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# VECTOR MODELS IN THE LARGE $N$ LIMIT: A FEW APPLICATIONS

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ABSTRACT

In these lecture notes prepared for the
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and updated for the Saalburg summer school 1998, we review the solutions of $O(N)$ or $U(N)$ models in the large $N$ limit and as $1/N$ expansions, in the case of vector representations. The general idea is that invariant composite fields have small fluctuations for $N$ large. Therefore the method relies on constructing effective field theories for these composite fields after integration over the initial degrees of freedom. We illustrate these ideas by showing that the large $N$ expansion allows to relate the $(\phi^2)^2$ theory and the non-linear $\sigma$-model, models which are renormalizable in different dimensions. In the same way large $N$ techniques allow to relate the Gross–Neveu, an example of a theory with four-fermi self-interaction, with a Yukawa-type theory renormalizable in four dimensions, a topic relevant for four dimensional field theory.

Among other issues for which large $N$ methods are also useful we will briefly discuss finite size effects and finite temperature field theory, because they involve a crossover between different dimensions.

Finally we consider the case of a general scalar $V(\phi^2)$ field theory, explain how the large $N$ techniques can be generalized, and discuss some connected issues like tricritical behaviour and double scaling limit. Some sections in these notes are directly adapted from the work

Zinn-Justin J., 1989, *Quantum Field Theory and Critical Phenomena*, Clarendon Press (Oxford third ed. 1996).

*These lecture notes are dedicated to Mrs. T.D. Lee, who recently passed away, as a testimony of gratitude for the long lasting friendship between our families.*

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## 1 Introduction

In these lectures we describe a few applications of large $N$ techniques to quantum field theories (QFT) with $O(N)$ or $U(N)$ symmetries, where the fields are in the vector representation. We want to show that large $N$ results nicely complement results obtained from more conventional perturbative renormalization group (RG). Indeed the shortcoming of the latter method is that it mainly applies to gaussian or near gaussian fixed points. This restricts space dimension to dimensions in which the corresponding effective QFT is renormalizable, or after dimensional continuation, to the neighbourhood of such dimensions. Large $N$ techniques in some cases allow a study in generic dimensions. They rely on noting that in the large $N$ limit scalar (in the group sense) composite fields have small fluctuations (central limit theorem). Therefore if we are able to construct an effective field theory for the scalars, integrating out the initial degrees of freedom, we can solve the field theory in a $1/N$ expansion. Note that for vector representations the number of independent scalars is finite and independent of $N$, unlike what happens for matrix representations. This explains why vector models have been solved much more generally than matrix models.

In these lectures we will in particular stress two points: first it is necessary to always check that the $1/N$ expansion is both IR finite and renormalizable. Some technical aspects of this question which will be described in section 4.6. This is essential for the stability of the large $N$ results and the existence of a $1/N$ expansion. Second, the large $N$ expansion is just a technique, with its own (often unknown) limitations. It should not be discussed in isolation. Instead, as we shall do in the following examples, it should be combined with other perturbative techniques and the reliability of the $1/N$ expansion should be inferred from the general consistency of all results.

Second-order phase transitions in classical statistical physics will provide us with the first illustration of the usefulness of the large $N$ expansion. Due to the divergence of the correlation length at the critical temperature, systems then have at and near $T_c$ universal properties which can be described by effective continuum quantum field theories. The $N$-vector model that we discuss below is the simplest example but it has many applications since it allows to describe the critical properties of systems like vapour–liquid, binary mixtures, superfluid Helium or ferromagnetic transitions as well as the statistical properties of polymers. Before showing what kind of information can be provided by large $N$ techniques we will first shortly recall what can be learned from perturbative renormalization group (RG) methods. Long distance properties can be described in $d=4-\varepsilon$ dimension by a $(\phi^2)^2$ field theory. Instead in $d=2+\varepsilon$ the relevant QFT model is the $O(N)$ non-linear $\sigma$ model. It is somewhat surprising that the same statistical model can be described by two different theories. Since the results derived in this way are valid *a priori* only for $\varepsilon$ small, there is no overlap to test the consistency. The large $N$ expansion will allow us to discuss generic dimensions and thus to understand the relation between both field theories.

Another domain of application of the large $N$ expansion is finite size effects and finite temperature field theory. In these situations a dimensional crossover occurs between the large size or zero temperature situation where the infinite volume theory is relevant to a dimensionally reduced theory in the small volume or high temperature limit. Both effective field theories being renormalizable in different dimensions, perturbative RG cannot describe correctly both situations. Again large $N$ techniques will help us to understand the crossover.

Four-fermi interactions have been proposed to generate a composite Higgs particle in four dimensions, as an alternative to a Yukawa-type theory, as one finds in the Standard Model. Again, using the specific example of the Gross–Neveu model, we will use large $N$ techniques to clarify the relations between these two approaches. We will finally briefly indicate that other models with chiral properties, like massless QED or the Thirring model, can be studied by similar techniques.

In the last section we return to scalar boson field theories, and examine multi-critical points (where the large $N$ technique will show some obvious limitations), and the double scaling limit, a toy model for discussing problems encountered in matrix models of 2D quantum gravity.

## 2 The $N$-vector model near dimension four: Renormalization Group (RG)

The $N$-vector model is a lattice model described in terms of $N$-vector spin variables $\mathbf{S}_i$ of unit length on each lattice site $i$, interacting through a short range ferromagnetic $O(N)$ symmetric two-body interaction $V_{ij}$. The partition function of such a model can be written:

$$Z = \int \prod_i \mathrm{d}\mathbf{S}_i \, \delta\left(\mathbf{S}_i^2 - 1\right) \exp\left[-\mathcal{E}\left(\mathbf{S}\right)/T\right], \tag{2.1}$$

in which the configuration energy $\mathcal{E}$ is:

$$\mathcal{E}\left(\mathbf{S}\right) = -\sum_{ij} V_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \,. \tag{2.2}$$

This model has a second order phase transition between a disordered phase at high temperature, and a low temperature ordered phase where the $O(N)$ symmetry is spontaneously broken, and the order parameter $\mathbf{S}_i$ has a non-vanishing expectation value. At a second order phase transition the correlation length diverges, and therefore a non-trivial long distance physics can be defined. Scaling and universality properties emerge which we want to study.

To generate correlation functions one can add to $\mathcal{E}\left(\mathbf{S}\right)$ a coupling to a space-dependent magnetic field

$$\mathcal{E}\left(\mathbf{S}\right) = -\sum_{ij} V_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_i \mathbf{H}_i \cdot \mathbf{S}_i \,. \tag{2.3}$$

*2.1 Mean field theory and the stability of the gaussian fixed point*

To derive the critical properties of the $N$-vector model one can proceed in the following way: one starts from the mean field approximation, valid in high dimensions. One then shows that the mean field approximation is the first term in a systematic expansion. One discovers that for dimensions $d > 4$ the successive terms in the expansion do not modify the leading mean field behaviour. For $d < 4$ instead IR divergences appear and the mean field approximation is no longer valid. Moreover a summation of the leading IR divergences to all orders in the expansion leads to an effective local $\phi^4$ field theory. The corresponding action is given by the first relevant terms of the Landau–Ginzburg–Wilson hamiltonian:

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}c\,(\nabla\phi)^2 + \tfrac{1}{2}a\phi^2(x) + b\frac{1}{4!}\left(\phi^2(x)\right)^2\right], \tag{2.4}$$

with $a$, $b$ and $c$ being *regular* functions of the temperature for $T$ close to $T_c$.

