Chiral Perturbation Theory

starting point :

QCD in chiral limit

$N_f = 2$ [or 3] massless quarks u, d [$, s$]

$$\mathcal{L}^0_{\rm QCD} = \overline{q} i\gamma^\mu \left(\partial_\mu + i g_s \frac{\lambda_\alpha}{2} G^\alpha_\mu \right) q + \mathcal{L}_{\rm heavy\ quarks} + \ldots$$

$$= \overline{q_L} i \not{D} q_L + \overline{q_R} i \not{D} q_R + \mathcal{L}_{\rm heavy\ quarks} + \ldots$$

$$q_{R,L} = \frac{1}{2}(1 \pm \gamma_5) q \ , \qquad q = \begin{pmatrix} u \\ d \\ [\mathrm{s}] \end{pmatrix}$$

exhibits global symmetry

$$\underbrace{SU(N_f)_L \times SU(N_f)_R}_{\text{chiral group } G} \times U(1)_V \times U(1)_A$$

$U(1)_V$: baryon number

$U(1)_A$: Abelian anomaly $\stackrel{N_f=3}{\Rightarrow}$ $M_{\eta'} \neq 0$

strong evidence (phenom. + theory) for

spontaneous chiral symmetry breaking

$$\boxed{G \longrightarrow H = SU(N_f)_V}$$

$N_f = 2$ isospin

$N_f = 3$ flavour $SU(3)$

Mechanism for SCSB ?

recall order parameter of SCSB

$$\lim_{V\to\infty}\langle 0|[Q^V(x^0), A]|0\rangle \neq 0$$

QCD : axial charges $Q_A^a = Q_R^a - Q_L^a$

$(a = 1, \ldots, N_f^2 - 1)$

A : colour singlet, pseudoscalar operators

unique possibility for operator dim $=3$:

$$A_b = \overline{q}\gamma_5\lambda_b q \quad \to \quad [Q_A^a, A_b] = -\frac{1}{2}\overline{q}\{\lambda_a, \lambda_b\}q$$

$SU(N_f)_V$ invariance of the vacuum $\to$

$$\langle 0|\overline{u}u|0\rangle = \langle 0|\overline{d}d|0\rangle \; [= \langle 0|\overline{s}s|0\rangle]$$

$\Rightarrow$ sufficient condition for SCSB :

quark condensate $\langle 0|\overline{q}q|0\rangle \neq 0$

certainly not necessary :

$d = 5$: $\langle 0|\overline{q}\sigma_{\mu\nu}\lambda_\alpha q G^{\alpha\mu\nu}|0\rangle \neq 0$

$d = 6$: many more possibilities

Important question :

$\langle 0|\overline{q}q|0\rangle$ dominant order parameter of SCSB ?

answer could depend on N_f :

$\langle 0|\overline{q}q|0\rangle$ probably decreases with N_f

Moussallam; Descotes, Girlanda, Stern

$N_f = 2$: answer from $\pi\pi$ scattering

Implementation of SCSB

dim $G/H = N_f^2 - 1$ Goldstone bosons

$\rightarrow$ pseudoscalar meson fields ϕ^a

parametrize coset space

$$L(\phi) = (u_L(\phi), u_R(\phi)) \in G/H$$

chiral transformation $g = (g_L, g_R) \in G$

nonlinear realization

$$u_A(\phi) \xrightarrow{g} g_A u_A(\phi) h(g,\phi)^{-1} \qquad (A = L, R)$$

Def.: matrix field $U(\phi)$ (transforming linearly)

$$U(\phi) := u_R(\phi) u_L(\phi)^\dagger \xrightarrow{g} g_R U(\phi) g_L^{-1}$$

no loss of information compared to (u_L, u_R) :

with standard choice of coset coordinates

$$u_R(\phi) = u_L(\phi)^\dagger := u(\phi) = \exp i\lambda_a \phi^a / 2F$$

$$\longrightarrow U(\phi) = u(\phi)^2$$

Goldstone matrix element defines F :

$$\langle 0 | \overline{q(x)} \gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x) | \phi_b(p) \rangle = i \delta_{ab} F p^\mu e^{-ipx}$$

meson decay constant in the chiral limit

Non-Goldstone fields (meson resonances , baryons , . . .)

