the end of this lecture, as it is the most basic and prototypical instance of a cohomological computation). When using these relations in (1.25), together with the assumption $\frac{\delta f}{\delta\phi^i} = 0$, we find

$$f = \partial_\mu j^\mu, \quad j^\mu = \xi^\mu + \int_0^1 \frac{d\lambda}{\lambda}\tilde{k}^\mu[x,\lambda\phi]. \tag{1.27}$$

□

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Note also that, according to (1.25), local functions can be decomposed as $f[x,\phi] = f[x,0] + \tilde{f}[x,\phi]$, with $\tilde{f}[x,0] = 0$, and that Lemma 1 holds in the space of local functions $\tilde{f}$ with no field-independent part.

[2] There is no actual integration involved, what we mean is the use of Leibniz' rule as appropriate in the context of integrations by parts.

1.3 Local functionals

A **local functional** is the integral, on the base manifold of the jet bundle, of a local function evaluated at sections $\phi^i(x)$ (of compact support or vanishing at the boundary),

$$F[\tilde{f}, \phi(x)] = \int_M d^n x\, \tilde{f}|_s. \tag{1.28}$$

Lemma 2. *For two local functionals F, G, we have*

$$F[\tilde{f}, \phi(x)] = G[\tilde{g}, \phi(x)] \tag{1.29}$$

$\forall \phi^i(x)$ *if and only if*

$$\tilde{f} = \tilde{g} + \partial_\mu \tilde{j}^\mu, \tag{1.30}$$

which in turn is equivalent to

$$\frac{\delta}{\delta \phi^i}(\tilde{f} - \tilde{g}) = 0, \tag{1.31}$$

because of Lemma 1.

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1.3.1 Exercise 2

Prove Lemma 2.

Proof. The implication

$$\tilde{f} = \tilde{g} + \partial_\mu \tilde{j}^\mu \qquad \Longrightarrow \qquad F[\tilde{f}, \phi(x)] = G[\tilde{g}, \phi(x)] \tag{1.32}$$

is almost obvious. In fact, we have

$$\begin{aligned} F[\tilde{f}, \phi(x)] = \int d^n x\, \tilde{f}|_s &= \int d^n x\, \tilde{g}|_s + \int d^n x\, (\partial_\mu \tilde{j}^\mu)|_s \\ &= G[\tilde{g}, \phi(x)] + \int d^n x\, \frac{d}{dx^\mu}(j^\mu|_s) = G[\tilde{g}, \phi(x)], \end{aligned} \tag{1.33}$$

where we have used property (1.16) of total derivatives evaluated at a section, together with the fact that we are assuming suitable fall-off or boundary conditions on the fields.

Proving the other implication is also straightforward: if $F[\tilde{f}, \phi(x)] = G[\tilde{g}, \phi(x)]$, we must have

$$\int d^n x\, (\tilde{f} - \tilde{g})|_s = 0. \tag{1.34}$$

Since this must be true for all $\phi^i(x)$, the functional $\int d^n x(\tilde{f} - \tilde{g})|_s$ is a constant in $\phi(x)$. This means that its functional derivative must vanish:

$$\frac{\delta}{\delta \phi^i(x)} \int d^n x(\tilde{f} - \tilde{g}) = 0. \tag{1.35}$$

But the vanishing of the functional derivative of a local functional means the vanishing of the Euler-Lagrange derivative of its associated local function when all boundary terms can be neglected, i.e.,

$$\frac{\delta \tilde{f}}{\delta \phi^i} = \frac{\delta \tilde{g}}{\delta \phi^i}, \tag{1.36}$$

which is what we wanted to prove. □

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This motivates the following algebraic definition of local functionals [14]: a local functional is an equivalence class of local functions

$$F[\tilde{f}, \phi(x)] \longleftrightarrow \left\{ [\tilde{f}] : \tilde{f} \sim \tilde{g} \Leftrightarrow \frac{\delta}{\delta \phi^i}(\tilde{f} - \tilde{g}) = 0 \right\}, \tag{1.37}$$

so that we can stop thinking about functionals and functional analysis, and concentrate on suitable equivalence classes of local functions instead.

1.4 The Horizontal Complex

Up to now we have mainly discussed vector fields on the jet bundle. As in the case of ordinary differential geometry, we will now introduce suitable differential forms. A **horizontal differential form** on the jet bundle is an object

$$\omega = \sum_{l=0}^{n} \sum_{0 \leqslant \mu_1 < \cdots < \mu_l \leqslant n-1} \omega_{\mu_1 \ldots \mu_l} dx^{\mu_1} \ldots dx^{\mu_l}, \tag{1.38}$$

where $\omega_{\mu_1 \ldots \mu_l}$ are local functions on the jet bundle. We will omit the wedge products between forms and treat the dx's as anticommuting Grassmann odd variables instead. In this local context, forms in top form degree n correspond to Lagrangian densities times the volume form. The **horizontal differential** d_H is then defined as

$$\begin{aligned} d_H :& \Omega^p \longrightarrow \Omega^{p+1} \\ d_H \omega^p &= dx^\mu \partial_\mu \omega^p. \end{aligned} \tag{1.39}$$

As a consequence of the fact that total derivatives commute, the horizontal differential is nilpotent:

$$[\partial_\mu, \partial_\nu] = 0 \qquad \Longrightarrow \qquad d_H^2 = 0. \tag{1.40}$$

This allows us to define the cohomology of d_H: we define p-**cocycles** Z^p and p-**coboundaries** B^p as

$$\omega^p \in Z^p \qquad \Longleftrightarrow \qquad d_H \omega^p = 0, \tag{1.41}$$

$$\omega^p \in B^p \qquad \Longleftrightarrow \qquad \omega^p = d_H \eta^{p-1}, \tag{1.42}$$

and as usual we define the cohomology groups H^p as

$$H^p(d_H, \Omega) \simeq Z^p/B^p. \tag{1.43}$$

The cohomology of differential forms on $\mathbb{R}^n$ is encoded in the Poincar lemma, which implies that any closed form with form degree at least 1 is exact. This theorem is generalized to the present case, that of a trivial jet bundle, but before discussing this generalization, let us review the ordinary Poincar lemma for the de Rham complex in $\mathbb{R}^n$.

Theorem 1. *(Poincar lemma) The only non-trivial de Rham cohomology group of $\mathbb{R}^n$ is in form degree 0:*

$$H^p(d, \Omega) = \delta_0^p \mathbb{R}. \tag{1.44}$$

Proof. Let us define the operator

$$\rho = x^\mu \frac{\partial}{\partial dx^\mu}. \tag{1.45}$$

Note that, if we define the vector field $v = x^\mu \frac{\partial}{\partial x^\mu}$, we have

$$\rho(dx^\mu) = x^\nu \frac{\partial}{\partial dx^\nu} dx^\mu = x^\mu = \iota_v(dx^\mu), \tag{1.46}$$

so that we are in fact representing the operation of contracting differential forms along vector fields by a suitable operator involving the Grassmann odd variables dx^μ. In this context, we have

$$N \equiv \{d, \rho\} = \left(x^\mu \frac{\partial}{\partial x^\mu} + dx^\mu \frac{\partial}{\partial dx^\mu} \right). \tag{1.47}$$

Using this, we can write for a generic p-form $\omega \in \Omega^p(\mathbb{R}^n)$,

$$\begin{aligned} \omega(x, dx) - \omega(0,0) &= \int_0^1 d\lambda \frac{d}{d\lambda} \omega(\lambda x, \lambda dx) \\ &= \int \frac{d\lambda}{\lambda} \left[\left(x^\mu \frac{\partial}{\partial x^\mu} + dx^\mu \frac{\partial}{\partial dx^\mu} \right) \omega \right] (\lambda x, \lambda dx) \\ &= \int_0^1 \frac{d\lambda}{\lambda} \left(\{d, \rho\} \omega \right) (\lambda x, \lambda dx) \end{aligned} \tag{1.48}$$

In particular, if $d\omega = 0$, equation (1.48) becomes

$$\omega(x, dx) = \omega(0,0) + dI(\omega), \tag{1.49}$$

where

$$I(\omega) = \int_0^1 \frac{d\lambda}{\lambda} (\rho\omega)(\lambda x, \lambda dx), \tag{1.50}$$

sends p-forms to $p-1$ forms. We now see that all closed forms in $\mathbb{R}^n$ are exact, except for the constant 0-forms,

$$H^p(d, \Omega) = \delta_0^p \mathbb{R}, \tag{1.51}$$

which is what we wanted to prove. □

Remark 4. *As a consequence of the Poincar lemma, every top form, which is necessarily closed, must be exact. If Lagrangian densities would correspond to top forms in the de Rham complex, they all would be given by a divergence. Since there are Euler-Lagrange equations giving rise to non-trivial dynamics, the Poincar lemma must be modified when we go from the de Rham to the horizontal complex.*

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1.4.1 Exercise 3

Show that $f(x) = \frac{\partial}{\partial x^\mu}\xi^\mu(x)$, with $\xi^\mu = \int_0^1 d\lambda \lambda^{n-1} x^\mu f(\lambda x)$ by using the Poincaré lemma in top form degree n. If $g_\mu(x)$ are n functions that satisfy the integrability conditions $\frac{\partial}{\partial x^\mu} g_\nu - \frac{\partial}{\partial x^\nu} g_\mu = 0$, show that $g_\mu = \frac{\partial}{\partial x^\mu} f(x)$ with $f = x^\mu \int_0^1 d\lambda g_\mu(\lambda x)$ by using the Poincaré lemma in form degree 1.

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We will now state a generalization of the Poincar lemma to the field theoretical case, giving an idea of how the proof goes in analogy to the ordinary Poincar lemma.

Theorem 2. *(Algebraic Poincar lemma) The cohomology groups of the horizontal complex are*

$$H^p(d_H, \Omega(E)) = \begin{cases} \mathbb{R}, & p = 0, \\ 0, & 0 < p < n, \\ [\omega^n], & p = n \end{cases} \tag{1.52}$$

where we have denoted by $[\omega^n]$*, suitable equivalence classes of top forms,*

$$[\omega^n] = \Big\{ \omega \in \Omega^n(E) : \omega \sim \omega' \Leftrightarrow \omega = \omega' + d_H j^{n-1} \Leftrightarrow \frac{\delta}{\delta \phi^i}(\omega - \omega') = 0 \Big\}. \tag{1.53}$$

Proof. The idea of the proof is the same as for the ordinary Poincar lemma. The analog of the number operator in the previous proof is the operator

$$\delta_Q \equiv \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}}, \tag{1.54}$$

which commutes with the total derivative and implements an **infinitesimal field variation**. Such a variation can be written in a unique way as

$$\delta_Q f = \sum_{|\mu| \leqslant k} \partial_{(\mu)} \left[Q^i \frac{\delta f}{\delta \phi^i_{(\mu)}} \right], \tag{1.55}$$

where

$$\frac{\delta f}{\delta \phi^i_{(\mu)}} = \sum_{|\nu| \leqslant k - |\mu|} \binom{|\mu| + |\nu|}{|\mu|} (-)^{|\nu|} \partial_{(\nu)} \frac{\partial f}{\partial \phi^i_{((\mu)(\nu))}} \tag{1.56}$$

are higher order Euler-Lagrange derivatives. One then defines the operator

$$I^p_Q(\omega^p) = \sum_{|\lambda|\leqslant k-1} \frac{|\lambda|+1}{n-p+|\lambda|+1}\partial_{(\lambda)}\left[Q^i \frac{\delta}{\delta\phi^i_{((\lambda)\rho)}}\frac{\partial\omega^p}{\partial dx^\rho}\right] \tag{1.57}$$

that sends p-forms to $p-1$-forms. The difficult part of the proof, which we omit here, is to show that

$$\begin{cases} \delta_Q\omega^p = I^{p+1}_Q(d_H\omega^p) + d_H(I^p_Q\omega^p), \quad 0\leqslant p<n, \\ \delta_Q\omega^n = Q^i\frac{\delta\omega^n}{\delta\phi^i} + d_H(I^n_Q\omega^n). \end{cases} \tag{1.58}$$

The proof proceeds then in the same way as in the ordinary Poincar lemma by using

$$\rho_H\tilde{\omega}^p = \int_0^\lambda \frac{d\lambda}{\lambda}(I_{H,\phi}\tilde{\omega}^p)[x,\lambda\phi], \qquad \{d_H,\rho_H\}\tilde{\omega}^p = \tilde{\omega}^p, \tag{1.59}$$

for $p<n$. □

Chapter 2

Dynamics and the Koszul-Tate complex

After having laid down the basic concepts, in this second lecture we start dealing with dynamics. Noether identities and the associated Koszul-Tate differential [15, 16] are introduced in the framework, allowing us to deal with gauge theories.