Note that the expression (2.4) , which in the sense of classical statistical physics is a configuration energy, is often called hamiltonian. The reason is that if one starts from a classical hamiltonian and a functional integral over phase space, the integral over conjugate momenta is gaussian and thus trivial. From the point of view of quantum field theory the expression (2.4) has the form of an euclidean action, analytic continuation to imaginary time of the classical field theory action. We shall thus generally call it the action.

Alternatively one can imagine starting from the configuration energy (2.2) and constructing Wilson's renormalization group by integrating out short distance degrees of freedom. The spin variable $\mathbf{S}_i$ is then replaced by a local average, a vector of continuous length of the type of the field $\phi(x)$.

Mean field theory corresponds to the gaussian fixed point of this renormalization group. At the critical temperature one finds a massless free field theory

$$\mathcal{H}_{\mathrm{G}}(\phi) = \int \mathrm{d}^d x\,\tfrac{1}{2}c\,(\nabla\phi)^2 .$$

One then performs an analysis of the stability of the gaussian fixed point. Mean field theory assumes that the order parameter, here the field $\phi(x)$, is small and varies only on macroscopic scales. Therefore a general action can be expanded in powers of the field $\phi(x)$ and derivatives.

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x\,\tfrac{1}{2}c\,(\nabla\phi)^2 + \sum_{\ell} \mathcal{H}_{\ell}(\phi),$$

where $\sum_{\ell}$ means sum over all space integrals $\mathcal{H}_{\ell}(\phi)$ of $O(N)$ symmetric monomials in $\phi$ of degree $n_{\ell}$ and containing $m_{\ell}$ derivatives (often below called operators, a language borrowed from quantum field theory).

A convenient way to understand the relevance of the $\mathcal{H}_\ell(\phi)$ terms in the large distance (infrared) limit is to rescale all space or momentum variables, and measure distances in units of the correlation length, or, at the critical temperature, in some arbitrary unit much larger than the lattice spacing and corresponding to the typical distances at which correlations are measured.

Let us perform such a rescaling here, and rescale also the field $\phi(x)$ in such a way that the coefficient of $[\nabla\phi(x)]^2$, to which all contributions will be compared, becomes the standard $1/2$:

$$x \mapsto \Lambda x, \tag{2.5}$$

$$\phi(x) \mapsto \zeta\phi(x). \tag{2.6}$$

After this rescaling all quantities have a dimension in units of $\Lambda$. Our choice of normalization for the gradient term implies:

$$\zeta = c^{-1/2}\Lambda^{1-d/2}, \tag{2.7}$$

which shows that $\phi$ now has in terms of $\Lambda$ its canonical dimension $d/2-1$.

A term $\mathcal{H}_\ell(\phi)$ then is multiplied by

$$\mathcal{H}_\ell(\phi) \mapsto \Lambda^{d-n_\ell(d-2)/2-m_\ell}\mathcal{H}_\ell(\phi).$$

For $\Lambda$ large we observe the following:

(i) The leading term is the term proportional to $\int \mathrm{d}^d x\, \phi^2(x)$, which is multiplied by $\Lambda^2$. This is not surprising since it gives a mass to the field and therefore the theory moves away from the massless critical theory (the term is called relevant).

(ii) If $d > 4$ all other terms are multiplied by negative powers and therefore become negligible in the long distance limit. They are called irrelevant. The gaussian fixed point is stable and mean field theory thus correct.

(iii) In four dimensions the $\phi^4$ interaction is independent of $\Lambda$: it is called marginal while all other interactions remain irrelevant. The analysis of the stability of the gaussian fixed point then requires a finer study which will be based on the field theory perturbative renormalization group.

(iv) Below four dimensions the $\phi^4$ interaction becomes relevant, the gaussian fixed point is certainly unstable. The question of the existence of another non-trivial fixed point is non-perturbative and cannot be easily answered. Partial answers are based upon the following assumption: the dimensions of operators are *continuous* functions of the space dimension. This means that we are going to look for a fixed point which, when $d$ approached four, coalesces with the gaussian fixed point. Moreover even at this new fixed point, at least in some neighbourhood of dimension four, all operators except $(\phi^2)^2$ should remain irrelevant. The action (2.4) should contain all relevant operators and therefore enough information about the non-trivial fixed point.

After the rescaling (2.5,2.6) the action $\mathcal{H}(\phi)$ then becomes:

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\nabla\phi(x)\right]^2 + \tfrac{1}{2} r\phi^2(x) + \frac{1}{4!} g\Lambda^{4-d} \left(\phi^2(x)\right)^2 \right\}, \tag{2.8}$$

with $r = a\Lambda^2/c$, $g = b/c^2$. The action (2.8) generates a perturbative expansion of field theory type which can be described in terms of Feynman diagrams. These have to be calculated with a momentum cut-off of order $\Lambda$, reflection of the initial microscopic structure. The corresponding theory is thus analogous to regularized quantum field theory. The precise cut-off procedure can be shown to be irrelevant except that it should satisfy some general regularity conditions. For example the propagator can be modified (as in Pauli–Villars's regularization) but the inverse propagator in momentum space must remain a regular function of momentum (the forces are short range).

Let us call $r_c$ the value of the parameter $r$ which corresponds, at $g$ fixed, to the critical temperature $T_c$ at which the correlation length $\xi$ diverges. In terms of the scale $\Lambda$ the critical domain is then defined by:

$$\begin{aligned} &\text{physical mass } = \xi^{-1} \ll \Lambda \;\Rightarrow\; |r - r_c| \ll \Lambda^2 \\ &\text{distances} \gg 1/\Lambda \quad \text{or momenta} \ll \Lambda, \\ &\text{magnetization } M \equiv |\langle\phi(x)\rangle| \ll \zeta^{-1} \sim \Lambda^{(d/2)-1}. \end{aligned} \tag{2.9}$$

Note that these conditions are met if $\Lambda$ is identified with the cut-off of a usual field theory. However an inspection of the action (2.8) also shows that, in contrast with conventional quantum field theory, the $\phi^4$ coupling constant has a dependence in $\Lambda$ given *a priori*. For $d < 4$ the $\phi^4$ coupling is very large in terms of the scale relevant for the critical domain. In the usual formulation of quantum field theory instead the *bare* coupling constant also is an adjustable parameter. This implies for instance that for $d < 4$ (super-renormalizable theory) the coupling constant varies when the correlation length changes. This is a somewhat artificial situation if one believes that that the initial bare or microscopic theory has a physical meaning.

The critical properties of the field theory (like the long distance behaviour of correlation functions) can then be analyzed by RG methods in $4 - \varepsilon$ dimension, i.e. near the so-called upper-critical dimension (and with some additional assumptions in $d < 4$).

