Message to participants, pre-school :

Begin with a disclaimer: I never lectured on these topics (really worked on them since 2008). these notes are an attempt to be at least somewhat pedagogical. there are, at places, too many details (that we'll skip) and, at other places, some "~" signs instead of "=" - some left for exercise on purpose...

Goals: I would be happy if, after the school, the students appreciate the beautiful physics of the Polyakov 3d model and have some qualitative idea of how it extends towards "real" QCD on $R^3 \times S^1$.

I expect feedback, questions, and guidance from students during lectures...

Background: ① please review monopole solutions of 't Hooft/Polyakov in particular BPS limit (J.Harvey 9603086 - Sect 1.3, 1.4, 2.1, 2.2, 2.3, 4.1, 4.2, for example).

② please also review the notion of "dilute instanton gas" as per Coleman's book, or "ABC of instantons" of Vainshtein et al. (found online).

③ the notion of a "'t Hooft vertex" in electroweak theory, for example: PRL 37 (1976) 8 ("Symmetry breaking through Bell-Jackiw anomalies", by 't Hooft)

Erich Poppitz, August 2016

P.S. Apologies for handwriting... may type up, one day. EP

"Confinement, YM, & SYM"

INTRO: What is this all about?

Well ... "confinement". This is old news, 40+ years, and we have "gotten used" to it (we'll try to more carefully define what we mean by confinement further below). Nonetheless, while we believe that it is a property of pure YM theory, we have no "proof" [see Clay Institute website], or, more modestly, an analytical understanding in continuum asymptotically free theory on $\mathbb{R}^4$ is lacking.

> N.B.: We have understanding on the lattice, analytical via the strong-coupling expansion [Wilson, 1970s], far from continuum limit, and, of course, numerical - glueball hadron etc spectra can be calculated. the latter part though, is "experiment" really.

Notice also that I can't review all analytical approaches to confinement that exist. For a fairly critical review see (the rather thin, unfortunately) book by Jeff Greensite, "the confinement problem" (Springer, 2009)

(*) using 4d asymptotically free QFT; not AdS/CFT where large 't Hooft coupling regime is rather far from this...

What I will concentrate upon is theoretical approaches under analytical control(*) [no uncontrolled approximations involved]. these are only a few:

- most celebrated is Seiberg-Witten theory $\mathcal{N}=2$ SYM w/ soft mass preserving $\mathcal{N}=1$ SUSY
  - confinement is due to the condensation of monopoles (particles!) or dyons & a dual Higgs mechanism, on $\mathbb{R}^4$
  - this is "Abelian" confinement (will become clear later)
- another example is the confinement of monopoles via nonabelian strings, also occurring in some $\mathcal{N}=2$ models: Hanany-Tong, Konishi, Bolognesi, Shifman, Yung - —

- **the topic of these lectures:**
  - YM theory w/ massive or massless adjoint Weyl fermions

    (universality class of YM)

    (if one adjoint ≡ SYM

  - on $\mathbb{R}^4$, w/out Higgs fields, this is analytically untractable — but on $\mathbb{R}^3 \times S^1$, w/ S^1 of size L & with periodic b.c. for fermions, this theory becomes analytically tractable,

(because fermions are periodic, a more apt name is $\mathbb{R}^{1,2} \times S^1$
← spatial not thermal circle)

provided $N_c \Lambda L \ll 1$ (to be explained!) (p.㊵!)

($SU(N_c)$; Λ is strong scale → so circle size is small, $\lesssim \frac{1}{N_c \Lambda}$)

- We will learn that the generation of mass gap & confinement in the calculable small-$N_c \Lambda L$ regime occur due to a locally-4d generalization of the 3d Polyakov mechanism of confinement. (to be explained!)

Discrete & abelian chiral symmetry breaking can also be studied; interesting relations to confinement/deconfinement transitions.

My goal is to, more or less pedagogically, review these developments. there are many connections to the other approaches, which are either under development or await understanding...

Before we begin, let me also spend some time adressing some "philosophical question"--

Qu.: -- "You study YM theories on $R^3 \times S^1_L$ at small L, as well as with unphysical representations of fermions. This is not the real world! Why bother?"

I don't want to oversell you anything. Here are my answers

A.1. the story about confinement, thermal transition, etc.., that comes about is rather pretty and elegant. To a theoretical physicist, this is rather satisfying, in itself.

A.2. One may expect that, while studying a theoretically controlled regime (here, weak coupling: perturbation theory + semiclassics), one may encounter surprising new features that are more generally valid. (ie. "honest work" might teach you something--)

↳ • the recent developments & interest in resurgence in QFT arose from these studies!

↳ Ünsal, Argyres, Dunne ... 2012–

• other surprises/unusual behavior --

⑤

A.3. Once a calculable regime is well understood, it is tempting to push it towards, or even beyond, its limits of validity — — one may end up w/ some interesting phenomenological models

↳ Shuryak et al 2012 — "MILM's of deconfinement [monopole-instanton-liquid models]

I don't expect you understood/believed/internalized all I said above - Come back after lectures/during discussions!

Plan: ① 3d Polyakov model pages ⑥-㉟

② from R^3 to $R^3 \times S^1$: self-dual instantons, properties, "GPY" potential, holonomy vev... pages (36)-(54)

③ "deformed YM" theory pages (55)-(63)

④ QCD(adj)/SYM, magnetic bions, & confinement ... SYM soft breaking & deconfinement pages (64)-(8

⑤ back to "Intro" (p. ①-⑤).

↓ pages (88)-(94)

Polyakov model on R^3

(e.g. • Polyakov: NPB 120 (1977) 429-458;
• "Gauge fields & strings" book
• Witten in 'Qu. Fields – for mathematicians', v2)

→ "Georgi-Glashow model" in 3d

SU(2) w/ a triplet Higgs, w/ Euclidean action

$$S = \int d^3x \left[\frac{1}{4g_3^2} F_{\mu\nu}^{a} F^{\mu\nu a} + \frac{1}{2g_3^2} (D_\mu A_4)^a (D^\mu A_4)^a + V(A_4^a) \right]$$

For future use, I called the triplet Higgs "A_4^a".

$$\left(\begin{array}{l} A_4 \equiv A_4^a T^a \;\; ; \;\; D_\mu A_4 \equiv \partial_\mu A_4 + i[A_\mu, A_4] \\ \mathrm{tr}\, T^a T^b = \tfrac{1}{2} \delta^{ab} \;\; -- \end{array} \right).$$

Imagine that V_{eff} is s.t. $\langle A_4^a \rangle = \delta^{a3} v$.

$[g_3^2] = 1$, gauge coupling in 3d is dimensionful.

then, we have:
- massless photon: A_μ^3
- massive W-bosons: $(A_\mu^1 \pm i A_\mu^2)$

$\underbrace{m_W = v}$, with our normalization of A_4 kinetic term.

- massive A_4^3 (radial Higgs mode) → mass depends on potential V

To be concrete, let's take the following form of $V(A_4)$:

dimensionless

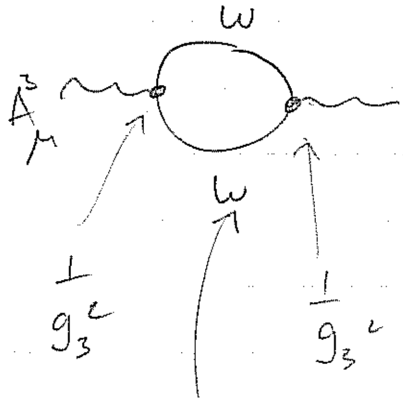
$$V(A_4) = \frac{\lambda}{g_3^2}\left(A_4^a A_4^a - v^2\right)^2$$

then $m_{A_4} \sim \sqrt{\lambda}\, v$ ← we'll see, on $R^3 \times S^1$

Notice we have two mass scales in this theory, v & g_3^2. v is the scale of "Abelianization": $SU(2) \xrightarrow{v} U(1)$.
If v were 0 we'd have a full nonabelian theory with dimless coupling $\left(g_3^2/\mu\right)$ ← energy scale which becomes strong @ $\mu \sim g_3$ → this 3d nonabelian pure YM theory + adjoint scalar is believed to confine, & is analytically non-tractable...
But for $v \gg g_3^2$ we have $m_W = v$ (& $\sqrt{\lambda}\, v$) & only an abelian theory @ scales $\ll v$, with no light fields charged under the unbroken U(1)

(this is important, as in 3d, even U(1) w/ charged light fields does become strong)

So the IR theory is essentially free. There's the kinetic term for the U(1) photon + higher dimensional operators due to exchange of heavy W bosons; (they also affect leading op'r:



$$\sim \underline{(F^3_{\mu\nu})^2 \frac{1}{v}}$$ @ one loop

W-propagator $\sim g_3^2 \frac{1}{k^2+v^2}$

(this is clearly $\ll \frac{1}{g_3^2}(F^3_{\mu\nu})^2$ for $\underbrace{v \gg g_3^2}$)

one loop $\ll$ tree level

Moral: perturbatively, the Polyakov model IR dynamics @ $v \gg g_3^2$ is rather boring:

$$\mathcal{L}_{eff} = \frac{1}{4g_3^2}(F^3_{\mu\nu})^2 + \underline{O}\left(\frac{g_3^2}{v}\right) \text{ corrections } + \underline{O}\left(\frac{\text{energy}}{v}\right) \text{ correction}$$

the amazing thing is that in this 3d model nonperturbative effects completely change the IR dynamics, generating a mass gap...

(Polyakov '77)

notice this is another way of seeing that as $v \to$ "$O(g_3^2)$" "we've got a problem", as tree level $\sim$ one loop $\Rightarrow$ strong coupling!

<u>the mass gap & confinement in this model is due to the proliferation in the vacuum of</u>

(the word "condensation" is not quite appropriat as what proliferates are instantons, objects localized in Euclidean time, or pseudo-particles, rather than physical objects, as in Seiberg-Witten theory)

<u>magnetic monopole-instantons</u>. Let's recall what these are, in a language that will be useful on $R^3 \times S^1$ as well.

Our action on R^3 (w/out $V(A_4)$) can be equivalently written as

$(\mu,\nu = 1,2,3)$

$$\frac{1}{4g_3^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2g_3^2} (D_\mu A_4)^a (D_\mu A_4)^a \equiv$$

$$\equiv \frac{1}{4g_3^2} F^a_{MN} F^a_{MN} \quad , \quad w/ \quad M,N = 1,2,3,4$$

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \epsilon^{abc} A^b_M A^c_N$$

$$\left(\& \; \partial_4 \equiv 0 : \quad F^a_{\mu 4} = \partial_\mu A^a_4 + \epsilon^{abc} A^b_\mu A^c_4 \right)$$

- let's imagine that $V(A_4)$ is absent $[\lambda = 0]$
this is called the "BPS limit"
(in our application on $R^3 \times S^1$, $V(A_4)$ is small, as we'll see)

- let's also introduce

$$B^a_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\lambda}\,F^a_{\nu\lambda} \qquad \text{"magnetic"}$$

$$E^a_\mu = (D_\mu A_4)^a = F^a_{\mu 4} \qquad \text{"electric"}$$

- then we have

$$\mathcal{L}_E = \frac{1}{4g_3^2}\left[(F^a_{\mu\nu})^2 + (F^a_{\mu 4})^2 2\right]$$

$$= \frac{1}{2g_3^2}\left[(B^a_\mu)^2 + (E^a_\mu)^2\right]$$

$$= \frac{1}{2g_3^2}\left[\underbrace{(E^a_\mu \mp B^a_\mu)^2}_{\geq 0} \pm 2E^a_\mu B^a_\mu\right]$$

$$\geq \pm\frac{1}{g_3^2}\,E^a_\mu\,B^a_\mu$$

so $$S_E \geq \pm\frac{1}{g_3^2}\int d^3x\,E^a_\mu\,B^a_\mu .$$

(just *names* !)
$F_{\mu 4}$ is *not* an electric field in any way -- this honor would go to the corresponding component of $F^a_{\mu\lambda}$ in Minkowski — here all is Euclidean !)

confusions still in literature !

⑪

the equal sign is obeyed by $\underset{\text{self-dual}}{\text{SD}}$ or $\overset{\text{anti}}{\text{ASD}}$ fields,

for (anti) BPS solutions, action is, then $S_{\substack{BPS\\BPS^*}} = \frac{1}{g_3^2}\left|\int B_\mu^{a\,2}\, d^3x\right.$

the BPS limit ($\lambda=0$) simplifies finding solutions; it's the only case explicit forms exist:

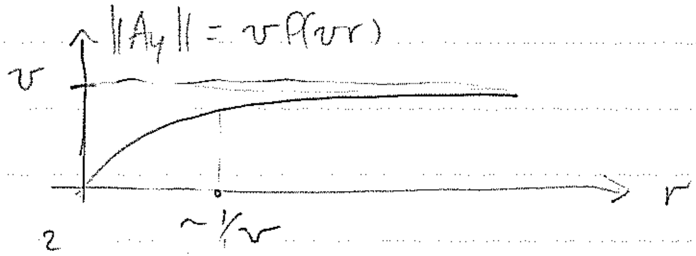
e.g. BPS (SD) monopole-instanton

see bottom of ⑫ for B_μ^a

$$A_4^a = -n_a v\, P(vr) \quad , \quad P(x) = \coth x - \frac{1}{x} \xrightarrow{\infty} 1-\frac{1}{x}$$

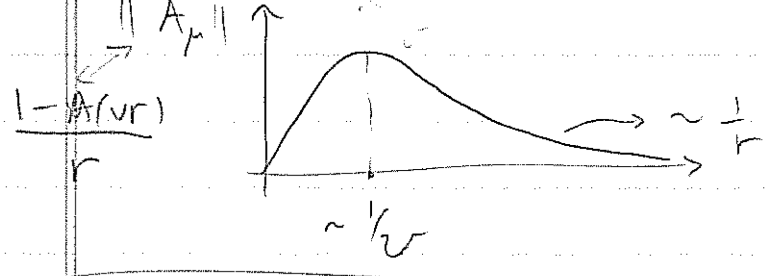
$$A_\mu^a = \epsilon_{a\mu\nu} n_\nu \frac{1-A(vr)}{r} \quad , \quad A(x) = \frac{x}{\sinh x} \xrightarrow{\infty} \underline{O}(x e^{-x})$$

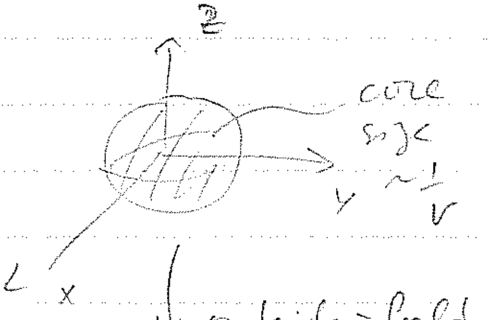
these are the regular, so-called "hedgehog gauge" solution



@ $r >> v$

both E & B $\sim \frac{1}{r}$





outside = field of pointlike charge $\vec{B}$

- in "hedgehog" gauge orientation of B_μ^a varies as fn of direction
- in "string" gauge (valid in "½-space"), it is in 3rd isospin direction, like that of Dirac monopole

⑫

When $\lambda \neq 0$ we don't have anymore $E^a_\mu = B^a_\mu$ & A^3_y has a mass from 'Higgs potential', so the long-range field for non-BPS solutions is only B^a_μ.

The "magnetic" charge of the solution is (use string gauge to write

$$\oint_{S^2_\infty} d^2\vec{\sigma} \cdot \vec{B}^{(3)} = 4\pi Q_m$$

$$\text{w/} \quad \vec{B}^3\Big|_\infty = Q_m \frac{\vec{r}}{|\vec{r}|^3}, \quad \text{for smallest action solution this is } \underline{\underline{Q_m = 1}}.$$

(explicit form of solution in string gauge, incl. $r \lesssim v$ region, eg in 1105.0940 $\longrightarrow$ gives $B_\mu dx^\mu = B_r dr + B_\theta d\theta + B_\varphi d\varphi$

$$A_y = v P(vr)\frac{\tau^3}{2} \to v\frac{\tau^3}{2}$$

$$B_r \xrightarrow[r\to\infty]{} \frac{1}{r^2}\frac{\tau^3}{2}, \quad B_\theta, B_\varphi \sim v^2_\times \underline{\mathcal{O}}(e^{-vr}) \xrightarrow[r\to\infty]{} 0 \quad)$$

Finally, the action is easy to compute for BPS solutions

$$S = \frac{1}{g_3^2}\int d^3x \, B^a_\mu B^a_\mu = \frac{4\pi v}{g_3^2}$$

DIY:

for BPS, $B^a_\mu = (\delta_{\mu a} - n_\mu n_a) F_1(vr) + n_\mu n_a F_2(vr)$

$$F_1(vr) = \frac{v^2}{\sinh vr}\left(\frac{1}{vr} - \coth vr\right) \xrightarrow{\infty} v^2 \underline{\mathcal{O}}(e^{-vr})$$

$$F_2(vr) = \frac{v^2}{\sinh^2 vr} - \frac{1}{r^2} \longrightarrow -\frac{1}{r^2}$$

(we'll find these useful, near p. 43!)

Ⓑ

Summary: • "magnetic monopole-instantons" are finite action Euclidean solutions of the classical EOM.

• they have $\sim \frac{1}{r^2}$ long-range "tails" of B^3_μ-fields, thus carry "magnetic charges" (integral of B^3_μ over an S^2 at infinity, surrounding the monopole-instanton is nonzero)

Euclidean action of these solutions, see p. 16/17 ⟶

• we found them in the BPS limit, where no potential for A_4 was present — in this limit these instantons thus carry also $D_\mu A_4$ - field $\sim \frac{1}{r^2}$ (and are self-dual or ASD in 4d sense); in 3d Polyakov model & in our $R^3 \times S^1$ applications SD is only approximate. In fact, since there will be an A_4-potential, A_4 is gapped & instead of $\vec{E} \sim \frac{1}{r^2}$ we have $\vec{E} \sim \frac{1}{r^2} e^{-m_4 r}$ — but $\vec{B}$ is not gapped & is still $\sim 1/r^2$.

NB: non BPS monopole instantons have to be studied numerically, no explicit soltn exist — "matched asymptotic expansions" etc —

Back to the physics of the Polyakov model.
Recall we found that, perturbatively, we had a U(1) ($\subset SU(2)$) massless photon & that's that.

Further, @ $v \gg g_3^2$ theory is weakly couple & we expect perturbation theory, expansion around saddle points in the path integral, to be a good guidance

expansion around trivial saddle point in path $\int$

We found also that nonperturbative saddle points exist as well → our monopole-instantons. (action is on p.⑫, see also bottom of p.⑯)
What do they "do"?

Consider, e.g. the long-distance correlation f-n of some gauge-invariant observable, e.g. B_μ^3.

$$\langle B_\mu^3(x)\, B_\mu^3(y)\rangle\Big|_{\text{in}} \sim \frac{g_3^2}{|x-y|^3}$$

perturbation theory: $\langle A_\mu^3(r) A_\nu^3(0)\rangle \sim \frac{g_3^2}{r}\,\delta_{\mu\nu} + \dots$

(to get result, act w/ two derivatives)
– never mind the coeffs. –

But we also found that there are nonperturbative field configurations, the monopole-instantons that carry long-range B^3-fields. So, one expects that these fluctuations can spontaneously appear anywhere in spacetime w/ some probability per unit spacetime volume -- we'll estimate this as

$$\sim v^3 e^{-S_0}$$

S_0 is the Euclidean action of the instanton (in the path $\int$ it controls how likely it is for such a fluctuation to occur)

So we have our correlation function

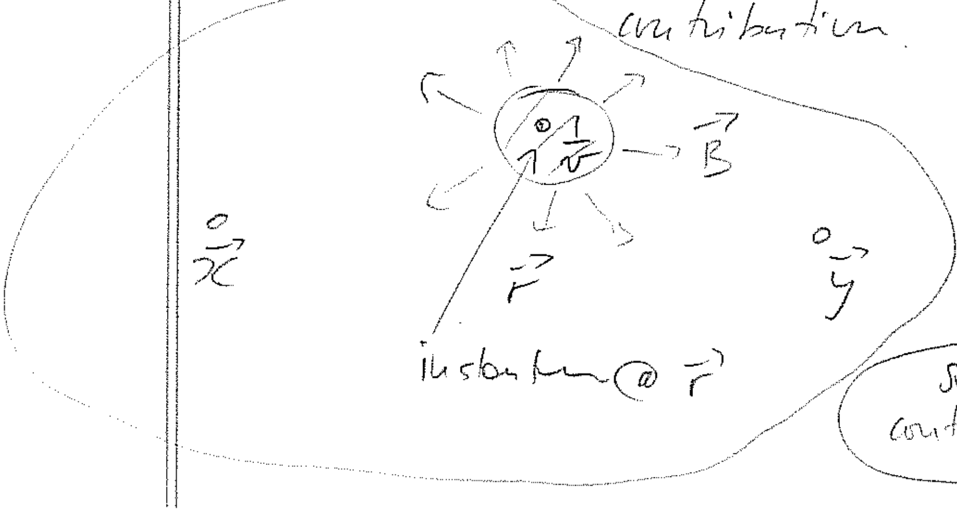
$$\left\langle B^3_r(x) \;\; B^3_r(y) \right\rangle \Big/ \sim v^3 e^{-S_0} \int d^3\vec{r} \; \frac{\bar{x}-\bar{r}}{|\bar{x}-\bar{r}|^3} \cdot \frac{\bar{y}-\bar{r}}{|\bar{y}-r|^3}$$

1-instanton contribution

$\vec{B}$ field of instanton @ $\vec{r}$ @ $\vec{x}$

field of instanton @ $\vec{r}$ (field @ y)

sum over contributions of M's @ all $\vec{r}$



$$\sim \frac{v^3 e^{-S_0}}{|\vec{x}-\vec{y}|}$$

to get result,
(think of that $\int$ differently:
this is the interaction energy
of two charges @ x & y)

S_0 — —

$$\langle B^3_r(x),\ B^3_r(y)\rangle \sim$$

$$\sim \left\{ \underbrace{\frac{g_3^2}{|x-y|^3}}_{\text{perturbative}} + \underbrace{v^3 e^{-S_0} \frac{1}{|\vec{x}-\vec{y}|}}_{\text{nonperturbative}} \right\}$$

more important in IR
due to long-range
tail of monopole-instanton

even though e^{-S_0} is tiny (we'll see)
such effects dominate the IR —
& need to be resummed — !