More details can be found in [1] and also in [17, 18].

2.1 Stationary surface

In field theory we usually deal with partial differential equations. In particular, we select those that arise from an action principle because we know how to quantize them. The action is a local functional on the jet bundle

$$S[L, \phi(x)] = \int_M d^n x \, L[x, \phi]|_{\phi(x)}, \tag{2.1}$$

and the equations of motion derived from the action principle are

$$\frac{\delta L}{\delta \phi^i} = 0. \tag{2.2}$$

In order to obtain solutions $\bar{\phi}^i(x)$, we have to evaluate the left hand side of (2.2) at sections and find those that solve the associated partial differential equations. For our purpose here, it is enough to exchange this task with that of studying some algebraic/geometric properties of hypersurfaces in the jet bundle. In fact, for most cases of physical interest, when the Euler-Lagrange equations do not contain derivatives of order higher than one, equation (2.2) defines a surface in the second jet space $J^2(E)$. However, even in these cases, we need to take into account higher derivatives of the equations of motion. More generally then, we consider the **stationary surface** defined in the jet-bundles $J^k(E)$ by the Euler-Lagrange equations and an appropriate number of their total derivatives,

$$\partial_{(\mu)} \frac{\delta L}{\delta \phi^i} = 0. \tag{2.3}$$

A local function f is **weakly zero** if it vanishes when one pulls it back to the stationary surface. In this case, one uses Dirac's notation and writes

$$f \approx 0. \tag{2.4}$$

In fact, because of standard regularity assumptions on the functions that define this surface, if an object vanishes weakly, then it must be a linear combination of the equations defining the surface. This means that

$$f \approx 0 \qquad \Longleftrightarrow \qquad f = k^{i(\mu)}\partial_{(\mu)}\frac{\delta L}{\delta \phi^i}. \tag{2.5}$$

Let us work out an example in order to clarify the content of this construction in a simple setting:

Example 1. *(Massive scalar field) Let us consider a free real scalar field ϕ with mass m. In this case the Lagrangian is*

$$L^{KG} = -(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2). \tag{2.6}$$

The equation of motion for this Lagrangian is the massive Klein-Gordon equation

$$-\frac{\delta L^{KG}}{\delta\phi} = -\partial^\mu\partial_\mu\phi + m^2\phi = \phi_{00} - \phi_{ii} + m^2\phi = 0 \tag{2.7}$$

In our context, this field equation is an algebraic equation between the local coordinates of the jet space ϕ, ϕ_μ, $\phi_{\mu_1\mu_2}$.

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2.1.1 Exercise 4

Compute the Euler-Lagrange derivatives of

(i) Yang-Mills theory

$$L^{YM}[A^\alpha_\mu] = -\frac{1}{4g^2}F^\alpha_{\mu\nu}F^{\beta\mu\nu}g_{\alpha\beta}, \quad F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^\alpha_{\beta\gamma}A^\beta_\mu A^\gamma_\nu, \tag{2.8}$$

where $g_{\alpha\beta}$ is an invariant symmetric tensor on a Lie algebra $\mathfrak{g}$ with structure constants $f^\alpha_{\beta\gamma}$,

(ii) Chern-Simons theory in 3 spacetime dimensions,

$$L^{CS}[A^\alpha_\mu] = \frac{k}{8\pi}g_{\alpha\beta}\epsilon^{\mu\nu\rho}A^\alpha_\mu(\partial_\nu A^\beta_\rho + \frac{1}{3}f^\beta_{\gamma\delta}A^\gamma_\nu A^\delta_\rho), \tag{2.9}$$

(iii) Einstein gravity

$$L^{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\sqrt{|g|}(R - 2\Lambda). \tag{2.10}$$

Hint: Use that

$$\delta L = \delta\phi^i\frac{\delta L}{\delta\phi^i} + \partial_\mu(\cdot)^\mu$$

defines the Euler-Lagrange equations uniquely, that

$$\delta\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(\delta g_{\rho\nu;\lambda} + \delta g_{\rho\lambda;\nu} - \delta g_{\nu\lambda;\rho}),$$

and that $\sqrt{|g|} j^{\mu}{}_{;\mu} = (\sqrt{|g|} j^{\mu})_{,\mu}$ implies

$$\sqrt{|g|} l^{\nu_1\dots\nu_n\mu} m_{\nu_1\dots\nu_n;\mu} = (\sqrt{|g|} l^{\nu_1\dots\nu_n\mu} m_{\nu_1\dots\nu_n})_{,\mu} - \sqrt{|g|} l^{\nu_1\dots\nu_n\mu}{}_{;\mu} m_{\nu_1\dots\nu_n}.$$

(iv) the Poisson-Sigma model [19, 20] in 2 spacetime dimensions,

$$L^{PSM}[\eta_{i\mu}, X^i] d^2x = \eta_i dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j, \tag{2.11}$$

where $\eta_i = \eta_{i\mu} dx^{\mu}$ and $\alpha^{ij}(X)$ defines a Poisson tensor on target space,

$$\alpha^{ij} = -\alpha^{ji}, \quad \alpha^{il}\frac{\partial\alpha^{jk}}{\partial X^l} + \text{cyclic}(i,j,k) = 0,$$

so that the Poisson bracket

$$\{f(X), g(X)\} = \frac{\partial f}{\partial X^i}\alpha^{ij}\frac{\partial g}{\partial X^j}$$

is skew symmetric, satisfies the Leibniz rule and the Jacobi identity.

2.2 Noether identities

Noether identities are relations between the equations determining the stationary surface:

$$N^{i(\mu)}\partial_{(\mu)}\frac{\delta L}{\delta\phi^i} \equiv N^i\left[\frac{\delta L}{\delta\phi^i}\right] = 0. \tag{2.12}$$

Here $N^{i(\mu)}$ are local functions. Let us consider an illustrative example, which may appear a little unusual:

Example 2. *Consider several scalar fields satisfying the Klein-Gordon equation. In this case, we have a set of Noether identities given by*

$$N^i = \frac{\delta L}{\delta\phi^j}\mu^{[ji]}, \tag{2.13}$$

where $\mu^{[ji]}$ are a collection of local functions that are antisymmetric in their indices:

$$\mu^{[ij]}\frac{\delta L}{\delta\phi^i}\frac{\delta L}{\delta\phi^j} = 0. \tag{2.14}$$

Note that the presence of Noether identities is not due to the fact that we are considering multiple scalar fields: in the case of a single field, in fact, we have for example a Noether identity determined by

$$N = \partial_{\nu}\left(\frac{\delta L}{\delta\phi}\right) M^{[\nu\mu]}\partial_{\mu}. \tag{2.15}$$

The ones we have just shown are examples of **trivial Noether identities**: a trivial Noether identity is a Noether identity with coefficients vanishing on-shell, i.e. characterized by

$$N^{i(\mu)}\partial_{(\mu)} \approx 0. \tag{2.16}$$

A **proper gauge theory** is defined as a theory for which there are nontrivial Noether identities. For example, in electromagnetism,

$$L^{Max}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \frac{\delta L^{Max}}{\delta A_\mu} = \partial_\nu F^{\nu\mu}, \tag{2.17}$$

we have

$$\partial_\mu\partial_\nu F^{\mu\nu} = 0. \tag{2.18}$$

with corresponding nontrivial Noether identity

$$N^{(\mu)\nu}\partial_{(\mu)} = \eta^{\mu\nu}\partial_\mu \neq 0. \tag{2.19}$$

Other examples are Yang-Mills and Chern-Simons theories for which the Noether identities take the form

$$D_\mu \frac{\delta L^{YM;CS}}{\delta A^\alpha_\mu} = 0, \tag{2.20}$$

where $D_\mu f^\alpha = \partial_\mu f^\alpha + A^\beta_\mu f^\alpha_{\beta\gamma} f^\gamma$ (adjoint representation), while $D_\mu g_\alpha = \partial_\mu g_\alpha - A^\beta_\mu f^\gamma_{\beta\alpha} g_\gamma$ (co-adjoint representation).

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2.2.1 Exercise 5

In the case of Einstein gravity, show that the Noether identities

$$\partial_\nu \frac{\delta L^{EH}}{\delta g_{\mu\nu}} + \Gamma^\mu_{\nu\lambda} \frac{\delta L^{EH}}{\delta g_{\nu\lambda}} = 0 \tag{2.21}$$

are equivalent to the contracted Bianchi identities

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{2.22}$$

2.3 Irreducible gauge theories

One typical problem in field theory is to find all the Noether identities of a given theory, a problem which can be naturally tackled in this framework. In particular, we say that we have an **irreducible gauge theory** if we have a set of Noether identities $\{R^{\dagger i(\mu)}_\alpha \partial_{(\mu)}\}$ such that

$$R^{\dagger i}_\alpha \left[\frac{\delta L}{\delta \phi^i}\right] = 0, \tag{2.23}$$

which is

- **Non-trivial**:
$$R_\alpha^{\dagger i(\mu)} \not\approx 0; \tag{2.24}$$

- **Irreducible**: this means that there is no combination of the Noether identities and their derivatives which is trivial:
$$Z^{\dagger\alpha(\nu)}\partial_{(\nu)} \circ R_\alpha^{\dagger i(\mu)}\partial_{(\mu)} \approx 0 \qquad \Longrightarrow \qquad Z^{\dagger\alpha(\beta)} = 0; \tag{2.25}$$

- Constitutes a **generating set**: this means that any Noether operator, $N^{i(\mu)}\partial_{(\mu)}$ satisfying (2.12), may be written in the form
$$N^{i(\lambda)}\partial_{(\lambda)} = Z^{\dagger\alpha(\nu)}\partial_{(\nu)} \circ R_\alpha^{\dagger i(\mu)}\partial_{(\mu)} + M^{[j(\nu)i(\lambda)]}\partial_{(\nu)}\frac{\delta L}{\delta\phi^i}\partial_{(\lambda)}, \tag{2.26}$$
i.e., as a sum of two terms, one given by the Noether identities of the generating set and the other vanishing on-shell (or, more precisely, an antisymmetric combination of the left hand sides of the equations of motion and their derivatives).

To find such a generating set is in general hard. In the examples above, a slightly tedious analysis of the left hand sides of the equations of motion of the theory and their derivatives in jet space shows that the standard ones given above constitute such a generating set. As we will see below, such an analysis is in fact equivalent to showing that one has found all the non-trivial gauge symmetries of the theory.

2.4 The Koszul-Tate complex

We have already introduced a nilpotent operator, the horizontal differential d_H, together with its cohomological complex in order to have algebraic control on functionals. Another differential, the **Koszul-Tate differential** δ, and its cohomology will take care of the dynamics of the theory. In the case of irreducible gauge theories, it is constructed as follows.

