Exercises Saalburg School 2015

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1. a. Let

$$[\Sigma_{\mu\nu}]_{\alpha}^{\;\;\beta} = \eta_{\mu\alpha}\delta_{\nu}^{\beta} - \eta_{\nu\alpha}\delta_{\mu}^{\beta}. \tag{1}$$

Show that

$$[\Sigma_{\mu\nu}, \Sigma_{\kappa\lambda}] = \eta_{\nu\kappa}\Sigma_{\mu\lambda} - \eta_{\nu\lambda}\Sigma_{\mu\kappa} - \eta_{\mu\kappa}\Sigma_{\nu\lambda} + \eta_{\mu\lambda}\Sigma_{\nu\kappa}. \tag{2}$$

b. Same for

$$[\Sigma_{\mu\nu}]_{\alpha\beta}^{\;\;\gamma\delta} = \left(\eta_{\mu\alpha}\delta_{\nu}^{\gamma} - \eta_{\nu\alpha}\delta_{\mu}^{\gamma}\right)\delta_{\beta}^{\delta} + \delta_{\alpha}^{\gamma}\left(\eta_{\mu\beta}\delta_{\nu}^{\delta} - \eta_{\nu\beta}\delta_{\mu}^{\delta}\right). \tag{3}$$

2. Define

$$Z^2 = P^{\mu}\Sigma_{\mu\lambda}P^{\nu}\Sigma_{\nu}^{\;\lambda}, \quad W^2 = \frac{1}{2}\,P^2\Sigma^2 - Z^2. \tag{4}$$

a. Show that for a vector field A_α

$$\begin{aligned} \left[Z^2 \cdot A\right]_{\alpha} &= -P^2A_{\alpha} - 2P_{\alpha}P^{\beta}A_{\beta}, \\ \left[W^2 \cdot A\right]_{\alpha} &= -2P^2A_{\alpha} + 2P_{\alpha}P^{\beta}A_{\beta}. \end{aligned} \tag{5}$$

b. Show that the eigenvalue equations

$$P^2A_{\alpha} = m^2A_{\alpha}, \quad \left[Z^2 \cdot A\right]_{\alpha} = \lambda A_{\alpha}, \qquad \left[W^2 \cdot A\right]_{\alpha} = \kappa A_{\alpha}, \tag{6}$$

have 2 solutions:
(i) a scalar solution (pure gradient)

$$\lambda = -3m^2, \quad \kappa = 0, \quad A_{\alpha} = P_{\alpha}\Phi, \quad P^2\Phi = m^2\Phi; \tag{7}$$

(ii) a transverse (divergence-free) vector solution

$$\lambda = -m^2, \quad \kappa = -2m^2, \quad P^2A_{\alpha} = m^2A_{\alpha}, \quad P^{\alpha}A_{\alpha} = 0. \tag{8}$$

c. Show that for a symmetric tensor field

$$\begin{aligned} [Z^2 \cdot A]_{\alpha\beta} &= -2P^2A_{\alpha\beta} - 4P_{\alpha}P^{\gamma}A_{\beta\gamma} - 4P_{\beta}P^{\gamma}A_{\alpha\gamma} + 2\eta_{\alpha\beta}P^{\gamma}P^{\delta}A_{\gamma\delta} + 2P_{\alpha}P_{\beta}A_{\gamma}^{\;\gamma}, \\ [W^2 \cdot A]_{\alpha\beta} &= -6P^2A_{\alpha\beta} + 4P_{\alpha}P^{\gamma}A_{\beta\gamma} + 4P_{\beta}P^{\gamma}A_{\alpha\gamma} - 2P_{\alpha}P_{\beta}A_{\gamma}^{\;\gamma} \\ &\quad + 2\eta_{\alpha\beta}\left(P^2A_{\gamma}^{\;\gamma} - P^{\gamma}P^{\delta}A_{\gamma\delta}\right). \end{aligned} \tag{9}$$

d. Show that in this case the eigenvalue equations appropriately generalized from (6) have 4 solutions:
(i) A pure-trace scalar

$$\lambda = 0, \quad \kappa = 0, \quad A_{\alpha\beta} = \eta_{\alpha\beta}\Phi, \quad P^2\Phi = m^2\Phi; \tag{10}$$