What's S_0? p.10: $E = B$, $S_E = \frac{1}{g_3^2}\int \vec{E}\cdot\vec{B}\, d^3x =$

only a f.n of v, for BPS solution

again $S_0 \gg 1$ for $v \gg g_3^2$

$$\boxed{S_0 = \frac{4\pi v}{g_3^2}} \quad \boxed{\& e^{-S_0}_{cc,}}$$

[use nonsingular hedgehog gauge to compute]

Before we continue, a comment on scales, resummation & 'physical' picture ...

+ monopole has size $\frac{1}{v^3}$ (space time volume)
& "Boltzmann" factor $e^{-4\pi v/g_3^2} \equiv e^{-S_0}$

+ in a box of size L^3 it can appear w/ probability

$$\frac{L^3}{\left(\frac{1}{v^3}\right)} e^{-S_0}$$

↖ "action suppression"

↖ "entropy enhancement"

+ of course, one has the probability to have more than one monopole --, for two:

$$\sim \left(\frac{L^3}{\frac{1}{v^3}}\right)^2 e^{-2S_0} \times e^{-S_{\text{inter.}}}$$

↑ in reality, the Boltzman factor for two $\neq e^{-2S_0}$ since they interact — & since interaction is long-range has to be accounted for!

+ the classical stat mech picture is that of a classical gas of Monopole-instantons & (anti-) – – where # of particles is not fixed, i.e. that of a grand partition function of M & M*

* we need to learn how to deal w/ this. – some technology needed!

NB for those familiar w/ 't Hooft Polyakov monopoles in 4d (particles)

$$\begin{pmatrix}\text{Energy of static}\\ \text{solution}\end{pmatrix} \equiv \frac{4\pi \overset{\text{scale}}{v}}{\underset{\substack{\uparrow\\ \text{4d coupling at scale } v \text{, say in GUT or 4d Georgi-Glashow model}}}{g_4^2}} \leftarrow \text{dim.less}$$

$$\downarrow$$

$$\text{in } R^3 \ \underbrace{\begin{pmatrix}\text{finite}\\ \text{Energy}\end{pmatrix} \rightarrow \begin{pmatrix}\text{finite}\\ \text{action}\end{pmatrix}}$$

dimension taken care of by $\underset{(\text{dimless})}{g_4^2} \rightarrow \underset{(\text{dimensionful})}{g_3^2}$.

Some more technology. To study the effects of the many $\underset{\substack{\uparrow\\ \text{BPS}\\ \text{monopole-instanton}}}{M}$ & $\underset{\substack{\uparrow\\ \text{BPS}^*}}{M^*}$'s

it is convenient to use a dual language.

In 3d, a photon has a single degree of

freedom (one polarization), which can be also described by means of a scalar, the so-called "dual photon".

Notice: the duality transform is performed in

① the long distance theory ($\mu \ll v$) where the Abelian photon (& the 3rd component of A_4, A_4^3, if $\lambda = 0$, as in SYM) is the only light d.o.f. & there are no dynamical charged fields.

(see his book) → ② originally, Polyakov discovered the "3d Polyakov mechanism" in compact-U(1) lattice gauge theory; the duality can be discussed on the lattice, as well as the mechanism of confinement, but we'll continue w/ our continuum discussion.

Our long-distance theory is defined by

$$S_E = \frac{1}{4g_3^2} F_{\mu\nu}^2 + \ldots$$

(we shall omit index "3" from now on, understanding that we deal w/ Abelian component)

(see p. ⑧)

& the path $\int$ over A_μ can be written as an $\int$ over $F_{\mu\nu}$ + imposing a constraint that $F_{\mu\nu}$ obeys the Bianchi identity $\epsilon^{\mu\nu\lambda}\partial_\lambda F_{\mu\nu} = 0$ (which, locally, guarantees that $F_{\mu\nu} = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$).

Impose the constraint via a lagrange multiplier field "σ".

[[we shall not be concerned w/ global issues, at least on $\mathbb{R}^3$, here - see Witten's lectures in "QFT for mathematicians", vol 2]]

So we have

temporarily in Minkowski w/ (+,-,-)

to avoid "i"s in ϵ

$$S_{\text{Mink}} = -\frac{1}{4g_3^2}\int d^3x\, F_{\mu\nu}F^{\mu\nu}$$

$$-\frac{1}{8\pi}\int d^3x\, \partial_\mu\sigma\, F_{\nu\lambda}\,\epsilon^{\mu\nu\lambda} \quad (\epsilon^{012} = +1)$$

& the path $\int$ is over σ & $F_{\mu\nu}$.

Integrating over σ, we find the constraint $dF = 0$ which is solved by $F = dA$, obtaining the original description.

On the other hand, $\int$ over $F_{\mu\nu}$, we have that $\frac{1}{2g_3^2}F^{\mu\nu} = -\frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\partial_\lambda\sigma$

& $S_{\text{Mink}} = -\int d^3x \Big[F_{\mu\nu}\Big(\frac{1}{4g_3^2}F^{\mu\nu} + \frac{1}{16\pi}\epsilon^{\mu\nu\lambda}\partial_\lambda\sigma\Big) + \frac{1}{16\pi}F_{\mu\nu}\epsilon^{\mu\nu\lambda}\partial_\lambda\sigma$

= (1st term above = 0 once F is subbed via σ)

$$- \frac{1}{16\pi} \int d^3x \left(- \frac{g_3^2}{4\pi} \right) \epsilon_{\mu\nu\lambda} \partial^\lambda \sigma \, \epsilon^{\mu\nu\rho} \partial_\rho \sigma$$

$$= \frac{1}{2} \frac{g_3^2}{(4\pi)^2} \int d^3x \, \partial_\lambda \sigma \, \partial^\lambda \sigma \equiv S[\sigma]$$

action of dual photon

Duality relation is

$$F^{\mu\nu} = - \frac{g_3^2}{4\pi} \epsilon^{\mu\nu\lambda} \partial_\lambda \sigma$$

$$\epsilon^{012} = +1$$

which clearly shows how local gauge invariant operators are to be mapped between the "electric" description

$$S_{el} = - \frac{1}{4 g_3^2} \int d^3x \, F_{\mu\nu} F^{\mu\nu}$$

& the "magnetic" one

$$S_{magn} = \frac{1}{2} \frac{g_3^2}{(4\pi)^2} \int d^3x \, \partial_\mu \sigma \, \partial^\mu \sigma$$

But there are other operators that are of interest. In the electric theory, one can have "line operators", which can be interpreted as the insertion of static nondynamical probes, e.g.

$$W_{Q_e}(\vec{x}_*) = e^{i Q_e \int_{-\infty}^{\infty} dx^0 A_0(\vec{x}_*, x^0)} \quad (*)$$

— a

<u>Wilson line operator</u> creating a static electric charge Qe

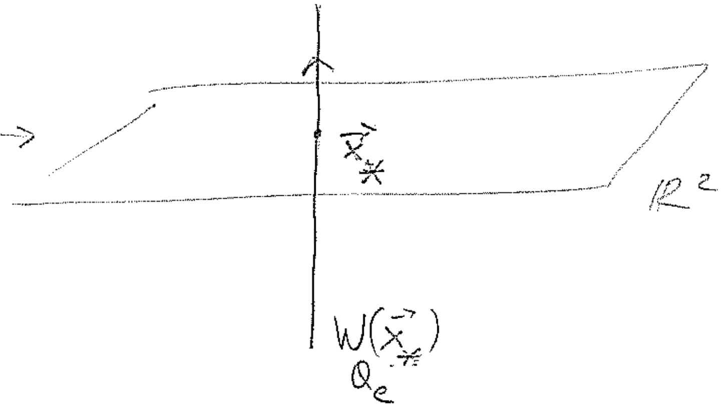
We will need to learn what is the image of that operator in the magnetic theory?

[(*) is meant to be inserted in the path integral: this is a usual $\int d^3x\, j^\mu(x) A_\mu(x)$ insertion of a source $j^\mu(x) = \delta^{\mu 0}\delta^{(2)}(\vec{x}-\vec{x}_*)$

↓

giving rise to the above Wilson line (*)]

picture of timeline (static) Wilson line



Conversely, in the magnetic (σ) theory one can have operators that create "fluxons". Let's elaborate as these may be less familiar:

notice σ theory has a symmetry $\sigma \to \sigma + \text{const}$ where the conserved current is

$$j_\mu^{[\sigma\text{-shift}]} = \frac{g_3^2}{(4\pi)^2}\partial_\mu \sigma \longleftrightarrow -\frac{1}{4\pi}\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$$

by the duality relation this is mapped to ↗, the topological current of the U(1) theory.

the Charge is $\sim \partial_0 \sigma \sim F_{12}$ = the true magnetic field (in 3d)

$$Q^{[\sigma\text{-shift}]} = \frac{g_3^2}{(4\pi)^2}\int \partial_0\sigma\, d^2x \longleftrightarrow -\frac{1}{4\pi}\int d^2x\, F_{12}$$

㉒

In canonical quantization of σ- theory, we then have that

$e^{i\sigma(\vec{x}_*)}|0\rangle$ is an eigenstate of $Q^{[\sigma\text{-shift}]}$: $Q^{[\sigma\text{-shift}]}\, e^{i\sigma(\vec{x}_*)}|0\rangle =$

$$= \left(\text{use } \Pi_\sigma = \frac{g_3^2}{(4\pi)^2}\partial_0\sigma\right)$$

$$= \int \Pi_\sigma(\vec{x})\, d^2x\ e^{i\sigma(\vec{x}_*)}|0\rangle$$

$$= e^{i\sigma(\bar{x}_*)}\underbrace{\int e^{-i\sigma(\bar{x}_*)}\,\Pi_\sigma(\bar{x})\, e^{i\sigma(\bar{x}_*)}}\, d^2x\ |0\rangle$$

$$= e^{i\sigma(\vec{x}_*)}\underbrace{\int d^2\vec{x}\ i\left[\Pi_\sigma(\vec{x}),\sigma(\vec{x}_*)\right]}\ |0\rangle$$

$$= e^{i\sigma(\vec{x}_*)}1\,|0\rangle$$

__Moral:__ $e^{i\sigma(\vec{x}_*)}$ creates unit "σ-shift" charge. In terms of the electric theory, this is the creation of a unit magnetic flux "$-\frac{1}{4\pi}\int d^2x\, F_{12} = 1$" when acting on "$e^{i\sigma}$"$|0\rangle$

This operator $(e^{i\sigma})$ does not have a simple description in terms of the original electric fields
$\equiv$ "disorder operator"

Notice also that the Wilson loop (^{eg} insertion of static electric source) is simple in electric theory but less trivial in magnetic theory — where it is a "disorder operator".

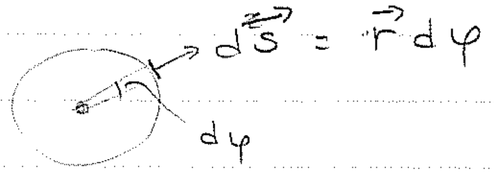
⊛ ← N.B ≡ one can spend quite some time studying how these operators are defined in various pictures (Euclidean, canonical, continuum, lattice..., & all is fun, but we'll try to cut things short & just go by w/ the results we'll need!

see (p23.1)

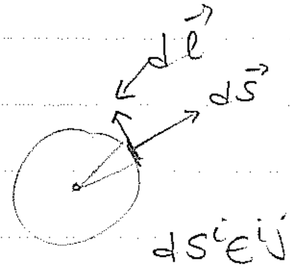
Begin w/ $\underset{Q_e}{W}(\vec{x}_*)$ – an insertion of static Q_e charge.

A static charge creates an $\underset{(F^{0i})}{\vec{\mathcal{E}}}$ - field : $\vec{\mathcal{E}} \sim Q_e \frac{\hat{\vec{r}}}{r}$ (in 2d space

By Gauss' law,



$$\oint \vec{\mathcal{E}}\cdot d\vec{s} \sim Q_e$$



but by duality $\mathcal{E}^i \sim F^{0i} \sim \epsilon^{ij}\partial_j \sigma$

$ds^i \epsilon^{ij} \sim dl^j$

so $\oint \vec{\mathcal{E}}\cdot d\vec{s} \sim \oint d\vec{l}\cdot\vec{\nabla}\sigma \sim Q_e$

these will be important to us

Moral: electric charge @ x_* ≡ monodromy of σ around x_*

Claim: σ monodromy around a minimal (SU(2)) charge ≡ 2π

I can't help but say that our exposition of this duality is rather "lightweight" — traditionally it has been presented this way, though!

But at least I should mention that recently important puzzles in similar dualities were addressed by Kapustin & Seiberg (1401.o...). Applied to the duality @ hand, a more precise statement would be

compact 3d U(1) gauge theory (w/ electric center 1-form symmetry)	dual to ⟷	compact 3d scalar ("dual photon") coupled to a 1-form U(1) gauge field topological (w/ 0 field strength), gauging shift symmetry
	or	
compact 3d U(1) gauge theory coupled to a 2 form U(1) gauge field topological (gauging the electric center symmetry)	⟷	compact 3d scalar w/ shift symmetry

In each case there is only one global symmetry (one form center, in top case, & shift U(1) in bottom case) & a well defined operator map ...

[above subtlety not really relevant to us → for one, we don't have a U(1) center...]

DIY: (Exercise ① In our theory, charge is quantized; as our U(1) is compact (eg, it comes from SU(2)'s 3rd component). Thus, there is a minimal electric charge Q_e ($=\frac{1}{2}$ in some normalization). Show that the monodromy of σ around such a charge is 2π.

② Also notice that this is consistent w/ $e^{i\sigma}$ being a well defined local operator (as it is natural to identify $\sigma \simeq \sigma + 2\pi$); σ is thus a compact scalar field. [We'll see that $e^{i\sigma}$ creates, if inserted in path $\int$, a 't Hooft Polyakov monopole – the minimal magnetic charge in a SU(2) theory].

Now on to "$e^{i\sigma}$". In canonical picture we saw it created a flux w/ "$-\frac{1}{4\pi}\int d^2x\, F_{12} = 1$".
Consider now the Euclidean picture of the insertion of $e^{i\sigma}$'s in the path integral, as follows:

$$\left\langle e^{i\int d^3x\, j(x)\sigma(x)} \right\rangle \equiv \frac{\int \mathcal{D}\sigma\, e^{-\frac{\varkappa}{2}\int d^3x (\partial\sigma)^2 + i\int d^3x\, j\sigma}}{\int e^{-\varkappa \int d^3x (\partial\sigma)^2}\mathcal{D}\sigma}$$

$\downarrow$

let's calculate this

(we have in mind $j(x) = \sum_a q_a \delta^{(3)}(x - x_a)$, but will study this later) the saddle point of integrand is

$$(**) \quad \varkappa \vec{\nabla}^2 \sigma(x) = -i j(x)$$, then up to plane waves

$$\sigma(x) = \frac{1}{\varkappa} \frac{i}{4\pi} \int d^3x' \frac{1}{|x-x'|} j(x')$$

so $$-\frac{\varkappa}{2}\int d^3x (\partial\sigma)^2 + i \int j(x)\sigma(x) d^3x =$$

$$= + \frac{\varkappa}{2}\int d^3x (\partial\sigma)^2_\sigma - \varkappa \int d^3x (\partial\sigma)^2 + i \int j(x)\sigma(x) d^3x$$

$$= -\frac{\varkappa}{2}\int d^3x \, \sigma \nabla^2 \sigma + \int d^3x \, \sigma \underbrace{\left(\varkappa \vec{\nabla}^2 \sigma + i j(x)\right)}_{\phi \quad \text{by} (**)}$$

$$= -\frac{1}{2}\int d^3x \, \sigma \, \varkappa \vec{\nabla}^2 \sigma =$$

$$= -\frac{1}{2}\int d^3x \left(\frac{1}{\varkappa}\frac{i}{4\pi}\int d^3x' \frac{1}{|x-x'|} j(x')\right)(-i) j(x)$$

$$= -\frac{1}{2}\frac{1}{4\pi\varkappa}\int d^3x \, d^3x' \, j(x) \frac{1}{|x-x'|} j(x')$$

so $$\langle \_ \rangle = e^{-\frac{1}{2}\frac{1}{4\pi\varkappa}\int d^3x \, d^3x' \, j(x) \frac{1}{|x-x'|} j(x')}$$

let $j(x) = \sum_a q_a \delta(x - x_a)$ -- then

$$\langle \_ \rangle = e^{-\frac{1}{2}\frac{1}{4\pi\varkappa}\sum'_{a,b} \frac{q_a q_b}{|x_a - x_b|}}$$

$$= e^{-\frac{1}{4\pi\varkappa}\sum_{a>b} \frac{q_a q_b}{|x_a - x_b|}}$$

(a=b diverge - - indicate need of UV def of operators, of course).

1st - notice signs' importance, if $q_a q_b > 0$, same sign, we have

$$e^{-\frac{1}{4\pi\varkappa}\frac{(q_a q_b)}{r_{ab}}}$$

& we have max probability when $r_{ab} \to \infty$ - repulsive - (& v.v.)

<u>Moral:</u>

$$\langle e^{i q_1 \sigma(x_1) + i q_2 \sigma(x_2)} \rangle = e^{-\frac{1}{4\pi\varkappa}\frac{q_1 q_2}{|x - x_2|}}$$

"master formula"

<u>in words:</u>

- $e^{iq\sigma(x)}$ in path integral inserts a magnetic charge q
- like charges repel & opposite ones attract, exactly as one expects

now recall $\varkappa = \frac{g_3^2}{(4\pi)^2}$ so we have

$$e^{-\underbrace{\frac{4\pi}{g_3^2}}\frac{q_1 q_2}{|x_1 - x_2|}} \quad \boxed{***}$$

(magnetic monopole-instantons interact w/ strength $\frac{1}{g_3^2}$, as they should)

<u>Exercise:</u> Show that, for two 't Hooft-Polyakov monopoles (or monopole-antimonopole),

taking their fields (in singular gauge & outside their nonabelian cores), one recovers exactly $\boxed{***}$, by computing the action

$$\frac{1}{2g_3^2}\int (B_\mu^3)^2\, d^3x$$

(exclude cores, take $|x-y| \gg \frac{1}{V}$ & B: sum of two fields - - -)

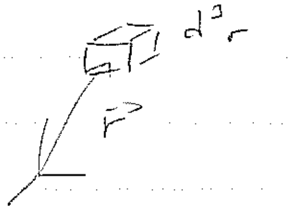
[Notice: BPS solutions also interact via $(E_\mu^3)^2$ — but as we discussed this is gapped & for now can ignore — (but not in SUSY!).]

Now ready to roll! (most of technology is done!)

i.e. perform the sum over monopole-instanton contributions. we have that an (anti-) monopole (M^*) M can appear w/ probability

$$V^3 d^3r\ e^{-S_0}$$

in



(prefactor can include also powers of g_3^2 → instanton zeromode Jacobians etc. — ignore!)

same for M & M^*

well do a parametric estimate "w/ exponential accuracy" only!

n monopoles & m antimonopoles @ $r_1 \dots r_n$ @ $r_{n+1} \dots r_{n+m}$ will appear w/ probability (n M's are indistinguishable):

(1 for M, −1 for M^*)

(this is $\sim$to their contribution to Z)

$$\frac{(V^3 e^{-S_0})^n}{n!}\frac{(V^3 e^{-S_0})^m}{m!}\int d^{3(n+m)}r\ e^{-\frac{4\pi}{g_3^2}\sum_{a>b}^{n+m}\frac{q_a q_b}{|r_a - r_b|}} =$$

$$= \frac{\int d^{3(n+m)} r}{\int \mathcal{D}\sigma\, e^{-\varkappa\int(\partial\sigma)^2 d^3x}}\ \frac{(V^3 e^{-S_0})^n}{n!}\ \frac{(V^3 e^{-S_0})^m}{m!}\int \mathcal{D}\sigma\, e^{-\varkappa\int d^3x(\partial\sigma)^2 + i\int d^3x\,\sigma(x)j(x)} \qquad \nabla_{***}$$

(p. 24 result)

$[\varkappa \equiv g_3^2/(4\pi)^2]$ where $j(x) = \sum_{a=1}^{n+m} q_a \delta^{(3)}(\vec{x}-\vec{r}_a)$

㉘

(**) you can ask whether prefactor of $v^3 e^{-S_0}$ has been "exactly" computed. NO. only recently, to 1-loop, only in $\mathcal{N}=1$ SYM ! [earlier works w/ higher SUSY] want confinement and mass gap !

Notice that $\overset{**}{\underset{*}{\nabla}}$ is only the contribution of <u>nonperturbative</u> fluctuations M & M^* ($n+m$ total) to the vacuum to vacuum amplitude (Z). In each case there are perturbative fluctuations, too - the photons / or their dual/.