For each equation of motion $\frac{\delta L}{\delta\phi^i}$, one introduces a Grassmann odd field ϕ^*_i, whereas for each non-trivial Noether operator of the generating set $R_\alpha^{+i(\mu)}\partial_{(\mu)}$ (if any), one introduces a Grassmann even field C^*_α. The definition of the total derivative operator ∂_μ, of local functions, the horizontal complex and local functionals is then extended so as to include (a polynomial dependence of) these additional fields and their derivatives. In terms of generators, the Koszul-Tate differential may then by defined through $\delta\phi^i = 0 = \delta x^\mu = \delta dx^\mu$, while
$$\begin{cases} \delta\phi^*_i = \frac{\delta L}{\delta\phi^i}, \\ \delta C^*_\alpha = R_\alpha^{\dagger i}[\phi^*_i], \end{cases} \tag{2.27}$$
with the understanding that
$$[\delta, \partial_\mu] = 0. \tag{2.28}$$

Equivalently, it may be represented by the odd vector field

$$\delta = \partial_{(\mu)} \frac{\delta L}{\delta \phi^i} \frac{\partial}{\partial \phi^*_{i(\mu)}} + \partial_{(\nu)} \left(R^{\dagger i}_{\alpha}[\phi^*_i] \right) \frac{\partial}{\partial C^*_{\alpha(\nu)}}. \tag{2.29}$$

Further, assuming that all odd variables anticommute between themselves (so that, for example, dx^μ's anticommute with ϕ^*_i's), (2.28) becomes

$$\{\delta, d_H\} = 0. \tag{2.30}$$

This vector field is nilpotent,

$$\delta^2 = \frac{1}{2}\{\delta, \delta\} = 0, \tag{2.31}$$

on account of (2.23).

The introduction of the additional fields provides an additional $\mathbb{Z}$-grading of the extended horizontal complex, the **antifield number**, which can be described by the vector field

$$\mathcal{A} = \phi^*_{i(\mu)} \frac{\partial}{\partial \phi^*_{i(\mu)}} + 2C^*_{\alpha(\beta)} \frac{\partial}{\partial C^*_{\alpha(\beta)}}. \tag{2.32}$$

Applied to a field, this operator gives back the same field multiplied by its antifield number: so, for example,

$$\mathcal{A}\phi^i = 0, \qquad \mathcal{A}\phi^{*2}_i = 2\phi^{*2}_i. \tag{2.33}$$

If Ω_k denote the horizontal forms with definite antifield number k, then

$$\delta : \Omega_k \longrightarrow \Omega_{k-1}. \tag{2.34}$$

Since δ reduces rather than increases the degree, one refers to it as boundary rather than a co-boundary operator. Since it is nilpotent, one may define its homology. This is the object of the following:

Theorem 3.

$$H_k(\delta, \Omega) = \begin{cases} 0, & k > 0, \\ \Omega(\Sigma) \simeq \Omega_0/I, & k = 0 \end{cases} \tag{2.35}$$

where we have defined the set of forms that vanish after the pull-back to Σ as

$$I = \{\omega \in \Omega : \omega \approx 0\}. \tag{2.36}$$

Remark 5. *A convenient way of describing functions (or horizontal forms) on a surface embedded in a larger space is to consider the functions defined on the whole space modulo those that vanish on the surface. It is in this sense that one says that the construction above provides a homological resolution of the horizontal complex $\Omega(\Sigma)$ associated to the stationary surface.*

Remark 6. *Crucial in the proof of the first line of the theorem (also referred to as "acyclicity of δ in higher antifield number") are properties* (2.25) *and* (2.26) *of the generating set. For reducible gauge theories (for instance theories involving $p-$ forms with $p > 1$), the construction has to be suitably modified in order for the theorem to continue to hold.*

2.5 Characteristic cohomology

Characteristic cohomology $H^p(d_H, \Omega(\Sigma))$ contains important physical information related to the equations of motion.

In the extended complex, it is encoded through

$$H^p(d_H, \Omega(\Sigma)) \simeq H_0^p(d_H|\delta). \tag{2.37}$$

Indeed, consider first a weakly closed p-form with zero antifield number

$$d_H \omega_0^p \approx 0. \tag{2.38}$$

Since any (weakly) vanishing function is a linear combination of the equations of motion and their total derivatives, we can write

$$d_H \omega_0^p = k^{i(\mu)} \partial_{(\mu)} \frac{\delta L}{\delta \phi^i}. \tag{2.39}$$

This in turn is equivalent to

$$d_H \omega_0^p + \delta \omega_1^{p+1} = 0, \tag{2.40}$$

where

$$\omega_1 = -k^{i(\mu)} \phi^*_{i(\mu)}. \tag{2.41}$$

For the coboundary condition, every weakly exact form

$$\omega_0^p \approx d_H \eta_0^{p-1}, \tag{2.42}$$

for the same reason as above, can be written as

$$\omega_0^p = d_H \eta_0^{p-1} + \delta \eta_1^p. \tag{2.43}$$

$H^{n-1}(d_H, \Omega(\Sigma))$, the characteristic cohomology in degree $n-1$, classifies the non-trivial conserved currents of the theory. Indeed, an $n-1$-form of zero antifield number can be written as

$$\omega_0^{n-1} = j^\mu \left(d^{n-1}x\right)_\mu, \tag{2.44}$$

with

$$\left(d^{n-k}x\right)_{\mu_1 \dots \mu_k} = \frac{1}{k!(n-k)!} \epsilon_{\mu_1 \dots \mu_k \nu_{k+1} \dots \nu_n} dx^{\nu_{k+2}} \dots dx^{\nu_n}. \tag{2.45}$$

This implies that

$$d_H \omega_0^{n-1} \approx 0, \qquad \Leftrightarrow \qquad \partial_\mu j^\mu \approx 0. \tag{2.46}$$

Similarly,

$$\eta_0^{n-2} = k^{[\mu\nu]} \left(d^{n-2}x\right)_{\mu\nu}, \tag{2.47}$$

so that a trivial conserved current as in (2.43) is explicitly given by

$$j^\mu = \partial_\nu k^{[\mu\nu]} + t^\mu, \tag{2.48}$$

with $t^\mu \approx 0$. In other words, trivial conserved currents are given by divergences of "superpotentials" $k^{[\mu\nu]}$ and by on-shell vanishing ones.

Example 3. *(Energy-momentum tensor) For a scalar field with Lagrangian given in* (2.6)*, the energy-momentum tensor*

$$T^{\mu}{}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \delta^{\mu}{}_{\nu}L^{KG}, \tag{2.49}$$

provides 4 non-trivial conserved currents since one may easily check that $\partial_{\mu}T^{\mu}{}_{\nu} \approx 0$*. The same holds for electromagnetism with Lagrangian given in* (2.17) *for which*

$$T^{\mu}{}_{\nu} = F^{\mu\sigma}\partial_{\nu}A_{\sigma} - \delta^{\mu}{}_{\nu}L^{Max}. \tag{2.50}$$

Remark 7. *A concrete expression for a non-trivial conserved current is merely a representative of an equivalence class defined up to the addition of trivial conserved currents. This is important for instance when one wants to couple matter fields to gravity in the context of general relativity, where a symmetric energy momentum-tensor is required so that it can be contracted with the metric. This is the case for the energy-momentum tensor* $T^{\mu\nu}$ *of the scalar field but not for the one of electromagnetism given in* (2.50)*. In this context, the* ***Belinfante procedure*** *consists in constructing a symmetric representative of the energy-momentum tensor by using trivial conserved currents.*

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2.5.1 Exercise 6

Show that the energy-momentum tensor (2.50) differs from the symmetric expression

$$T^{\mu}{}_{\nu} = -\left(F^{\mu\sigma}F_{\sigma\nu} + \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\delta^{\mu}{}_{\nu}\right), \tag{2.51}$$

by trivial terms.

———-

For theories with no non-trivial Noether identities, like scalar field theory for example, there is no additional characteristic cohomology besides the conserved currents in form degree $n-1$,

$$H^{p}\left(d_H, \Omega(\Sigma)\right) = 0, \qquad p < n-1. \tag{2.52}$$

For proper irreducible gauge theories, it can be shown that there is no characteristic cohomology in form degree $p < n-2$. The only additional characteristic cohomology is thus in form degree $p = n-2$. The conditions that a form be closed but not exact become in this case

$$\partial_{\nu}k^{[\mu\nu]} \approx 0, \qquad k^{\mu\nu} \not\approx \partial_{\sigma}\eta^{[\mu\nu\sigma]}. \tag{2.53}$$

In electromagnetism, we will show that there is a single class with representative given by $k^{\mu\nu} = F^{\mu\nu}$, which indeed satisfies

$$\partial_{\nu}F^{\mu\nu} \approx 0. \tag{2.54}$$

This cohomology class captures electric charge contained in a closed surface S through

$$Q = \oint_{t=cte,S} F^{\mu\nu}(d^2x)_{\mu\nu} = \int_{t=cte,S} F^{0i} d\sigma_i. \tag{2.55}$$

In linearized gravity, ADM charges fall into this class.

The proofs of these statements, as well as of the generalized Noether theorems to be discussed in the next section, rely on the following isomorphisms between cohomology classes:

Theorem 4 (inversion & descent equations)**.**

$$H^{n-p}_0(d_H|\delta)/\delta^n_p\mathbb{R} \simeq H^{n-p+1}_1(\delta|d_H) \simeq \cdots \simeq H^n_p(\delta|d_H), \tag{2.56}$$

which in turn can be proved by elementary homological techniques of “diagram chasing”.

Chapter 3

Symmetries and longitudinal differential

Up to now we have discussed Noether identities and conservation laws from the perspective of the geometry and topology of the surface of the equations of motion. For a stationary surface originating from a variational principle, we will now relate the former to gauge and the latter to global symmetries in terms of complete and generalized Noether theorems.

3.1 Prolongation of transformations

Given a section

$$(x^\mu, \phi^i(x)) \tag{3.1}$$

of the jet bundle, a generalized vector field on $J^0(E)$

$$v = a^\mu \frac{\partial}{\partial x^\mu} + b^i \frac{\partial}{\partial \phi^i}, \tag{3.2}$$

where a^μ and b^i are local functions, defines an infinitesimal transformation of a section, as

$$\left(x^\mu + \varepsilon a^\mu|_s, \phi^i(x) + \varepsilon b^i|_s\right) . \tag{3.3}$$

The first term thus encodes the action of the transformation on the spacetime coordinates, while the second term encodes the variation of the field. The transformation then produces the new section

$$\phi'^i(x + \varepsilon a) = \phi^i(x) + \varepsilon b^i|_s, \tag{3.4}$$

so that the variation of the field will have not only the intrinsic term b^i, but also a drag term including its derivative

$$\delta_Q \phi^i \equiv Q^i, \qquad\qquad Q^i = \left(b^i - a^\nu \phi^i_\nu\right) . \tag{3.5}$$

Q^i is called the **characteristic representative** of the transformation. Through the characteristic representative we can define the **prolongation** of the transformation from the fields to their derivatives, i.e. to the infinite jet space, in terms of a so-called **evolutionary vector field**,

$$\delta_Q = \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}}. \tag{3.6}$$

The prolongation is defined in such a way that the transformation commutes with total derivatives:

$$[\delta_Q, \partial_\mu] = 0. \tag{3.7}$$

3.2 Symmetries of the equations of motion

When studying field theories, we can encounter two types of symmetries: symmetries of the equations of motion and symmetries of the action. A symmetry of the equations of motion is defined by

$$\delta_Q \frac{\delta L}{\delta \phi^i} \approx 0. \tag{3.8}$$

In particular, this is equivalent to saying that given a section $\bar{\phi}^i$ solving the Euler-Lagrange equations, then the transformed section

$$\bar{\phi}^i(x) + \epsilon Q^i|_{\phi^i(x)} \tag{3.9}$$

is a solution of the Euler-Lagrange equations to order ϵ^2.