(ii) A traceless scalar

$$\lambda = -8m^2, \quad \kappa = 0, \quad A_{\alpha\beta} = \left(P_\alpha P_\beta - \frac{1}{4}\,\eta_{\alpha\beta}P^2 \right)\Omega, \quad P^2\Omega = m^2\Omega; \tag{11}$$

(iii) A transverse (divergence-free) vector

$$\lambda = -6m^2, \quad \kappa = -2m^2, \quad A_{\alpha\beta} = P_\alpha V_\beta + P_\beta V_\alpha, \quad P^\alpha V_\alpha = 0, \quad P^2 V_\alpha = m^2 V_\alpha; \tag{12}$$

(iv) A traceless transverse tensor

$$\lambda = -2m^2, \quad \kappa = -6m^2, \quad A_\alpha^{\;\alpha} = 0, \quad P^\alpha A_{\alpha\beta} = 0, \quad P^2 A_{\alpha\beta} = m^2 A_{\alpha\beta}. \tag{13}$$

e. Check that in all cases the eigenvalues of W^2 satisfy

$$-\frac{\kappa}{m^2} = s(s+1), \tag{14}$$

where $s = 0$ for scalars, $s = 1$ for vectors and $s = 2$ for pure symmetric tensors.

3. *Anti-symmetric tensor fields*
Let $t_{\mu\nu} = -t_{\nu\mu}$ be an anti-symmetric tensor field. Define

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}, \quad \omega_{\mu\nu} = \frac{P_\mu P_\nu}{P^2}. \tag{15}$$

a. Show that one can define 2 projection operators for $t_{\mu\nu}$:

$$\Pi^{(0)\,\kappa\lambda}_{\mu\nu} = \frac{1}{2}\left(\theta_\mu^{\;\kappa}\theta_\nu^{\;\lambda} - \theta_\mu^{\;\lambda}\theta_\nu^{\;\kappa} \right), \quad \Pi^{(1)\,\kappa\lambda}_{\mu\nu} = \frac{1}{2}\left(\theta_\mu^{\;\kappa}\omega_\nu^{\;\lambda} - \theta_\nu^{\;\kappa}\omega_\mu^{\;\lambda} - \theta_\mu^{\;\lambda}\omega_\nu^{\;\kappa} + \theta_\nu^{\;\lambda}\omega_\mu^{\;\kappa} \right), \tag{16}$$

with the properties

$$\Pi^{(A)} \cdot \Pi^{(B)} = \delta^{AB}\Pi^{(A)}, \quad \Pi^{(0)} + \Pi^{(1)} = 1, \tag{17}$$

where the unit operator on anti-symmetric tensors is

$$1 \;\rightarrow\; \frac{1}{2}\left(\delta_\mu^{\;\kappa}\delta_\nu^{\;\lambda} - \delta_\nu^{\;\kappa}\delta_\mu^{\;\lambda} \right).$$

b. Consider the field equation

$$\Box\Pi^{(0)\,\kappa\lambda}_{\mu\nu} t_{\kappa\lambda} = m^2 t_{\mu\nu}. \tag{18}$$

Check that it is regular (no non-local poles), and that for $m^2 > 0$ it implies

$$\partial^\lambda t_{\lambda\mu} = 0, \qquad \Box t_{\mu\nu} = m^2 t_{\mu\nu}, \tag{19}$$

i.e., the field is divergence-free and satisfies the Klein-Gordon equation.
c. Show that the field eqn. (18) can be derived from an action

$$S = \int d^4x \left[-\frac{1}{3!} \left(\partial_\mu t_{\nu\lambda} + \partial_\nu t_{\lambda\mu} + \partial_\lambda t_{\mu\nu} \right)^2 - \frac{m^2}{2} t_{\mu\nu}^2 \right]. \tag{20}$$

d. Check that in the massless case $m^2 = 0$, both the action and the field equation are invariant under abelian gauge transformations

$$t'_{\mu\nu} = t_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{21}$$

e. Explain how these gauge transformations can be used to reobtain the condition that $t_{\mu\nu}$ is divergence-free also in the massless case.