*Dimensions of fields.* Because we deal with translation invariant theories, we will generally discuss the scaling behaviour of correlation functions in momentum variables. Let us relate the scaling behaviour of connected correlation functions expressed in terms of space and momentum variables. When functions have a scaling behaviour, one defines

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i/\lambda) \right\rangle_{\mathrm{c}} = \lambda^D \left\langle \prod_i \mathcal{O}_i(x_i) \right\rangle_{\mathrm{c}} \quad \text{with } D = \sum_i d_{\mathcal{O}_i}\,, \tag{2.10}$$

where $\mathcal{O}_i$, sometimes called operator, is a local polynomial in the basic fields (associated with the order parameter), and the quantity $d_{\mathcal{O}_i}$, which we sometimes also denote $[\mathcal{O}_i]$, is called the dimension of the field (operator) $\mathcal{O}_i$.

After Fourier transformation and factorization of the $\delta$-function of momentum conservation, one then finds, in $d$ space dimension,

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(\lambda p_i \right\rangle_{\mathrm{c}} = \lambda^{D'} \left\langle \prod_{i} \mathcal{O}_i(p_i) \right\rangle_{\mathrm{c}} \quad \text{with now } D' = d + \sum_i (d_{\mathcal{O}_i} - d)\,. \quad (2.11)$$

Finally it is convenient to introduce the Legendre transform $\Gamma(\phi)$ of the generating functional $W(H) = T \ln Z$ of $\phi$-field connected correlation functions. We denote by $W^{(n)}$ and $\Gamma^{(n)}$ the corresponding connected and 1PI functions. One verifies that if one performs a Legendre transformation on the source associated with the field (operator) $\mathcal{O}_i$, the quantity $d_{\mathcal{O}_i} - d$ in equation (2.11) is replaced by $-d_{\mathcal{O}_i}$.

### *2.2 RG equations for the critical (massless) theory*

The field theory with the action (2.8) can now be studied by field theoretical methods. From simple power counting arguments one concludes that the critical (or massless) theory does not exist in perturbation theory for any dimension smaller than 4. If we define, by dimensional continuation, a critical theory in $d = 4 - \varepsilon$ dimensions, even for arbitrarily small $\varepsilon$ there always exists an order in perturbation ($\sim 2/\varepsilon$) at which IR (infrared) divergences appear. Therefore the idea, originally due to Wilson and Fisher, is to perform a double series expansion in powers of the coupling constant $g$ and $\varepsilon$. Order by order in this expansion, the critical behaviour differs from the mean field behaviour only by powers of logarithm, and we can construct a perturbative critical theory by adjusting $r$ to its critical value $r_c\,(T = T_c)$.

To study the large cut-off limit we then use methods developed for the construction of the renormalized massless $\phi^4$ field theory. We introduce rescaled (renormalized) correlation functions, defined by renormalization conditions at a new scale $\mu \ll \Lambda$, and functions of a renormalized coupling constant $g_{\mathrm{r}}$. We write here equations for Ising-like systems, the field $\phi$ having only one component. The generalization to the $N$-vector model with $O(N)$ symmetry, is straightforward except in the low temperature phase or in a symmetry breaking field, a situation which will be examined in section 2.6. Then:

$$\begin{cases} \Gamma_{\mathrm{r}}^{(2)}\,(p; g_{\mathrm{r}}, \mu, \Lambda)\big|_{p^2=0} = 0\,, \\ \dfrac{\partial}{\partial p^2}\Gamma_{\mathrm{r}}^{(2)}\,(p; g_{\mathrm{r}}, \mu, \Lambda)\big|_{p^2=\mu^2} = 1\,, \\ \Gamma_{\mathrm{r}}^{(4)}\,(p_i = \mu\theta_i; g_{\mathrm{r}}, \mu, \Lambda) = \mu^{\varepsilon} g_{\mathrm{r}}\,, \end{cases} \quad (2.12)$$

in which $\theta_i$ is a numerical vector. These correlation functions are related to the original ones by the equations:

$$\Gamma_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = Z^{n/2}(g, \Lambda/\mu)\, \Gamma^{(n)}(p_i; g, \Lambda) . \tag{2.13}$$

Renormalization theory tells us that the functions $\Gamma_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda)$ of equation (2.13) have at $p_i$, $g_{\mathrm{r}}$ and $\mu$ fixed, a large cut-off limit which are the renormalized correlation functions $\Gamma_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu)$. A detailed analysis actually shows that at any finite order in perturbation theory:

$$\Gamma_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = \Gamma_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu) + O\left(\Lambda^{-2}(\ln \Lambda)^L\right) , \tag{2.14}$$

in which the power $L$ of $\ln \Lambda$ increases with the order in $g$ and $\varepsilon$. Furthermore the renormalized functions $\Gamma_{\mathrm{r}}^{(n)}$ do not depend on the specific cut-off procedure and, given the normalization conditions (2.12), are therefore universal. Since the renormalized functions $\Gamma_{\mathrm{r}}^{(n)}$ and the initial ones $\Gamma^{(n)}$ are asymptotically proportional, both functions have the same small momentum or large distance behaviour. To determine the universal critical behaviour it is thus sufficient to study the renormalized field theory. And indeed most peturbative calculations of universal quantities have been performed in this framework. However, it is interesting to determine not only the asymptotic critical behaviour, but also the corrections to the asymptotic theory. Furthermore, renormalized quantities are not directly obtained in non-perturbative calculations. For these reasons it is also useful to express the implications of equation (2.13) directly on the initial theory.

*Bare RG equations.* Let us differentiate equation (2.13) with respect to $\Lambda$ at $g_{\mathrm{r}}$ and $\mu$ fixed, taking into account (2.14):

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}}, \mu \text{ fixed}} Z^{n/2}(g, \Lambda/\mu)\, \Gamma^{(n)}(p_i; g, \Lambda) = O\left(\Lambda^{-2}(\ln \Lambda)^L\right) . \tag{2.15}$$

We now neglect corrections subleading (in perturbation theory) by powers of $\Lambda$. Then, using chain rule, we can rewrite equation (2.15) as:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g, \Lambda/\mu) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g, \Lambda/\mu)\right] \Gamma^{(n)}(p_i; g, \Lambda) = 0 . \tag{2.16}$$

The functions $\beta$ and $\eta$, which are dimensionless and may thus depend only on the dimensionless quantities $g$ and $\Lambda/\mu$, are defined by:

$$\beta(g, \Lambda/\mu) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}}, \mu} g , \tag{2.17a}$$

$$\eta(g, \Lambda/\mu) = -\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}}, \mu} \ln Z(g, \Lambda/\mu) . \tag{2.17b}$$

However, the functions $\beta$ and $\eta$ can also be directly calculated from equation (2.16) in terms of functions $\Gamma^{(n)}$ which do not depend on $\mu$. Therefore the functions $\beta$ and $\eta$ cannot depend on the ratio $\Lambda/\mu$ (in the definitions (2.17) consistency requires that contributions which goes to zero like some power of $\mu/\Lambda$, should be neglected, as in equation (2.15)). Then equation (2.16) can be rewritten:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2}\eta(g)\right] \Gamma^{(n)}(p_i; g, \Lambda) = 0 \,. \tag{2.18}$$

Equation (2.18) is an equation satisfied when the cut-off is large by the physical correlation functions of statistical mechanics which are also the bare correlation functions of quantum field theory. It expresses the existence of a renormalized theory.