3.3 Variational symmetries

Symmetries of the action, also called **variational symmetries** in this algebraic context, are defined by

$$\delta_Q L = \partial_\mu k^\mu_Q, \tag{3.10}$$

for some local functions k^μ_Q. Notice that since we have defined Lagrangians as equivalence classes up to total derivatives this definition does not depend on the representative. We can rearrange the above condition, using

$$\delta_Q L = \partial_{(\mu)} Q^i \frac{\partial L}{\partial \phi^i_{(\mu)}} \tag{3.11}$$

together with the definition of Euler-Lagrange derivative (1.18) in order to write

$$Q^i \frac{\delta L}{\delta \phi^i} = \partial_\mu \left(k^\mu_Q - \frac{\partial L}{\partial \phi^i_\mu} Q^i + \dots \right) \equiv \partial_\mu j^\mu_Q, \tag{3.12}$$

which is the usual statement of Noether's first theorem (to wit, that a variational symmetry Q^i gives rise to a conserved current j^μ_Q; we will return to that later).

A question one is led to ask is whether every variational symmetry is also a symmetry of the equations of motion and vice versa. In fact, only the direct implication is true. Indeed, for general local functions f, g, Q^i, one can show that

$$(-)^{|\mu|} \partial_{(\mu)} \left[\frac{\partial(\partial_\nu f)}{\partial \phi^i_{(\mu)}} g \right] = -(-)^{|\mu|} \partial_{(\mu)} \left[\frac{\partial f}{\partial \phi^i_{(\mu)}} \partial_\nu g \right], \tag{3.13}$$

$$\delta_Q \frac{\delta f}{\delta \phi^i} = \frac{\delta}{\delta \phi^i}(\delta_Q f) - (-)^{|\mu|}\partial_{(\mu)}\left[\frac{\partial Q^j}{\partial \phi^i_{(\mu)}}\frac{\delta f}{\delta \phi^j}\right]. \tag{3.14}$$

Applying the latter equation to the case $f = L$, with variational Q^i, the first term on the right-hand side vanishes since the Euler-Lagrange derivative of a total derivative is zero,

$$\frac{\delta}{\delta \phi^i}(\partial_\mu k^\mu_Q) = 0. \tag{3.15}$$

The second term, on the other hand, is weakly zero since it is proportional to the equations of motion and their derivatives. Thus we find

$$\delta_Q \frac{\delta L}{\delta \phi^i} \approx 0 \tag{3.16}$$

i.e., that Q^i is a symmetry of the equations of motion.

In order to see that the other implication does not hold in general, it suffices to see a simple counterexample. For the massless Klein-Gordon Lagrangian,

$$L = -\frac{1}{2}(\partial_\mu \phi)^2, \tag{3.17}$$

the field equation is determined by

$$\frac{\delta L}{\delta \phi} = \Box \phi. \tag{3.18}$$

An equations of motion symmetry is given by

$$Q = \lambda \phi, \tag{3.19}$$

with λ constant. However,

$$\delta_Q L = 2\lambda L, \tag{3.20}$$

which cannot be a total derivative since the Euler-Lagrange equations of motion are non-trivial, while the Euler-Lagrange derivative would annihilate any total derivative.

Remark 8. *Non-Noetherian symmetries, i.e., symmetries of variational equations which are not variational symmetries, do play an important role in the context of integrable systems.*

Example 4. *(Translations) For a translation in the direction of x^ν, we have*

$$a^\mu_\nu = \delta^\mu_\nu, \qquad b^i_\nu = 0. \tag{3.21}$$

Equation (3.3) *then implies that*

$$Q^i_\nu = -\phi^i_\nu, \tag{3.22}$$

with prolongation given by

$$\delta_{Q_\nu} = -\phi^i_{((\mu)\nu)}\frac{\partial}{\partial \phi^i_{(\mu)}} = -\partial_\nu + \frac{\partial}{\partial x^\nu}. \tag{3.23}$$

We see in particular that a Lagrangian with no explicit dependence on space-time coordinates is translation invariant,

$$\frac{\partial L}{\partial x^\nu} = 0 \qquad \Longrightarrow \qquad \delta_{Q_\nu} L = -\partial_\nu L = \partial_\mu(-L\delta^\mu_\nu), \tag{3.24}$$

with associated Noether current $j^\mu_\nu = \frac{\partial L}{\phi^i_\mu}\phi^i_\nu - \delta^\mu_\nu L \equiv T^\mu{}_\nu$*, the (canonical) energy-momentum tensor.*

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3.3.1 Exercise 7

Show that $\delta_\xi \phi = \xi^\nu \partial_\nu \phi$ (i) is a variational symmetry of the Klein-Gordon Lagrangian if (i) ξ^ν is a Killing vector of flat space; (ii) ξ^ν is a conformal Killing vector in 2 spacetime dimensions in the massless case. Show that $\delta_\xi A^a_\mu = \xi^\nu \partial_\nu A^a_\mu + \partial_\mu \xi^\nu A^a_\nu$ is a variational symmetry of the Yang-Mills Lagrangian if (i) ξ^ν is a Killing vector of flat space; (ii) ξ^ν is a conformal Killing vector of flat space in 4 spacetime dimensions.

Hint: Contract the equation defining the energy-momentum tensor by ξ^ν and conclude using the properties of the latter.

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An important property of evolutionary vector fields is that they form an algebra under the commutator of vector vector fields,

$$\begin{aligned}[\delta_{Q_1}, \delta_{Q_2}] &= \partial_{(\mu)}\left(\delta_{Q_1} Q^i_2 - \delta_{Q_2} Q^i_1\right)\frac{\partial}{\partial \phi^i_{(\mu)}} \\ &\equiv \partial_{(\mu)}[Q_1, Q_2]^i \frac{\partial}{\partial \phi^i_{(\mu)}},\end{aligned} \tag{3.25}$$

where we have used (3.7) repeatedly.

Both equations of motion symmetries and variational symmetries form a sub-algebra thereof. The latter follows because

$$\begin{aligned}[\delta_{Q_1}, \delta_{Q_2}]L &= \delta_{Q_1}\partial_\mu k^\mu_{Q_2} - \delta_{Q_2}\partial_\mu k^\mu_{Q_1} \\ &= \partial_\mu\left(\delta_{Q_1} k^\mu_{Q_2} - \delta_{Q_2} k^\mu_{Q_1}\right).\end{aligned} \tag{3.26}$$

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3.3.2 Exercise 8

Let j^μ_Q be the Noether current associated to a variational symmetry Q^i with $j_Q = j^\mu (d^{n-1}x)_\mu$ the associated $n-1$ form. Define the (covariant) Dickey bracket through

$$\{j_{Q_1}, j_{Q_2}\} = \delta_{Q_1} j_{Q_2}. \tag{3.27}$$

Prove that

$$\{j_{Q_1}, j_{Q_2}\} = j_{[Q_1,Q_2]} + \text{trivial}, \tag{3.28}$$

where trivial includes constants in spacetime dimension 1. Hint: apply δ_{Q_1} to $dj_{Q_2} = Q^i_2 \frac{\delta L}{\delta \phi^i} d^n x$ and use (3.14).

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3.4 Gauge Symmetries

A gauge symmetry corresponds to an (infinite-dimensional) sub-set of variational symmetries that depend on an arbitrary local function f and its derivatives. It is defined by

$$\delta_f \phi^i = Q^i(f) = Q^{i(\mu)}\partial_{(\mu)} f, \qquad \delta_f L = \partial_\mu k_f^\mu. \tag{3.29}$$

The most simple example that comes to mind is that of electromagnetism, where

$$\delta_f A_\mu = \partial_\mu f. \tag{3.30}$$

Gauge symmetries can also be characterized through the important

Theorem 5. *(Noether's second theorem) There is a one-to-one correspondence between Noether identities and gauge symmetries.*

Proof. Let us first prove that we can associate a Noether identity to a gauge symmetry. Indeed, we have

$$Q^i(f)\frac{\delta L}{\delta \phi^i} = \partial_\mu j_f^\mu. \tag{3.31}$$

By doing multiple integration by parts, we can rewrite this equation as

$$(-)^{|\mu|} f \partial_{(\mu)}\left[Q^{i(\mu)}\frac{\delta L}{\delta \phi^i}\right] \equiv f Q^{\dagger i}\left[\frac{\delta L}{\delta \phi^i}\right] = \partial_\mu\left(j_f^\mu - t_f^\mu\right), \tag{3.32}$$

where t_f^μ vanishes on-shell, and we have defined the **Noether operator** $Q^{\dagger i}$, which is the formal adjoint of $Q^{i(\mu)}\partial_{(\mu)}$. In particular, since f is arbitrary, we can replace it by a new independent field on the jet-space. This allows one to take the Euler-Lagrange derivative with respect to f. Since the Euler-Lagrange derivative annihilates total derivatives, we thus arrive at the Noether identity

$$Q^{\dagger i}\left(\frac{\delta L}{\delta \phi^i}\right) = 0. \tag{3.33}$$

To prove the converse, let us start by multiplying a Noether identity by an arbitrary local function f,

$$f N^i\left(\frac{\delta L}{\delta \phi^i}\right) = 0. \tag{3.34}$$

It is then again only a matter of multiple integrations by parts to arrive at an equation of the form (3.31), with $Q^i(f) = N^{\dagger i}(f)$, and $j_f^\mu = t_f'^\mu \approx 0$. □

Remark 9. *Let us note also that, when combining* (3.32) *with the Noether identity* (3.33)*,*

$$\partial_\mu\left(j_f^\mu - t_f^\mu\right) = 0, \qquad \Longrightarrow \qquad d_H(j_f - t_f) = 0, \tag{3.35}$$

so that $j_f - t_f$ is a closed $n-1$-form. However, we know from Theorem 2 that

$$H^{n-1}(d_H) = \delta_0^{n-1}\mathbb{R}. \tag{3.36}$$

This means that

$$j_f = t_f + d_H \eta^{(n-2)} + \delta_0^{n-1} C, \tag{3.37}$$

which is the sum of a term which vanishes on-shell and an exact term, up to a constant C in 1 spacetime dimension, i.e., in classical mechanics. We thus obtain as a corollary the result that ***the Noether current associated to a gauge symmetry is always trivial.***

Remark 10. *Because all theories, including for instance scalar field theories, admit trivial Noether identities, they also admit gauge symmetries. Note however that to a trivial Noether identity corresponds a trivial gauge symmetry in the sense that the characteristic $Q^i[f] \approx 0$. More generally, again as a corollary, any gauge symmetry in an irreducible gauge theory can be written in terms of a generating set as*

$$Q^i(f) = R^i_\alpha \left(Z^\alpha(f) \right) + (-)^{|\mu|} \partial_{(\mu)} \left(M^{[j(\nu) i(\mu)]} \partial_{(\nu)} \frac{\delta L}{\delta \phi^j} f \right). \tag{3.38}$$

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3.4.1 Exercise 9

Study the global symmetries and the conserved currents of the first order Hamiltonian action

$$S_H = \int dt [p_i \dot{q}^i - H(q,p,t)]. \tag{3.39}$$

Hint: Use the fact that one can show that every symmetry that vanishes on-shell is trivial, i.e., is an antisymmetric combination of the equations of motion and thus a trivial gauge symmetry. The Hamiltonian equations of motion can then be used to remove all time derivatives in the characteristic of global symmetries, so that one may assume

$$\delta q^i = Q^i(q,p,t), \quad \delta p_i = P_i(q,p,t). \tag{3.40}$$

With this starting point, write the out the condition that such symmetry is a variational symmetry in Noether form

$$Z^A \frac{\delta L_H}{\delta z^A} = \frac{d}{dt} j, \quad z^A = (q^i, p_i), \tag{3.41}$$

by identifying time-derivatives to conclude that, up to constants, conserved currents are determined by $j(z^A, t)$, such that

$$\frac{\partial}{\partial t} j + \{j, H\} = 0, \tag{3.42}$$

and the associated global symmetries are given by the Hamitlonian vector field generated by j,

$$Z^A = \omega^{BA} \frac{\partial j}{\partial z^B}, \quad \omega^{AB} = \begin{pmatrix} 0 & \delta^i_l \\ -\delta^k_j & 0 \end{pmatrix} \tag{3.43}$$

Show that the associated Dickey bracket is the Poisson bracket.