4. The Einstein form of the action for GR is

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\Gamma_{\mu\lambda}^{\ \ \kappa} \Gamma_{\nu\kappa}^{\ \ \lambda} - \Gamma_{\mu\nu}^{\ \ \lambda} \Gamma_{\lambda\kappa}^{\ \ \kappa} \right). \tag{22}$$

The metric connection is

$$\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2}\, g^{\lambda\kappa} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right). \tag{23}$$

a. Show that

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\lambda\mu}^{\ \ \kappa}\, g_{\kappa\nu} + \Gamma_{\lambda\nu}^{\ \ \kappa}\, g_{\mu\kappa}. \tag{24}$$

and that

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\, \Gamma_{\mu\nu}^{\ \ \nu}. \tag{25}$$

b. By partial integration derive the identities

$$\begin{aligned} \int d^4x \sqrt{-g}\, g^{\mu\nu}\, \Gamma_{\mu\kappa}^{\ \ \lambda} \Gamma_{\nu\lambda}^{\ \ \kappa} &\simeq \frac{1}{2} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\partial_\lambda \Gamma_{\mu\nu}^{\ \ \lambda} + \Gamma_{\mu\nu}^{\ \ \lambda} \Gamma_{\lambda\kappa}^{\ \ \kappa} \right), \\ \int d^4x \sqrt{-g}\, g^{\mu\nu} \Gamma_{\mu\nu}^{\ \ \lambda} \Gamma_{\lambda\kappa}^{\ \ \kappa} &\simeq \int d^4x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \Gamma_{\nu\lambda}^{\ \ \lambda}, \end{aligned} \tag{26}$$

up to boundary terms.

c. Using these results show that

$$\begin{aligned} S &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\partial_\lambda \Gamma_{\mu\nu}^{\ \ \lambda} - \partial_\mu \Gamma_{\nu\lambda}^{\ \ \lambda} - \Gamma_{\mu\kappa}^{\ \ \lambda} \Gamma_{\nu\lambda}^{\ \ \kappa} + \Gamma_{\mu\nu}^{\ \ \lambda} \Gamma_{\lambda\kappa}^{\ \ \kappa} \right) \\ &\simeq -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, R. \end{aligned} \tag{27}$$

Conventions:

$$R_{\mu\nu\kappa}^{\ \ \ \ \lambda} \equiv \partial_\mu \Gamma_{\nu\kappa}^{\ \ \lambda} - \partial_\nu \Gamma_{\mu\kappa}^{\ \ \lambda} - \Gamma_{\mu\kappa}^{\ \ \sigma} \Gamma_{\nu\sigma}^{\ \ \lambda} + \Gamma_{\nu\kappa}^{\ \ \sigma} \Gamma_{\mu\sigma}^{\ \ \lambda}.$$

5. The field equation for the massless spin-2 field in Minkowski space-time is

$$\Box h_{\mu\nu} - \partial_\mu \partial_\lambda h^\lambda_{\ \nu} - \partial_\nu \partial_\lambda h^\lambda_{\ \mu} + \partial_\mu \partial_\nu h^\lambda_{\ \lambda} - \eta_{\mu\nu} \left(\Box h^\lambda_{\ \lambda} - \partial_\kappa \partial_\lambda h^{\kappa\lambda} \right) = -\kappa T_{\mu\nu}. \tag{28}$$

a. Check that the equation is invariant under gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{29}$$

b. The Einstein equations can be simplified by switching to different field variables

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} h^\lambda_{\ \lambda}. \tag{30}$$

Show that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\bar{h}^{\lambda}_{\ \lambda}, \tag{31}$$

and rewrite the Einstein equations in terms of $\bar{h}_{\mu\nu}$:

$$\Box\bar{h}_{\mu\nu} - \partial_\mu\partial_\lambda\bar{h}^{\lambda}_{\ \nu} - \partial_\nu\partial_\lambda\bar{h}^{\lambda}_{\ \mu} + \eta_{\mu\nu}\partial_\kappa\partial_\lambda\bar{h}^{\kappa\lambda} = -8\pi G T_{\mu\nu}. \tag{32}$$

Check the invariance under the modified gauge transformations

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\,\partial_\lambda\xi^\lambda. \tag{33}$$

c. Define the momentum components $\varepsilon_{\mu\nu}(k)$ of the redfined spin-2 field $\bar{h}_{\mu\nu}$ by

$$\bar{h}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^2}\,\varepsilon_{\mu\nu}(k)e^{-ik\cdot x}. \tag{34}$$