But these are divided out in $\overset{**}{\underset{*}{\nabla}}$ — the denominator — & including them simply means dropping A. So we

(gives $\det^{-1/2} \vec{\nabla}^2$) the one physical d.o.f. of photon

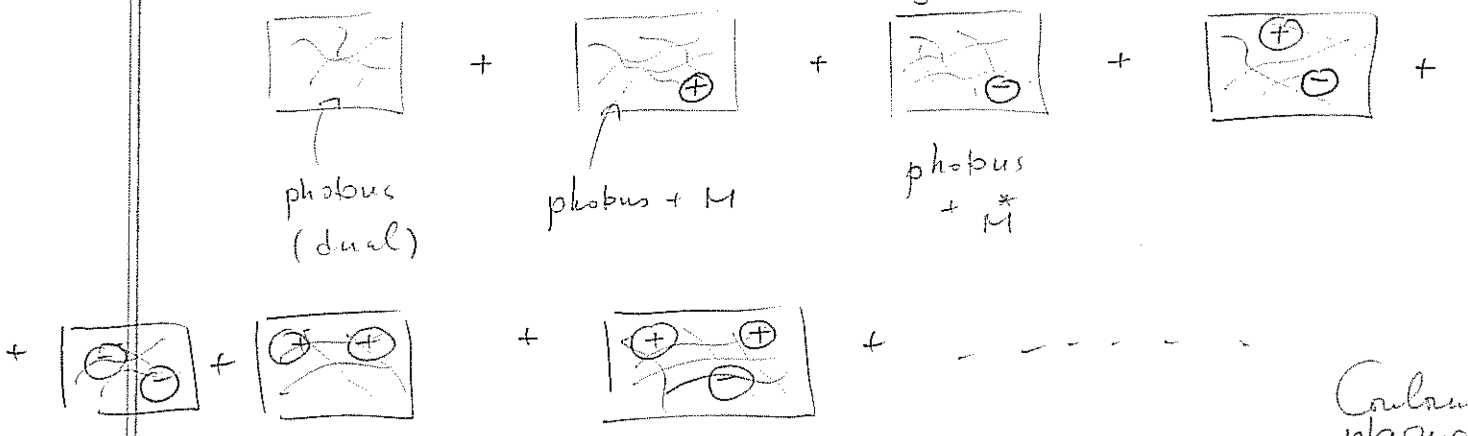
(**) see comment above

have

$$Z = \sum_{n,m=0}^{\infty} \int \mathcal{D}\sigma \, e^{-\varkappa \int d^3x \, (\partial\sigma)^2} \int d^{3(n+m)} r \, \frac{(v^3 e^{-S_0})^n}{n!} \frac{(v^3 e^{-S_0})^m}{m!} \times$$

$$\times \, e^{i\int d^3x \, \sigma(x) \sum_{a=1}^{n+m} q_a \delta^{(3)}(\bar{x}-\bar{r}_a)} \Rightarrow (\text{next page})$$

pictorially; we are summing over following configurations contributing to Z:



Coulomb plasma

$Z \equiv$ grand partition function of a classical gas of 3d Coulomb charges ⊕ & ⊖ (M, $\overset{*}{M}$)

by physical analogy, this should produce a length scale — Debye screening length — determined by the density of the plasma . . . .

$$Z = \int \mathcal{D}\sigma\, e^{-\varkappa\int d^3x(\partial\sigma)^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(v^3e^{-S_0})^n}{n!}\,\frac{(v^3e^{-S_0})^m}{m!}\times$$

$$\times\int d^3r_1\dots d^3r_m\int d^3r_{n+1}\dots d^3r_{n+m}\times$$

(I used the form of $j(x)$) ⟶

$$\times\, e^{i(\sigma(r_1)+\dots+\sigma(r_n))}\times e^{-i(\sigma(r_{n+1})+\dots+\sigma(r_{n}}$$

$$=\int\mathcal{D}\sigma\, e^{-\varkappa\int d^3x(\partial\sigma)^2}\sum_{n=0}^{\infty}\frac{\left(d^3r_1\, v^3e^{-S_0}e^{i\sigma(r_1)}\right)^n}{n!}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\left(d^3r_m\, v^3e^{-S_0}\,e^{-i\sigma(r_m)}\right)^m}{m!}$$

$$=\int\mathcal{D}\sigma\, e^{-\varkappa\int d^3x(\partial\sigma)^2}\times e^{\int d^3x\, v^3e^{-S_0}(e^{i\sigma(x)}+e^{-i\sigma(x)})}$$

$$Z=\int\mathcal{D}\sigma\, e^{-\int d^3x\left[\underbrace{\frac{g_3^2}{(4\pi)^2}}_{\varkappa}\left((\partial\sigma)^2+\frac{v^3}{\varkappa}e^{-S_0}\cos\sigma(x)\right)\right]}$$

↑ We've gone a long way — !

- at v: $SU(2)\to U(1)$, $\int$out W-bosons, left w/ σ
- including Σ of nonperturbative saddle points of Euclidean path $\int$ (M & M^*) we arrive @ above Z
- the Σ over M & M^* has done something dramatic: it has induced a mass gap (potential) for the <u>dual photon</u> — due to an Euclidean analogue of Debye screening

⟶ the proliferation of M & M^* (not "condensation") in vacuum has led to mass gap = due to Σ over nonperturbative saddles.

Let's look at scales, again:

$$\underbrace{v \gg g_3^2}_{} \gg \text{mass gap} \equiv m_\sigma$$

responsible for weak coupling & applicability of semiclassics

What is m_σ: clearly $m_\sigma^2 \sim \frac{v^3}{x} e^{-S_0} =$

$$m_\sigma^2 \sim \frac{v^3}{g_3^2} e^{-\frac{4\pi v}{g_3^2}}$$

$$m_\sigma \sim v\, e^{-\frac{S_0}{2}}$$

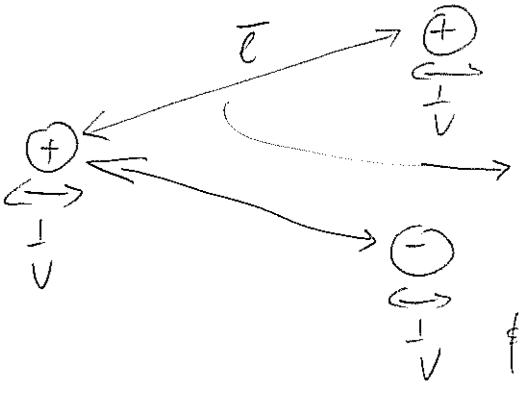
exponentially smaller scale

never mind the exact prefactor (there are a couple more inverse powers of g_3 --)

hierarchy: $v \gg g_3^2 \gg v\, f\left(\frac{v}{g_3^2}\right) e^{-\frac{1}{2}\frac{4\pi v}{g_3^2}} \equiv m_\sigma$

---

the M/M* density is $v^3 e^{-S_0}$, so their typical separation is $\frac{1}{v} e^{+S_0/3}$ – exponentially large compared to W-boson mass.



$\bar{\ell} \sim \frac{1}{v} e^{+S/3}$: so, a very dilute gas

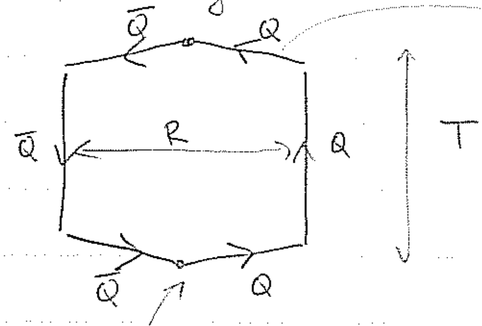
& $\frac{1}{m_\sigma} \sim \ell_{Debye} \sim \frac{1}{v} e^{S_0/2} \gg \bar{\ell}$

(so there are many charges/Debye volume)

One last thing before we go $R^3 \to R^3 \times S^1$.

So we've got ourselves a mass gap.
I promised confinement... <u>linear potential between probe charges</u> → where does this come from?

The way one looks for a confining potential is to study the following:



(contour C in spacetime)

{ create QQ pair, separate by R, propagate by T, annihilate

- for ∞ heavy $Q\bar{Q}$'s this is the insertion of a $W(C)$
- in full nonabelian theory

$$W_R(C) = \mathrm{tr}_R\, P\, e^{i\oint_C A^\mu dx^\mu}$$

↑ representation ↑ path ordered exponent

- if $\langle W_R(C)\rangle = e^{-\Sigma \cdot A(C)}$ × (local renormaliz'n of W)

we say there's "area law"

↑ area spanned by C as "$C \to \infty$" (min.)

for C as in picture: $\langle W_R \rangle \sim e^{-\underbrace{(\Sigma_R R)}_{}T}$ as $R, T \to \infty$

distance

energy of static $Q\bar{Q}$ ~ R

representation [sorry!]

$$\boxed{\Sigma_R = \text{confining string tension (depends on } R)}$$

Recall now our earlier discussion (p 23) of W(C) in σ - theory ("magnetic dual" description), where we argued that inserting of W(C) in path integral — — for a fundamental quark of SU(2) — means that one should ∫ over σ configurations that have monodromy 2π around C. (recall this is what is called a 'disorder' operator — it is an instruction of what to ∫ over, rather than a local expression).

So to find $< W(C) > \approx$

$$\approx \int \mathcal{D}\sigma \, "W(C)" \, e^{-\frac{g_3^2}{(4\pi)^2} \int d^3x \left((\partial\sigma)^2 + m_\sigma^2 \cos\sigma\right)}$$

(forgive me a factor of 2; we don't care now!)

In the weak coupling limit, the path ∫ is dominated by classical configurations. So the calculation of ⟨W(C)⟩ can be translated, to leading semiclassical order as:

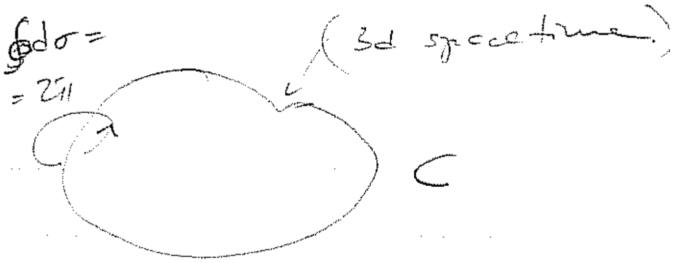
(a) — find a classical σ - field configuration that has the property that it has monodromy 2π around C and solves the equations of motion (so gives the largest contribution to the ⟨W(C)⟩).

(b) — if we do (find it) we can argue that $\langle W(C)\rangle \sim e^{-S_{cl}}$

the action found in (a).

(N.B. this can be formalized & computerized, for arbitrary C. At this point, "life's too short"... [ask me if you must].)

In pictures: $\langle W(C) \rangle =$

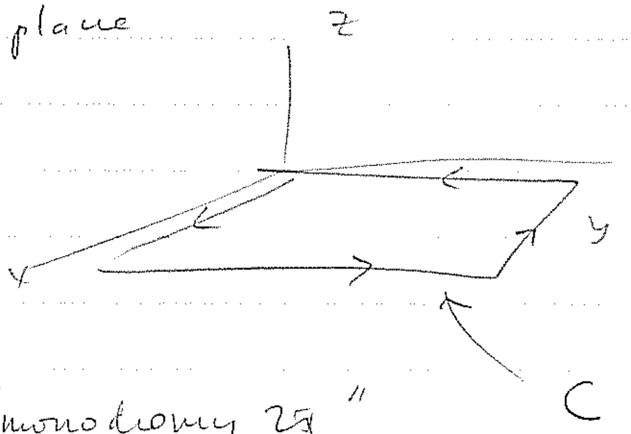


find $\bar\sigma$: $\left.\frac{\delta S}{\delta\sigma}\right|_{\bar\sigma} = 0$ subject to monodromy(C) = 2π

& then

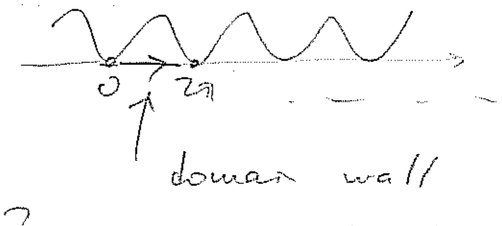
$$\langle W \rangle \sim e^{-S[\bar\sigma]}.$$

We'll take a huge rectangular contour, say in x-y plane



Having "monodromy 2π" as C → entire xy plane means that upon crossing the xy plane (from $z<0$ to >0) σ should change by 2π. But we know this configuration:

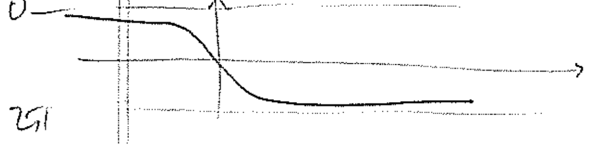
$$S[\sigma] \sim g_3^2 \int d^3x \left[(\partial\sigma)^2 + \underbrace{m_\sigma^2 \cos\sigma}_{} \right]$$



$\bar\sigma$ is xy-indep. as $C \to \infty$

$\bar\sigma(z)$ is 0 as $z \to -\infty$

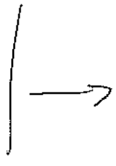
2π as $z \to +\infty$



What's the action? (per unit area) (xy)

thickness $\sim \frac{1}{m_\sigma}$

$\Delta\sigma \sim 2\pi$



$$\left(\frac{\Delta\sigma}{\Delta z}\right)^2 \sim \left(\frac{2\pi}{1/m_\sigma}\right)^2 \sim \frac{1}{(1/m_\sigma)^2}$$

$$S[\bar\sigma] \sim \underset{\text{prefactor}}{g_3^2}\; \underset{\substack{\text{area of } C\\ (xy \text{ plane})}}{A_{(xy)}}\; \underset{\text{thickness}}{\frac{1}{m_\sigma}}\; \overbrace{\frac{1}{(1/m_\sigma)^2}}$$

$$S[\bar\sigma] \sim (\text{Area}_{(xy)} \text{ of } C) \times g_3^2 m_\sigma$$

so $\langle W(C)\rangle \sim e^{-\text{Area}\,\underbrace{(g_3^2 m_\sigma)}_{\text{string tension}}}$

this is correct up to factor of 2, π or so (which can easily be found 'cause domain wall solution easy!)

$$\Sigma \sim \underbrace{\# g_3^2 m_\sigma}_{\text{very interesting formula!}} \sim g_3^2 v e^{-S_0/2}$$

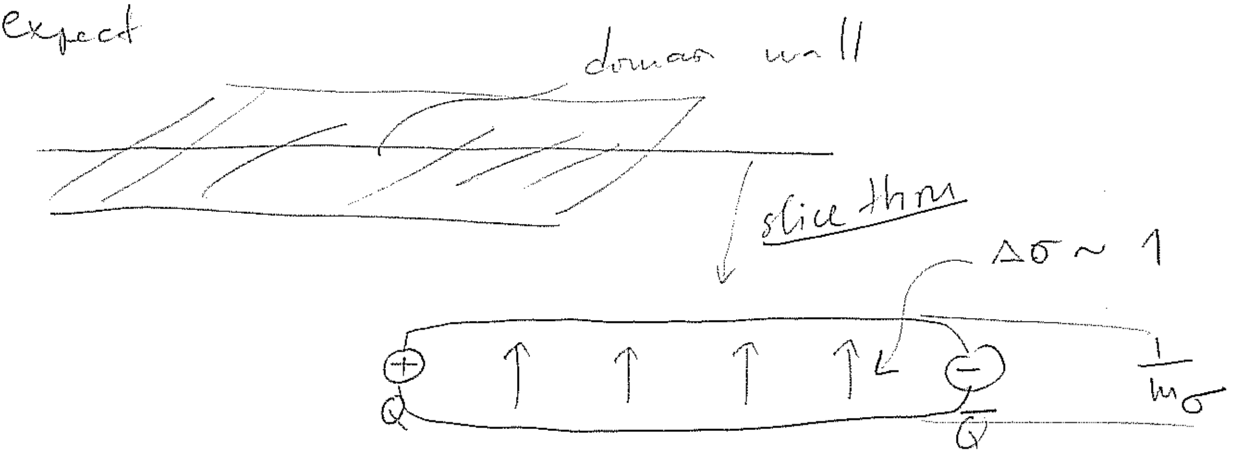
- recall all ignorance of M/M* prefactor is in m_σ

* thickness of domain wall /string/ $\sim 1/m_\sigma$

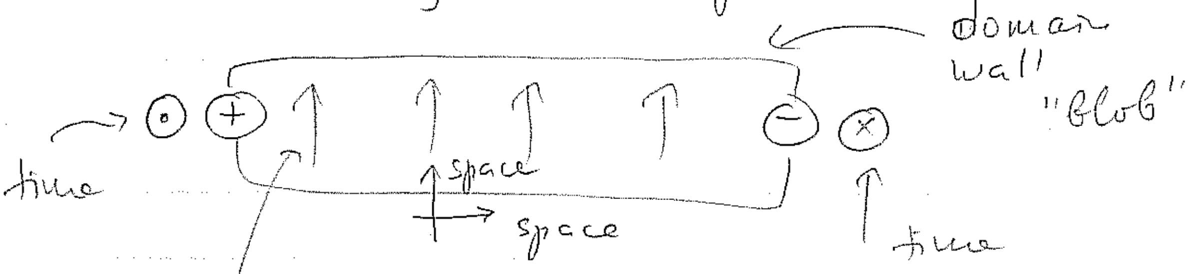
* tension $>> \frac{1}{(\text{thickness})^2}$, since $g_3^2 >> m_\sigma$

not a "thin" string!

Pictorially, for finite C we expect



this was all Euclidean — interpret now in Minkowski: — then my slice is spatial

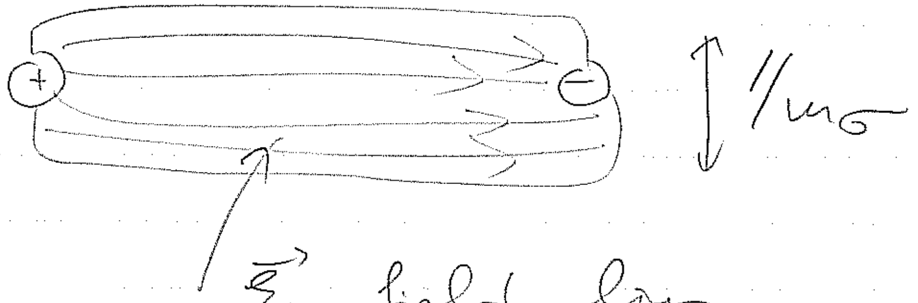


gradient of σ is spatial —

— by duality $\mathcal{E}^i \sim \epsilon^{ij} \partial_j \sigma$

— so this is spatial $\vec{\mathcal{E}}$ field in direction from $\oplus$ to $\ominus$

Notice: this is "abelian" confinement: at length scales where string formation occurs ($R \gg l_{\text{Debye}}$) only abelian fields are relevant !



$\vec{\mathcal{E}}$ field lines.
(as one expects)

- instead of spreading in a dipole-like fashion (each charge has $1/r$ fall off in 2+1d) field is collimated in a $\sim 1/m_\sigma$ tube of El. flux.
- reason is the Debye screening due to the monopole instanton plasma!
- & we (OK, Polyakov...) can understand it!

[pretty cool. remember my A.I., more to come, we're still 3d (p.④) (!).

From R^3 to $R^3 \times S^1_L$ ← size of S^1

- we are making our way "up" to $\mathbb{R}^4$
- 1st thing that we can do is dispose of the Higgs field — and replace it by A_4^a's zero mode on S^1
- the action, keeping only 0 modes on S^1, is (we'll do $\Lambda L \ll 1$ limit — later) same as the one on p. 6, w/ $g_3^2 \to \frac{g_4^2}{L}$

in other words, this is the dim-reduced action of 0 KK modes

$$S = \int d^3x \left[\frac{L}{4g_4^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{L}{2g_4^2} (D_\mu A_4^a)^2 + \ldots \right]$$

(g_4 should be taken @ $1/L$ — one-loop calculation —) etc.

- possible potential for A_4
- there's no such thing @ tree level, but may be induced by loops of W-bosons & whatever other fields there may be

- in theories w/ $\mathcal{N}=1$ SUSY, $V(A_4)$ is absent to all orders in pert. theory (recall periodic SUSY preserving b.c.!)
  ↳ so, there, A_4 is a "flat direction" (... let's explain more ...)

- for now, do not commit to SUSY or QCD w/ many Weyl adjoint fermions, but study the bosonic sector under the assumption that A_4^a 0 mode has a vev, just like an adjoint Higgs in the Polyakov model would, breaking $SU(2) \xrightarrow[v]{} U(1)$.

What is the range of possible v's for $\langle A_4^a \rangle$?
To answer that, let's introduce the "holonomy" –
– the noncontractible Wilson loop around S^1:

* $\Omega(\vec{x}) = \mathcal{P}\, e^{i \oint_{S^1_L} A_4^a T^a dx^4}$; $\det \Omega = 1$, an SU(2) element

$\uparrow$

clearly this is a f-n of $\vec{x} \in R^3$ only, not on S^1_L coordinate

* under gauge trf's [trfs-always periodic]

$$\Omega(\vec{x}) \to g(\vec{x})\, \Omega(\vec{x})\, g^{-1}(\vec{x})$$

(for a Wilson line

$$\mathcal{P} e^{i\int_a^b A_\mu dx^\mu} \to g(a) \left(\mathcal{P} e^{i\int_a^b A_\mu dx^\mu}\right) g^{-1}(b)$$

so $\Omega(\vec{x})$ transforms as an adjoint Higgs field on R^3 $(\equiv \varphi^a T^a)$ under gauge trfs

[in thermal field theory, if x_4 is Euclidean time, Ω is known as "Polyakov loop" –
– important for later] $\nearrow$

(we'll use interchangeably "Polyakov" & "Wilson" loop w/out being pedantic)

* Clearly a vev of A_4 is a vev of Ω & v.v. (@ weak coupling, where small fluctuations)

Think i.t.o. Ω & consider its vev, $\langle \Omega \rangle$.
this is $\vec{x}$ - independent, by assumption.
But $\langle \Omega \rangle$ is not gauge invariant – it is a 2×2 SU(2) matrix which transforms under gauge transformations. However, the eigenvalues of $\langle \Omega \rangle$ are gauge invariant (eg they are the solutions of $\det(\lambda \mathbb{1} - \langle \Omega \rangle) = 0$, which is a gauge inv't eqn).

In other words, one can always diagonalize $\langle \Omega \rangle$:

$$\langle \Omega \rangle = \begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix}$$

(EV's are c.c. since det = 1)

$$= e^{i \langle A_4^3 \rangle L \frac{\tau^3}{2}}$$

(recall def of $\Omega \equiv$ $\equiv \mathcal{P} e^{i \oint A_4^a T^a dx^4}$)

$$= \begin{pmatrix} e^{i\frac{\langle A_4^3\rangle L}{2}} & 0 \\ 0 & e^{-i\frac{\langle A_4^3\rangle L}{2}} \end{pmatrix}$$

① notice that $\langle A_4^3 \rangle$ & $\langle A_4^3 \rangle + \frac{2\pi}{L} n$, $\forall n \in \mathbb{Z}$ lead to the same eigenvalues (ordering is a Weyl trfm $\in SU(2)$)

$\Rightarrow A_4^3$ is a periodic variable, period $\frac{2\pi}{L}$

② notice if $\langle A_4^3 \rangle \equiv \frac{\pi}{L}$, smthg special happens:
$\mathrm{tr}\langle \Omega \rangle$ = gauge inv't! $= 0$ $\left(\langle \Omega \rangle = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}\right)$

② the point $\langle A_4^3 \rangle \equiv \frac{\pi}{L}$ where $\mathrm{tr}\langle \Omega \rangle = 0$
(contd.)
is a very special one, called
"center symmetric" ← whether $\mathrm{tr}\langle \Omega \rangle = 0$ or not determines the realization of a global symmetry the YM theory (w/ adjoint matter) has: center symmetry

→ in modern parlance, this is called a "1-form global symmetry" (as it acts on line operators, such as Ω, rather than on local fields — see Gaiotto, Kapustin, Seiberg ≈1501.... —— -- but this has been rather well known to lattice people)

the $\mathbb{Z}_2$ center symmetry of SU(2) QCD(adj) acts on $\mathrm{tr}\Omega$ as: $\mathrm{tr}\Omega \to -\mathrm{tr}\Omega$.
(leave it as this for now – later: order parameter for deconfinement transition, if Ω is Polyakov loop...)