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3.5 Generalized Noether theorems

The tools developed so far also allow us to give a more complete version of Noether's first theorem than can be found in Noether's original paper or in field theory textbooks.

Theorem 6. *(Noether's first theorem) There is a one-to-one correspondence between non-trivial Noether symmetries and non-trivial Noether currents, which we can write as*

$$[Q^i] \longleftrightarrow [j^{n-1}], \tag{3.44}$$

where the equivalence classes $[Q^i]$ *are defined by*

$$Q^i \sim Q^i + R^i{}_\alpha(f^\alpha) + (-)^{|\mu|}\partial_{(\mu)}\left[M^{[j(\nu)i(\mu)]}\partial_{(\nu)}\frac{\delta L}{\delta\phi^i}\right], \tag{3.45}$$

while $j^{n-1} \sim j^{n-1} + t^{n-1} + d_H\eta^{n-2} + \delta^{n-1}C$ *with* $t^{n-2} \approx 0$.

Proof. The proof follows by spelling out the details of Theorem 4 in form degree $p = n-1$,

$$H^n_1(\delta|d_H) \simeq H^{n-1}_0(d_H|\delta)/\delta^{n-1}\mathbb{R}. \tag{3.46}$$

Indeed, the RHS has already been treated in the context of characteristic cohomology and its Koszul-Tate resolution. For the LHS, we note that in maximal form degree n, by uniquely fixing the d_H-exact term in the coboundary condition, canonical representatives in antifield number 1 and 2 are given by

$$\begin{aligned} \omega^n_1 &= \phi^*_i Q^i d^n x, \\ \omega^n_2 &= \left(f^\alpha C^*_\alpha + \frac{1}{2}(-)^{|\mu|}\partial_{(\mu)}[M^{j(\nu)[i(\mu)]}\phi^*_{j(\nu)}]\phi^*_i\right) d^n x, \end{aligned} \tag{3.47}$$

with $Q^i, f^\alpha, M^{[j(\nu)i(\mu)]}$ local functions. For such a representative in antifield number 1, the cocycle condition $\delta\omega^n_1 + d_H\omega^{n-1}_0 = 0$ reduces to the condition that Q^i defines a variational symmetry,

$$Q^i\frac{\delta L}{\delta\phi^i} = \partial_\mu j^\mu. \tag{3.48}$$

For the coboundary condition $\omega^n_1 = \delta\eta^n_2 + d_H(\cdot)$, one may assume that η^n_2 is of the form of ω^n_2 above by suitably adjusting the d_H-exact term. Taking an Euler-Lagrange derivative of the resulting equation with respect to ϕ^*_i then leads to $Q^i = R^i_\alpha(f^\alpha) + (-)^{|\mu|}\partial_{(\mu)}[M^{[i(\mu)j(\nu)]}\frac{\delta L}{\delta\phi^j}]$. □

Remark 11. *From the viewpoint of Noether's first theorem, all gauge symmetries should be considered as trivial; physically distinct "global" symmetries are thus described by equivalence classes of variational symetries, up to gauge symmetries.*

Under suitable assumptions, one may then also show that

$$H^n_p(\delta|d_H) = 0, \quad p > 2, \tag{3.49}$$

in irreducible gauge theories. Theorem 4 then allows one to conclude that there is no characteristic cohomology, up to constants in spacetime dimensions p.

There remains the case of $p = 2$,

$$H^n_2(\delta|d_H) \simeq H^{n-2}_0(d_H|\delta)/\delta^{n-2}_0\mathbb{R}, \tag{3.50}$$

By a similar, but more involved, reasoning to that used for $p = 1$, it can be shown that the computation of characteristic cohomology in degree $n-2$ (up to constants in spacetime dimensions 2) reduces to the problem of finding physically distinct **global reducibility parameters**, that is to say equivalence classes of $[f^\alpha]$'s such that

$$R^i_\alpha(f^\alpha) \approx 0, \quad f^\alpha \sim f^\alpha + t^\alpha, \quad t^\alpha \approx 0. \tag{3.51}$$

In a large class of theories, such as Yang-Mills theories or general relativity in spacetime dimensions greater or equal to three, one may show that the equivalence classes of f^α's are determined by field independent ordinary functions $\bar{f}^\alpha(x)$ satisfying the strong equality

$$R^i_\alpha(\bar{f}^\alpha) = 0. \tag{3.52}$$

In this case, like in the case of Noether's first theorem, there is an explicit formula for the associated "surface charges", that is to say representatives for the corresponding characteristic cohomology. Indeed, by repeated integrations by part, one has

$$R^i_\alpha(f^\alpha)\frac{\delta L}{\delta\phi^i} = f^\alpha R^{\dagger i}_\alpha\left(\frac{\delta L}{\delta\phi^i}\right) + \partial_\mu S^{\mu i}_f\left(\frac{\delta L}{\delta\phi^i}\right). \tag{3.53}$$

The first term on the right-hand side is zero because of the Noether identities. This means that $S^{\mu i}_f(\frac{\delta L}{\delta\phi^i})$ represent the weakly vanishing Noether currents associated to the gauge symmetries $R^i_\alpha(f^\alpha)$, that can readily be worked out by keeping the boundary terms in (3.53). When using field independent reducibility parameters $\bar{f}^\alpha$ satisfying (3.52), the right hand side vanishes as well. It follows that the $n-1$ form $S_{\bar{f}} = S^{\mu i}_{\bar{f}}[\frac{\delta L}{\delta\phi^i}](d^{n-1}x)_\mu$ satisfies

$$d_H S_{\bar{f}} = 0. \tag{3.54}$$

Because the horizontal cohomology of degree $n-1$ is trivial (cf. Theorem 2), this means that

$$(0 \approx) S_{\bar{f}} = d_H k_{\bar{f}}, \quad k_{\bar{f}} = \rho_H(S_{\bar{f}}). \tag{3.55}$$

In other words, all linearly independent surface charges can be explicitly constructed by applying the contracting homotopy of the horizontal complex to the weakly vanishing Noether current associated with linearly independent solutions of (3.52).

In the general case, where one has a basis $[f^\alpha_A]$ of possibly field dependent global reducibility parameters, one may show that $\rho_H(S_{f_A})$ still provide a basis of characteristic cohomology in degree $n-2$ (up to constants in spacetime dimension 2).

Working out explicit expressions for the surface charges is direct but quite lenghty in the case of second order field equations, that is why we will not do so here.

Example 5. *Let us give some examples of field-independent reducibility parameters. In the case of electromagnetism, where*

$$\delta_f A_\mu = \partial_\mu f, \tag{3.56}$$

we find

$$\partial_\mu \bar{f} = 0, \qquad \Longrightarrow \qquad \bar{f} = const, \tag{3.57}$$

which characterizes the electric charge.

For the case of Yang-Mills and Chern-Simons theory based on a sem-simple Lie group, and of General Relativity, condition (3.52) *reads*

$$D_\mu \bar{f} = 0, \qquad \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0, \tag{3.58}$$

both of which admit only the trivial solutions $\bar{f}^\alpha = \bar{\xi}^\mu = 0$ because the equations need to hold for arbitrary gauge potentials or metrics.

In the case of Yang-Mills theory or gravity linearized around a background solution $\bar{g}_{\mu\nu}$ or $\bar{A}$, a generating set of gauge transformations is given by

$$\delta_f a_\mu = D^{\bar{A}}_\mu f, \qquad \delta h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}. \tag{3.59}$$

In case the background solution is flat for instance, $\bar{A} = g^{-1}dg$, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the solutions to equation (3.52) *are given by*

$$\bar{f} = \lambda^\alpha g^{-1} T_\alpha g, \quad \bar{\xi}_\mu = a_\mu + \omega_{[\mu\nu]} x^\nu, \tag{3.60}$$

and the associated surface charges are related to color charges, respectively the ADM charges in general relativity [21, 22].

Remark 12. *The above results on surface charges associated to global reducibility parameters can be extended to the case of* **asymptotic symmetries** *and the associated current algebras.*

3.6 Gauge symmetry algebra

From their definitions, it follows with little work that gauge symmetries form an ideal in the algebra of variational symmetries, while on-shell vanishing gauge symmetries in turn form an ideal in the sub-algebra of gauge symmetries. The latter may thus be quotiented away and the resulting quotient is the algebra of non-trivial gauge symmetries we are interested in.

The information on non-trivial gauge symmetries is contained in the generating set. We may start by considering gauge symmetries of the form $\delta_\epsilon \phi^i = R^i_\alpha(\epsilon^\alpha)$ with $\epsilon^\alpha(x)$ arbitrary functions. The commutator $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ is a variational symmetry that depends on arbitrary functions. Up to on-shell vanishing gauge symmetries, it may thus be written in terms of the generating set,

$$\delta_{\varepsilon_1} R^i_\alpha(\varepsilon^\alpha_2) - \delta_{\varepsilon_2} R^i_\alpha(\varepsilon^\alpha_1) \approx R^i_\gamma\left(f^\gamma_{\alpha\beta}(\varepsilon^\alpha_1, \varepsilon^\beta_2)\right), \tag{3.61}$$

for some bi-differential, skew-symmetric operators "structure operators"

$$f^{\gamma(\mu)((\nu)}_{\alpha\beta} \partial_{(\mu)}\epsilon^\alpha_1 \partial_{(\nu)}\epsilon^\beta_2. \tag{3.62}$$

From the identity

$$[\delta_{\varepsilon_1}, [\delta_{\varepsilon_2}, \delta_{\varepsilon_3}]] + \text{cyclic}(1,2,3) = 0, \tag{3.63}$$

where "cyclic" stands for cyclic permutations, and the irreducibility assumption for the generating set, it then follows that the structure operators satisfy the generalized Jacobi identities,

$$\delta_{\varepsilon_1} f^\gamma_{\alpha\beta}(\varepsilon^\alpha_2, \varepsilon^\beta_3) + f^\gamma_{\delta\rho}(\varepsilon^\delta_1, f^\rho_{\alpha\beta}(\varepsilon^\alpha_2, \varepsilon^\beta_3)) + \text{cyclic}(1,2,3) \approx 0. \tag{3.64}$$

This algebraic structure of vector fields in involution may be captured through a suitable differential. Instead of the arbitrary functions $\epsilon^\alpha(x)$ one introduces so-called ghosts, that is to say Grassmann odd fields $C^\alpha(x)$, that are promoted to additional coordinates in the fiber of the jet-bundle. Defining the action on generators as

$$\gamma\phi^i = R^i_\alpha(C^\alpha), \quad \gamma C^\gamma = -\frac{1}{2} f^\gamma_{\alpha\beta}(C^\alpha, C^\beta), \tag{3.65}$$

and extending to the jet-bundle as

$$\gamma = \partial_{(\mu)}\left(R^i_\alpha(C^\alpha)\right)\frac{\partial}{\partial\phi^i_{(\mu)}} - \frac{1}{2}\partial_{(\mu)}\left(f^\gamma_{\alpha\beta}(C^\alpha, C^\beta)\right)\frac{\partial}{\partial C^\alpha_{(\mu)}}, \tag{3.66}$$

it follows from (3.61) and (3.64) that

$$\gamma^2 \approx 0. \tag{3.67}$$

Remark 13. *When using general non-trivial gauge symmetries with arbitrary local function f^α instead of arbitrary functions ϵ^α, equation* (3.61) *becomes*

$$\begin{aligned} &\delta_{f_1} R^i_\alpha(f^\alpha_2) - \delta_{f_2} R^i_\alpha(f^\alpha_1) \approx R^i_\gamma\left([f_1, f_2]^\gamma_A\right), \\ &[f_1, f_2]^\gamma_A = f^\gamma_{\alpha\beta}(f^\alpha_1, f^\beta_2) + \delta_{f_1} f^\gamma_2 - \delta_{f_2} f^\gamma_3, \end{aligned} \tag{3.68}$$

while (3.64) *turns into the statement that, on-shell, the bracket $[f_1, f_2]_A$ satisfies the Jacobi identity. On-shell, the algebraic structure that emerges in this way for irreducible gauge theories is that of a Lie algebroid (compare for instance to section 2.1 of [23]).*

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3.6.1 Exercise 10

Show that

$$\delta_\epsilon X^i = -\alpha^{ij}\epsilon_j, \quad \delta_\epsilon \eta_{i\mu} = \partial_\mu \epsilon_i + \partial_i \alpha^{jk} \eta_{j\mu} \epsilon_k, \tag{3.69}$$

are gauge symmetries of the Poisson-Sigma model. Work out the structure functions and show that the gauge algebra is open. Work out the coefficient of the on-shell vanishing terms.