Prove that the reality of $\bar{h}_{\mu\nu}$ requires $\varepsilon^*_{\mu\nu}(k) = \varepsilon_{\mu\nu}(-k)$, and show that in empty space ($T_{\mu\nu} = 0$)

$$k^2\varepsilon_{\mu\nu} - k_\mu k^\lambda\varepsilon_{\lambda\nu} - k_\nu k^\lambda\varepsilon_{\lambda\mu} + \eta_{\mu\nu}k^\kappa k^\lambda\varepsilon_{\kappa\lambda} = 0, \tag{35}$$

and derive the corresponding gauge transformations in momentum space:

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu\alpha_\nu + k_\nu\alpha_\mu - \eta_{\mu\nu}\,k^\lambda\alpha_\lambda, \tag{36}$$

with

$$\xi_\mu = i\int \frac{d^4k}{(2\pi)^2}\,\alpha_\mu(k)e^{-ik\cdot x}, \qquad \alpha^*_\mu(k) = -\alpha_\mu(-k).$$

d. Show that one can find a gauge transformation parameter α_μ such that

$$k^\mu\varepsilon'_{\mu\nu} = 0 \quad \text{and} \quad k^2\varepsilon'_{\mu\nu} = 0. \tag{37}$$

This implies that $\varepsilon'_{\mu\nu}(k) \neq 0$ only on the light cone $k^2 = 0$:

$$\varepsilon'_{\mu\nu}(k) = e_{\mu\nu}(\mathbf{k},\omega_{\mathbf{k}})\,\delta(k^2), \tag{38}$$

and that the metric perturbation can be expanded as

$$\bar{h}_{\mu\nu}(x) = \int \frac{d^3\mathbf{k}}{8\pi^2\omega_{\mathbf{k}}}\left(e_{\mu\nu}(\mathbf{k},\omega_{\mathbf{k}})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + e^*_{\mu\nu}(\mathbf{k},\omega_{\mathbf{k}})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right), \tag{39}$$

with the convention $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$.
e. Check, that the momentum amplitude $e_{\mu\nu}(\mathbf{k})$ satisfies

$$k_i e_{i\mu} = \omega_{\mathbf{k}} e_{0\mu}, \tag{40}$$

and this condition is respected by gauge transformations on the light cone:

$$\alpha_\mu = a_\mu(\mathbf{k}, \omega_\mathbf{k})\,\delta(k^2),$$

such that

$$\begin{aligned} e'_{00} &= e_{00} + \omega_\mathbf{k} a_0 + \mathbf{k}\cdot\mathbf{a}, \\ e'_{i0} &= e_{i0} + k_i a_0 + \omega_\mathbf{k} a_i, \\ e'_{ij} &= e_{ij} + k_i a_j + k_j a_i - \delta_{ij}\left(\mathbf{k}\cdot\mathbf{a} - \omega_\mathbf{k} a_0\right). \end{aligned} \tag{41}$$

Find $(a_0, \mathbf{a})$ such that

$$e'_{00} = e'_{i0} = \sum_{k=1}^{3} e'_{kk} = 0. \tag{42}$$

Answer:

$$a_0 = -\frac{1}{4\omega_\mathbf{k}}\left(e_{00} + \sum_k e_{kk}\right), \quad a_i = -\frac{1}{\omega_\mathbf{k}} e_{i0} + \frac{k_i}{4\omega_\mathbf{k}^2}\left(e_{00} + \sum_k e_{kk}\right). \tag{43}$$

f. Explain that there are only 2 polarization modes for a perturbation with wave vector $\mathbf{k}$, and that these can be taken to be space-like, transverse and traceless:

$$e_{00} = e_{i0} = 0, \qquad \sum_i k_i e_{ij} = 0, \qquad \sum_i e_{ii} = 0. \tag{44}$$

In particular explain that for a pertubation mode moving in the z-direction one can take

$$e_{\mu\nu}(k_z, \omega_\mathbf{k}) = A_+(k_z) e^+_{\mu\nu} + A_\times(k_z) e^\times_{\mu\nu}, \tag{45}$$

with

$$e^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

g. Prove, that these gauge transformations turn the solutions for the metric perturbations $\bar{h}_{\mu\nu}$ into a solution of the linearized Einstein equations with the properties

$$\bar{h}_{00} = \bar{h}_{i0} = 0, \qquad \sum_i \bar{h}_{ii} = 0, \qquad \sum_i \partial_i \bar{h}_{ij} = 0, \qquad \Box \bar{h}_{ij} = 0. \tag{47}$$