Let us work at the center-symmetric point $\langle A_4^3 \rangle = \frac{\pi}{L}$ from now on. (this requires justification as we must explain what potential causes this...)

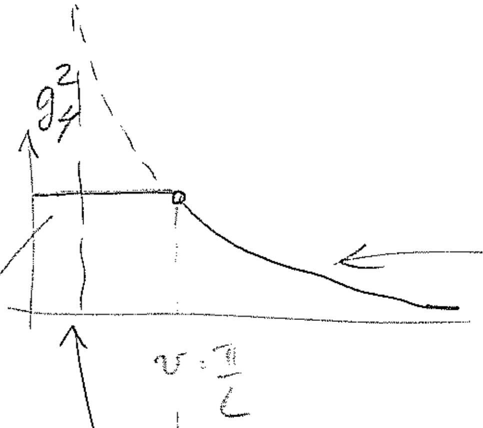
W/ $\langle A_4^3 \rangle = \frac{\pi}{L}$, all is as before ...

– replace $v = \frac{\pi}{L}$

$m_W = \frac{\pi}{L}$ (in $SU(N)$: $\frac{2\pi}{NL}$)

So we have $SU(2) \xrightarrow{\frac{\pi}{L}} U(1)$

↓ w/ adjoint fields only, nothing is charged under $U(1)$



(asymptotic freedom, log-running in 4d)

(coupling frozen)

↓ Below v, only $U(1)$, free

Λ : strong coupling scale, no semiclassics reliable

But if $v \gg \Lambda$, i.e. $\frac{1}{L} \gg \Lambda$, the blow up of the coupling is not reached & weak coupling OK (e.g. electroweak theory vs QCD!)

Origin of $NL\Lambda \ll 1$ condition is $m_W \sim \frac{1}{NL} \gg \Lambda$, for $SU(N)$.,) weak-coupling regime validity

What about the nonperturbative physics, confinement, mass gap, etc? Does it all go thru?

(clearly, assuming that $\langle A_4^3 \rangle = \frac{\pi}{L} \gg \Lambda$ is somehow justified, at scales $\ll v \sim \frac{1}{L}$, the weakly coupled perturbative theory is as described in the Polyakov model)

It turns out that there are interesting twists regarding the monopole solutions on $R^3 \times S^1$:
instanton

Discovered in late 1990's, independently by lattice-oriented people (Kraan, van Baal) & string theorists (Piljin Yi, Kimyeong Lee)

The latter's discovery is clearer & easier to explain, especially using D-branes [ask me]; I'll follow a purely QFT path (it also stresses that the various monopole-instantons that we discuss below are the fundamental ingredients, rather than the "calorons" & their 'constituents' found by van Baal et al.).

First recall the BPS M solution (p. 10-12).

use $v = \frac{\pi}{L}$

$$S_0 = \frac{4\pi v}{g_3^2} = \frac{4\pi^2}{L g_3^2} = \frac{4\pi^2}{g_4^2} \longleftarrow$$ already very interesting result ($\longrightarrow$

— For those of you who remember that 4d BPST instantons have action $S_{BPST} = \frac{8\pi^2}{g_4^2}$

note that we just found that at the center-symmetric vev $\langle A_4^3 \rangle \equiv v = \frac{\pi}{L}$, the monopole-instanton action is

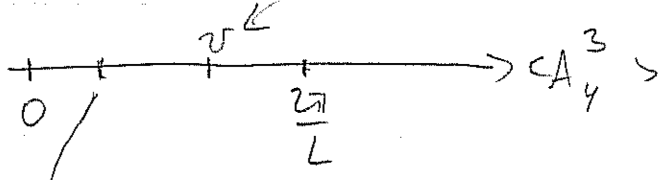
$$\underset{\displaystyle \frac{4\pi^2}{g_4^2}}{\underset{\shortparallel}{S_0}} = \frac{S_{BPST}}{2}$$

— intriguing ?!

Indeed, there is a connection... recall that $\langle A_4^3 \rangle$ is periodic, period $\frac{2\pi}{L}$.

(one simple explanation: $\partial_4 + \langle A_4^3 \rangle$ is the only way $\langle A_4^3 \rangle$ ever appears in the action of the theory, but ∂_4, in Fourier space is $\frac{2\pi k}{L}$, so really we have $\frac{2\pi k}{L} + \langle A_4^3 \rangle$, where k is the Matsubara (or Kaluza-Klein #); by relabeling k's one can shift $\langle A_4^3 \rangle$, giving rise to said periodicity)

some generic $0 < v < \frac{2\pi}{L}$



$\to v' = \frac{2\pi}{L} - v$ is another vev, in same fundamental domain

((one can also consider more general $v_k = \frac{2\pi k}{L} + v$))

[see comment on p 45; our construction is for the two lowest action solutions, the 'constituents' of the BPST $Q=1$ instanton]<

Recall that in stringy gauge, at $r \gg v$, we have for the BPS (self dual) solutions

$\underline{M}$: (SD)

$$\begin{cases} A_4\big|_\infty \to v\,\frac{\tau^3}{2} \\ B_r\big|_\infty \to \frac{1}{r^2}\frac{\tau^3}{2}\,, \quad B_\theta, B_\varphi \to \emptyset \quad \text{at } \infty \end{cases}$$

Consider now the SD solutions, but w/ vev. (*), $v' = \frac{2\pi}{L} - v$, we have:

$\underline{M'}$: (SD, v' vacuum)

$$\begin{cases} A_4'\big|_\infty \to \frac{2\pi}{L}\frac{\tau^3}{2} - v\frac{\tau^3}{2} \\ B_r'\big|_\infty \to \frac{1}{r^2}\frac{\tau^3}{2} \end{cases}$$

- these are x_4-independent
- still charge $Q_4 = +1$
- action is, however, $S' = \frac{4\pi v'}{g_3^2} = \frac{4\pi}{g_3^2}\left(\frac{2\pi}{L} - v\right)$

to construct solutions w/ same $A_4|_\infty$ as M, we can use following trick. Consider $U = e^{-i\frac{\tau^3}{2}\frac{2\pi x_4}{L}}$. this is an $SU(2)$ group element[*], so under

$$A_4 \to U A_4 U^+ + i U \partial_4 U^+$$

$$A_\mu \to U A_\mu U^+ + i U \partial_\mu U^+$$

[*] but NOT a gauge trf (not periodic)

the action is invariant & the EOM are as well. So U maps solutions into solutions.

$(\overset{\checkmark}{x_4}\text{-}\underline{\underline{\text{independent}}}) \to (\overset{\checkmark}{x_4}\text{-dependent})$

Moreover $A_4''\big|_\infty \equiv U\, A_4\big|_\infty U^+ + i\, U \partial_4 U^+ = \frac{2\pi}{L}\frac{\tau^3}{2} - v\frac{\tau^3}{2} - \frac{2\pi}{L}\frac{\tau^3}{2}$

$\underline{M''}$: (SD, $-v$-vacuum)

$$A_4''\big|_\infty = -v\,\frac{\tau^3}{2}$$

$$B_r''\big|_\infty = \frac{1}{r^2}\frac{\tau^3}{2}$$

(**↑) I am only giving asymptotics of solution on S^2_∞ – – inside it is a horrible x^4-dependent mess!!

Finally, to get a SD sol'n in same vacuum, use (a Weyl trsf'n)

$$\tilde{U} = i\sigma^2$$

, another SU(2) element that keeps solutions → solutions to obtain (**↑)

$$(\overbrace{i\sigma^2}^{\tilde{U}} \tau^3 \overbrace{(-i\sigma^2)}^{\tilde{U}^\dagger} = -\tau^3$$

"twisted" or "Kaluza-Klein" monopole-instanton

$$\left\{ \underline{\underline{M'''}} \equiv \underline{\underline{KK}} \quad \underline{\text{SD}, v\text{-vacuum}} \right. \qquad A_4'''\big|_\infty = v\frac{\tau^3}{2} \qquad B_r'''\big|_\infty = -\frac{1}{r^2}\frac{\tau^3}{2}.$$

$$\left\{ \begin{array}{l} Q_m = -1 \\ S_{KK} = \frac{4\pi}{g_3^2}\left(\frac{2\pi}{L} - v\right) \\ Q_T \equiv 1 - \frac{Lv}{2\pi} \end{array} \right.$$

for SD solutions $S_{eucl.} = \frac{8\pi^2}{g^2} Q_T$ compare w/ ⟶

BPS monopole instanton

$$\left\{ \underline{\underline{M}} \quad \underline{\text{SD}, v\text{-vacuum}} \right. \qquad A_4\big|_\infty = v\frac{\tau^3}{2} \qquad B_r\big|_\infty = \frac{1}{r^2}\frac{\tau^3}{2}.$$

$$\left\{ \begin{array}{l} Q_m = +1 \\ S_M = \frac{4\pi}{g_3^2} v \\ Q_T = \frac{Lv}{2\pi} \end{array} \right.$$

- both are SD (SD eqn is invt under U, Ũ)
- M has $Q_m = +1$, $Q_T = Lv/2\pi$
- KK has $Q_m = -1$; $Q_T \equiv 1 - Lv/2\pi$

] see p. (67.1) for further refs

- SD also means that action ~ 4d $\int F\tilde{F}$ = topological charge

If these solutions were to be combined, we'd have

$$\left| \begin{array}{l} Q_m = -1 + 1 = 0 \\ Q_T = 1 \end{array} \right. \quad \& \quad S_{KK+M} = \frac{4\pi}{g_3^2} \times \left[\left(\frac{2\pi}{L} - v\right) + v\right] =$$

$$= \frac{8\pi^2}{g_3^2 L} = \frac{8\pi^2}{g_4^2} \;!$$

⟵ indeed in some well defined sense, they can be thought of as constituents of the 4d BPST...

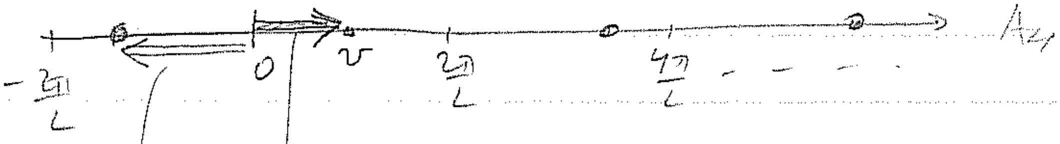
(48)

So, we now moved from 1977 to 1997 - - -.

One more note on these extra (twisted) monopole-instantons on $R^3 \times S^1$

(Piljin Yi, Kimeyong Lee ... "Instantons & monopoles on partially compactified D-branes")

What we did shouldn't sound so mysterious.



→ usual 3d BPS sol'n w/ $A_4|_\infty = v$

another 3d (x_4-indep) BPS sol'n w/ $A_4|_\infty = -\frac{2\pi}{L} + v$

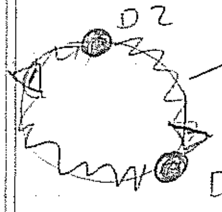
-- one can have x_4-indep BPS solutions w/ $A_4|_\infty = \frac{2\pi k}{L} + v$, $k \in \mathbb{Z}$

then one uses appropriate $U = e^{\pm i \frac{\tau^3}{2} k \frac{2\pi x_4}{L}}$

to make an x_4-dependent solution w/ action $\frac{4\pi}{g_3^2}\left|\frac{2\pi k}{L} + v\right|$ & appropriate Q_m (mindful of various signs --)

we did, really, $k = -1$ here -

- giving the lowest action twisted monopole instanton



Euclidean D0 = monopole-instanton [ask me!]

(etc. ---)

(separation of D2's ~ v!)

(46)

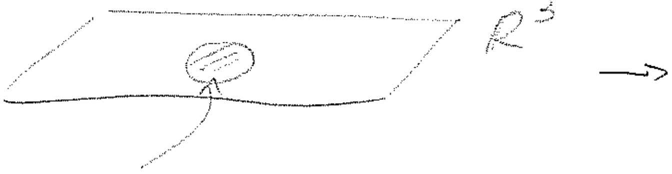
Summary: on $R^3 \times S^1$ assuming $\langle A_4^3 \rangle = v \frac{\tau^3}{2}$

$$0 < v < \frac{2\pi}{L}$$

(special point w/ $\underset{\overset{\nearrow}{L}}{\mathrm{tr}\,\Omega} = 0$ is $v = \frac{\pi}{L}$)

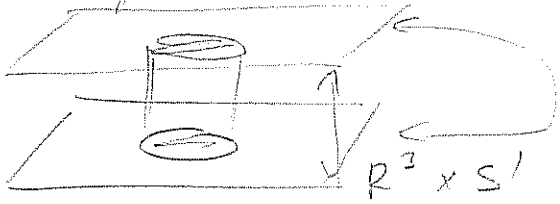
("center symmetric")

there are magnetic monopole-instanton solution of a new type - which are x_4 - dependent.



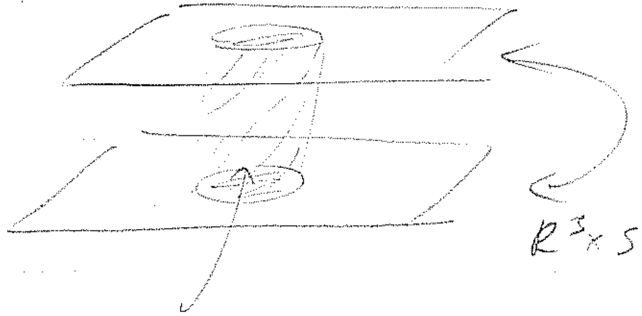
M (BPS solution)

$$S = \frac{4\pi v}{g_3^2}$$



M (BPS) trivially embedded in $R^3 \times S^1$

$$S_M = \frac{4\pi L v}{g_4^2}, \quad \frac{L}{g_4^2} = \frac{1}{g_3^2}$$



KK (BPS) solution, twisted on S^1

$$S_{KK} = \frac{4\pi L}{g_4^2}\left(\frac{2\pi}{L} - v\right)$$

$S_M + S_{KK} = S_{BPST}$

$Q_M = -Q_{KK}$

- both self-dual
- opposite magnetic charge
- at center symmetric point $\frac{1}{2}$ BPST action ("instanton quarks")

Now we need to bite the bullet & address the question: what dynamics is responsible for $\langle A_4^3 \rangle = v \left(= \frac{\pi}{L}, \text{say} \right)$?

the answer is theory-dependent. (Let's imagine we consider QCD(adj) w/ n_f adjoint Weyl fermions, for concreteness - and for some cheating -) the tree level action of the theory on $R^3 \times S^1$ is the usual one

$$S_{\substack{QCD\\(adj)}} = \frac{1}{2g^2} \int_{R^3 \times S^1_L} d^4x \; \mathrm{tr}\, F_{MN} F^{MN} + \frac{2i}{g^2} \int_{R^3 \times S^1_L} d^4x \; \bar{\lambda}^I \left(\bar{\sigma}^\mu D_\mu - i \bar{\sigma}_3 [A_4, \cdot\,] \right) \lambda_I$$

- asymptotic freedom for $n_f \leq 5$.
- SUSY for $n_f = 1$

- $I = 1 \ldots n_f$ flavor index
- $\lambda_I \equiv \lambda_{I\alpha}$, $\bar{\lambda}^I = \bar{\lambda}^I_{\dot{\alpha}}$ $\uparrow$ (SL(2,C) indices)
- $\mathrm{tr}\, t^a t^b = \frac{1}{2} \delta^{ab}$, as usual

if fermions have mass

$$S_m = \frac{m}{g^2} \int_{R^3 \times S^1_L} d^4x \; \mathrm{tr}\, \lambda^\alpha_I \lambda_{\alpha I} + h.c.$$

(the $1/g^2$ normalization of λ's is SUSY-inspired)

On R^4, no gauge field can get vev w/out breaking Lorentz invariance, but on $R^3 \times S^1_L$, as we discussed, $\langle A_4^3 \rangle$ can be nonzero. At tree level in $S_{QCD(adj)} + S_m$ this is a "flat direction", i.e. we can assign any value to $\langle A_4^3 \rangle = v$ ($0 < v < \frac{2\pi}{L}$, as discussed).

In the quantum theory, there are, of course loop corrections, and an effective potential for v will be generated (the vacuum energy, as a function of v, will generally be $\neq 0$, unless there is a symmetry ensuring vanishing).

SUSY is exactly such a symmetry - if not broken by boundary conditions! But as we stated @ very beginning, we shall study periodic b.c. for both fermions & bosons, i.e.

$$\lambda_{\alpha_I}(x_4 + L) = \lambda_{\alpha_I}(x_4)$$

↑

(more general b.c. are possible
- thermal - λ is antiperiodic
- various phases, so long as they correspond to anomaly free symmetries)

For periodic λ_I's, nonrenormalization theorems in SUSY guarantee that the perturbative contribution to the vacuum energy will vanish (including its v-dependence, of course) - so there will be no potential for v... At one loop (& this will suffice for us, as we work for $\Lambda L \ll 1$) this vanishing is due to the cancellation of the boson & fermion determinants (for $n_f = 1$).

We'll use this to "cheat" – in other words, we'll compute the single Weyl fermion determinant on $R^3 \times S^1$. By SUSY, we know it $\equiv$ the gauge boson determinant. This will help us avoid computing the latter [see Gross, Pisarski, Yaffe – it involves gauge fixing, ghosts & the like – things we can't be bothered with now].

So, the answer for the one-loop vacuum energy as a function of "v" will be:

$$V_{eff}(v) = (n_f - 1) \times \Big[\log\det(\overset{\text{single}}{\text{fermion}}\ \text{determinant}(v))$$

gauge boson det. contribution.

because, for $n_f = 1$, we should get $\emptyset$

N.B. for $n_f = 0$, $V_{eff}(v)$ is known as "the GPY"-potential ... 1980's! (as we'll see, it shows that in the high-T (≡small-L) phase, $\mathrm{tr}\,\Omega/2 = 1$, i.e. center symmetry is broken in the deconfined phase)

this will be largely omitted — read on your own...
(did it "for sports")
— jump to (50.6) & (51), then (53.2)

we're trying to compute the one-adjoint-Weyl fermion determinant in the nontrivial holonomy background

use Wess & Bagger ($SL(2,\mathbb{C})$) spinor notation

$$S_\lambda = \frac{2}{g^2}\int_0^L dx^3 \int_{\mathbb{R}^3} d^3x \; \mathrm{tr}\Big(i\bar\lambda_{\dot\alpha}\, \bar\sigma^{M\dot\alpha\alpha} D_M \lambda_\alpha + \frac{m}{2}\lambda_\alpha \lambda_\beta \epsilon^{\beta\alpha} + \frac{m^*}{2}\bar\lambda_{\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}\Big)$$

here $\underline{M = 0,1,2,3}$ $\xrightarrow{}$ compact direction $\overset{\|}{0 - L = 2\pi R}$ & $\underline{\mu = 0,1,2}$

$$\bar\lambda_{\dot\alpha} = (\lambda_\alpha)^*$$

(we'll write det. in Minkowski & then continue)

$(\sigma^0 \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix})$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} \qquad \bar\sigma^0 = -\sigma^0, \quad \bar\sigma^{1,2,3} = -\sigma^{1,2,3}$$

use a decomposition using $\frac{\tau^3}{2}, \frac{\tau^+}{2}, \frac{\tau^-}{2}$

$\tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau_2)$ [generalizes to any G, using $H^a, E_\beta, E_\beta^\dagger$ ...]

$$\lambda_\alpha(x^\mu, x^3) = \frac{\tau^3}{2}\lambda^3_\alpha(x^\mu, x^3) + \frac{\tau^1}{2}\lambda^1_\alpha(x^\mu, x^3) + \frac{\tau^2}{2}\lambda^2_\alpha(x^\mu, x^3)$$

50.2

$$= \frac{\tau^3}{2}\lambda^3 + \frac{1}{2}(\tau^+ + \tau^-)\lambda^1 + \frac{1}{2}\frac{\tau^+ - \tau^-}{i}\lambda^2$$

$$\lambda_\alpha(x^\mu, x^3) = \frac{\tau^3}{2}\lambda^3_\alpha + \frac{\tau^+}{2}(\lambda^1_\alpha - i\lambda^2_\alpha) + \frac{\tau^-}{2}(\lambda^1_\alpha + i\lambda^2_\alpha)$$

$$\equiv \frac{\tau^3}{2}\lambda^3_\alpha + \frac{\tau^+}{2}\lambda^+_\alpha + \frac{\tau^-}{2}\lambda^-_\alpha(x^\mu, x^3)$$

$$\bar\lambda_{\dot\alpha}(x^\mu x^3) = \frac{\tau^3}{2}\bar\lambda^3_{\dot\alpha} + \frac{\tau^1}{2}\bar\lambda^1_{\dot\alpha} + \frac{\tau^2}{2}\bar\lambda^2_{\dot\alpha} = \text{--- same}$$

$$= \frac{\tau^3}{2}\bar\lambda^3_{\dot\alpha} + \frac{\tau^+}{2}\underset{\bar\lambda^1_{\dot\alpha} - i\bar\lambda^2_{\dot\alpha}}{\underset{\parallel}{\bar\lambda^+_{\dot\alpha}}} + \frac{\tau^-}{2}\underset{\bar\lambda^1_{\dot\alpha} + i\bar\lambda^2_{\dot\alpha}}{\underset{\parallel}{\bar\lambda^-_{\dot\alpha}}}$$

also def FT : periodic λ's !