$$\psi \xrightarrow{g \in G} \underbrace{h_\psi(g,\phi)}_{SU(N_f)_V \text{ repr. of } \psi} \psi$$

CHPT : 2 choices of basis

U-basis	u-basis
mesons only	mesons, baryons, ...
	vielbein
$\partial_\mu U$	$u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger)$
$\partial_\mu U \to g_R \partial_\mu U g_L^\dagger$	$u_\mu \to h u_\mu h^{-1}$

lowest-order invariant (nonlinear σ model)

$$\langle \partial_\mu U \partial^\mu U^\dagger \rangle \qquad = \qquad \langle u_\mu u^\mu \rangle$$

$$\langle \quad \rangle = \mathrm{tr}_{\mathrm{N_f}}$$

$\to$ derivative coupling of Goldstone fields

$\to$ $\mathcal{L}_{\mathrm{eff}}$ nonrenormalizable

However

no chiral symmetry in nature !

- explicit breaking : $m_q \neq 0$

$$m_u, m_d \quad \ll \quad \Lambda \simeq M_\rho \qquad N_f = 2 : \pi$$

$$m_s \quad < \quad M_\rho \qquad N_f = 3 : \pi, K, \eta$$

- electroweak interactions :
  included perturbatively in α, G_F

Main assumption of CHPT

expansion around chiral limit meaningful

therefore (even for $\alpha = G_F = 0$) :

2-fold expansion of chiral Lagrangians

i. derivatives $\sim$ momenta

ii. quark masses

$$\mathcal{L}_{\rm eff} = \sum_{i,j} \mathcal{L}_{ij} \ , \qquad \mathcal{L}_{ij} = O(\partial^i m_q^j)$$

relation through meson masses :

$$\begin{aligned} M_M^2 &\sim Bm_q + O(m_q^2) \\ B &= -\langle 0|\overline{u}u|0\rangle / F^2 \end{aligned}$$

$\rightarrow$ 2 different schemes depending on

size of quark condensate $\sim B$

Standard CHPT

Weinberg, Gasser, Leutwyler

Ass.: $M_M^2 \sim B m_q$ dominant contribution

corresponds to $B(\nu = 1\ \mathrm{GeV}) \simeq 1.4\ \mathrm{GeV}$

$\rightarrow \quad 3M_{\eta_8}^2 = 4M_K^2 - M_\pi^2$ Gell-Mann, Okubo

$\rightarrow \quad$ chiral counting : $m_q = O(M^2) = O(p^2)$

$$\mathcal{L}_{\text{eff}} = \sum_n \mathcal{L}_n \ , \qquad \mathcal{L}_n = \sum_{i+2j=n} \mathcal{L}_{ij}$$

mesons	$n = 2, 4, 6, \ldots$
mesons + baryons	$n = 1, 2, 3, \ldots$

Generalized CHPT

Stern, Knecht, Moussallam

allow for possibility of small quark condensate
e.g.: $B(\nu = 1\ \mathrm{GeV}) = O(F_\pi = 92.4\ \mathrm{MeV})$

$\rightarrow \quad M_M^2 \sim O(m_q^2)$ dominant more natural counting : $m_q = O(p)$

(same) effective Lagrangian is reordered

e.g., for mesons

$$\mathcal{L}_{\text{eff}} = \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_3 + \tilde{\mathcal{L}}_4 + \ldots$$