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Chapter 4

Batalin-Vilkovisky formalism

The (original) aim of the **Batalin-Vilkovisky** (BV) **formalism** is to control gauge invariance during the perturbative quantization of gauge theories. It builds on the methods of Faddeev and Popov, Slavnov, Taylor, Zinn-Justin, Becchi-Rouet-Stora, and Tyutin designed for Yang-Mills type theories with closed algebras involving structure constants (see [24] for a thorough review), and extends them to generic gauge theories with field-dependent structure operators and open algebras. Besides [1] and the references in the introduction of [17], useful reviews include [25] and [26], section 15.9.

4.1 Motivation

Up to now we have dealt with classical gauge field theories. If one is interested in the perturbative quantization of field theories, the main formula, encoding the Feynman rules for Green's functions, is

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}_0[0]} = \frac{\int [d\phi] e^{\frac{i}{\hbar}\left(S_0[\phi]+S_I[\phi]+J_A\phi^A\right)}}{\int [d\phi] e^{\frac{i}{\hbar}S_0[\phi]}} = e^{\frac{i}{\hbar}S_I\left[\frac{\hbar}{i}\frac{\delta}{\delta J}\right]} e^{\frac{i}{2\hbar}J_A(\mathcal{D}^{-1})^{AB}J_B} \tag{4.1}$$

Here

$$S_0 = -\frac{1}{2}\phi^A \mathcal{D}_{AB}\phi^B, \tag{4.2}$$

is the quadratic part of the Larangian, including $i\epsilon$ terms so that a unique inverse $(\mathcal{D}^{-1})^{AB}$ exists, while S_I encodes cubic and higher order vertices. In the case of gauge theories, the non-trivial aspects we have treated at the classical level have direct counterparts at the quantum level: a consequence of non-trivial gauge symmetries and Noether identities is that the quadratic kernel of the action is no longer invertible.

Remark 14. *Note that in this notation (due to DeWitt), the index $A = (i, x^\mu)$ includes the spacetime (or momentum) dependence, summation over A then includes a spacetime (or momentum) integral and δ^A_B includes Dirac deltas.*

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4.1.1 Exercise 11

In the case of the Lagrangian for free electromagnetism, show that the non-invertibility of the quadratic kernel in momentum space is directly related to gauge invariance. Hint: What is the eigenvector of eigenvalue zero of the quadratic kernel ?

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This implies that we cannot directly define a propagator. In order to perform perturbative calculations, we then have to gauge-fix the system in such a way that the quadratic kernel becomes invertible. However, if one gauge-fixes the theory in a pedestrian way one looses all information about the original gauge invariance of the system: how can we be sure then that a different gauge-fixing would not give us different physical results?

The (original) aim of the **Batalin-Vilkovisky** (BV) **formalism** is to make the quadratic kernel invertible while retaining the consequences of gauge invariance.

4.2 BV antibracket, master action and BRST differential

The core of the formalism is the introduction of an anti-canonical structure. This is possible because during our classical treatment we have introduced, besides the original fields ϕ^i, ghosts C^α for an irreducible generating set of gauge symmetries. This extended set of fields is denoted by $\Phi^a = (\phi^i, C^\alpha)$. Associated antifields $\Phi^*_a = (\phi^*_i, C^*_\alpha)$ have been introduced for the Koszul-Tate resolution of the equations of motion. It is then possible to consider fields and antifields as conjugated variables through an odd graded Lie bracket called the **antibracket**. The antibracket is defined, for two arbitrary functionals F, G, as

$$(F, G) = \int d^n x \left(\frac{\delta^R F}{\delta \Phi^a(x)} \frac{\delta^L G}{\delta \Phi^*_a(x)} - \frac{\delta^R F}{\delta \Phi^*_a(x)} \frac{\delta^L G}{\delta \Phi^a(x)} \right), \tag{4.3}$$

where the L/R superscripts on the functional derivatives denote that they are taken respectively from the left or from the right. Remember that when dealing with anticommuting variables right and left derivatives can differ by sign factors. Remember also that in the case of a local functional

$$F = \int d^n x\, f, \tag{4.4}$$

the functional derivative is given by the Euler-Lagrange derivative of its integrand

$$\frac{\delta^R F}{\delta \Phi^a(x)} = \left. \frac{\delta^R f}{\delta \Phi^a} \right|_{\phi(x)}, \tag{4.5}$$

evaluated at a section, with similar relations for antifields. With respect to the $\mathbb{Z}$-grading, called **ghost number**,

gh	0	1	-1	-2
	ϕ^i	C^α	ϕ^*_i	C^*_α

,

the antibracket has ghost number 1, in the sense that if $\mathcal{F}^{g_i}$ is a functional of ghost number g_1, then

$$(\cdot,\cdot):\mathcal{F}^{g_1}\times\mathcal{F}^{g_2}\longrightarrow\mathcal{F}^{g_1+g_2+1}. \tag{4.6}$$

The antibracket has the following properties:
graded antisymmetry:

$$(F,G)=-(-)^{(|F|+1)(|G|+1)}(G,F), \tag{4.7}$$

graded Jacobi identity:

$$(F,(G,H))=((F,G),H)+(-)^{(|F|+1)(|G|+1)}(G,(F,H)), \tag{4.8}$$

where $|G|$ denotes the $\mathbb{Z}_2$-grading of G, which is 1 for fermionic quantities and 0 for bosonic ones.

Note also that for a bosonic functional B, one can easily show that

$$\frac{1}{2}(B,B)=\int d^nx\,\frac{\delta^R B}{\delta\Phi^a(x)}\frac{\delta^L B}{\delta\Phi^*_a(x)}=-\int d^nx\,\frac{\delta^R B}{\delta\Phi^*_a(x)}\frac{\delta^L B}{\delta\Phi^a(x)}, \tag{4.9}$$

when using that

$$\frac{\delta^R F}{\delta\Phi^a(x)}=(-)^{|a|(|a|+|F|)}\frac{\delta^L F}{\delta\Phi^a(x)}, \tag{4.10}$$

and similar relations for antifields, with $|\phi^a|=|a|$ and $|\Phi^*_a|=|a|+1$ and $\mathrm{gh}\,\Phi^*_a=-\mathrm{gh}\,\phi^a-1$.

For an irreducible gauge theory, the BV master action is a functional of ghost number 0 that starts with the classical action to which one couples though antifields an irreducible generating set of gauge transformations with gauge parameters replaced by ghosts,

$$S=\int d^nx\left[\mathcal{L}+\phi^*_i R^i_\alpha(C^\alpha)+\dots\right], \tag{4.11}$$

where $\mathcal{L}$ is the classical Lagrangian density. The higher order terms hidden in $\dots$ are completely determined by requiring that S satisfies the **master equation**

$$\frac{1}{2}(S,S)=0. \tag{4.12}$$

The (antifield-dependent) **BRST differential** is canonically generated through the master action (in the same sense as in classical mechanics the Hamiltonian is the canonical generator of time translations in the Poisson bracket),

$$s=(S,\cdot). \tag{4.13}$$

It raises the ghost number by 1 and is nilpotent, $s^2=0$. Indeed, the graded Jacoby identity implies that

$$(S,(S,\cdot))=((S,S),\cdot)-(S,(S,\cdot))=-(S,(S,\cdot)), \tag{4.14}$$

when using the master equation.

With these ingredients, it can be proven that

Theorem 7. *The solution to the master equation exists, it is uniquely defined up to anticanonical transformations*[1]*, and the solution is a local functional.*

An important additional degree to prove this theorem on the existence of the master action and to analyse the antifield dependent BRST differential is the antifield number. As we have seen before, for irreducible gauge theories, it assigns 1 to the ϕ_i^* and their derivatives, 2 to the C_α^* and their derivatives, and 0 to all other variables. It then follows that the classical action is of antifield number 0, the second term in (4.11) is of antifield number 1. More precisely, the theorem states that the $\ldots$ in (4.11) are terms of antifield number higher or equal to 2. Accordingly, the expansion of BRST differential in terms of the antifield number is

$$s = \delta + \gamma + \sum_{k \geqslant 0} s_k, \tag{4.15}$$

where δ lowers the antifield number by 1, γ preserves the antifield number, while s_k raises the antifield number by k.

From (4.13), it follows that the action of the BRST differential on fields and antifields is explicitly given by

$$\begin{cases} s\Phi^a(x) = -\frac{\delta^R S}{\delta \Phi_a^*(x)}, \\ s\Phi_a^*(x) = \frac{\delta^R S}{\delta \Phi^a(x)}. \end{cases} \tag{4.16}$$

Since S is a local functional, the BRST differential can be extended to the derivatives of the fields and antifields and written as a generalized vector field on the jet-bundle that commutes with total derivatives.

It also follows from (4.11) and the definition of the antifield number that δ coincides with the Koszul-Tate differential analysed previously,

$$\delta \phi_i^* = \frac{\delta L}{\delta \phi^i}, \quad \delta C_\alpha^* = R_\alpha^{+i}[\phi_i^*], \quad \delta \Phi^a = 0, \tag{4.17}$$

while the BRST differential encodes gauge invariance in the sense that

$$\gamma \phi^i = R_\alpha^i(C^\alpha), \tag{4.18}$$

i.e., to lowest order in antifield number, the BRST transformation of the original fields is a gauge transformation with gauge parameters replaced by anticommuting ghosts.

The higher order terms in the master action encode how complicated the algebra of gauge symmetries actually is. For instance, the next term is given in terms of the structure operators of the algebra,

$$\int d^n x \frac{1}{2} C_\alpha^* f_{\beta\gamma}^\alpha (C^\beta, C^\gamma). \tag{4.19}$$

This terms determines the BRST transformations of the ghosts to lowest order in antifield number,

$$\gamma C^\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma}(C^\beta, C^\gamma). \tag{4.20}$$

[1] Anticanonical transformations are the obvious generalizations of the canonical transformations of Hamiltonian mechanics.

For theories with closed algebras involving structure constants or field-independent structure operators, this is the only additionl term needed in the master action.

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4.2.1 Exercise 12

Check that in this particular case, the expansion of the BRST differential according to antifield number stops in degree 0, $s = \delta + \gamma$, and work out $\gamma\phi^*_i$, γC^*_α.