$$\lambda^{3,+,-}_\alpha(x^\mu, x^3) = \frac{1}{L}\sum_{p\in\mathbb{Z}}\int\frac{d^3k}{(2\pi)^3}\, e^{i\frac{2\pi p}{L}x^3 + ik_\mu x^\mu}\,\lambda^{3,+,-}_{\alpha(p,k)}$$

$$\bar\lambda^{3,+,-}_{\dot\alpha}(x^\mu, x^3) = \frac{1}{L}\sum_{p\in\mathbb{Z}}\int\frac{d^3k}{(2\pi)^3}\, e^{-i\frac{2\pi p}{L}x^3 - ik_\mu x^\mu}\,\bar\lambda^{3,+,-}_{\dot\alpha(p,k)}$$

$$iD_M = i\left(\partial_0, \partial_1, \partial_2, \partial_3 + iv\left[\frac{\tau^3}{2}, \cdot\right]\right) ; \quad \left[\frac{\tau^3}{2}, \frac{\tau^\pm}{2}\right] = \pm\frac{\tau^\pm}{2}$$

$$= \left(-k_0, -k_1, -k_2, -\frac{2\pi p}{L} - v\left[\frac{\tau^3}{2}, \cdot\right]\right)$$

(acting on λ_α)

$$\bar\sigma^M D_M = \left(\sigma^\mu k_\mu, \sigma^3\left(\frac{2\pi p}{L} + v\left[\frac{\tau^3}{2}, \cdot\right]\right)\right)$$

(50·3) $\mathrm{tr}\,T^+T^+ = 0 \quad \mathrm{tr}\left(\frac{T^3}{2}\frac{T^3}{2}\right) = 1 \quad , \quad \mathrm{tr}\left(\frac{T^3}{2}\frac{T^\pm}{2}\right) = 0 \quad \mathrm{tr}\,\frac{T^+}{2}\frac{T^-}{2} = \frac{1}{4}$

Then

$$S_\lambda = \frac{2}{g^2}\frac{1}{L}\sum_{p\in\mathbb{Z}}\int\frac{d^3k}{(2\pi)^3}\Big[\underline{\frac{1}{2}\bar\lambda^3_{\dot\alpha(k,p)}\left(\sigma^{\mu\dot\alpha\alpha}k_\mu + \sigma^{3\dot\alpha\alpha}\frac{2\pi p}{L}\right)\lambda^3_\alpha(k,p)}$$

$$+\frac{1}{4}\bar\lambda^+_{\dot\alpha(k,p)}\left(\bar\sigma^{\mu\dot\alpha\alpha}k_\mu + \sigma^{3\dot\alpha\alpha}\left(\frac{2\pi p}{L} - v\right)\right)\lambda^-_{\alpha(k,p)}$$

$$+\frac{1}{4}\bar\lambda^-_{\dot\alpha(k,p)}\left(\sigma^{\mu\dot\alpha\alpha}k_\mu + \sigma^{3\dot\alpha\alpha}\left(\frac{2\pi p}{L} + v\right)\lambda^+_{\alpha(k,p)}\right.$$

$$\underline{+\frac{m}{2}\cdot\frac{1}{2}\left(\lambda^3_{\alpha(k,p)}\lambda^3_{\beta(-k,-p)}\epsilon^{\beta\alpha} + \bar\lambda^3_{\dot\alpha(k,p)}\bar\lambda_{\dot\beta(-k,-p)}\epsilon^{\dot\alpha\dot\beta}\right)}$$

$$+\frac{m}{2}\cdot\frac{1}{4}\left(\lambda^+_{\alpha(k,p)}\lambda^-_{\beta(-k-p)}\epsilon^{\beta\alpha} + \lambda^-_{\alpha(k,p)}\lambda^+_{\beta(-k,-p)}\epsilon^{\beta\alpha}\right)$$

$$+\frac{m^*}{2}\frac{1}{4}\left(\bar\lambda^+_{\dot\alpha(kp)}\bar\lambda^-_{\dot\beta(-k-p)}\epsilon^{\dot\alpha\dot\beta} + \bar\lambda^-_{\dot\alpha(k,p)}\bar\lambda_{\dot\beta(-k,-p)}\epsilon^{\dot\alpha\dot\beta}\right)\Big]$$

underlined: $\lambda^3, \bar\lambda^3$ - determinant is v-independent

((physics: λ^3 ~ partner (SUSY) of U(1) massless photon - does not couple to holonomy & so no contribution to $V_{eff}(v)$ ⇒ likewise in gauge boson sector only $W^\pm$ couple to v, while W^3 remains massless))

⇒ we need only compute $\lambda^\pm, \bar\lambda^\pm$ determinant.
(only one that's v-dependent)

notice also that the overall normalization of S_λ is irrelevant for det calculation ($\frac{2}{g^2} \times \frac{1}{4}$ just gives some additive constant to V_{eff})

So,

$$S_\lambda \sim \frac{1}{L}\sum_{p\in\mathbb{Z}} \int \frac{d^3k}{(2\pi)^3} \Big\{ \bar\lambda^{+}_{\dot\alpha(k,p)} \begin{pmatrix} k^0 + (\frac{2\pi p}{L} - v) & k^1 - ik^2 \\ k^1 + ik^2 & k^0 - (\frac{2\pi p}{L} - v) \end{pmatrix}^{\dot\alpha\alpha} \times \lambda^{-}_{\alpha(k,p)}$$

$$+ \bar\lambda^{-}_{\dot\alpha(k,p)} \begin{pmatrix} k^0 + (\frac{2\pi p}{L} + v) & k^1 - ik^2 \\ k^1 + ik^2 & k^0 - (\frac{2\pi p}{L} + v) \end{pmatrix}^{\dot\alpha\alpha} \lambda^{+}_{\alpha(k,p)}$$

$$+ \lambda^{+}_{\alpha(k,p)} \frac{m}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\alpha\beta} \lambda^{-}_{\alpha(-k-p)}$$

$$+ \lambda^{-}_{\alpha(k,p)} \frac{m}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\alpha\beta} \lambda^{+}_{\alpha(-k,-p)}$$

$$+ \bar\lambda^{+}_{\dot\alpha(k,p)} \frac{m^*}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\dot\alpha\dot\beta} \bar\lambda^{-}_{\dot\beta(-k,-p)}$$

$$+ \bar\lambda^{-}_{\dot\alpha(k,p)} \frac{m^*}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\dot\alpha\dot\beta} \bar\lambda^{+}_{\dot\beta(-k,-p)} \Big\}$$

we're after ⤵.

$$\prod_{\substack{(k,p)\\ \alpha,\dot\alpha}} d\lambda^{+}_{\alpha(k,p)}\, d\lambda^{-}_{\alpha(k,p)}\, d\bar\lambda^{+}_{\dot\alpha(k,p)}\, d\bar\lambda^{-}_{\dot\alpha(k,p)}\; e^{-S_\lambda} \equiv \text{const.}\, e^{-V_3 V_{eff}(v)}$$

(50.5)

So "$-V_{eff}(v) = \ln\det$ (the quadratic operator)"

especially straightforward if $m=0$:

$$e^{-V_{eff}(v)\cdot V_3} \sim \prod_{\substack{\vec{k} \\ \cap \\ \mathbb{Z}^3}} \prod_{\substack{p \\ \cap \\ \mathbb{Z}}} \det\begin{pmatrix} k^0+(\frac{2\pi p}{L}-v) & k^1-ik^2 \\ k^1+ik^2 & k^0-(\frac{2\pi p}{L}-v)\end{pmatrix} \times$$

(imagine discretizing k^μ in V_3)

$$\times \det\begin{pmatrix} k^0+(\frac{2\pi p}{L}+v) & k^1-ik^2 \\ k^1+ik^2 & k^0-(\frac{2\pi p}{L}-v)\end{pmatrix}$$

$$= \prod_{\substack{\vec{k} \\ \cap \\ \mathbb{Z}^3}} \prod_{\substack{p \\ \cap \\ \mathbb{Z}}} \left(k^{0\,2} - \left(\tfrac{2\pi p}{L}-v\right)^2 - k^{1\,2} - k^{2\,2}\right) \times \left(k^{0\,2} - \left(\tfrac{2\pi p}{L}+v\right)^2 - k^{1\,2} - k^{2\,2}\right) =$$

$$= \prod_{\substack{\vec{k} \\ \text{(Euclid.)}}} \prod_{p} \left(\vec{k}^2 + \left(\tfrac{2\pi p}{L}-v\right)^2\right)\left(\vec{k}^2 + \left(\tfrac{2\pi p}{L}+v\right)^2\right) =$$

(two terms give identical contributions after Σ_p)

$$= \text{const} \times e^{-2\underbrace{V_3}\int \frac{d^3k}{(2\pi)^3} \sum_{p\in\mathbb{Z}} \ln\left(k^2 + \left(\frac{2\pi p}{L}-v\right)^2\right)}$$

(standard: V_3 appears after converting discrete $\sum_{\vec{k}}$ to $\int$)

<u>Upshot</u>: $V_{eff}(v) = -2 \int \frac{d^3k}{(2\pi)^3} \sum_{p=-\infty}^{\infty} \ln\left(k^2 + \left(\frac{2\pi p}{L}-v\right)^2\right).$

<u>Claim</u>: $\int \frac{d^3k}{(2\pi)^3} \sum_p \ln\left(k^2 + \left(\frac{2\pi p}{L}-v\right)^2\right) = \xrightarrow{\hspace{3em}}$

(define via ζ fn)

(50.6)

$$= - \lim_{s\to 0} \frac{d}{ds} \sum_p \int \frac{d^3k}{(2\pi)^3} \ln (\dots)^{-s} = \dots$$

$$\dots = (\text{all equivalent expressions})$$

$$= \text{const} + \frac{1}{L^3} \frac{1}{24\pi^2} [vL]^2 \left(2\pi - [vL] \right)^2$$

$$\left([vL] = vL \bmod 2\pi \right)$$

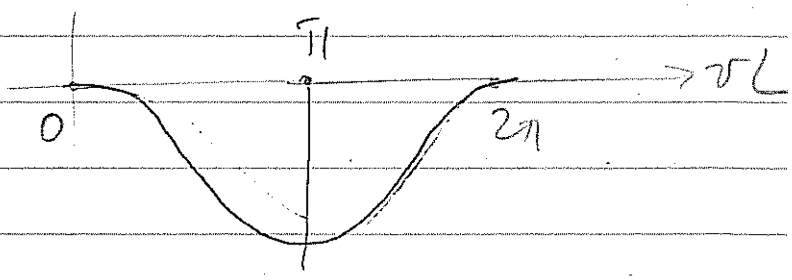
$$= -\frac{1}{L^3} \frac{4\pi^2}{3} \left(\zeta\left(-3, \frac{vL}{2\pi}\right) + \zeta\left(-3, -\frac{vL}{2\pi}\right) - \left(\frac{vL}{2\pi}\right)^3 \right)$$

$$= -\frac{1}{\pi^2} \sum_{m\neq 0} \frac{e^{ivLm}}{m^4} \frac{1}{L^3}$$

$$= -\frac{1}{L^3\pi^2} \left(Li_4\left(e^{ivL}\right) + Li_4\left(e^{-ivL}\right) \right)$$

the top one is most transparent to plot

it says $$V_{eff}\Bigg|_{\left\| \begin{matrix} n_f = 1 \\ \underline{\lambda \text{ contribution}} \\ \underline{m = 0} \end{matrix} \right\|} = -\frac{1}{12\pi^2 L^3} (vL)^2 (2\pi - vL)^2$$



(51) from calculation on pp 50.1 – 50.6
& discussion on p. 49, we have

$$V_{eff}(v) = -\frac{n_f - 1}{12\pi^2 L^3}(vL)^2(2\pi - vL)^2$$

$$\left(\text{for } 0 \le vL \le 2\pi\right)$$

* one loop, ok for $L \ll 1/\Lambda$; massless λ

* vanishes for $n_f = 1$ (SUSY)

* for $n_f > 1$, fermions dominate

& prefer center-symmetric vev

notice: @ one loop, in pure YM theory on $R^3 \times S^1$ — which is to say, thermal theory, as for Euclidean bosonic theory S^1 can always be thought as thermal — we have the opposite situation:

$$V_{eff}^{\text{pure YM}} = \frac{T^3}{12\pi^2}\left(\frac{v}{T}\right)^2\left(2\pi - \frac{v}{T}\right)^2$$

- Gross Pisarski Yaffe 1980's, high T pert. theory
- $v = 0$ or $v = 2\pi$: center symmetry

deconfined phase → broken

W/ massless $\overset{n_f > 1}{\vee}$ adjoint fermions on $R^3 \times S^1_L$, w/ periodic b.c., center symmetry is preserved @ small $L \ll 1/\Lambda$.

Dynamical abelianization occurs @ scale $v \simeq \frac{\pi}{L}$, $SU(2) \to U(1)$, & semiclassical description of vacuum & nonperturbative IR physics should hold. How's the physics of confinement changed wrt Polyakov model?

Answer: in an interesting way.

What about $n_f = 1$, SYM?

↳ By susy, no $V_{eff}(v)$ generated perturbatively - instead there is a classical flat direction parameterized by v $\equiv$ classical Coulomb branch.

Answer: Remarkably in pure $\mathcal{N}=1$ SYM, it is nonperturbative effects that single out the center symmetric vev $v = \frac{\pi}{L}$, rather than the perturbative ones!

53.1

before we continue one more excersyse: include "$m\lambda\lambda$" in our calculation of V_{eff}. there are some lessons there, too!

because of $\vec{k} \leftrightarrow -\vec{k}$ & $p \leftrightarrow -p$ mixing, do this: (I will be more schematic here, omitting some subtleties ...)

$$\mathcal{M} = \left(\begin{array}{cc|cc|cc|cc}
k^0+(\frac{2\pi}{L}p-v) & k^1-ik^2 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 & +m^*/2 \\
k^1+ik^2 & k^0-(\frac{2\pi}{L}p-v) & & & & & -m^*/2 & 0 \\
\hline
\multicolumn{2}{c|}{0} & -k^0-(\frac{2\pi}{L}p+v) & -k^1-ik^2 & 0 & -m/2 & \multicolumn{2}{c}{0} \\
 & & -k^1+ik^2 & -k^0+(\frac{2\pi}{L}p+v) & m/2 & 0 & & \\
\hline
\multicolumn{2}{c|}{0} & 0 & +m^*/2 & -k^0-(\frac{2\pi}{L}p+v) & -k^1+ik^2 & \multicolumn{2}{c}{0} \\
 & & -m^*/2 & 0 & -k^1-ik^2 & -k^0+(\frac{2\pi}{L}p+v) & & \\
\hline
0 & -m/2 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & k^0+(\frac{2\pi}{L}p-v) & k^1+ik^2 \\
+m/2 & 0 & & & & & k^1-ik^2 & k^0-(\frac{2\pi}{L}p-v)
\end{array}\right)$$

basis $\left(\bar{\lambda}^+_{(k,p)}\ \lambda^+_{(k,p)}\ \bar{\lambda}^+_{(-k,-p)}\ \lambda^+_{(-k,-p)}\right) \mathcal{M} \begin{pmatrix} \lambda^-_{(k,p)} \\ \bar{\lambda}^-_{(k,p)} \\ \lambda^-_{(-k,-p)} \\ \bar{\lambda}^-_{(-k,-p)} \end{pmatrix}$

$$\det \mathcal{M} = \left(-k^{0^2}+k^{1^2}+k^{2^2}+m^*m+\left(\frac{2\pi}{L}p-v\right)^2\right)^2\left(-k^{0^2}+k^{1^2}+k^{2^2}+mm^*+\left(\frac{2\pi}{L}p+v\right)^2\right)^2$$

by writing in above bases we assumed $\sum\limits_{p>0}$ & $\sum\limits_{k_1>0,k_2k_3}$, so should take $\sqrt{}$

So instead we have

$$V_{eff}(v) = -2 \int \frac{d^3k}{(2\pi)^3} \sum_{p=-\infty}^{\infty} \ln\left(k^2 + m^2 + \left(\frac{2\pi}{L}p - v\right)^2\right)$$

$$= \ldots$$

$$= \frac{m^2}{\pi^3 R} \sum_{p>0} \frac{K_2(pmL)}{p^2} \cos(pvL)$$

$K_2(x) \sim \frac{2}{x^2} - \frac{1}{2}$ @ small x, so $\frac{1}{2}\left(\mathrm{Li}_4(e^{ivL}) + \mathrm{Li}_4(e^{-ivL})\right)$

This gives

$$V_{eff}(v)\Big|_{m\to 0} = \frac{4}{\pi^2 L^3} \overbrace{\sum_{p=1}^{\infty} \frac{\cos(pvL)}{p^4}}$$

(which is exactly $(-2) \times$ 3rd line on p (50.6))

Take n_f massive QCD(adj.):

$$V_{eff} = n_f \frac{2m^2}{\pi^2 L} \sum_{p>0} \frac{K_2(pmL)}{p^2} \cos(pvL) - \frac{4}{\pi^2 L^3} \sum_{p>0} \frac{\cos(pvL)}{p^4}$$

$$= \frac{2}{\pi^2 L^3} \sum_{p>0} \left(n_f (mLp)^2 K_2(mLp) - 2\right) \frac{\cos(pvL)}{p^4}.$$

~~(for $mL \leq 1$ little difference, $p \sim 1$ dominn)~~

— if you play w/ this $\Longrightarrow$

for $(mL) \lesssim 1$, $n_f \geq 2$ $\|$

$vL = \pi$ is the minimum,

still. So even w/ *massive* adjoint fermions, provided $m \sim \frac{(\text{say } 1)}{L}$

one still has center symmetry & abelianization, $v = \pi/L$

- At energy $\leq v$, still abelian dynamics.
- this set up is now known as "deformed YM" theory $\Longrightarrow$

$\Rightarrow$ adjoint fermions decouple from IR, but ensure center symmetry & calculable IR dynamics is intact.

Summary of the many pages between ㊾ & here:

① $n_f=1$, $m=0$ QCD(adj.)

- v is arbitrary ("flat direction", in pert. theory "Coulomb branch")
- $v=\frac{\pi}{L}$, for SYM is assured nonperturbatively !?

similar

② $n_f>1$, $m=0$ QCD(adj)

- $v=\frac{\pi}{L}$ due to one-loop "GPY" potential $SU(2) \to U(1)$ @ v, abelianization, semiclassical calculability (at $L \ll 1/\Lambda$)
- nonperturbative dynamics?

③ $n_f>1$, $m\neq 0$ QCD(adj)

- if $m \gg 1/L$ (for $n_f=2$, about $\frac{1.3}{L}$)
  - at small L fermions decouple
  - theory is essentially pure YM – w/ broken center & strong coupling
- if $m \lesssim 1/L$ (for $n_f=2$, $mL \sim 1$ is OK) there is center stability & semiclassical calculability
- nonperturbative dynamics?

③ is closest to Polyakov model – yet captures essential features of 4d theory, in calculable regime!

(55)

"deformed" Yang-Mills theory

So, let's start w/ case ③: dYM

Üusal, Yaffe 0803.0344
Shifman, Üusal 0802.1232
Üusal 1201.6426, ...
Anber, EP 1508.00910

As presented in those works (early ones) center stability is due to "double-trace deformation": to relate to p (53.2), bottom, notice that

$$\cos(pvL) = \frac{1}{2}\left(e^{ipvL} + e^{-ipvL}\right)$$

$$= \frac{1}{2}\left(\left(e^{ip\frac{vL}{2}} + e^{-ip\frac{vL}{2}}\right)^2 - 2\right)$$

$$= \underset{(v\text{-indep.})}{\text{const.}} + \frac{1}{2}\left(e^{ip\frac{vL}{2}} + e^{-ip\frac{vL}{2}}\right)^2$$

then recall p. (38)

$$\underset{\substack{(\text{Fundamental})\\ \downarrow \\ (\text{Wilson loop around } S^1_L \text{ in fundamental rep.})}}{\Omega} = \mathrm{diag}\left(e^{i\frac{vL}{2}}, e^{-i\frac{vL}{2}}\right)$$

so $\cos(pvL) = \text{const} + \frac{1}{2}|\mathrm{tr}\,\Omega_F^p|^2$

therefore $V_{\text{eff}}\Big|_{\underline{\text{fermions}}}$ of p (53.2) can be written as

$$V_{\text{eff}}\Big|_{\underline{\text{fermions}}} = \frac{\cancel{n_f}}{\pi^2 L^3}\sum_{p>0} n_f \, \frac{mL^2}{p^2} K_2(mLp)\,|\mathrm{tr}\,\Omega_F^p|^2.$$

((the advantage of this form: valid for SU(N)!))

the "double-trace" deformation is due to massive adjoints.

(56)

▽ <u>nature of small-L limit(s)</u>: Ⓐ "3d limit" $L \to 0$ $g_3^2 = \frac{g_4^2(L)}{L}$ fixed,

Ⓑ $\Lambda L \ll 1$, fixed, $g_4^2(\tfrac{1}{L}) \sim \frac{1}{\log \frac{1}{\Lambda L}}$ $S_M = S_{KK}$, in Ⓐ: $\Lambda \to 0$!

$S_M = \frac{4\pi v}{g_3^2}$ - fix, $S_{KK} = \underbrace{\frac{8\pi^2}{g_4^2}}_{\to\infty} - \frac{4\pi v}{g_3^2}$ v - arbitr fixed

irrelevant!

(the reason for this discussion is to dispel fears that a deformation of the form

$$\Delta S = \frac{1}{L^3} \int_{R^3} d^3x \sum_n a_n |tr_F \Omega^n|^2 \quad \boxed{*}$$

is nonrenormalizable --)

A final remark, albeit tangential to our discussion, is that those refs (Unsal, Yaffe) introduced $\boxed{*}$ motivated by the large-N limit & "volume independence @ large-N" -- we can't discuss this in detail here...

Let's discuss the nonperturbative IR physics of dYM on $R^3 \times S^1_L$ @ $L\Lambda \ll 1$ ▽↑ At distances $\gg \frac{1}{L}$ we have an effective 3d theory of the dual photon σ, just like in the Polyakov model

$$\mathcal{L}_{eff} = \frac{g_4^2}{(4\pi)^2 L} (\partial_\mu \sigma)^2 \quad \text{in pert. theory} \quad p.⑧$$

← p.(20)

+ corrections $\left(\sim \frac{g_4^2}{\underbrace{Lv}_{(=\pi)}}, \frac{energy}{v}\right)$

Non-perturbatively, all is like in Polyakov model, <u>except</u> there are two kinds of lowest action monopole-instantons:

M: $Q_M = +1$ $Q_T = \frac{1}{2}$ $S_0 = \frac{4\pi^2}{g_4^2}$

KK $Q_M = -1$ $Q_T = \frac{1}{2}$ $S_0 = \frac{4\pi^2}{g_4^2}$ ← (actions @ center symmetric point) $\boxed{**}$

+ their "antiparticles"

$\boxed{**}$ since $m_{A_4} \neq 0$ in dYM, there are corrections to S_0 (but small!) while Q_T same.