Drawback : more unknown LECs at given order

$\rightarrow$ less predictive

present status

no compelling evidence

against standard chiral counting

crucial test $(N_f = 2)$: $\pi\pi \rightarrow \pi\pi$

Construction of $\mathcal{L}_{\text{eff}}$

couple external matrix fields v_μ, a_μ, s, p

$$\mathcal{L}^0_{\text{QCD}} \rightarrow \mathcal{L}^0_{\text{QCD}} + \overline{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \overline{q}(s - ip\gamma_5)q$$

$\rightarrow$ $\mathcal{L}_{\text{eff}}$ inherits

local chiral symmetry

advantages

- $\mathcal{L}_{\rm eff}$, Green functions (of quark currents)

  manifestly chiral invariant

- phys. amplitudes : $v_\mu = a_\mu = p = 0$

  $s = \text{diag}(m_u, m_d[, m_s])$

Effective chiral Lagrangian

strong interactions of mesons (standard CHPT)

$$\mathcal{L}_{\rm eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

$$\begin{aligned} \mathcal{L}_2 &= \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle \\ &= \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle \end{aligned}$$

gauge-covariant derivative

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)$$

$$\chi = 2B(s + ip) \ , \qquad \chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u$$

2 parameters :

$$\begin{aligned} F_\pi &= F\,[1 + O(m_q)] \\ \langle 0|\overline{u}u|0\rangle &= -F^2 B\,[1 + O(m_q)] \end{aligned}$$

CHPT at lowest order : $O(p^2)$

$\mathcal{L}_2$ at tree level $\simeq$ current algebra

amplitudes depend only on F_π, M_M^2

e.g.: $\pi\pi \to \pi\pi$

$$A_2(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} \qquad \text{Weinberg}$$

$\to$ absolute prediction from pure symmetry !

CHPT at $O(p^4)$

chiral Lagrangian $\mathcal{L}_4$ contains

10 (7) measurable LECs for $N_f = 3$ (2) T

$\mathcal{L}_{\rm eff}$ hermitian $\to$ tree amplitudes real

BUT

unitarity + analyticity $\to$ complex amps.

example : $\pi\pi \to \pi\pi$

$$\text{Im } t_l^I(s) \geq (1 - \frac{4M_\pi^2}{s})^{\frac{1}{2}} |t_l^I(s)|^2$$

partial waves $t_l^I(s)$ start at $O(p^2)$

Chiral Lagrangian of $O(p^4)$ for $N_f = 3$

Gasser, Leutwyler

$$\mathcal{L}_4 = L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle$$

$$+ L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle$$

$$+ L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2$$

$$+ L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle$$

$$- i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle$$

$+ 2$ contact terms

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \,, \qquad r^\mu = v^\mu + a^\mu$$

$$F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] \,, \qquad l^\mu = v^\mu - a^\mu$$

$$L_i = L_i^r(\mu) + \Gamma_i \Lambda(\mu)$$

$$\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2}[\log 4\pi + 1 + \Gamma'(1)] \right\}$$

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \frac{\mu_1}{\mu_2}$$

$\rightarrow \quad \mathrm{Im}T \neq 0$ at $O(p^4)$

systematic low-energy expansion
$\rightarrow \quad$ loop expansion for $n > 2$

loop amps. are in general divergent

regularization and renormalization

Consequences :

- renormalization guarantees independence of regularization (no cutoff dependence)
- divergences absorbed by coupling constants in $\mathcal{L}_4, \mathcal{L}_6, \ldots$
- renormalized LECs scale dependent : contain effects of (heavy) states not included in $\mathcal{L}_{\rm eff}$ as explicit fields

example : CHPT for $N_f = 2$

only pions in $\mathcal{L}_{\rm eff}$

$\rightarrow \quad$ LECs contain effects of $K, \eta, \ldots$

In principle :

LECs dimensionless functions of

$\Lambda_{\text{QCD}}, m_c, m_b, m_t$

In practice :

matching (at $\Lambda \simeq M_\rho$) not possible
in perturbation theory $\rightarrow$

need to bridge gap between

CHPT	$\longleftrightarrow$	pert. QCD
$E < M_\rho$		$E > 1.5$ GeV

Different approaches:

most successful or promising $\rightarrow$

phenomenological	:	resonance saturation
theoretical	:	lattice, large N_c

Loop Expansion

$\sim$ expansion around classical solution

of which equation ?