*2.3 RG equations and large distance behaviour: the ε-expansion*

Equation (2.18) can be solved by the method of characteristics: one introduces a dilatation parameter $\lambda$, together with a running coupling constant $g(\lambda)$ and a scale dependent field renormalization $Z(\lambda)$ satisfying the flow equations

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = \beta(g(\lambda)) \,, \qquad g(1) = g \,; \tag{2.19a}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = \eta(g(\lambda)) \,, \qquad Z(1) = 1 \,. \tag{2.19b}$$

The behaviour of correlation functions for $|p_i| \ll \Lambda$ ($\lambda \to 0$) is then governed by IR fixed points, zeros of the RG $\beta$-function with a positive slope.

The RG functions $\beta$ and $\eta$ can be calculated in perturbation theory. From the relation between bare and renormalized coupling constant and the definition (2.17$a$) it follows that ($\varepsilon = 4 - d$):

$$\beta(g, \varepsilon) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O\left(g^3, g^2\varepsilon\right) . \tag{2.20}$$

Let us now assume that $g$ initially is sufficiently small, so that perturbation theory is applicable. We see that above or at four dimensions, i.e. $\varepsilon \le 0$, the function $\beta$ is positive and $g(\lambda)$ decreases approaching the origin $g = 0$. We recover that the gaussian fixed point is IR stable for $d > 4$, and find that it ia also stable at $d = 4$.

Below four dimensions, instead, the gaussian fixed point $g = 0$ is IR repulsive. However, expression (2.20) shows that, for $\varepsilon$ small, $\beta(g)$ now has a non-trivial zero $g^*$:

$$\beta(g^*) = 0, \quad g^* = \frac{48\pi^2}{N+8}\varepsilon + O\left(\varepsilon^2\right), \quad \text{with } \beta'(g^*) \equiv \omega = \varepsilon + O\left(\varepsilon^2\right) . \tag{2.21}$$

The slope $\omega$ at the zero is positive. This non-gaussian fixed point thus is IR stable, at least in the sense of an $\varepsilon$-expansion. In four dimensions it merges with the gaussian fixed point and the eigenvalue $\omega$ vanishes, indicating the appearance of a marginal operator.

The solution of the RG equation then determines the behaviour of $\Gamma^{(n)}(p_i; g, \Lambda)$ for $|p_i| \ll \Lambda$:

$$\Gamma^{(n)}(\lambda p_i; g, \Lambda) \underset{\lambda\to 0}{\sim} \lambda^{d-(n/2)(d-2+\eta)} \Gamma^{(n)}(p_i; g^*, \Lambda), \tag{2.22}$$

where $\eta = \eta(g^*)$. Critical correlation functions have a power law behaviour for small momenta, independent of the initial value of the $\phi^4$ coupling constant $g$.

In particular the small momentum behaviour of the inverse two-point function is obtained for $n = 2$. For the two-point function $W^{(2)}(p)$ this yields:

$$W^{(2)}(p) = \left[\Gamma^{(2)}(p)\right]^{-1} \underset{|p|\to 0}{\sim} 1/p^{2-\eta}. \tag{2.23}$$

The spectral representation of the two-point function implies $\eta > 0$. A short calculation yields:

$$\eta = \frac{N+2}{2(N+8)^2}\varepsilon^2 + O\left(\varepsilon^3\right). \tag{2.24}$$

The scaling in equation (2.22) indicates that the field $\phi(x)$, which had at the gaussian fixed point a canonical dimension $(d-2)/2$, has now acquired an "anomalous" dimension $d_\phi$ (see the discussion of the end of section 2.1):

$$d_\phi = \tfrac{1}{2}(d - 2 + \eta).$$

These results call for a few comments. Within the framework of the $\varepsilon$-expansion, one thus proves that all correlation functions have, for $d < 4$, a long distance behaviour different from the one predicted by mean field theory. In addition the critical behaviour does not depend on the initial value of the $\phi^4$ coupling constant $g$. At least for $\varepsilon$ small one may hope that the analysis of leading IR singularities remains valid and thus it does not depend on any other coupling either. Therefore the critical behaviour is *universal*, although less universal than in mean field theory, in the sense that it depends only on some small number of qualitative features of the system under consideration.

### *2.4 Critical correlation functions with $\phi^2(x)$ insertions*

RG equations for critical correlation functions with $\int \mathrm{d}^d x\, \phi^2(x)$ insertions can also be derived. The operator $\phi^2(x)$ has a direct physical interpretation. It is the most singular part (i.e. the most relevant) of the energy density (2.4). Long distance scaling properties follow. Moreover these RG equations can be used to derive RG equations for correlation functions in the whole critical domain.

We denote by $\Gamma^{(l,n)}(q_1,\ldots,q_l;p_1,\ldots,p_n;g,\Lambda)$ the mixed 1PI correlation functions of the order parameter $\phi(x)$ and the energy density $\frac{1}{2}\phi^2(x)$ ($n$ $\phi$ fields and $l$ $\frac{1}{2}\phi^2$ operators, with $(l+n)\geq 2$). Renormalization theory tells us that we can define renormalized correlation functions $\Gamma_{\rm r}^{(l,n)}(q_i;p_j;g_{\rm r},\mu)$ which, in addition to conditions (2.12), satisfy:

$$\begin{aligned} \Gamma_{\rm r}^{(1,2)}(q;p_1,p_2;g_{\rm r},\mu)\Big|_{p_1^2=p_2^2=\mu^2,\ p_1\cdot p_2=-\frac{1}{3}\mu^2} &= 1\,, \\ \Gamma_{\rm r}^{(2,0)}(q,-q;g_{\rm r},\mu)\Big|_{q^2=\frac{4}{3}\mu^2} &= 0\,, \end{aligned} \tag{2.25}$$

and are related to the original ones by:

$$\lim_{\Lambda\to\infty} Z^{n/2}(Z_2/Z)^l\left[\Gamma^{(l,n)}(q_i;p_j;g,\Lambda)-\delta_{n0}\delta_{l2}\Lambda^{-\varepsilon}A\right]=\Gamma_{\rm r}^{(l,n)}(q_i;p_j;g_{\rm r},\mu)\,. \tag{2.26}$$

$Z_2(g,\Lambda/\mu)$ and $A(g,\Lambda/\mu)$ are two new renormalization constants.