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The first, simple example is as usual electromagnetism, for which

$$\phi^i = A_\mu, \qquad R_\mu(C) = \partial_\mu C. \tag{4.21}$$

In this case, no additional nonlinear terms have to be added to the master action, which then is

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A^{*\mu} \partial_\mu C \right]. \tag{4.22}$$

For **nonabelian gauge theories** based on semisimple Lie algebras, a case which includes both Yang-Mills and Chern-Simons theories, the only additional nonlinear term required is the one involving the structure constants,

$$S = S_{YM/CS}[A] + \int d^n x \left[A^{*\mu}_\alpha D_\mu C^\alpha + \frac{1}{2} C^*_\alpha f^\alpha{}_{\beta\gamma} C^\beta C^\gamma \right]. \tag{4.23}$$

Remark 15. *For electromagnetism, Yang-Mills and Chern-Simons theories, couplings to matter fields y^i, which can be (complex) scalars, Dirac or Weyl fermions that transform under a matrix resesentation representation ${T_\alpha}^i{}_j$ of the gauge algebra,*

$$[T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma, \tag{4.24}$$

are introduced as follows. In the Lagrangian $L_M[y, \partial y]$, supposed to be invariant under $\delta_k y^i = -k^\alpha {T_\alpha}^i{}_j y^j$ with k^α constant, one replaces $\partial_\mu y^i$ by $D_\mu y^i = \partial_\mu y^i + A^\alpha_\mu {T_\alpha}^i{}_j y^j$. It the follows that $L_M[y, Dy]$ is invariant under the gauge transformations $\delta_\epsilon A^\alpha_\mu$ and $\delta_\epsilon y^i = -\epsilon^\alpha {T_\alpha}^i{}_j y^j$ with spacetime dependent gauge parameters $\epsilon^\alpha(x)$. The master action for the complete theory is then the one for electromagnetism, Yang-Mills and Chern-Simons theories to which one adds

$$\int d^n x \left(L_M[y, Dy] - C^\alpha {T_\alpha}^i{}_j y^j y^*_i \right). \tag{4.25}$$

For **General Relativity**, one introduces diffeomorphism ghosts ξ^μ, and instead of structure constants, one deals with field-independent structure operators, since

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}, \tag{4.26}$$

and

$$[\delta_{\xi_1}, \delta_{\xi_2}]g_{\mu\nu} = -\mathcal{L}_{[\xi_1,\xi_2]}g_{\mu\nu}., \tag{4.27}$$

The BV master action in this case is

$$S = \int d^n x \left[\sqrt{|g|}R + g^{*\mu\nu}\mathcal{L}_\xi g_{\mu\nu} + \xi^*_\mu \partial_\nu \xi^\mu \xi^\nu\right]. \tag{4.28}$$

Finally for the Poisson-Sigma model, based on Exercise 10, one can show that the BV master action is given by [27]

$$S = S^{PSM} + \int d^2x \Big[- X^*_i \alpha^{ij} C_j + \eta^{*i\mu}(\partial_\mu C_i + \partial_i \alpha^{jk}\eta_{j\mu}C_k) + \frac{1}{2}C^{*i}\partial_i\alpha^{jk}C_jC_k + \frac{1}{4}\epsilon_{\mu\nu}\eta^{*i\mu}\eta^{*j\nu}\partial_i\partial_j\alpha^{kl}C_kC_l\Big], \tag{4.29}$$

with $\epsilon^{01} = 1$.

4.3 Gauge fixation

We already discussed how, because of gauge symmetry, the quadratic kernel of the action is non-invertible. In fact, this is not changed in the master action (4.11), since we have not introduced any additional terms quadratic in the classical fields. In order to have an invertible quadratic kernel and thus well-defined free propagators, allowing us to perform computations in perturbation theory, we still need to gauge-fix the theory.

In order to do so, one usually has to introduce more fields and antifields, belonging to the so called **nonminimal sector**: for theories of Yang-Mills type, these will be the fermionic **antighosts** $\bar{C}^\alpha$ of ghost number -1, the bosonic **Lagrange multiplier** (also called **Nakanishi-Lautrup auxiliary field**) B^α of gost number 0 and their respective antifields $\bar{C}^*_\alpha$, B^*_α of respective ghost numbers $0, -1$. The complete set of fields is then $\Phi^a = (\phi^i, C^\alpha, \bar{C}^\alpha, B^\alpha)$.

The master action is extended to the non-minimal sector through

$$S_{NM} = S - \int d^n x \bar{C}^*_\alpha B^\alpha. \tag{4.30}$$

The BRST transformation of these new fields generated by S are then

$$s\bar{C}^\alpha = B^\alpha, \qquad sB^\alpha = 0, \tag{4.31}$$

$$s\bar{C}^*_\alpha = 0, \qquad sB^*_\alpha = -\bar{C}^*_\alpha, \tag{4.32}$$

so that it is simply a shift symmetry, not affecting the physics. From the point of view of the cohomology of s, $\bar{C}^\alpha$ and B^α, their antifields and all their derivatives form so-called **contractible pairs**, which means that they do not contribute to any of the relevant cohomology groups. They are only needed as a convenient means to fix the gauge.

This is done through a so-called **gauge-fixing fermion** $\Psi[\Phi]$, which is a fermionic functional of the fields alone of ghost number -1 (and thus necessarily dependent on fields from the non-minimal sector). For a large class of gauges in Yang-Mills type theories for example, it may be choosen as

$$\Psi = \int d^n x \bar{C}^\alpha \left(\partial_\mu A^{\mu\beta} - \frac{\xi}{2} B^\beta \right) g_{\alpha\beta}, \tag{4.33}$$

where $g_{\alpha\beta}$ is the Cartan-Killing metric of the gauge group, that can be used, together with its inverse, to lower and raise Lie algebra indices.

The gauge-fixed action is then defined by a shift of the antifields by a "functional gradient" term of the gauge-fixing fermion:

$$S_\Psi[\Phi, \tilde{\Phi}^*] = S_{NM}\left[\Phi, \tilde{\Phi}^* + \frac{\delta \Psi}{\delta \phi} \right]. \tag{4.34}$$

In the specific case of Yang-Mills type theories in which Ψ is defined by (4.33) and the solution of the master action is linear in antifields, the gauge-fixed action is

$$S_\Psi[\Phi, \tilde{\Phi}^*] = S^{YM,CS} - s\Psi - \int d^n x \, s\phi^a \tilde{\Phi}^*_a, \tag{4.35}$$

with

$$-s\Psi = \int d^n x \left[-\partial_\mu \bar{C}_\alpha D^\mu C^\alpha - (\partial_\mu A^\mu_\alpha) B^\alpha + \frac{\xi}{2} B_\alpha B^\alpha \right]. \tag{4.36}$$

A crucial point of the BV construction is that the gauge-fixing is a **canonical transformation**, which means that it leaves the canonical antibracket relations invariant. In fact, it is a canonical transformation of the second type (following the classification of Goldstein's textbook), for which Ψ is the generator. This can be easily seen by identifying the fields with the coordinates of classical mechanics, and the antifields with the momenta:

$$q \leftrightarrow \Phi, \qquad p \leftrightarrow \Phi^*. \tag{4.37}$$

We recall that a canonical transformation $qdp = QdP + dF$ of the second type is generated by a function F of the form

$$F = F_2(q, P) - QP, \tag{4.38}$$

and transforms the canonical variables as

$$p = \frac{\partial F_2}{\partial q}, \qquad Q = \frac{\partial F_2}{\partial P}. \tag{4.39}$$

Gauge-fixing is a transformation of exactly this type, with

$$F_2 = qP + \psi(q) \qquad \Longrightarrow \qquad \begin{cases} p = P + \frac{\partial \psi}{\partial q} \\ q = Q. \end{cases} \tag{4.40}$$

This fact has the important consequence that the gauge-fixed master action is still a solution of the master equation in the new variables,

$$(S_\Psi, S_\Psi)_{\Phi, \tilde{\Phi}^*} = 0. \tag{4.41}$$

The gauge fixed (antifield-dependent) BRST differential is defined by $s_\psi = (S_\Psi, \cdot)_{\Phi,\tilde{\Phi}*}$ and is nilpotent off-shell, $s_\Psi^2 = 0$.

Note that in theories with a master action that is linear in antifields, the BRST transformations of the fields coincide before and after gauge fixing, $s_\Psi \Phi^a = s\Phi^a$. In the general case, defining the gauge fixed BRST differential without antifields by $s^g\Phi^a = s_\Psi\Phi^a[\Phi, \tilde{\Phi}^* = 0]$, we have

$$s^g S_\Psi[\Phi, 0] = 0, \tag{4.42}$$

which follows by putting the antifields $\tilde{\Phi}_a^*$ to zero in $\frac{1}{2}(S_\Psi, S_\Psi) = 0$. Except for theories with a linear dependence in antifields, s^g is in general only nilpotent when the equations of motion defined by $S_\Psi(\Phi, 0)$ hold, $(s^g)^2\Phi^a \approx 0$. This follows by putting the antifields $\tilde{\Phi}_a^*$ to zero in $(S_\Psi, (S_\Psi, \phi^a)_{\Phi,\tilde{\Phi}*})_{\Phi,\tilde{\Phi}*} = 0$.

In order to avoid this and to better control the original gauge invariance of the theory, it is most convenient not to put the antifields $\tilde{\Phi}^*$ to zero during renormalization. These antifields then act as **sources for the (non-linear) BRST transformation** whose renormalization is then controlled together with the renormalization of the (gauge fixed) action.

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4.3.1 Exercise 13

Eliminate the auxiliary fields B^α in the gauge fixed master action with antifields for Yang-Mills type theories. Show that $s^g\bar{C}^a$ is only nilpotent on-shell. Work out the momentum space propagators in Yang-Mills and Chern-Simons theories with and without auxiliary B^α fields.

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4.4 Independence of gauge fixing

The statement that we didn't spoil the physical content of gauge invariance after gauge-fixing is captured by the following:

Theorem 8 (Fradkin-Vilkovisky theorem). *Expectation values of BRST-closed operators are independent of the choice of gauge-fixing.*

Indeed, let Ψ and $\Psi + \delta\Psi$ be two different gauge-fixing fermions, and $X[\Phi, \tilde{\Phi}^*]$ be BRST closed, i.e., such that

$$(S_\Psi, X) = 0. \tag{4.43}$$

In particular, if X only depends on the original fields, this condition means that

X is gauge invariant. we have,

$$\begin{aligned}
&\langle 0,+\infty|T\hat{X}|0,-\infty\rangle_{\Psi+\delta\Psi}-\langle 0,+\infty|T\hat{X}|0,-\infty\rangle_{\Psi}\\
=&\int[d\Phi]\left(e^{\frac{i}{\hbar}S_{\Psi+\delta\Psi}}-e^{\frac{i}{\hbar}S_{\Psi}}\right)X\\
=&\frac{i}{\hbar}\int[d\Phi]\frac{\delta^R S_\Psi}{\delta\tilde{\Phi}^*_A}\frac{\delta^L\delta\Psi}{\delta\Phi^A}e^{\frac{i}{\hbar}S_\Psi}X+O((\delta\Psi)^2)\\
=&(-)^{|A|+1}\frac{i}{\hbar}\int[d\Phi]\frac{\delta^L(\delta\Psi)}{\delta\Phi^A}\frac{\delta^R S_\Psi}{\delta\tilde{\Phi}^*_A}e^{\frac{i}{\hbar}S_\Psi}X+O((\delta\Psi)^2)\\
=&\frac{i}{\hbar}\int[d\Phi]\delta\Psi\frac{\delta^L}{\delta\Phi^A}\left[(s_\psi\Phi^A)e^{\frac{i}{\hbar}S_\Psi}X\right]+O((\delta\Psi)^2).
\end{aligned}\tag{4.44}$$

Here in the second equality we have expanded the exponential of the action to first order in the deformation of the gauge-fixing, while in the fourth equality we have performed an integration by parts inside the functional integral, and we have noted that $\frac{\delta^R S_\Psi}{\delta\tilde{\Phi}^*_A}=-s_\psi\Phi^A$. Neglecting the contact term contribution $\frac{\delta^L}{\delta\Phi^A}(s_\Psi\Phi^A)$ which is proportional to $\delta(0)$ and taking into account antifield dependent BRST invariance of the gauge fixed action in the form

$$s_\Psi\Phi^A\frac{\delta^L S_\Psi}{\delta\phi^A}=\frac{1}{2}(S_\Psi,S_\Psi)=0,\tag{4.45}$$

and also (4.43), we find that

$$\begin{aligned}
&\langle 0,+\infty|T\hat{X}|0,-\infty\rangle_{\Psi+\delta\Psi}-\langle 0,+\infty|T\hat{X}|0,-\infty\rangle_{\Psi}\\
=&\frac{i}{\hbar}\int[d\Phi]\delta\Psi e^{\frac{i}{\hbar}S_\Psi}(s_\psi\phi^A)\frac{\delta^L X}{\delta\Phi^A}+O((\delta\Psi)^2)\\
=&\frac{i}{\hbar}\int[d\Phi]\delta\Psi e^{\frac{i}{\hbar}S_\Psi}\frac{\delta^R S_\Psi}{\delta\Phi^A}\frac{\delta^L X}{\delta\tilde{\Phi}^*_A}+O((\delta\Psi)^2),
\end{aligned}\tag{4.46}$$

The term of first order in $\delta\Psi$ then vanishes on account of the Schwinger-Dyson equations of the theory.