Chiral Lagrangian (mesons, standard CHPT)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

to ensure systematic chiral expansion :

expand around solution of

EOM of lowest-order Lagrangian $\mathcal{L}_2$

$\rightarrow$ perturbative propagator always $O(p^2)$

Successive orders in the chiral expansion characterized by

chiral dimension D_L

= degree of homogeneity in

external momenta and meson masses

Meson sector (standard CHPT)

general loop diagram with

L : number of loops

N_n : number of vertices from $\mathcal{L}_n$

I : number of internal meson lines

$$\rightarrow \quad D_L = 4L + \sum_{n\geq 2} nN_n - 2I$$

connected diagram : $L = I - \sum_n N_n + 1$

$$\rightarrow \quad D_L = 2L + 2 + \sum_{n\geq 4} (n-2)N_n \geq 2L + 2$$

scale of chiral expansion

D_L increases with L , but phys. dim. fixed

$\rightarrow \quad \frac{1}{F^2}\frac{1}{(4\pi)^2}$ for each loop

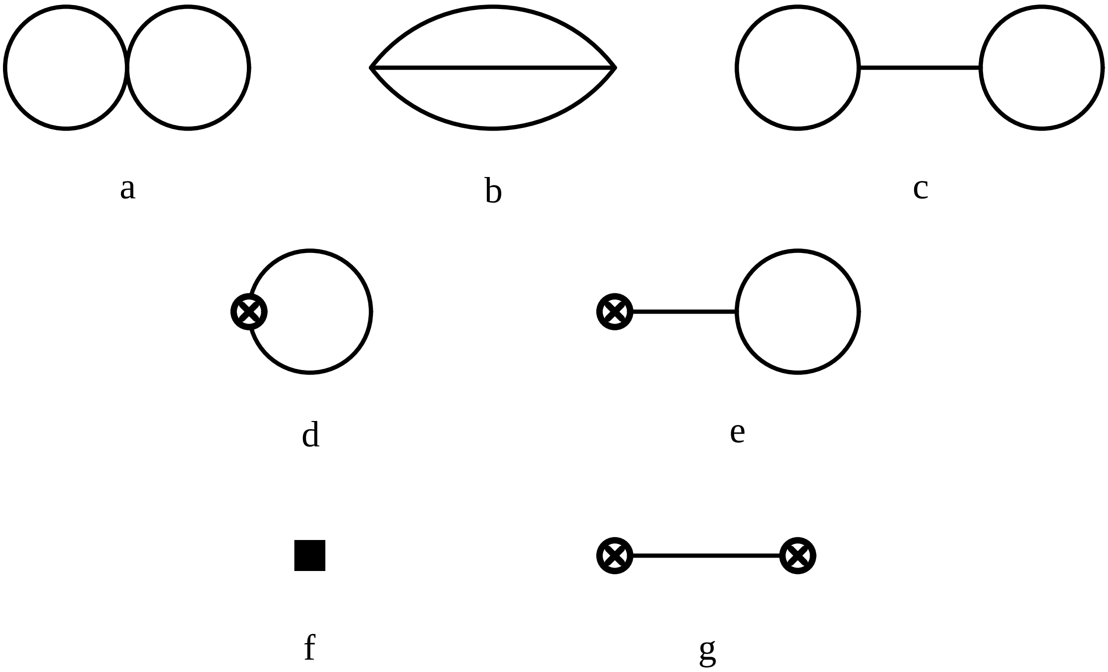
for mesons $\rightarrow$ expansion in $\frac{p^2}{(4\pi F)^2}$

$N_f = 3$: $\frac{p^2}{(4\pi F)^2} = 0.18 \frac{p^2}{M_K^2}$

CHPT at $O(p^6)$

skeleton diagrams for $D_L = 6$:

$L = 2$		a,b,c
$L = 1$	$: N_4 = 1$	d,e
$L = 0$	$: N_6 = 1$ or $N_4 = 2$	f,g



Exercise : all $O(p^6)$ diagrams for $\pi\pi \to \pi\pi$

Complete calculations to $O(p^6)$
(standard CHPT)

$N_f = 2$	
$\gamma\gamma \to \pi^0\pi^0$	Bellucci et al. (1994)
$\gamma\gamma \to \pi^+\pi^-$	Bürgi (1996)
$\pi \to l\nu_l\gamma$	Bijnens and Talavera (1997)
$\pi\pi \to \pi\pi$	Bijnens et al. (1996, 1997)
π form factors	Bijnens et al. (1998)

$N_f = 3$	
VV, AA	Golowich and Kambor (1995, 1997)
	Amoros et al. (1999)
form factors	Post and Schilcher (1997)
$K \to \pi\pi l\nu_l$	Amoros et al. (1999)

Renormalization

process independent procedure useful $\to$

divergent part of $Z[v, a, s, p]$

method : heat kernel expansion

advantages

- nontrivial check for explicit loop calculations (discussion)
- RGE for renormalized, scale dep. LECs
- leading infrared singularities : chiral logs

$O(p^4)$: $L = 0, 1$

divergences in dim. regularization

$$\frac{\mu^{d-4}}{(4\pi)^2}\left[\frac{1}{d-4} + \frac{1}{2}\ln M^2/\mu^2 + \ldots\right]$$

independent of arbitrary scale μ

M : typical scale

$$M = M_\pi (M_K) \text{ for } N_f = 2(3)$$

Renormalization :

$$L_i(d) = \mu^{d-4}\left[\frac{\Gamma_i}{(4\pi)^2(d-4)} + \underbrace{L_i^r(\mu)}_{\text{ren. LEC}} + \ldots\right]$$

again scale independent

coefficients Γ_i chosen to cancel divergences $\rightarrow$

amps. depend on scale-indep. combinations

$$\boxed{L_i^r(\mu) - \tfrac{1}{2}\Gamma_i l}$$

with chiral log $\quad l = \frac{1}{(4\pi)^2}\ln M^2/\mu^2$

→ **renormalization group equations**

$$\mu \frac{dL_i^r(\mu)}{d\mu} = -\frac{\Gamma_i}{(4\pi)^2}$$

$$\rightarrow \quad L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \frac{\mu_1}{\mu_2}$$

Phenomenological values of $L_i^r(M_\rho)$

i	$L_i^r(M_\rho) \times 10^3$ 1995	2000 (Amoros et al.)	main source
1	0.4 ± 0.3	0.52 ± 0.23	$K_{e4}, (\pi\pi \rightarrow \pi\pi)$
2	1.35 ± 0.3	0.72 ± 0.24	$K_{e4}, (\pi\pi \rightarrow \pi\pi)$
3	-3.5 ± 1.1	-2.70 ± 0.99	$K_{e4}, (\pi\pi \rightarrow \pi\pi)$
4	-0.3 ± 0.5		Zweig rule
5	1.4 ± 0.5	0.65 ± 0.12	F_K / F_π
6	-0.2 ± 0.3		Zweig rule
7	-0.4 ± 0.2	-0.26 ± 0.15	Gell-Mann–Okubo, L_5, L_8
8	0.9 ± 0.3	0.47 ± 0.18	$M_{K^0} - M_{K^+}, L_5$
9	6.9 ± 0.7		$\langle r^2 \rangle_V^\pi$
10	-5.5 ± 0.7		$\pi \rightarrow e\nu\gamma$