Differentiating with respect to $\Lambda$ at $g_{\rm r}$ and $\mu$ fixed, as has been done in section 2.2, and using chain rule one obtains a set of RG equations:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)-l\eta_2(g)\right]\Gamma^{(l,n)}=\delta_{n0}\delta_{l2}\Lambda^{-\varepsilon}B(g)\,. \tag{2.27}$$

In addition to $\beta$ and $\eta$ (equations (2.17)) two new RG functions, $\eta_2(g)$ and $B(g)$, appear:

$$\eta_2(g)=-\Lambda\left.\frac{\partial}{\partial\Lambda}\right|_{g_{\rm r},\mu}\ln\left[Z_2(g,\Lambda/\mu)/Z(g,\Lambda/\mu)\right], \tag{2.28}$$

$$B(g)=\left[\Lambda\left.\frac{\partial}{\partial\Lambda}\right|_{g_{\rm r},\mu}-2\eta_2(g)-\varepsilon\right]A(g,\Lambda/\mu)\,. \tag{2.29}$$

Note that for $n=0$, $l=2$, the RG equation (2.27) is not homogeneous. This is a consequence of the non-multiplicative character of renormalization in this case.

In the homogeneous case, equation (2.27) can be solved exactly in the same way as equation (2.16). A new function $\zeta_2(\lambda)$ has to be introduced, associated with the RG function $\eta_2(g)$. Again the solution of equation (2.27) combined with simple dimensional analysis leads to the scaling behaviour

$$\Gamma^{(l,n)}(\lambda q_i;\lambda p_j;g,\Lambda)\underset{\lambda\to 0}{\propto}\lambda^{d-n(d-2+\eta)/2-l/\nu}, \tag{2.30}$$

where the correlation length exponent $\nu$ is related to $\eta_2(g^*)$ by:

$$\nu=\left[\eta_2(g^*)+2\right]^{-1}. \tag{2.31}$$

The dimension of the field $\phi^2$ follows (see section 2.1)

$$d_{\phi^2} = d - 1/\nu . \tag{2.32}$$

Using equations (2.28,2.31) it is easy to calculate $\eta_2(g)$ at one-loop order. At the fixed point $g = g^*$ (equation (2.21)) one then obtains the exponent $\nu$:

$$2\nu = 1 + \tfrac{(N+2)}{2(N+8)}\varepsilon + O\left(\varepsilon^2\right).$$

*The* $\left\langle \phi^2 \phi^2 \right\rangle$ *correlation function.* The $\phi^2$ (energy density) two-point function $\Gamma^{(2,0)}$ satisfies an inhomogeneous RG equation. To solve it one first looks for a particular solution, which can be chosen of the form $\Lambda^{-\varepsilon} C_2(g)$:

$$\beta(g)\, C_2'(g) - \left[\varepsilon + 2\eta_2(g)\right] C_2(g) = B(g) . \tag{2.33}$$

The solution is uniquely determined by imposing its *regularity* at $g = g^*$.

The general solution of equation (2.27) is then the sum of this particular solution and of the general solution of the homogeneous equation which has a behaviour given by equation (2.30):

$$\Gamma^{(2,0)}(\lambda q; g, \Lambda) - \Lambda^{-\varepsilon} C_2(g) \underset{\lambda \to 0}{\sim} \lambda^{d-2/\nu} . \tag{2.34}$$

*Remarks.*

(i) The physics we intend to describe corresponds to integer values of $\varepsilon$, $\varepsilon = 1, 2$. Although we can only prove the validity of all RG results within the framework of the $\varepsilon$-expansion, we shall eventually assume that their validity extends beyond an infinitesimal neighbourhood of dimension 4. The large $N$-expansion provides a test of the plausibility of this assumption. The decisive test comes, of course, from the comparison with experimental or numerical data.

(ii) In four dimensions the $\phi^4$ interaction is marginally irrelevant; the renormalized coupling constant of the $\phi^4$ field theory goes to zero only logarithmically when the cut-off becomes infinite. This induces logarithmic corrections to mean field theory. Moreover, since no other fixed point seems to exist, this leads to the so-called *triviality property* (see section 2.7) of the $\phi^4$ quantum field theory.

### *2.5 Scaling behaviour in the critical domain*

We have described the scaling behaviour of correlation functions at criticality, $T = T_c$. We now consider the critical domain which is defined by the property that the correlation length is large with respect to the microscopic scale, but finite.

*Remark.* The temperature is coupled to the total configuration energy. Therefore a variation of the temperature generates a variation of all terms contributing to the effective action. However the most relevant contribution (the most IR singular) corresponds to the $\phi^2(x)$ operator. We can therefore take the difference $t = r - r_c \propto T - T_c$ between the coefficient of $\phi^2$ in (2.8) and its critical value as a linear measure of the deviation from the critical temperature. Dimensional analysis then yields the relation

$$\Gamma^{(n)}(p_i; t, g, \Lambda) = \Lambda^{d-n(d-2)/2}\Gamma^{(n)}\left(p_i\Lambda^{-1}; t\Lambda^{-2}, g, 1\right). \qquad (2.35)$$

With this parametrization the critical domain corresponds to $|t| \ll \Lambda^2$.

*Expansion around the critical theory.* One thus adds to the critical action a term of the form $\frac{1}{2}t\int \mathrm{d}^d x\, \phi^2(x)$. To derive RG equations in the critical domain one expands correlation functions in formal power series of $t$. The coefficients are critical correlation functions involving $\phi^2(x)$, for which RG equations have been derived in section 2.4, inserted at zero momentum. Some care has to be taken to avoid obvious IR problems. Summing the expansion, one obtains RG equations valid for $T \neq T_c$, $|T - T_c| \ll 1$.

After summation of the $t$-expansion one finally obtains the RG equation:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \eta_2(g)\, t\frac{\partial}{\partial t}\right]\Gamma^{(n)}(p_i; t, g, \Lambda) = 0. \qquad (2.36)$$

*Scaling laws above $T_c$.* As for previous RG equations, equation (2.36) can be integrated by using the method of characteristics. In addition to the functions $g(\lambda)$ and $Z(\lambda)$, one needs a running temperature $t(\lambda)$. Taking the large $\Lambda$, or the small $\lambda$ limit one finally obtains:

$$\Gamma^{(n)}(p_i; t, g, \Lambda = 1) \underset{\substack{t\ll 1\\ |p_i|\ll 1}}{\sim} m^{(d-n(d-2+\eta)/2)} F_+^{(n)}(p_i/m), \qquad (2.37)$$

with:

$$m(\Lambda = 1) = \xi^{-1} \sim t^{\nu}. \qquad (2.38)$$

From equation (2.37) we infer that the quantity $m$ is proportional to the physical mass or inverse correlation length. Equation (2.38) then shows that the divergence of the correlation length $\xi = m^{-1}$ at $T_c$ is characterized by the exponent $\nu$.

For $t \neq 0$, the correlation functions are finite at zero momentum and behave as:

$$\Gamma^{(n)}(0; t, g, \Lambda) \propto t^{\nu(d-n(d-2+\eta)/2)}. \qquad (2.39)$$

In particular for $n = 2$ we obtain the inverse magnetic susceptibility:

$$\chi^{-1} = \Gamma^{(2)}(p = 0; t, g, \Lambda) \propto t^{\nu(2-\eta)}. \qquad (2.40)$$

The exponent which characterizes the divergence of $\chi$ is usually called $\gamma$. The equation (2.39) establishes the relation between exponents:

$$\gamma = \nu(2 - \eta). \qquad (2.41)$$

*2.6 Scaling laws in a magnetic field and below $T_c$*

In order to pass continuously from the disordered ($T > T_c$) to the ordered phase ($T < T_c$), avoiding the critical singularities at $T_c$, it is necessary to add to the action an interaction which explicitly breaks its symmetry. One thus add a small magnetic field to the spin interactions. One then derives RG equations in a field, or at fixed magnetization. In this way correlation functions above and below $T_c$ can be continuously connected, and scaling laws established in the whole critical domain. The first example is provided by the relation between field and magnetization, i.e. the equation of state.