So our "Coulomb gas" partition function will have two different kinds of particles. For the long-distance physics, the only distinction between KK & M^* is Q_T, the topological charge.

Now in YM on $R^3\times S^1$ we can have a θ-angle, so let's add in dYM, too. then the Boltzmann factors of M, KK, M^*, KK^* are

here θ includes phase of adjoint mass ...

$M \quad e^{-S_0}\times e^{i\frac{\theta}{2}} \quad (Q_m=+1)$

$KK \quad e^{-S_0}\times e^{i\theta/2} \quad (Q_m=-1)$

$M^* \quad e^{-S_0}\times e^{-i\theta/2} \quad (Q_m=-1)$

$KK^* \quad e^{-S_0}\times e^{-i\theta/2} \quad (Q_m=+1)$, each $\times\left(\int d^3r\, v^3\right)$

+ interactions

(Recall for an instanton on R^4: $e^{i\theta}$)

Next, we pick up the derivation of the Polyakov model from p. (29), 2nd equal sign, including these new "particles":

$$Z=\int \mathcal{D}\sigma\, e^{-\frac{g_4^2}{(4\pi)^2 L}\int d^3x(\partial\sigma)^2}\times\sum_{n=0}^{\infty}\frac{\left(d^3r\, v^3 e^{-S_0}e^{i\frac{\theta}{2}}e^{i\sigma(r)}\right)^n}{n!}$$

$$\times\sum_{m=0}^{\infty}\frac{\left(d^3r'\, v^3 e^{-S_0}e^{-\frac{i\theta}{2}}e^{-i\sigma(r')}\right)^m}{m!}\times$$

$$\times\sum_{l=0}^{\infty}\frac{\left(d^3r''\, v^3 e^{-S_0}e^{+i\frac{\theta}{2}}e^{-i\sigma(r'')}\right)^l}{l!}\times\sum_{k=0}^{\infty}\frac{\left(d^3r'''\, v^3 e^{-S_0}e^{-\frac{i\theta}{2}}e^{i\sigma(r''')}\right)^k}{k!}$$

$\uparrow$ (M)

$\leftarrow$ (M^*)

$\uparrow$ (KK)

$\nearrow$ (KK^*)

(58)

$$= \int \mathcal{D}\sigma\, e^{-\frac{g_4^2}{(4\pi)^2 L}\int d^3x (\partial\sigma)^2 + 2\int d^3x\, v^3 e^{-S_0}\left(\cos(\sigma+\frac{\theta}{2}) + \cos(\sigma-\frac{\theta}{2})\right)}$$

$$= \int \mathcal{D}\sigma\, e^{-\frac{g_4^2}{(4\pi)^2 L}\int d^3x (\partial\sigma)^2 + \int d^3x\, v^3 e^{-S_0} 2\left(\overset{M}{e^{i\sigma} e^{i\frac{\theta}{2}}} + \overset{M^*}{e^{-i\sigma} e^{-i\frac{\theta}{2}}} + e^{i\sigma} e^{-i\frac{\theta}{2}} \xleftarrow{KK^*} + \overset{KK}{e^{-i\sigma} e^{i\frac{\theta}{2}}}\right)}$$

M

KK*

't Hooft vertex

$\boxed{\text{IR lagrangian } (\mu \gg \tfrac{1}{L}) = \text{kinetic term} + \text{'t Hooft vertices!}}$

regroup:

$$e^{i\theta/2}\cos\sigma + e^{-i\theta/2}\cos\sigma$$

$$\sim \cos\frac{\theta}{2}\cos\sigma$$

$$= \int \mathcal{D}\sigma\, e^{-\int d^3x \frac{g_4^2}{(4\pi)^2 L}\left[(\partial\sigma)^2 + v^3 (\#) e^{-S_0} \cos\frac{\theta}{2}\cos\sigma\right]}$$

$(\#) \to \underline{\mathcal{O}}\left(\frac{vL}{g_4^2}\right)$

$\cos\frac{\theta}{2}\cos\sigma$ ← "topological interference"

* when $\theta \neq 0$ M & KK contributions have phases that can interfere (qualitative difference from Polyakov model)

* notice special case $\theta = \pi$: it appears ~~that~~ mass term vanishes, then -- this is, however, an illusion, most likely; evidence:

① For SU(2), in next order of semiclassical expansion ($\underline{\mathcal{O}}(e^{-2S_0})$), there are contributions of charge 2 monopole instantons – those go like $e^{\pm 2i\sigma}$ (e.g. KK & M^*, or M & KK* close together...)!

$$\sim e^{-2S_0}\cos 2\sigma$$

59

this term can be written on symmetry grounds — its coeff!, though, has not been calculated... Notice an interesting phenomenon, though: at $\theta = \pi$ we have

$$V(\sigma) = \left(0 \times \cos\sigma + \cos 2\sigma \, e^{-S_0}\right)$$

this has unique min

2 minima

these two minima correspond to spontaneous breaking of CP. YM w/ θ-term is CP invariant for $\theta = 0, \pi$ & CP is spontaneously broken @ $\theta = \pi$: these are long standing conjectured properties of YM which can be seen here in an explicitly calculable setting.

[for SU(N), indeed, where leading term (e^{-S}) does not vanish @ $\theta = \pi$]

② 2nd piece of evidence is that, for SU(N), the leading M, KK... order does not vanish, but the appearance of two vacua @ $\theta = \pi$ persists, as expected.

→ within QFT; [see e.g. work of Zhitnitsky 1109.2608] + later

another one, via AdS/CFT see Witten 9707109

Let's dwell on how CP is realized in the long-distance theory... First, in YM

$$e^{i\theta \underbrace{\int F\tilde{F}}} \xrightarrow{CP} e^{-i\theta\int F\tilde{F}} \qquad CP: \left(\begin{array}{l}\vec{E}\to\vec{E}\\ \vec{B}\to-\vec{B}\end{array}\right)$$

(normalized s.t. = 1 for BPST)

since θ-dependence is 2π-periodic, only $\theta=0$ & $\theta=\pi$ (mapped to $\theta=-\pi$ by CP) are CP invariant ($-\pi\sim\pi$). In long distance theory, CP acts as $\sigma \overset{CP}{\to} \sigma+\pi$. (+ time reversal). It maps

"fluxon" $1 \overset{CP}{\to} -1$: recall p ㉒!

$$e^{i\sigma} \xrightarrow{CP} e^{-i\sigma}$$

$$e^{i\frac{\theta}{2}}e^{i\sigma} \xrightarrow{CP} -e^{i\frac{\theta}{2}}e^{-i\sigma}$$

etc...

for $\theta=\pi$, CP maps $M \Leftrightarrow KK^*$
$M^* \Leftrightarrow KK$

- $\cos\frac{\theta}{2}\cos\sigma$ - term is NOT invt., unless $\theta=\pi$, when it vanishes.
- $\cos 2\sigma$ - term is INVT.

So the moral, so far, from dYM is:

1 — vs. Polyakov model: KK & M, phases(θ), "topological interference", qualitatively different

2 — string tension has θ-dependence — decreases as θ increases away from $\emptyset$; this is a qualitative prediction borne out by lattice studies (learned how to include small θ, still far from $\theta=\pi$) (dell Debbio et al 0603041 th)

3 — T_c for deconfinement also depends on θ... (decreases)

[skip?]
(?)

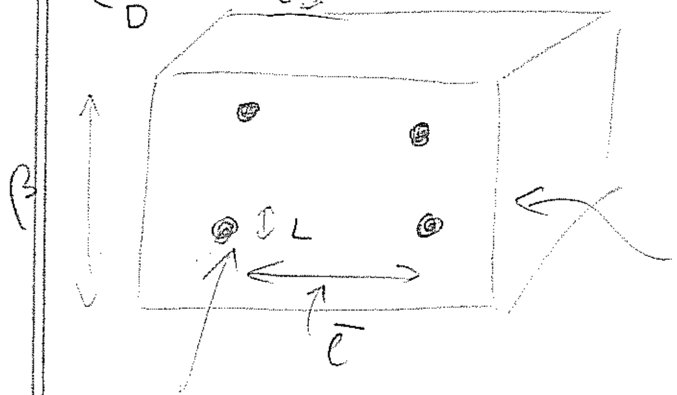
A qualitative picture of deconfinement transition in SU(2) dYM is as follows

$$R^{1,2} \times S^1_L \longrightarrow S^1_\beta \times R^2 \times S^1_L$$

$\uparrow$ size of thermal circle

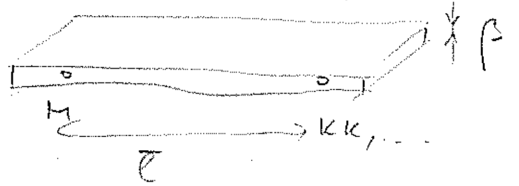
consider the following regime

$$\frac{1}{\ell_D} >> \frac{1}{\bar{\ell}} \gg \beta \gg L \qquad \left(T \ll \frac{1}{L}\right)$$



R^3 = Euclidean $R^{1,2}$
(S^1_L not shown)

- monopole-instanton gas
- since $\beta >> L$, M, M^*, KK, KK^* all 'fit' in the 'time' direction
- if $\bar{\ell} >> \beta$ picture looks more like this

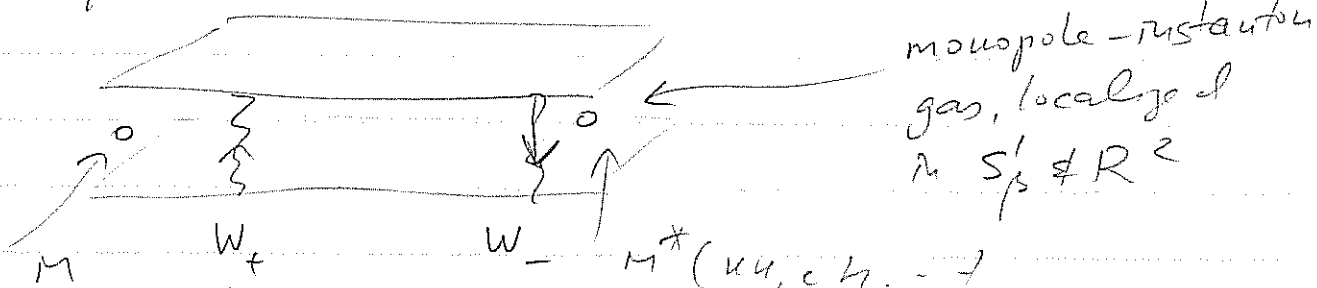


$\Rightarrow$ gas looks ≈ 2D, then

So if just M, KK + hc kept our Z(thermal) would be a gas of 2d (magnetic) charges.
(known to exhibit a Kosterlitz-Thouless transition)

but, while this is correct, it's not all:
@ finite β ($1/T$) W bosons can be excited thermally; recall $m_W = \frac{1}{L}$
& since $T \ll \frac{1}{L}$ ($\beta \gg L$) these will be highly nonrelativistic: they'll form a nonrelativistic gas of electrically charged particles w/ fugacity $\sim e^{-\frac{m_W}{T}}$.

So, qualitatively we have



static W bosons

$$\left(\text{fugacity of magnetic charges} \sim e^{-\frac{4\pi^2}{g_4^2}}\right)$$

$$\left(\text{fugacity of electric charges} \sim e^{-\frac{1}{LT}}\right) \quad \left(\frac{m_W}{T}\right)$$

magnetic charges interact via 2d Coulomb law w/ $\frac{LT}{g_4^2} \log r$, while electric ones — w/ $\frac{g_4^2}{LT} \log r$

A transition occurs when the two fugacities are equal:

$$e^{-\frac{4\pi^2}{g^2}} \simeq e^{-\frac{1}{LT}}$$

i.e.

$$T_c \approx \frac{g^2}{4\pi^2 L}$$

$T < T_c$	$T > T_c$
- magnetic charges dominate	- electric charges dominate
- electric charge confined	- magnetic charge confined
- magnetic charge screened	- electric charge screened

(the physics near T_c is described by this 2d electric-magnetic Coulomb gas & has interesting properties ... @ $T=T_c$ can be solved exactly, etc.)

$\uparrow$ [end skip ?] $\uparrow$

remarks from p.60, contd:

4 – I will mention the nature & property of confining strings in dYM later (together w/ QCD(adj)), including comparison w/ strings in Seiberg-Witten theory (the other calculable theory with abelian confinement). We'll see that there are important interesting differences.

5 – results suggest (!?) continuous connection to $\mathbb{R}^4$:

$$\underbrace{m \sim \frac{1}{L} \gg \Lambda}_{}\ \overset{\boxed{\mathbb{R}^3 \times S^1_L \text{ calculable}}}{} \xleftrightarrow[\text{no phase transition}]{?} \overset{\boxed{\approx \mathbb{R}^4}}{} m \gg \Lambda \gg \frac{1}{L} \to 0$$

$\boxed{\text{lattice!}}$? future!

QCD(adj)/SYM (w/ $m=0$ n_f Weyl fermions)

As we already discussed, for $n_f > 0$ $m=0$ Weyl fermions, we have center stability due to "GPY", p. (51). So, abelianization will occur @ $v = \frac{1}{2}$. The perturbative physics is modified by the addition of $\lambda^3_{\alpha_{I}}$ $\left(\overset{\displaystyle I}{\underset{1 \ldots n_f}{\shortparallel}}\right)$ (the Cartan component of the adjoint fermions) to A^3_μ (or σ). But $\lambda^3_{\alpha_I}$ are uncharged & so do not affect the freezing of the coupling & the weak-coupling nature of the IR theory. So, in perturbation theory, we have

$$\mathcal{L}_{ff} = \frac{g_4^2}{(4\pi)^2 L} (\partial_\mu \sigma)^2 + \sum_{I=1}^{n_f} \bar{\lambda}_{\dot{\alpha}_I} \bar{\sigma}^{\mu \dot{\alpha}\alpha} \partial_\mu \lambda_{\alpha_I} + \ldots$$

where I normalized the λ's in a non-susy way. Before we continue, let us mention that the global symmetries of the $m=0$ n_f theory are enlarged by adding flavor symmetry, classically: $U(n_f)$ are acting on "I"-index ($\lambda_{\alpha_I} \to U^I{}_J \lambda_{\alpha_I}$). Quantum mechanically, the $U(1)$ part of $U(n_f)$ is anomalous. Easiest way to see this is from the BPST-instanton-induced 't Hooft vertex. For $SU(2)$ w/ $n_f = 1$ (SYM) it has the form $e^{-S_{BPST}} tr(\lambda^\alpha \lambda_\alpha \lambda^\beta \lambda_\beta) e^{i\theta}$. Under $U(1)$ $\lambda \to e^{i\alpha}\lambda$ this is multiplied by $e^{4i\alpha}$ → so it's broken $U(1) \to \mathbb{Z}_4$ ($\alpha = \frac{2\pi}{4}$ only).

I wrote the 't Hooft v-x for SU(2) 1-flavor w/ a θ-angle (even though in massless theory it can be rotated away) just to emphasize that the anomalous symmetry can be compensated by a shift of θ (here: by -4α); in fact an identical way to see this is to compute the U(1)-anomaly like Fujikawa, deduce that the change of the measure is equivalent to $\theta \to \theta + 4\alpha$, & then observe that for $\alpha = \frac{2\pi}{4}$, i.e. $\mathbb{Z}_4 \subset U(1)$, the change of θ is 2π, i.e. no change.

Moral: for $n_f = 1$, a $\mathbb{Z}_4$ subgroup of the $U(1)$ flavor symmetry is anomaly free.

Ditto, for $n_f > 1$, we have that a $\mathbb{Z}_{4n_f}$ subgroup of $U(1)$ is anomaly free.

A $\mathbb{Z}_2$ subgroup of that is just F-number $(-1)^F$; $\lambda_\alpha \to -\lambda_\alpha$, so the $\mathbb{Z}_{2n_f} = \mathbb{Z}_{N_c n_f}$ is the anomaly free one. Now for $n_f > 1$ we have $\mathbb{Z}_{N_c n_f} \times SU(n_f)$ — but $SU(n_f)$ has a $\mathbb{Z}_{n_f}$ center $(\sqrt[n_f]{1} \times \mathbb{1})$, so really it's only $\mathbb{Z}_{N_c}$ transforms that constitute the true discrete anomaly-free symmetry.

[More precisely: a $\mathbb{Z}_{N_c n_f}$ transform w/

$$e^{i\frac{2\pi}{N_c n_f}k} \quad \text{w/ } \overset{\in\mathbb{Z}}{k} = N_c \overset{\in\mathbb{Z}}{l}$$

is equivalent to one in the center of $SU(n_f)$, so only the ones w/ $k = 1 \ldots N_c$ — generating $\mathbb{Z}_{N_c}$ are independent ones; this includes unit element, of course]

(This is important to us, because, as we will see, the nonabelian $SU(n_f)$ part remains unbroken - no continuous (nonabelian) chiral symmetry broken - but the $\mathbb{Z}_{N_c}$ spontaneously breaks, leading to N_c vacua.)

The fact that the 't Hooft v-x has the (schematic) form indicated:

for BPST: $e^{-S_{BPST}}\lambda^4$ for $SU(2)$ $n_f=1$

also follows from index theorem, saying that in an I-BPST background a violation of $U(1)$ charge by 4 units should occur —

[recall B+L in electroweak:

$$e^{-S(BPST)}\underbrace{(qqq\ell)^3}_{\Delta B = \Delta L = 3}$$
]

— therefore BPST has 4 fermion zero modes (for $n_f=1$). For $n_f>1$: $4n_f$ zero modes ($SU(2)$), or $2N_c n_f$ for $SU(N_c)$. So the 't Hooft vertex for BPST should be

$$\sim e^{-S_{BPST}}\lambda^{4n_f} \quad \left(\text{e.g.} \sim \left(\det_{I,J}\left(\lambda_I^{a\alpha}\lambda^a_{\alpha J}\right)\right)^2\right)$$

note precise contraction of indices won't matter to us.

Now what happens w/ our M & KK, the I's

(67)

constituents? A quick & dirty argument is to argue that (# fermion zero modes) ~ ~ (topological charge). For I's $= 4 n_f \underset{1}{\underset{\parallel}{Q_T}}$

Hence, since our solutions have $Q_T = 1/2$ we expect that for M's & KK's, # 0 modes $= 4 n_f \overset{(1/2)}{\overset{\parallel}{Q_T}}$ $= 2 n_f$.

This argument is too quick, but it is true for adjoint reps fermions (index theorem for M & KK, see Unsal, EP, 0812...).

So we expect that 't Hooft vertex for M & KK will be like(*)

$$e^{-S_0} e^{i\sigma} (\lambda\lambda)^{n_f} \quad M \quad \left((\lambda\lambda)^{n_f} \sim \sim \det_{IJ} \lambda_I^{(3)\alpha} \lambda_{\alpha J}^{(3)} \right)$$

$$e^{-S_0} e^{-i\sigma} (\lambda\lambda)^{n_f} \quad KK$$

$$e^{-S_0} e^{-i\sigma} (\bar\lambda\bar\lambda)^{n_f} \quad M^*$$

$$e^{-S_0} e^{+i\sigma} (\bar\lambda\bar\lambda)^{n_f} \quad KK^*$$

(*) see §5.1 in 1307.1317 or 1406.1199 for full glory derivation & proper notation in SYM framework — here, sloppy, OK, ...

The fact that it is $\mathbb{Z}_{4n_f}$ which is anomaly free means that σ has to transform under $\mathbb{Z}_{4n_f}$:

$$(\lambda\lambda)^{n_f} \xrightarrow[\lambda \to e^{i\frac{2\pi}{4n_f}}\lambda]{} e^{i\pi} (\lambda\lambda)^{n_f}, \text{ so } \underline{\sigma \to \sigma + \pi}$$

under $\mathbb{Z}_{4n_f}$. (σ transforms like $\mathbb{Z}_2$)

SIDE REMARK

For the mathematically minded, a few refs:

- * gauge bundles on T^4 discussed by
  - 't Hooft CMP 81 (1981) 267-275
  - van Baal CMP 85 (1982) 529-547
- * canonical quantization on T^3, eg in
  - Witten, 0006010 (hep-th), van Baal's PhD thesis (online)

one subtlety that arises is that the "topological charge" can be noninteger in the presence of magnetic fluxes... (see above papers)

But we saw an example of this!

(**)

"M": $Q_T = \frac{vL}{2\pi}$ ⟵ magnetic charge contribution

"KK": $Q_T = 1 - \frac{vL}{2\pi}$ ⟵

↑ $\pi_3(G)$ winding # contribution

see p(44), middle

See Appx. of Gross, Pisarski, Yaffe RMP 53 (1981) 43 on classification of finite action field configurations on $R^3 \times S^1$, essentially boiling down to our eqns. (*).

Notice that despite this nonintegrality of Q_T, the θ-dependence of the physics is still 2π periodic. (e.g. explicit result for dYM w/ $\theta \neq 0$).

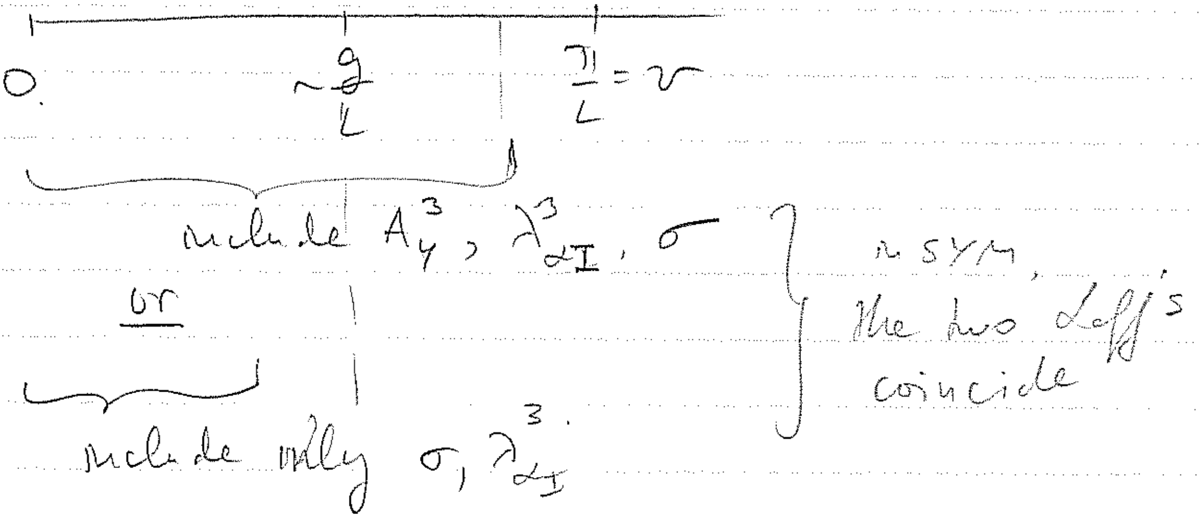
END

68.