$O(p^6)\ :\ L = 0, 1, 2$

reducible diagrams c,e,g :

with appropriate choice of $\mathcal{L}_4$ (discussion) :

sum c+e+g
finite and scale independent

irreducible diagrams a,b,d :

divergences should be polynomials in

masses and momenta of $O(p^6)$

However : diagrams a,d involve

Green function $G(x,x)$ (and derivatives)

$$G(x,x) = \frac{2\mu^{d-4}}{(4\pi)^2}\left(\frac{1}{d-4} + \frac{1}{2}\ln M^2/\mu^2\right) a_1(x,x) + \underbrace{\overline{G}(x,x)}_{\text{finite,nonlocal}}$$

$a_1(x,x)$: (local) Seeley-DeWitt coefficient

$\rightarrow$ diagrams a,b,d separately have

nonlocal divergences

of the type

$$\frac{1}{d-4}\overline{G}(x,x) \quad \text{and} \quad \frac{1}{d-4}\ln M^2/\mu^2$$

proper renormalization at $O(p^4)$ $\quad\rightarrow$

nonlocal divergences cancel in

sum a+b+d

remaining divergences : polynomials of $O(p^6)$

in coordinate space :

dimensionless, divergent coefficients C_i

of chiral Lagrangian

$$\mathcal{L}_6 = \sum_{i=1}^{53(90)} \underbrace{C_i}_{\text{coeffs.}} \underbrace{O_i}_{\text{monomials}} \qquad \text{for} \quad N_f = 2(3)$$

tree diagram f :

divergent parts of C_i chosen to make

a+b+d+f finite $\quad\Rightarrow$

complete functional $Z_6[v, a, s, p]$
finite and scale independent
with renormalized LECs $C_i^r(\mu)$

renormalization group equations

$$\mu \frac{dC_i^r(\mu)}{d\mu} = \frac{1}{(4\pi)^2}\left[2\hat{\Gamma}_i^{(1)} + \hat{\Gamma}_i^{(L)}(\mu)\right]$$

$\hat{\Gamma}_i^{(1)}$: constants

$\hat{\Gamma}_i^{(L)}(\mu)$: linear combs. of $L_i^r(\mu)$

Chiral logs

often numerically important or even dominant

($\rightarrow$ $\pi\pi$ scattering)

trivial observation :

total amplitudes are scale independent

$\rightarrow$ chiral logs disappear for $\mu = M$

$$l = \frac{1}{(4\pi)^2} \ln M^2/\mu^2 = 0$$

However

$\mu = M$ generates "unnaturally " big $L_i^r(\mu)$ $\rightarrow$

infrared singularities shifted into LECs

natural size : $L_i^r(\mu \simeq M_\rho)$

Chiral logs at $O(p^6)$

double chiral logs appear in following combinations (exercise)

irreducible $\quad 4L_i^r(\mu)l - \Gamma_i l^2 := k_i$

reducible $\quad [L_i^r(\mu) - \frac{1}{2}\Gamma_i l][L_j^r(\mu) - \frac{1}{2}\Gamma_j l]$

$$= L_i^r(\mu)L_j^r(\mu) - \frac{1}{8}(\Gamma_i k_j + \Gamma_j k_i)$$

$\Rightarrow$ full dependence on $l^2, lL_i^r, L_i^r L_j^r$ in terms of k_i and $L_i^r L_j^r$

Generalized double-log approximation

Bijnens, Colangelo, E.

Remarks :

- coeffs. calculable from diagrams with $L \leq 1$ (exercise) Weinberg
- indication for size of $O(p^6)$ corrections
  no substitute for full calculation

Example : $F_K/F_\pi = 1.22 \pm 0.01$

GDLA : corrections big $(6 \div 12\%)$

$F_K/F_\pi - 1$ is used to fix LEC L_5^r :

$$L_5^r(M_\rho) = (1.4 \pm 0.5) \times 10^{-3}$$

recent fit with full $O(p^6)$ calculation (Amoros, Bijnens, Talavera) :

$$L_5^r(M_\rho) = (0.65 \pm 0.12) \times 10^{-3}$$

Exercise : summing chiral logs ?