RENORMALIZATION GROUP: EXERCISES

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Lectures delivered at the 21th Saalburg School, Wolfersdorf September 2015

Exercises

Exercise 1

Study the local stability of the Gaussian fixed point

$$\rho_{\mathrm{G}}(q) = \mathrm{e}^{-q^2/2}\,/\sqrt{2\pi}\,,$$

by starting directly from the equation

$$[\mathcal{T}_\lambda \rho](q) = \lambda \int \mathrm{d}q' \rho(q')\rho(\lambda q - q'). \tag{1}$$

Determine the value of the renormalization factor λ for which the Gaussian probability distribution ρ_{G} is a fixed point of $\mathcal{T}_\lambda$.

Setting $\rho = \rho_{\mathrm{G}} + \delta\rho$, expand equation (1) to first order in $\delta\rho$. Show that the eigenvectors of the linear operator acting on $\delta\rho$ have the form

$$\delta\rho_p(q) = (\mathrm{d}/\mathrm{d}q)^p\,\rho_{\mathrm{G}}(q), \quad p > 0\,.$$

Infer the corresponding eigenvalues.

Solution: a few indications. One recovers $\lambda = \sqrt{2}$. One linearizes the equation. One notes that probability conservation implies

$$1 = \int \mathrm{d}q \, \rho(q) = \int \mathrm{d}q \, [\rho_{\mathrm{G}} + \delta\rho(q)] = 1 + \int \mathrm{d}q \, \delta\rho(q) \;\Rightarrow\; \int \mathrm{d}q \, \delta\rho(q) = 0 \, .$$

Setting

$$[\mathcal{T}_\lambda(\rho_{\mathrm{G}} + \delta\rho)] = \rho_{\mathrm{G}} + \mathcal{L}\delta\rho + O\left(\|\delta\rho\|^2\right) ,$$

where the action of the linear operator $\mathcal{L}$ on a function $\delta\rho$ is given by

$$[\mathcal{L}\delta\rho](q) = 2\lambda \int \mathrm{d}q' \, \rho_{\mathrm{G}}(q')\delta\rho(\lambda q - q'),$$

one verifies that the eigenvectors of $\mathcal{L}$ have the proposed form by integrating several times by parts.

Exercise 2

Random walk on a circle. To exhibit the somewhat different asymptotic properties of a random walk on compact manifolds, it is proposed to study random walk on a circle. One still assumes translation invariance. The random walk is then specified by a transition function $\rho(q-q')$, where q and q' are two angles corresponding to positions on the circle. Moreover, the function $\rho(q)$ is assumed to be periodic and continuous. Determine the asymptotic distribution of the walker position. At initial time $n=0$, the walker is at the point $q=0$.

Solution. Due to translation invariance, the evolution equation is still a convolution, which simplifies in the Fourier representation. But, since the function $\rho(q)$ is periodic and continuous, it admits a Fourier series expansion of the form

$$\rho(q) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{iq\ell}\, \tilde{\rho}_\ell$$

with

$$\tilde{\rho}_\ell = \int_{-\pi}^{+\pi} \mathrm{d}q \; \mathrm{e}^{-iq\ell} \, \rho(q)$$

and, thus taking the absolute value on both side,

$$\tilde{\rho}_0 = 1\,, \quad |\tilde{\rho}_\ell| < 1 \quad \text{for} \quad \ell \neq 0\,.$$

Thus, at time n the distribution of the walker position can be written as

$$P_n(q) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{iq\ell} \, \tilde{\rho}_\ell^n\,.$$

For $n \to \infty$, the sum converges exponentially toward the contribution $\ell = 0$ and, thus,

$$P_n(q) \underset{n\to\infty}{=} \frac{1}{2\pi}\,,$$

which is a uniform distribution on the circle.

The maximum value of $|\tilde{\rho}_\ell|$ for $\ell \neq 0$, which yields the leading correction, defines a time

$$\tau = \max_{\ell \neq 0} - \frac{1}{\ln|\tilde{\rho}_\ell|},$$

which characterizes the exponential decay of corrections, called the relaxation time.

Exercise 3

Another universality class

One considers now the transition probability $\rho(q-q')$ with

$$\rho(q) = \frac{2}{3\pi} \frac{2+q^2}{(1+q^2)^2}.$$

The initial distribution is again

$$P_0(q) = \delta(q).$$

Evaluate the asymptotic distribution $P_n(q)$ for $n \to \infty$.

Solution. The Fourier transform $\tilde{\rho}(k)$ of $\rho(q)$ is

$$\tilde{\rho}(k) = \int dq \; \mathrm{e}^{ikq} \, \rho(q) = (1 + |k|/3) \, \mathrm{e}^{-|k|}$$

and, thus,

$$w(k) = \ln(1 + |k|/3) - |k| = -\tfrac{2}{3}|k| - \tfrac{1}{18}k^2 + O(|k|^3).$$

The non-analytic behaviour is a direct consequence of the slow decay of the transition function, which has no second moment.

Introducing the transformation

$$[\mathcal{T}_\lambda](k) = 2w(k/\lambda),$$

and expanding for $k \to 0$, we see that the only fixed point solution corresponds to $\lambda = 2$ and is

$$w_*(k) = -\tfrac{2}{3}|k|.$$

The asymptotic distribution is then

$$P_\infty(q) = \frac{1}{\pi}\int \mathrm{d}q\ \mathrm{e}^{ikq}\,\mathrm{e}^{-w_*(k)} = \frac{12}{\pi(9q^2+4)},$$

which is a Cauchy–Lorentz distribution.

Exercise 4

Random walk and detailed balance. One considers a Markovian process in continuum space with, as transition probability, the Gaussian function

$$\rho(q,q') = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(q-\lambda q')^2\right],$$

where λ is a real parameter with $|\lambda| < 1$.

Show that $\rho(q,q')$ satisfies the condition of detailed balance (10). Infer the asymptotic distribution at time n when $n \to \infty$. Associate to $\rho(q,q')$ a real symmetric operator as in equation (11).

Solution. The ratio $\rho(q,q')/\rho(q',q)$ can be factorized into a ratio of two functions of q and q' that correspond to normalizable distributions. One infers from the detailed balance condition the asymptotic distribution

$$P_\infty(q) = \sqrt{(1-\lambda^2)/2\pi}\, \mathrm{e}^{-(1-\lambda^2)q^2/2} \,.$$

One then introduces the real symmetric operator that has the same spectrum as ρ:

$$\mathcal{T}(q,q') = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{4}(1+\lambda^2)(q^2+q'^2) + \lambda q q'\right] .$$

For $\lambda > 0$, one can set $\lambda = \mathrm{e}^{-\theta}$, $\theta > 0$ and, after a linear transformation on q, q' one recovers, after some rescaling, the density matrix at thermal equilibrium of the harmonic oscillator. One infers the eigenvalues

$$\tau_k = \mathrm{e}^{-k\theta} \,, \quad k \ge 0 \,.$$

The case $\lambda < 0$ can be studied, for example, by setting $\lambda = -\,\mathrm{e}^{-\theta}$ and the spectrum is

$$\tau_k = (-1)^k\, \mathrm{e}^{-k\theta}\,, \quad k \geq 0\,.$$

More directly, one can study the action of ρ on the functions

$$\psi_k(q) = \frac{\mathrm{d}^k}{(\mathrm{d}q)^k} P_\infty(q).$$

The result is then obtained by successive integrations by parts.