*The equation of state.* Let us call $M$ the expectation value of $\phi(x)$ in a constant field $H$ (for $N > 1$ the quantities $M$ and $H$ should be regarded as the length of the corresponding vectors). The thermodynamic potential per unit volume, as a function of $M$, is by definition:

$$\Omega^{-1}\Gamma(M,t,g,\Lambda) = \sum_{n=0}^{\infty} \frac{M^n}{n!}\Gamma^{(n)}(p_i = 0; t, g, \Lambda) . \tag{2.42}$$

The magnetic field $H$ is given by:

$$H = \Omega^{-1}\frac{\partial\Gamma}{\partial M} = \sum_{n=1}^{\infty} \frac{M^n}{n!}\Gamma^{(n+1)}(p_i = 0; t, g, \Lambda) . \tag{2.43}$$

Noting that $n \equiv M\,(\partial/\partial M)$, we immediately derive from the RG equation (2.36), the RG equation satisfied by $H(M,t,g,\Lambda)$:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{1}{2}\eta(g)\left(1 + M\frac{\partial}{\partial M}\right) - \eta_2(g)\,t\frac{\partial}{\partial t}\right] H(M,t,g,\Lambda) = 0 . \tag{2.44}$$

To integrate equation (2.44) by the method of characteristics we have to introduce, in addition to the functions $g(\lambda)$, $t(\lambda)$ and $Z(\lambda)$, a new function $M(\lambda)$. However one verifies that $M(\lambda)$ is given by $M(\lambda) = MZ^{-1/2}(\lambda)$.

Then from the arguments outlined in previous sections one derives the scaling form

$$H(M,t,g,1) \sim M^{\delta} f\left(tM^{-1/\beta}\right), \tag{2.45}$$

with:

$$\beta = \tfrac{1}{2}\nu(d-2+\eta) = \nu d_{\phi}\,, \quad \delta = \frac{d+2-\eta}{d-2+\eta} = \frac{d}{d_{\phi}} - 1 . \tag{2.46}$$

Equation (2.45) exhibits the scaling properties of the equation of state. Moreover equations (2.46) relate the traditional critical exponents which characterize the vanishing of the spontaneous magnetization and the singular relation between magnetic field and magnetization at $T_c$ respectively to the exponents $\eta$ and $\nu$ introduced previously.

The universal function $f(x)$ is infinitely differentiable at $x = 0$. because when $M$ is different from zero the theory remains massive even at $t = 0$. The magnetic field $H$ has a regular expansion in odd powers of $M$ for $t > 0$. This implies that when the variable $x$ becomes large and positive, $f(x)$ has the expansion (Griffith's analyticity):

$$f(x) = \sum_{p=0}^{\infty} a_p x^{\gamma - 2p\beta}. \tag{2.47}$$

The appearance of a spontaneous magnetization, below $T_c$, implies that the function $f(x)$ has a negative zero $x_0$. Then equation (2.45) leads to the relation:

$$M = |x_0|^{-\beta} (-t)^{\beta} \quad \text{for} \quad H = 0\,, \quad t < 0\,. \tag{2.48}$$

Equation (2.48) gives the behaviour of the spontaneous magnetization when the temperature approaches the critical temperature from below.

*Correlation functions in a field.* We now examine the behaviour of correlation functions in a field. We write expressions for Ising-like systems. In the ordered phase some qualitative differences appear between systems which have a discrete and a continuous symmetry. We illustrate these differences with an example at the end of the section.

The correlation functions at fixed magnetization $M$ are obtained by expanding the generating functional $\Gamma(M(x))$ of 1PI correlation functions, around $M(x) = M$. From the RG equations satisfied by the correlation functions in zero magnetization (equations (2.36)) it is then easy to derive:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{1}{2}\eta(g)\left(n + M\frac{\partial}{\partial M}\right) - \eta_2(g)\, t \frac{\partial}{\partial t}\right] \Gamma^{(n)}(p_i; t, M, g, \Lambda) = 0\,. \tag{2.49}$$

This equation can be solved by exactly the same method as equation (2.44). One finds

$$\Gamma^{(n)}(p_i; t, M, g, \Lambda = 1) \sim m^{[d-(d-2+\eta)/2]} F^{(n)}\left(p_i/m, t m^{-1/\nu}\right), \tag{2.50}$$

for $|p_i| \ll 1$, $|t| \ll 1$, $M \ll 1$ and with the definition:

$$m = M^{\nu/\beta}. \tag{2.51}$$

The r.h.s. of equation (2.50) now depends on two different mass scales: $m = M^{\nu/\beta}$ and $t^{\nu}$.

*Correlation functions below $T_c$.* We have argued above that correlation functions are regular functions of $t$ for small $t$, provided $M$ does not vanish. It is therefore possible to cross the critical point and to then take the zero external magnetic field limit. In the limit $M$ becomes the spontaneous magnetization which is given, as a function of $t$, by equation (2.48). After elimination of $M$ in favour of $t$ in equation (2.50), one finds the critical behaviour below $T_c$:

$$\Gamma^{(n)}\left(p_i; t, M\left(t, H=0\right), g, 1\right) \sim m^{d-n(d-2+\eta)/2} F_-^{(n)}\left(p_i/m\right), \qquad (2.52)$$

with:

$$m = \left|x_0\right|^{-\nu}\left(-t\right)^{\nu}, \qquad H=0, \qquad t<0. \qquad (2.53)$$

We conclude that the correlation functions have exactly the same scaling behaviour above and below $T_c$.

The extension of these considerations to the functions with $\phi^2$ insertions, $\Gamma^{(l,n)}$ is straightforward. In particular the same method yields the behaviour of the specific heat below $T_c$:

$$\Gamma^{(2,0)}\left(q=0, M\left(H=0,t\right)\right) - \Lambda^{-\varepsilon} C_2\left(g\right) \underset{\text{for } t<0}{\sim} A^-\left(-t\right)^{-\alpha}, \qquad (2.54)$$

which similarly proves that the exponent above and below $T_c$ are the same.

Note that the constant term $\Lambda^{-\varepsilon} C_2\left(g\right)$ which depends explicitly on $g$ is the same above and below $T_c$, in contrast with the coefficient of the singular part.

The derivation of the equality of exponents above and below $T_c$, relies on the existence of a path which avoids the critical point, along which the correlation functions are regular, and the RG equations everywhere satisfied.

*Remark.* The universal functions characterizing the behaviour of correlation functions in the critical domain still depend on the normalization of physical parameters $t$, $H$, $M$, distances or momenta. Quantities which are independent of these normalizations are universal pure numbers. Simple examples are provided by the ratios of the amplitudes of the singularities above and below $T_c$ like $A^+/A^-$ for the specific heat.