Now that we understand the symmetries of QCD(adj), we can use them to see how the long-distance $\mathcal{L}_{eff}$ on $R^3 \times S^1$ is constrained.
We have the fields:

- $\sigma, \lambda^{(3)}_{\alpha \underset{1 - - n_f}{\underset{\parallel}{I}}}$ - perturbatively massless
- $A_4^{(3)}$ - massive, $m_{A_4} \sim \frac{g}{L}$ for $n_f > 1$ (massless for $n_f = 1$)
  (from GPY) ⟶ no GPY
- $W^\pm, \lambda^\pm_I$ - massive, $v = \frac{\pi}{L}$

since g is small, even for QCD(adj) we can write two kinds of $\mathcal{L}_{eff}$



include $A_4^3, \lambda^3_{\alpha I}, \sigma$

or

include only $\sigma, \lambda^3_{\alpha I}$

~ SYM, the two $\mathcal{L}_{eff}$'s coincide

Let's concentrate on the deep IR & just include σ & λ_I (omit "3").

Perturbatively, we have the $\mathcal{L}_{eff}$ from p. (64):

$$\frac{g_4^2}{(4\pi)^2 L}(\partial_\mu \sigma)^2 + \sum_{I=1}^{n_f} \bar{\lambda}_{\dot{\alpha} I}\, \bar{\sigma}^{\mu \dot{\alpha} \alpha} \lambda_{\alpha I} + \cdots$$

We have $\mathbb{Z}_{4n_f} \times SU(n_f)$ (modulo remark on Bottom of (65))

nonpert. $\mathcal{L}_{eff}$ should respect these symmetries.

$$\left.\begin{aligned} \lambda_I &\to e^{i\frac{\pi}{2n_f}} \lambda_I \\ &\underbrace{\sigma \to \sigma + \pi} \end{aligned}\right\} \text{ under } \mathbb{Z}_{4n_f}$$

$\downarrow$

= this symmetry [intertwining of σ-shift symmetry w/ discrete chiral symmetry] forbids a purely bosonic $\sim \cos\sigma$ term in the potential (which would come from M's, ... in dYM).

= but it allows a $\cos 2\sigma$ potential. } What would be its cause?

= terms like the 't Hooft operators written on p. (67) are, of course allowed: they are the ones induced by M & KK, like the $(qqql)^3$ term in the SM!

So what would give rise to $\cos 2\sigma$?
(& $\cos 4\sigma$, $\cos 6\sigma$ etc. ?)

Note: The reason we care so much about the purely bosonic term is that it is the one that can generate a mass gap for the gauge fluctuations (σ, the dual photon) & it is the one that determines the symmetry properties of the ground state — notice that since we are at weak coupling, multifermion operators like

$$(\lambda\lambda)^{n_f} \times e^{-S_0}$$

can not, via the NJL-mechanism lead to spontaneous $SU(n_f)$ chiral symmetry breaking — so we don't expect nonabelian chiral symmetry breaking @ small L.

We first note that $\cos 2\sigma \sim e^{2i\sigma} + e^{-2i\sigma}$ so whatever generates it has magnetic charge 2. Our basic (lowest action) instantons (p. (67)) are (their 't Hooft op's)

$$M \quad e^{-S_0} e^{i\sigma} (\lambda\lambda)^{n_f} \qquad M^* \quad e^{-S_0} e^{-i\sigma} (\bar\lambda\bar\lambda)^{n_f}$$

$$KK \quad e^{-S_0} e^{-i\sigma} (\lambda\lambda)^{n_f} \qquad KK^* \quad e^{-S_0} e^{i\sigma} (\bar\lambda\bar\lambda)^{n_f}$$

So if we're to form a "composite" object, there's only one way to obtain it from these

$$e^{-S_0}\, e^{i\sigma} (\lambda\lambda)^{n_f} \text{ "+" } e^{-S_0}\, e^{+i\sigma} (\bar\lambda\bar\lambda)^{n_f}$$

$$(M) \qquad\qquad\qquad (KK^*)$$

<u>or</u>

$$e^{-S_0}\, e^{-i\sigma} (\bar\lambda\bar\lambda)^{n_f} \text{ "+" } e^{-S_0}\, e^{-i\sigma} (\lambda\lambda)^{n_f}$$

$$(M^*) \qquad\qquad\qquad (KK)$$

(2nd line is "antiparticle" of 1st line)

△ <u>In</u> terms of the equivalent stat mech, (of grand canonical gas), we'd have "bound states" of M & KK* (& anti-)

Now M & KK* have repulsive magnetic interactions – how can they form "bound states" (more precisely: configurations contributing to Euclidean path $\int$ (that are more likely than others), containing a close-by M & KK*)?

<u>The answer:</u> fermion zero modes generate attraction ⟶

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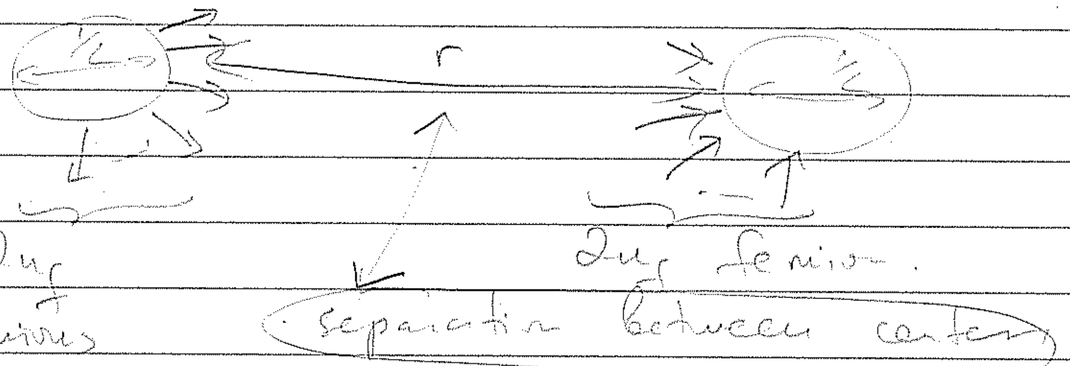
$M: \ e^{i\sigma}(\lambda\lambda)^{n_f}$ $\qquad$ $KK^{*} \ e^{+i\sigma}(\bar\lambda\bar\lambda)^{n_f}$



- recall we think of magnetic repulsion as

$$\overbrace{e^{i\sigma(0)} \ e^{i\sigma(r)}} = e^{-4\pi\frac{L}{g^2}\frac{1}{r}}$$

our "master formula" of p ㉖!

- fermion 0-mode exchange can similarly be obtained by contracting fermion legs of 't Hooft v-x:

$$\overbrace{(\lambda\lambda)^{n_f}(0) \ (\bar\lambda\bar\lambda)^{n_f}(r)} \quad \text{contraction w/ massless } \lambda \text{ propagator}$$

$$\sim \left(\frac{L^2}{r^2}\right)^{2n_f} = e^{-4n_f \log\frac{r}{L}} \qquad \langle \lambda \sigma \bar\lambda(r) \rangle \sim \partial_r \frac{1}{r} \sim \frac{1}{r^2}$$

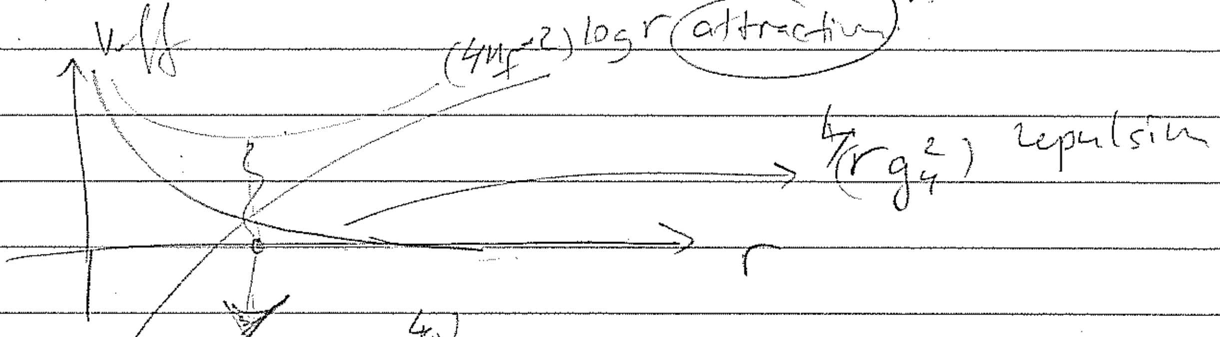
here, we don't care about prefactor (see Anber EP 1105.0940, Argyres Unsal 1206.1890)

So, we have
$$\int_{\text{"}0\text{"}}^{\infty} dr \, e^{-V_{eff}(r)} = \int_{\text{"}0\text{"}}^{\infty} e^{-\left[+\frac{4\pi L}{g^2}\frac{1}{r} + (4n_f - 2)\log\frac{r}{L}\right]} dr$$

(from measure $dr\,r^2$)

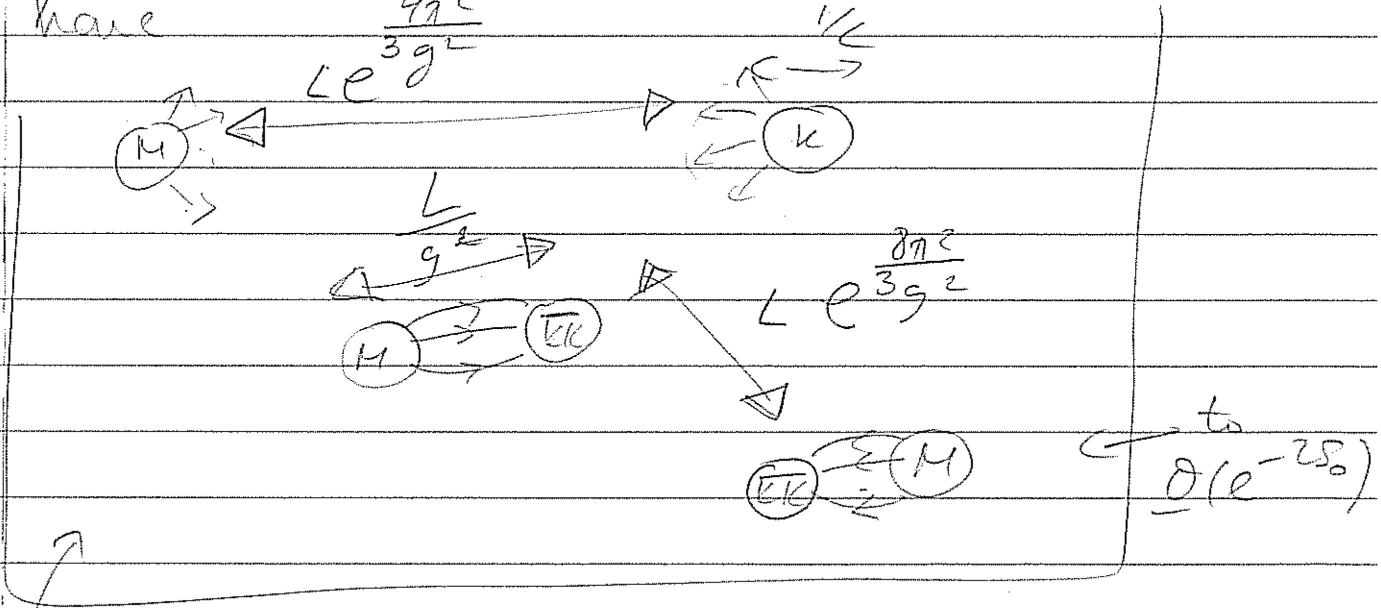
(integral over the "quasizeromode" M/KK* separation)

Clearly this has a minimum (V_{eff})



$$r_* \simeq \underbrace{\frac{4\pi L}{g^2(4n_f-2)}}_{\text{size of "bound state"}}$$

So, if we're to draw a "picture" of a typical configuration contributing to the vacuum-to-vacuum (Z) amplitude, we'd have



where we Σ w/ fugacities $\sim v e^{-S_0}$ for M, KK...
$\sim v^3 e^{-2S_0}$ for (M-KK) & $\bar{M}$KK)

Again, notice the hierarchy of scales, owing to L & $g^2(1/L) \ll 1$
↳ allows calculability!

So, our $\mathcal{L}_{eff}$ to $\underline{O}(e^{-2S_0})$, reads, w/ exponential-only accuracy:

$$\int d^3x \left[\frac{g_4^2}{(4\pi)^2 L} (\partial\sigma)^2 + \bar{\lambda}^I \not{\partial} \lambda_I + \underset{\uparrow}{L^{\#}} \cdot e^{-S_0} (\cos\sigma)\left((\lambda\lambda)^{n_f} + h.c.\right) \right.$$

(appropiate power to get dim.)

$$\left. + \underbrace{e^{-2S_0} \frac{1}{L^3} \cos 2\sigma}_{\uparrow} + \underline{O}(e^{-4S_0}) \right].$$

this is the effect of the "correlated tunneling events" of M-KK* (+ h.c.) called

"magnetic bions" by Unsal 0709.3269

[charge-2 "composite" objects]

$$V_{bion} = \frac{1}{L^3} \cos 2\sigma \; e^{-2S_0}$$

↓ there are 2 vacua: $\sigma = 0$ & $\sigma = \pi$.

(recall $\sigma \simeq \sigma + 2\pi$) related by the broken discrete chiral symmetry →

$\Rightarrow \mathbb{Z}_{4n_f} \to \mathbb{Z}_{2n_f}$

(for $SU(N_c)$: N_c vacua: $\mathbb{Z}_{2N_cN_f} \to \mathbb{Z}_{2N_f}$)

For $n_f > 1$ QCD(adj) we have a σ-mass gap!

$$m_\sigma^2 \sim \frac{1}{L^2} e^{-\frac{8\pi^2}{g_4^2}}, \quad m_\sigma \sim \frac{1}{L} e^{-\frac{4\pi^2}{g_4^2}}$$

but the λ_σ's remain massless they couple to σ only thru the higher-dimensional

$$e^{-S_0} \cos\sigma \, (\lambda\lambda)^4 + \quad \text{- multi-fermion term.}$$

see SYM part →

The confining strings in this theory have interesting properties - giving a field theory realization of properties 1st seen in M-theory QCD (Witten & (Soo-Jong Rey), 1997)), But I will discuss them after I go (briefly) through the main properties of SYM w/ $m=0$ (i.e. w/ unbroken SUSY).

As in the discussion of QCD(adj), I will stress symmetry aspects, rather than detailed calculations [which exist in SYM! (Davies, Hollowood, Khoze 0006011; Anber EP Teeple 1406.1199)

From the point of view of energy scales, the main change SYM gives is that A_4 has to be included in L_{eff} - since it is exactly massless!

Instead of A_4, let's use $\phi \equiv A_4 L - \pi$ - dimensionless. This will turn out to be the bosonic superpartner of σ ($\phi \equiv$ deviation of A_4 from center symm. value).

The main feature of SYM which distinguishes it from dYM & QCD adj. is that at the perturbative level ϕ $(\equiv A_4 L - \pi)$ is also massless (this is because of SUSY, no potential for ϕ).

Since ϕ is not gapped, monopole-instantons will also have a long-range interaction due to massless ϕ quanta exchange. The slickest way to obtain the form & strength of this interaction is to look at the 't Hooft vertices ....

we had $M \sim e^{i\sigma} e^{-\frac{4\pi v L}{g_4^2}} \lambda\cdot\lambda$ (SU(2))

but now $v (\equiv A_4^3)$ is not fixed but a flat direction (massless field) So let's write

$$vL = A_4 L = \phi + \pi$$

so the 't Hooft v-x is

$$M \sim e^{i\sigma} e^{-\frac{4\pi^2}{g_4^2}} e^{-\frac{4\pi}{g_4^2}\phi} \lambda\lambda$$

While for KK: $e^{-i\sigma} e^{-\left(\frac{8\pi^2}{g_4^2} - \frac{4\pi v L}{g_4^2}\right)} \lambda\lambda =$

$$KK \sim e^{-i\sigma} e^{-\frac{4\pi^2}{g_4^2}} e^{+\frac{4\pi}{g_4^2}\phi} \lambda\lambda$$

(w/our def. of ϕ, at $\phi = 0$ ($\equiv$ center symm. point), actions

(77)

of M & KK are identical ; as ϕ moves away from ϕ, action of M increases and of KK decreases, keeping sum constant ($\equiv$ BPST action)).

So, w/ $S_0 \equiv \frac{4\pi^2}{g_4^2}$, we have

$$M \quad e^{-S_0} \; e^{i\sigma} \; e^{-\frac{4\pi}{g_4^2}\phi} \quad \lambda\lambda$$

$$\overset{*}{M} \quad e^{-S_0} \; e^{-i\sigma} \; e^{-\frac{4\pi}{g_4^2}\phi} \quad \bar{\lambda}\bar{\lambda}$$

$$KK \quad e^{-S_0} \; e^{-i\sigma} \; e^{+\frac{4\pi}{g_4^2}\phi} \quad \lambda\lambda$$

$$\overset{*}{KK} \quad e^{-S_0} \; e^{i\sigma} \; e^{+\frac{4\pi}{g_4^2}\phi} \quad \bar{\lambda}\bar{\lambda}$$

We already used $\langle e^{i\sigma(0)} e^{\pm i\sigma(r)} \rangle = e^{\mp \frac{4\pi L}{g_4^2}\frac{1}{r}}$

which followed from $e^{-S_\sigma^{KM}} = e^{-\frac{1}{2}\frac{g_4^2}{(4\pi)^2 L}\int d^3x (\partial\sigma)^2}$

We need $\langle e^{-\frac{4\pi}{g_4^2}\phi(0)} e^{\mp \frac{4\pi}{g_4^2}\phi(r)} \rangle \underset{\boxed{**}}{=} e^{\pm \frac{4\pi L}{g_4^2}\frac{1}{r}}$

which follows from $e^{-S_\phi^{KM}} = e^{-\frac{L}{2 g_4^2}\int d^3x (\partial_\mu A_4)^2} =$

(p.(36), fixed typo →)
Tr$(D_\mu A_4)^2$ coeff

$$= e^{-\frac{1}{2 g_4^2 L}\int d^3x (\partial_\mu \phi)^2}$$

$\boxed{**}$ is obtained by rescaling ϕ by $\frac{g_4^2}{4\pi}$ & noticing that correlator is same as for σ except of "no i's".

If one declares that $e^{\pm\phi}$ corresponds to a unit "scalar charge", +1 or −1, then above result means that like charges attract:

$$\langle e^{-\phi(0)} e^{-\phi(r)} \rangle \sim e^{+\frac{\#}{r}}$$

$\uparrow$ highest probability @ $r \to 0$

& opposite charges repel.. In fact this is a well known property of BPS monopoles.

e.g. for two M monopoles, the σ (magnetic) repulsion is cancelled by the ϕ (scalar, "dilaton") attraction, ditto for two KK monopoles; similarly the M & KK exhibit no force due to σ & ϕ exchange (σ is attractive, ϕ is repulsive). The above 'calculation' using monopole-instanton's 't Hooft vertices is a nice shortcut. One can get same result, of course, by computing classical action of two well-separated BPS solutions – see 4.10 & 4.18–4.21 of 1105.0940.

So now we are on to some of the most intriguing features of SYM compared w/ the theories we studied so far. We have the four elementary 't Hooft vertices on p. (72); they all have fermion zero modes and so do not contribute to the generation of a scalar potential for the bosonic fields ϕ & σ — which will determine the vacuum of the IR theory. We ask, again, what "bound" states can be formed by these objects?

Clearly, only SD + ASD ones can be combined to form only bosonic (there is no $\overline{\lambda\lambda}$ or $\overline{\bar\lambda\bar\lambda}$ contraction — due to unbroken chiral symmetry). As in $n_f > 1$, we can have

$M + KK^*$ — the only difference is that the "Coulomb" repulsion will be twice as strong as on p. (72)/(73), due to the ϕ-contribution (& one has to put $n_f = 1$ in the fermion part). So "magnetic bions" contribute — what terms?

— clearly, $e^{\pm 2i\sigma}$: an M & KK^* will have no ϕ-charge (as one is positive, the other — negative), but two units of magnetic charge

So these "magnetic bions" will contribute

$$\sim v^3 e^{-2S_0} e^{\pm 2i\sigma(r)} d^3 r \, ,$$

& their contributions will have to be summed up. Now, an interesting puzzle arises. If magnetic bions are all there is, we would expect, similar to QCD adj., to have a potential of the form

$$\int d^3r\, V_{eff}(\sigma) = - v^3 e^{-S_0 2}\int d^3r \left(e^{2i\sigma} + e^{-2i\sigma}\right)$$

$\uparrow$ now I am more careful about signs than I was before (I didn't have to!)

the 'Coulomb' gas of magnetic bions has a contribution to Z

$$\sim \sum_n \frac{\left(e^{-2S_0}\int d^3r\, v^3 e^{2i\sigma(r)}\right)^n}{n!} \times \left(\text{anti-bions}\right)$$

$$= e^{-\underbrace{\int d^3r\left(-e^{-S_0 2} v^3 e^{2i\sigma(r)} + h.c.\right)}_{\text{I called this } V_{eff}.}}$$

this is minimized @ $\sigma = 0$ & the ground state energy is <u>negative</u>. $\sim \underbrace{-V_3 v^3 e^{-2S_0}}$.

this is a disaster, since SUSY (SYM, $N=1$ in 4d) has $\not 0$ Witten index and zero vacuum energy.

So something has been missed.... but going back to p. (77), it is clear what it is: the other non-BPS possible 'bound state' = the M-M* & KK-KK* are ...