Remark 16. *These are formal arguments that hold at tree level. They may be violated by ħ-correction when taking renormalization into account. Some of these ħ-corrections are captured in an elegant way by the so-called quantum BV formalism. However, this formalism remains formal as well unless due care is devoted to renormalization.*

4.5 Slavnov-Taylor identities and Zinn-Justin equation

BRST invariance of the gauge fixed action (with antifields) leads to relations among correlation functions, generalizing the **Slavnov-Taylor (ST) identities** of non-abelian gauge theories. They are most economically stated in terms of generating functionals. Indeed, starting from translation invariance of the path integral, neglecting contact terms and using (4.45), we get

$$\begin{aligned} 0 &= \frac{1}{Z[J,\tilde{\Phi}^*]}\int [d\Phi]\frac{\delta^L}{\delta\Phi^A}\left(s_\Psi\Phi^A e^{\frac{i}{\hbar}(S_\Psi + J_A\Phi^A)}\right) \\ &= \frac{i(-)^{|A|}J_A}{\hbar Z[J,\tilde{\Phi}^*]}\int [d\Phi] s_\Psi\Phi^A e^{\frac{i}{\hbar}(S_\Psi + J_A\cdot\Phi^A)} = \frac{i}{\hbar}(-)^{|A|}J_A\langle\widehat{s_\Psi\Phi^A}\rangle^{J,\tilde{\Phi}^*}. \end{aligned} \tag{4.47}$$

Defining the normalized generating functional for connected Green's functions $W[J,\tilde{\Phi}^*]$ through

$$e^{\frac{i}{\hbar}W[J,\tilde{\Phi}^*]} = \frac{Z[J,\tilde{\Phi}^*]}{Z[0,0]}, \tag{4.48}$$

this equation can be written as

$$(-)^{|A|}J_A\frac{\delta^R\mathcal{W}}{\delta\Phi^*_A} = 0. \tag{4.49}$$

Finally, performing the Legendre transform with respect to J_A,

$$\tilde{\Phi}^A = \langle\Phi^A\rangle^{J,\tilde{\Phi}^*} = \frac{\delta^L W}{\delta J^A}, \tag{4.50}$$

with inverse relation $J_A = J_A[\tilde{\Phi},\tilde{\Phi}^*]$, we can define the **quantum effective action**,

$$\Gamma[\tilde{\Phi},\tilde{\Phi}^*] = \left[\mathcal{W}[J,\tilde{\Phi}^*] - J_A\tilde{\Phi}^A\right]\Bigg|_{J=J[\tilde{\Phi},\tilde{\Phi}^*]}, \tag{4.51}$$

which can be shown to be the generating functional of 1PI diagrams. In particular, the fact that the effective action is defined as a Legendre transformation implies

$$\frac{\delta^R\Gamma}{\delta\tilde{\Phi}^A} = -J_A[\tilde{\Phi},\tilde{\Phi}^*], \qquad \frac{\delta^R\Gamma}{\delta\Phi^*_A} = \frac{\delta^R\mathcal{W}}{\delta\tilde{\Phi}^*_A}\Big|_{J=J[\tilde{\Phi},\tilde{\Phi}^*]}. \tag{4.52}$$

In terms of the effective action, the ST identity then takes the form of the generalized **Zinn Justin equation**, i.e., the master equation for the effective action,

$$\frac{1}{2}(\Gamma,\Gamma)_{\tilde{\Phi},\tilde{\Phi}^*} = 0. \tag{4.53}$$

4.6 Elements of renormalization

The effective action is crucial for the renormalized theory: once its is well-defined so is the complete theory. This is because connected Green's functions can be constructed by using the effective action in order to derive the Fyenman rules while summing over connected tree graphs alone. The passage from connected to all Green's functions is purely combinatorial, while the passage from Green's functions to S-matrix elements through reduction formulas does not involve ultraviolet issues either.

Gauge invariance is encoded in the master equation for the effective action, whose derivation above was formal. In order to go beyond such formal considerations, one may start by assuming the existence of a regularisation scheme that is consistent with gauge invariance, such as dimensional regularization in the absence of chiral fermions and the Levi-Civita (pseudo-)tensor. In this case, the regularized effective action $\Gamma_{\rm reg}$ continues to satisfy the master action,

$$\frac{1}{2}(\Gamma_{\rm reg}, \Gamma_{\rm reg})_{\tilde{\Phi},\tilde{\Phi}*} = 0. \tag{4.54}$$

When using that

$$\Gamma_{\rm reg}[\tilde{\Phi}, \tilde{\Phi}^*] = S_\Psi[\tilde{\Phi}, \tilde{\Phi}^*] + \hbar\Gamma^{(1)} + O(\hbar^2), \tag{4.55}$$

with $\hbar$ keeping track of the loop order in the perturbative expansion, and taking into account the master equation for S_Ψ, the master equation for $\Gamma_{\rm reg}$ reduces at order $\hbar$ to

$$(S_\Psi, \Gamma^{(1)})_{\tilde{\Phi},\tilde{\Phi}*} = 0. \tag{4.56}$$

The 1-loop contribution is made of a divergent and a finite part,

$$\Gamma^{(1)} = \frac{1}{\varepsilon}\Gamma^{(1)}_{\rm div} + \Gamma^{(1)}_{\rm fin}, \tag{4.57}$$

where we implicitly are assuming dimensional regularization but $1/\varepsilon$ could be replaced by another ultraviolet regulator regulator. A crucial property is that the one-loop divergences are **local functionals**. We then get

$$(S_\Psi, \Gamma^{(1)}_{\rm fin}) = 0, \qquad\qquad (S_\Psi, \Gamma^{(1)}_{\rm div}) = 0, \tag{4.58}$$

because the two terms are of different orders in ε. The latter equation thus means that one-loop divergences are BRST-closed local functionals. If furthermore, they can be expressed as

$$\Gamma^{(1)}_{\rm div} = (S_\Psi, \Xi), \tag{4.59}$$

for some local function Ξ of ghost number -1, then they are trivial divergences in the sense that they can be absorbed by a canonical redefinition of fields and antifields, while the absorption of nontrivial divergences requires the presence of appropriate coupling constants in the starting point action.

Non trivial divergences are thus decribed by antifield-dependent BRST cohomology in ghost number 0 in the space of local functionals,

$$H^0(s_\Psi) \simeq H^0(s), \tag{4.60}$$

where we stress that this cohomology is isomorphic to the gauge fixing independent cohomology associated to the minimal solution of the master action since both are related by an anticanonical transformation and the trivial pairs of the gauge fixing sector drop out of cohomology.

In renormalizable theories one can show that, once lower order divergences have been absorbed to a given order, the divergences at the next order are again BRST closed.

Remark 17. *The classical problem of constructing interactions consistent with gauge invariance (sometimes also refered to as "Noether procedure") can be formulated as a classical deformation problem using the BV formalism [28]. Similarily to divergences which consitute a quantum deformation, infinitesimal deformations in that context are also controlled by antifield-dependent BRST cohomology in ghost number* 0 *in the space of local functionals. The deformation parameter is no longer* $\hbar$ *but can for instance be related to the number of fields involved in the interactions or to the couplings constants. In the context of non-commutative Yang-Mills models, the existence of Seiberg-witten maps can also be discussed in these terms [29, 30].*

Up to now, we have assumed that the regularization used was compatible with gauge invariance, so that the master equation was still satisfied after regularization. It is however possible that no such regulator exists: in such a case the generating functional Γ no longer satisfies the master equation, but rather

$$\frac{1}{2}(\Gamma,\Gamma) = \hbar\mathcal{A}\circ\Gamma, \tag{4.61}$$

where the insertion $\mathcal{A}$ corresponds again to a local functional, called the **anomaly**. In this equation, Γ either refers to the renormalized effective action in the case of of power counting renormalizable theories and the equation is derived using the so-called quantum action principle in the framework of BPHZ renormalization (see [24] for a review), or it refers to the regularized effective action for instance in the context of dimensional renormalization, where so-called evanescent terms proportional to ϵ and due to the regularization, produce the right hand side (see [31, 32] for more details and also [33] for related considerations). Note that in both these frameworks divergences, respectively the associated finite amiguities in the BPHZ approach, are still controlled by $H^0(s)$, even in the presence of anomalies.

The generalized **Wess-Zumino consistency conditions** then follows:

$$(\Gamma,(\Gamma,\Gamma)) = 0 \qquad \Longrightarrow \qquad (\Gamma,\mathcal{A}\circ\Gamma) = 0. \tag{4.62}$$

To the lowest order in $\hbar$, this means that

$$(S_\Psi,\mathcal{A}) = 0. \tag{4.63}$$

In this case, if the anomaly is exact,

$$\mathcal{A} = (S_\Psi,\Sigma), \tag{4.64}$$

it can be reabsorbed by adding a local, BRST breaking counterterm, Σ, $S_\Psi \rightarrow S_\Psi - \Sigma$. Hence, to lowest nontrivial order, anomalies are characterized by antifield-dependent BRST cohomology in ghost number 1 in the space of local functionals,

$$H^1(s_\Psi) \simeq H^1(s), \tag{4.65}$$

the determination of which is a gauge-invariant, computable problem.

In Yang-Mills theory, the **Adler-Bardeen anomaly** constitues a famous example of such a cohomology class. It is computed in the so-called universal algebra and corresponds to $\omega^{1,4}$ in the chain of descent equations that relates a characteristic class in form degree 6 to a primitive element in ghost number 3,

$$\begin{aligned} \mathrm{Tr}\, F^3 &= d\omega^{0,5}, \\ s\omega^{0,5} + d_H \omega^{1,4} &= 0, \\ s\omega^{1,4} + d_H \omega^{2,3} &= 0, \\ &\vdots \\ s\omega^{4,1} + d_H \mathrm{Tr}\, C^3 &= 0, \\ s\mathrm{Tr}\, C^3 &= 0. \end{aligned} \tag{4.66}$$

When imposing power-counting restrictions, semi-simple Yang-Mills theories can be shown to be renormalizable quite easily in this framework. Indeed, in this case, it is almost straightforward to show that the only BRST cohomology class in ghost number 0 in the space of local functionals corresponds to the classical, gauge invariant Yang-Mills action itself: all divergences which are not absorbable by canonical field-antifield redefinitions can thus be absorbed by redefinitions of the couplings associated with the different simple factor of the gauge algebra.

The problem of completely characterizing antifield-dependent BRST cohomology in various ghost numbers, independently of power-counting restrictions, has been addressed for Yang-Mills type theories in [34, 35] and reviewed in [17]. The inclusion of free abelian vector fields has been completed more recently [36, 37]. Einstein gravity has been considered in [38].

Based on these results, it has been argued [39] that these theories are renormalizable in the modern sense, that is to say, in the sense of effective field theories: for semi-simple Yang-Mills theory or Einstein gravity, the antifield-dependent BRST cohomology is exhausted by integrated, on-shell gauge invariant observables. Once these are included with independent couplings from the very beginning, all divergences can be absorbed, either by anticanonical field-antifield redefinitions or by redefinitions of these couplings.

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Summary In this set of lectures, we have tried to set up the relevant background material to compute antifield-dependent BRST cohomology classes in the space of local functionals for generic irreducible gauge theories. What physical information they provide may be summarized in the following table:

g	$H^g(s)$
$\vdots$	$\varnothing$;
-3	$\varnothing$;
-2	reducibility parameters (Killing vectors) / ADM-type surface charges
-1	conserved currents and global symmetries
0	loop divergences, consistent deformations
1	anomalies
$\vdots$	

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