*The $O(N)$-symmetric $N$-vector model.* We now indicate a few specific properties of models in which the action has a continuous $O(N)$ symmetry.

The differences concern correlation functions in a field or below $T_c$. The addition of a magnetic field term in an $O(N)$ symmetric action has various effects.

First, the magnetization and the magnetic field are now vectors. The RG equations have exactly the same form as the Ising-like $N=1$ case but the scaling forms derived previously apply to the modulus of these vectors.

Second, since magnetic field or magnetization distinguish a direction in vector space, there now exist $2^n$ $n$-point functions, each spin being either along the magnetization or orthogonal to it. When the continuous $O(N)$ symmetry of the action is broken linearly in the dynamical variables (as in the case of a magnetic field) these different correlation functions are related by a set of identities, called WT identities. The simplest one involves the 2-point function $\Gamma_{\mathrm{T}}^{-1}$, at zero momentum, of the components orthogonal to $\mathbf{M}$, i.e. the transverse susceptibility $\chi_{\mathrm{T}}$

$$\Gamma_{\mathrm{T}}\,(p=0)=\chi_{\mathrm{T}}^{-1}=H/M\,. \tag{2.55}$$

It follows that if $H$ goes to zero below $T_c$, $H/M$ and therefore $\Gamma_{\mathrm{T}}$ at zero momentum vanish. The latter property implies the existence of $N-1$ (massless) Goldstone modes corresponding to the spontaneous breaking of the $O(N)$ symmetry.

Note finally that the inverse longitudinal 2-point function $\Gamma_{\mathrm{L}}(p)$ has IR singularities at zero momentum in zero field below $T_c$ generated by the Goldstone modes. This is characteristic of continuous symmetries, and will play an essential role in next section.

### *2.7 Four dimensions: logarithmic corrections and triviality*

Let us just briefly comment about the situation in four dimensions. If we solve the RG equation

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda)=\beta\bigl(g(\lambda)\bigr),$$

for the running coupling constant, assuming that $\beta(g)$ remains positive for all $g>0$ (no non-trivial fixed point), we find that $g(\lambda)$ goes to zero logarithmically; the operator $\phi^4$ is marginally irrelevant. Writing generally

$$\beta(g)=\beta_2 g^2+\beta_3 g^3+O(g^4),\quad \beta_2>0\,,$$

we find for $\lambda\to 0$:

$$\ln\lambda=-\frac{1}{\beta_2 g(\lambda)}-\frac{\beta_3}{\beta_2^2}\ln g(\lambda)+K(g), \tag{2.56}$$

with:

$$K(g)=\frac{1}{\beta_2 g}-\frac{\beta_3}{\beta_2^2}\ln g-\int_0^g \mathrm{d}g'\left(\frac{1}{\beta(g')}-\frac{1}{\beta_2 g'^2}+\frac{\beta_3}{\beta_2^2 g'}\right).$$

Since the running coupling constant goes to zero in the long distance limit, quantities can be calculated from perturbation theory. From the point of view of critical phenomena logarithmic corrections to mean field theory are generated.

Finally let us note that empirical evidence coming from lattice calculations strongly suggests the absence of any other fixed point.

From the point of view of particle physics one faces the *triviality* problem: for any initial bare coupling constant $g$ the renormalized coupling $g(\mu/\Lambda)$ at scale $\mu$ much smaller than the cut-off $\Lambda$ behaves like

$$g(\mu/\Lambda)\sim\frac{1}{\beta_2\ln(\Lambda/\mu)}, \tag{2.57}$$

Therefore if one insists in sending the cut-off to infinity one finds a free (trivial) field theory. However in the modern point of view of *effective* field theories, one accepts the idea that quantum field theories may not be consistent on all scales but only in a limited range. Then the larger is the range the smaller is the low energy effective coupling constant. In the standard model these comments may apply to the weak-electromagnetic sector which contains a $\phi^4$ interaction and trivial QED.

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## 3 The $O(N)$ Spin Model at Low Temperature: the Non-Linear $\sigma$-Model

Let us again consider the lattice model (2.1,2.2) of section 2 with partition function:

$$Z = \int \prod_i \mathrm{d}\mathbf{S}_i \, \delta\left(\mathbf{S}_i^2 - 1\right) \exp \sum_{ij} V_{ij} \mathbf{S}_i \cdot \mathbf{S}_j / T \,.$$

We will now discuss this model from the point of view of a low temperature expansion. The methods we employ, however, apply only to continuous symmetries, here to $N \geq 2$. They rely on the property that models with continuous symmetries, in contrast to models with discrete symmetries, have a non-trivial long distance physics at any temperature below $T_c$, due to the massless Goldstone modes.

We first prove universal properties of the low temperature, ordered, phase at fixed temperature. Then, in the non-abelian case, $N > 2$, we show that additional information about critical properties can be obtained, by analyzing the instability of the ordered phase at low temperature and near two dimensions, due to Goldstone mode interactions.

The analysis is based on the following observation: The $N$-vector model (2.1,2.2) can be considered as a lattice regularization of the non-linear $\sigma$-model (note $2\mathbf{S}_i \cdot \mathbf{S}_j = 2 - (\mathbf{S}_i - \mathbf{S}_j)^2$). The low temperature expansion of the lattice model is the perturbative expansion of the regularized field theory. The field theory is renormalizable in dimension two. RG equations, valid in two and more generally $2 + \varepsilon$ dimension follow. Their solutions will help us to understand the long distance behaviour of correlation functions.

It is somewhat surprising that two different continuum field theories, the $(\phi^2)^2$ and the non-linear $\sigma$-model describe the long distance physics of the same lattice model. This point will be clarified by an analysis of the $1/N$-expansion of both field theories. This property, totally mysterious at the classical level, emphasizes the essential nature of quantum (or statistical) fluctuations.

### *3.1 The non-linear $\sigma$-model*

We now study the non-linear $\sigma$-model from the point of view of renormalization and renormalization group. In continuum notation the field $\mathbf{S}(x)$ has unit length and the action is

$$\mathcal{S}(\mathbf{S}) = \frac{1}{2t} \int \mathrm{d}^d x \, \partial_\mu \mathbf{S}(x) \cdot \partial_\mu \mathbf{S}(x),$$

where $t$ is proportional to the temperature $T$. To generate perturbation theory we parametrize the field $\mathbf{S}(x)$:

$$\mathbf{S}(x) = \{\sigma(x), \boldsymbol{\pi}(x)\},$$

and eliminate locally the field $\sigma(x)$ by:

$$\sigma(x) = \left(1 - \boldsymbol{\pi}^2(x)\right)^{1/2} .$$

This parametrization is singular but this does not show up in perturbation theory which assumes $\pi(x)$ small.

*The $O(N)$ symmetry.* The $O(N-1)$ subgroup which leaves the component $\sigma$ invariant acts linearly on the $N-1$ component vector $\pi$. However a general $O(N)$ transformation will transform $\pi$ into a linear combination of $\pi$ and $\sqrt{1-\pi^2}$. The $O(N)$ symmetry is realized non-linearly. An infinitesimal transformation corresponding to the generators of $O(N)$ not belonging to $O(N-1)$ takes the form

$$\delta\pi = \omega\sqrt{1-\pi^2},$$

where $\omega$ is a $N-1$ component vector of parameters corresponding to these generators.