Consider an M-M* 'event": from far away, this has no magnetic charge & 2 units of scalar charge -- so we expect a contribution like

$$e^{-2S_0} v^3 d^3 R_{CM}\, e^{-\frac{4\pi}{g_4^2}2\phi(R_{CM})} \times$$

$$\boxed{***} \quad \times \int v^3 dr \quad \times \quad e^{\frac{4\pi L}{g_4^2}\frac{2}{r} - 2\log\frac{r}{L}}$$

(integration over relative coordinate)

Both σ & φ are attractive (opposite sign σ, same charge φ, as per p. (77))

fermion attraction, p. (72) w/ $n_f = 1$

(contribution of KK/KK* same except φ → −φ)

as both interactions are attractive, one would conclude that M/M* would collapse into the vacuum – where $\boxed{***}$ makes no sense – all terms are good for $r \gg L$ ONLY!!

On the other hand, we have a weakly coupled theory, where semiclassics should make sense. In addition, SUSY should be on, too. So we need some contribution to cancel (in the ground state) $V_{eff}(\sigma)$ of p. (80).

Way out suggested by both old & recent works! — complexification.

Bogomolnyi/Zinn-Justin 1980's
Unsal EP 2011
Argyres Unsal 2012

Balitsky-Yung '86
Unsal et al 2015
↓
see p. (84)

sounds like magic: let $g_4^2 \to -g_4^2$, this turns Coulomb attraction into repulsion; one obtains the same answer as for magnetic bions, which when analytically continued back gives a crucial "-1" factor: let's follow this one:

$$\int_0^\infty dr\, e^{-\frac{4\pi L}{g_4^2}\frac{2}{r} - 2\log\frac{r}{L}} = \int_0^\infty dr\, \frac{L^2}{r^2}\, e^{-\frac{8\pi L}{g_4^2 r}} =$$

$$= \frac{L g_4^2}{8\pi}$$, for the analytically continued value $\Longrightarrow$

going back to the 'physical' g_4^2 &
combining the M/M^* + KK/KK^* w/
the $M-KK^*$ & M^*-KK contributions,

we find

$$V_{eff} \sim v^3 e^{-S_0 2}\left(\cosh\left(\frac{4\pi}{g_4^2}\phi\right) - \cos 2\sigma\right)$$

w/

$$\frac{1}{2}\frac{g_4^2}{(4\pi)^2 L}(\partial_\mu \sigma)^2 + \frac{1}{2}\frac{1}{g_4^2 L}(\partial_\mu \phi)^2$$ kinetic term.

remarkably, this is the same potential that follows from "exact superpotential"

Seiberg-Witten 9607163
Aharony et al 9703310
Davies et al 9905015
also 1406... Anber et al

Potential is minimized @ $\underline{\phi = 0}$ & $\underline{\sigma = 0/\pi}$

& $V_{eff}(0,0) = 0$, in agreement w/ SUSY.

How was SUSY preserved? → there were contributions of "negative fugacity" to the partition function: the M/M^* & K/K^* molecules!

these are called "neutral bions"

(essentially "molecules", $I-I^*$)

& can only be defined via analytic continuation.

They are crucial to ensure that $\langle \phi \rangle = 0$, i.e. center symmetry is preserved (recall ϕ was defined as the deviation of A_y away from center symmetry).

A better (really! - but no time to explain all, see 1507.04063, §3.5) way to think of the complexification involved is via an analytical continuation in the path $\int$ over the relative distance "r" — if this integral is considered as one over a complex contour, which is deformed according to the steepest descent rules (for multi-dim integrals — "Lefschetz thimbles"). Notice that "r" describes a one dimensional subspace of the ∞ dimensional path $\int$ expanded around an $\approx$ saddle point consisting of an M & M^* very far apart.

to sketch this type of calculation, revert to normal-sign g_4^2 on p.81:

$$\int_{C_1} dr \; e^{\frac{8\pi L}{g_4^2 r} - 2\log\frac{r}{L}}$$

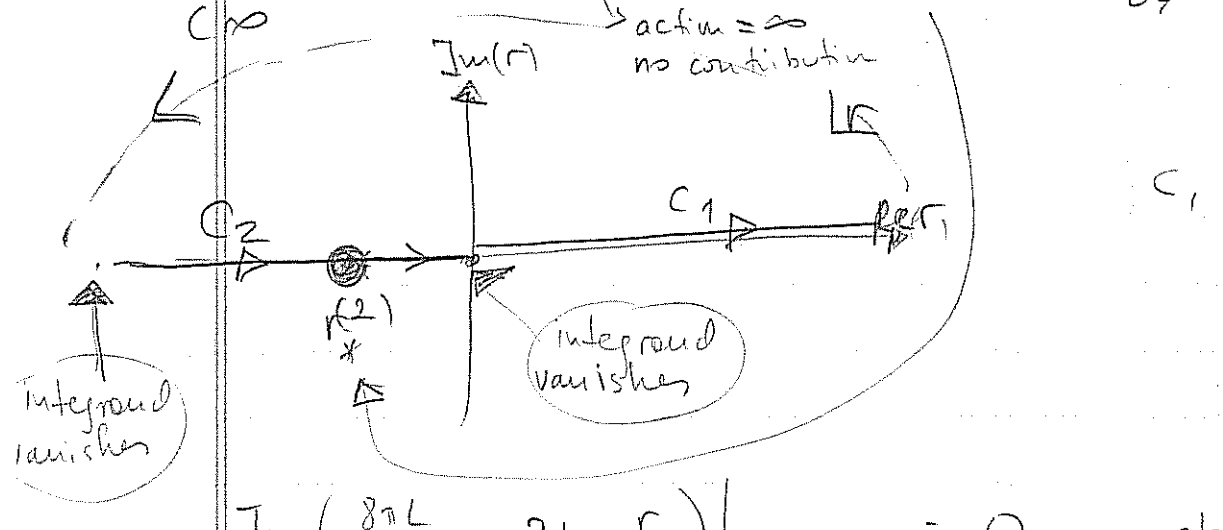
viewed as a $\mathbb{C}$ integral

$$-\frac{8\pi L}{g_4^2}\frac{1}{r_*^2} - \frac{2}{r_*} = 0$$

$$\frac{2}{r_*}\left(\frac{4\pi L}{g_4^2}\frac{1}{r^*} + 1\right) = 0$$

$r_*^{(1)} = \infty$ → action = ∞ no contribution

$r_*^{(2)} = -\frac{4\pi L}{g_4^2}$



$$\mathrm{Im}\left(\frac{8\pi L}{g_4^2 r} - 2\log\frac{r}{L}\right)\Bigg|_{r = r_*^{(2)}} = 0$$, so steepest descent is real axis.

$(\mathrm{Im}\; 2\log(-1) = 0)$

Assuming $\int_{C_1} = -\int_{C_2} \quad \left(\int_{C_\infty} = 0\right)$

↳ ((not a hard science, yet,... studies it all still...!))

we have for the prefactor of M/M* (say)

$$-\int_{-\infty}^{-0} dr\, e^{\frac{8\pi L}{g_4^2}\frac{1}{r} - 2\log\frac{r}{L}} =$$

$$= \underbrace{-\int_{0}^{\infty} dr\, e^{-\frac{8\pi L}{g_4^2}\frac{1}{r} - 2\log\frac{r}{L}}}_{\text{our answer for M-KK*}} =$$

($\int_{C_1} = -\int_{C_2}$) as advertised.

(Note: this accomplishes, in this ex., same as $g_4^2 \to -g_4^2$, but as I said, it is much preferred in other cases, see literature or ask me!)

Also note, there's quite a bit of current interest in formulating path∫ on "Lefshetz thimbles" (multi-dim generalization of steepest descent)

sign problem in, say, finite density QCD

cancellation of perturbative & nonperturbative ambiguities, "resurgence"

In SYM, we could've played the "exact superpotential" game - but it is, hopefully, instructive, to see another point of view that carries, perhaps, some interesting new ideas along...

Let's summarize what we learned about SYM on $\underline{R^3 \times S^1_L}$:

(Euclidean)

$$\mathcal{L}_{eff} = \frac{1}{2}\frac{g_4^2}{(4\pi)^2 L}(\partial_\mu \sigma)^2 + \frac{1}{2}\frac{1}{g_4^2 L}(\partial_\mu \phi)^2$$

$$+ v^3 e^{-2S_0}\left(\cosh\left(\frac{4\pi\phi}{g_4^2}\right) - \cos 2\sigma\right)$$

$$+ v^3 e^{-S_0}\left(\left(e^{i\sigma} e^{-\frac{4\pi}{g_4^2}\phi} + e^{-i\sigma} e^{\frac{4\pi}{g_4^2}\phi}\right)\lambda\lambda\right.$$

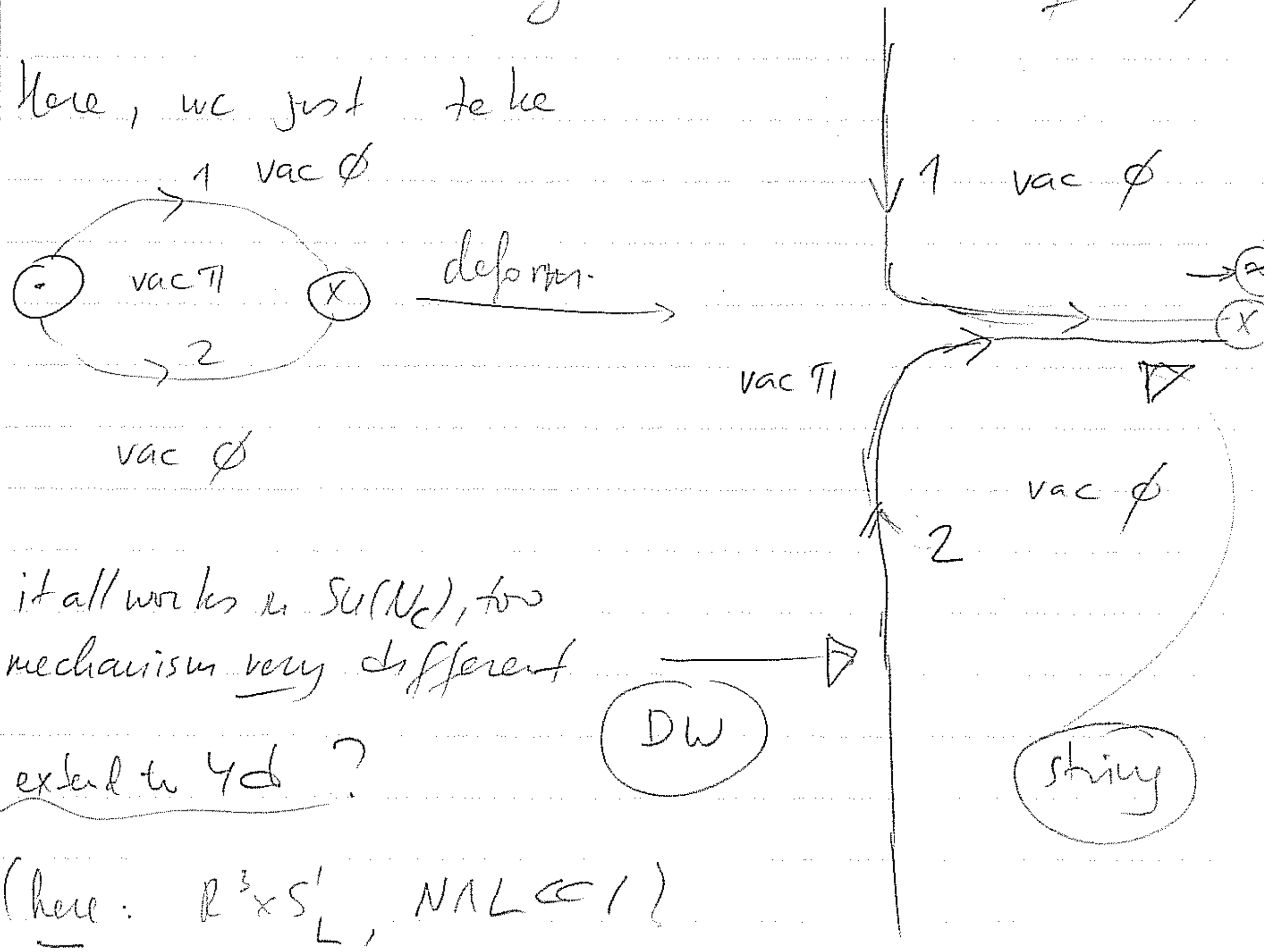
$$\left. + h.c.\right)$$

- $\mathbb{Z}_2$ discrete chiral symmetry broken $\sigma = 0, \pi$
- $\mathbb{Z}_2$ center symmetry unbroken $\phi = 0$ $(A_4 L = \pi)$.
- mass gap for $\underline{\phi, \sigma, \lambda}$ – all in same supermultiplet; as opposed to QCD(adj.) where λ_I were massless

(90)

But most amusingly, they give the 1st field theoretic (electric) setting where the idea that confining strings in SYM (QCD(adj)), (as well as dYM / YM w/ $\theta = \pi$) can end on DWs. This was proposed by Witten / Soo-Jong Rey / in 1997 using MQCD, where DWs & strings are all made of M5/M2

Here, we just take



- it all works in $SU(N_c)$, too
- mechanism very different

Issue: extend to 4d ?

(here: $\mathbb{R}^3 \times S^1_L$, $N\Lambda L \ll 1$)

(88)

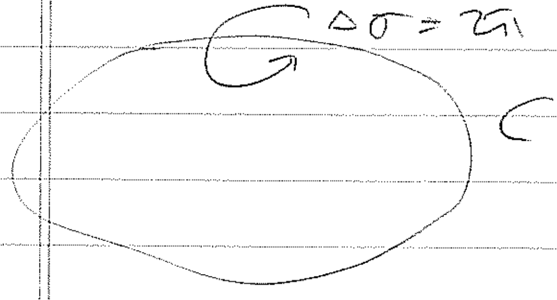
Back to the beginning ---

* hopefully got an idea that this weak-coupling semiclassical story about confinement is pretty (my A.1 of p ⑦)

* before we continue w/ a list of things I don't understand, two brief 'applications' (and issues, included):

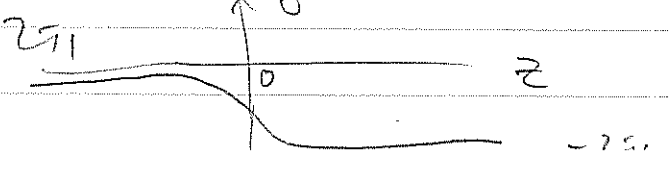
(I.) strings in QCD (adj)/SYM ...

as we discussed in Polyakov model (p. ㉝-㉟) the worldsheet of a confining string is a configuration of σ-field w/ monodromy 2π around the Wilson loop C:



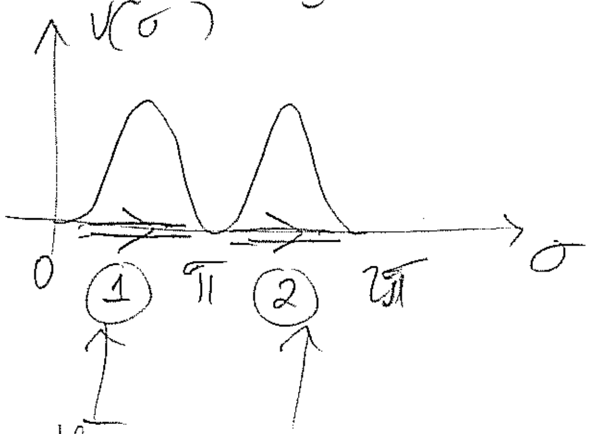
When $V(\sigma) \sim \cos\sigma$, as in Polyakov / dYM, w/ SU(2) gauge group, for $C \to$ (xy)-plane boundary, this configuration is the domain wall (DW)

$\sigma : 0 \to 2\pi$



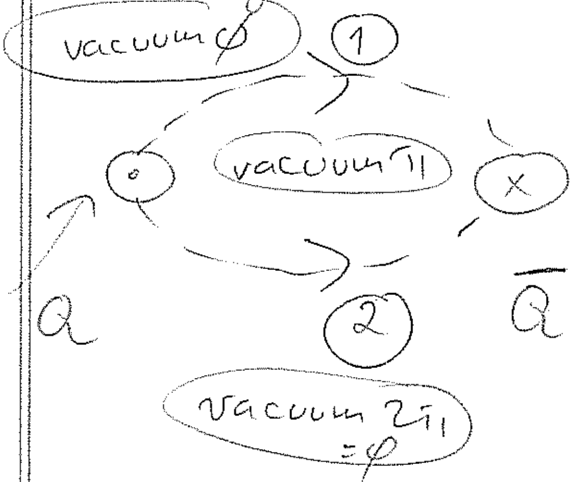
For QCD(adj)/SYM, we have $V(\sigma) \sim \cos 2\sigma$

A DW between $\sigma = 0$ & $\sigma = \pi$ has (obviously) monodromy π & two different vacua on the two sides — so it can not be the confining string.



there are two DWs, each has monodromy π, interpolating from vacuum 0 to vacuum π & from π to $2\pi = 0$.

To build a confining string, one can arrange a configuration like this (Wilson loop is $\perp$ to page, crossing it at ⊙ & ⊗).



these confining strings, especially for $SU(N_c > 2)$ have many interesting properties (many to be studied.)

Anber, Suleimanpasic EP 1507....

final word on confining strings & this setup —
— evident for $SU(N_c)$ <u>only</u> — structure very different from Seiberg-Witten theory, especially degeneracies & baryons.
$N_c > 2$

<u>Issue</u>: – how related to SW?
(as $L \to \infty$ SYM, say?)
– how related to lattice ??

$$dYM \to YM \quad L \to \infty \; ?$$

II SYM, soft breaking ...

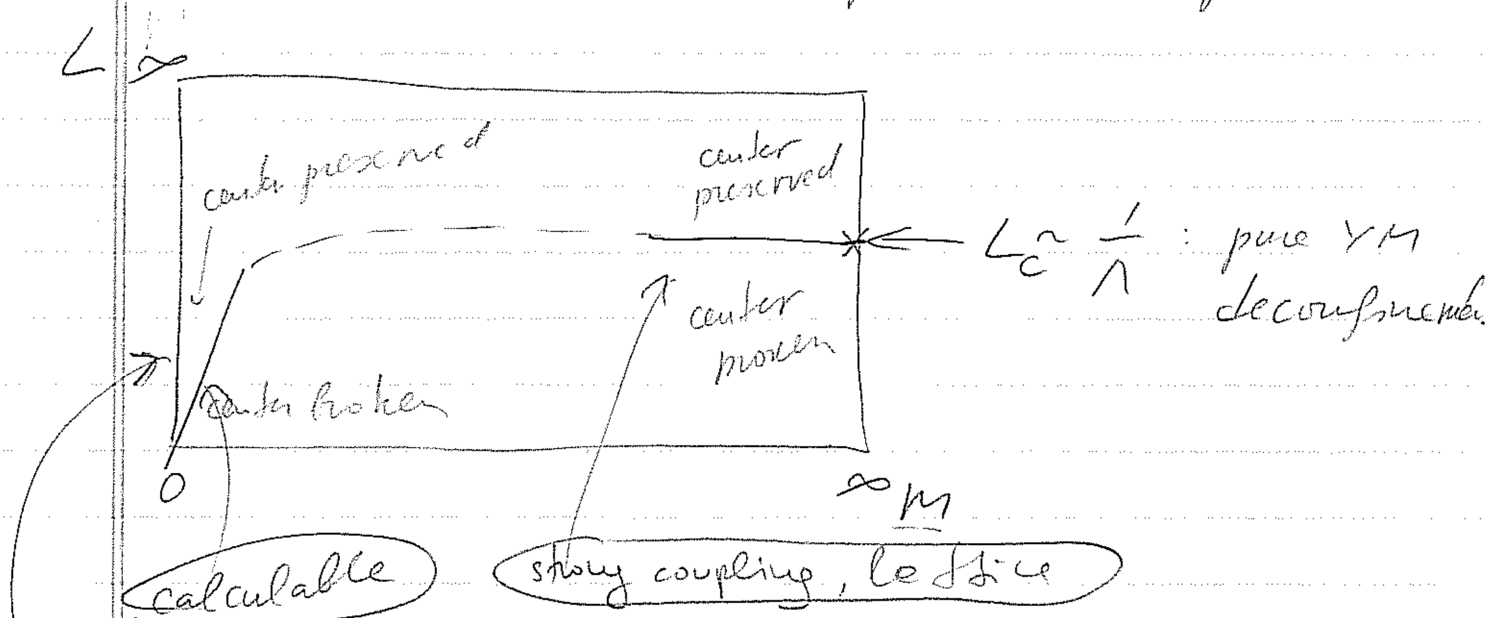
<u>in words/phase diagram</u>

Schäfer, Ünsal, EP 2012
Anber, EP Teeple 2014

- take SYM w/ $m \neq 0$ λ's.
on $R^3 \times S^1$ w/ periodic b.c.
- as $m \gg \Lambda$, λ's decouple, one ends up w/ pure YM

pure YM on $R^3 \times_S S^1$ = thermal YM

it has a deconfinement transition —
— center breaking (low T/large L
confined & center unbroken
while high-T/small-L
deconfined & center broken
as per GPY in pure YM)



at small L, increase m away from 0 : monopole contribution (in addition to Bion), center breaks when:

$$\frac{m}{L^2 \Lambda^3} \sim 1.$$

- for all gauge groups order of transition & other properties agree w/ lattice! (suggest continuity)
- some predictions (assuming continuity) verified (see refs)

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physics of semiclassically calculable transition: neutral bions (center stabilizing) vs magnetic monopole-instanton (& KK) vs GPY ⇒ center breaking.

- lattice has begun to verify phase diagram (Bergner et al 2015)

Issues: Of course connecting calculable semiclassics to the "real thing" is hard, if ever possible. theory of "resurgence" may give a glimpse of hope, but largely in its infancy.

You see that semiclassics beyond leading (dilute I-gas only) is hard. Molecules not greatly understood, even in QM, not to speak in QFT...

Complexification seems to be necessary. (many ex's by now in QM, on much better footing than what I showed you)

Issues: Already mentioned, mechanism of confinement here is abelian... while many features studied on lattice are reproduced, many are not. Require including heavy fields (W-bosons) that were ignored. I'll stop here.