As we have done for the $\left(\phi^2\right)^2$ model, we scale all distances in order to measure momenta in units of the inverse lattice spacing $\Lambda$. We thus write the partition function:

$$Z = \int \left[\left(1-\pi^2(x)\right)^{-1/2} \mathrm{d}\pi(x)\right] \exp\left[-\mathcal{S}(\pi)\right], \tag{3.1}$$

with

$$\mathcal{S}(\pi) = \frac{\Lambda^{d-2}}{2t} \int \mathrm{d}^d x \, g_{ij}(\pi) \partial_\mu \pi_i(x) \partial_\mu \pi_j(x), \tag{3.2}$$

where $g_{ij}$ is the metric on the sphere

$$g_{ij} = \delta_{ij} + \frac{\pi_i \pi_j}{1-\pi^2}. \tag{3.3}$$

Moreover, as expected, the functional measure is related to the metric by

$$\sqrt{\det(g_{ij})} = \frac{1}{\sqrt{1-\pi^2}}\,.$$

*Propagator, perturbation theory and power counting.* Unlike the $\phi^4$ field theory, the action is non-polynomial in the fields. An expansion of the action in powers of $\pi$ generates an infinite number of interactions. However we note that the power of $t$ in front of a diagram counts the number of loops. Therefore at a finite loop order, only a finite number of interactions contribute.

The $\pi$ propagator is proportional to:

$$\Delta_\pi(k) = \frac{t\Lambda^{2-d}}{k^2},$$

The $\pi$ thus has the usual canonical dimension $(d-2)/2$. Since we have interactions with arbitrary powers of $\pi$, the model is renormalizable in two dimensions, where all interactions have dimension two.

*The role of the functional measure.* If we try to write the functional measure as an additional interaction we find

$$\prod_x \frac{1}{\sqrt{1-\pi^2(x)}} = \exp\bigl(-\tfrac{1}{2}\sum_x \ln(1-\pi^2(x))\bigr).$$

This quantity is well-defined on the lattice but not in the continuum. This problem, which already appears in quantum mechanics ($d=1$) reflects the necessity of a lattice regularization to precisely define the quantum hamiltonian in the presence of interactions with derivatives. A perturbative solution is provided by dimensional regularization, where this term can simply be omitted. In lattice regularization it cancels quadratic divergences.

*IR divergences, spontaneous symmetry breaking and the role of dimension two.* We see that the perturbative phase of the non-linear $\sigma$ model is automatically a phase in which the $O(N)$ symmetry is spontaneously broken, and the $(N-1)$ components of $\mathbf{S}(x)$, $\pi(x)$, are massless Goldstone modes.

(i) For $d \le 2$ we know from the Mermin–Wagner theorem that SSB with ordering ($\langle \mathbf{S} \rangle \neq 0$) is impossible in a model with continuous symmetry and short range forces. Correspondingly IR divergences appear in the perturbative expansion of the non-linear $\sigma$ model for $d \le 2$, for example $\langle \sigma \rangle$ diverges at order $t$ as $\int \mathrm{d}^d p/p^2$. For $d \le 2$ the critical temperature $T_c$ vanishes and perturbation theory makes sense only in presence of an IR cut-off which breaks explicitly the symmetry and orders the spins (thus selecting a classical minimum of the action). Therefore nothing can be said about the long distance properties of the unbroken theory directly from perturbation theory.

(ii) For $d > 2$ instead, perturbation theory which predicts spontaneous symmetry breaking (SSB), is not IR divergent. This is consistent with the property that in the $N$-vector model, for $d > 2$, the $O(N)$ symmetry is spontaneously broken at low temperature. At $T < T_c$ fixed, the large distance behaviour of the theory is dominated by the massless or spin wave excitations. On the other hand nothing can be said, in perturbation theory, of a possible critical region $T \sim T_c$.

To go somewhat beyond perturbation theory we shall use field theory RG methods. It is therefore necessary to first define the model in two dimensions where it is renormalizable. There IR divergences have to be dealt with. We thus introduce an IR cut-off in the form of a magnetic field in the $\sigma$ direction (a constant source for the $\sigma$ field)

$$\mathcal{S}(\pi,h) = \frac{\Lambda^{d-2}}{t}\int \mathrm{d}^d x \left\{ \frac{1}{2}\left[ (\partial_\mu \pi(x))^2 + \frac{(\pi\cdot\partial_\mu\pi(x))^2}{1-\pi^2(x)} \right] - h\sqrt{1-\pi^2(x)} \right\}. \tag{3.4}$$

Expanding the additional term in powers of $\pi$ we see that it generates a mass term

$$\Delta_\pi(k) = \frac{t\Lambda^{2-d}}{k^2+h},$$

and additional interactions of dimension 0 in $d = 2$.

We then proceed in formal analogy with the case of the $(\phi^2)^2$ field theory, i.e. study the theory in $2+\varepsilon$ dimension as a double series expansion in the temperature $t$ and $\varepsilon$. In this way the perturbative expansion is renormalizable and RG equations follow.

*3.2 RG equations*

Using power counting and some non-trivial WT identies (quadratic in the 1PI functional) one can show that the renormalized action takes the form:

$$\mathcal{S}_{\mathrm{r}}(\pi_{\mathrm{r}}, h_{\mathrm{r}}) = \frac{\mu^{d-2} Z}{2t_{\mathrm{r}} Z_t} \int \mathrm{d}^d x \left[ (\partial_\mu \pi_{\mathrm{r}})^2 + (\partial_\mu \sigma_{\mathrm{r}})^2 \right] - \frac{\mu^{d-2}}{t_{\mathrm{r}}} h_{\mathrm{r}} \int \sigma_{\mathrm{r}}(x) \mathrm{d}^d x \,, \tag{3.5}$$

in which $\mu$ is the renormalization scale and:

$$\sigma_{\mathrm{r}}(x) = \left[ Z^{-1} - \pi_{\mathrm{r}}^2 \right]^{1/2}. \tag{3.6}$$

Note that the renormalization constants can and thus will be chosen $h$ independent. This is automatically realized in the minimal subtraction scheme.

The relation:

$$\pi_{\mathrm{r}}(x) = Z^{-1/2} \pi(x), \tag{3.7}$$

implies

$$\mu^{d-2} \frac{h_{\mathrm{r}}}{t_{\mathrm{r}}} = \Lambda^{d-2} Z^{1/2} \frac{h}{t}. \tag{3.8}$$

With our conventions the coupling constant, which is proportional to the temperature, is dimensionless. The relation between the cut-off dependent and the renormalized correlation functions is:

$$Z^{n/2}(\Lambda/\mu, t)\, \Gamma^{(n)}(p_i; t, h, \Lambda) = \Gamma_{\mathrm{r}}^{(n)}(p_i; t_{\mathrm{r}}, h_{\mathrm{r}}, \mu)\,. \tag{3.9}$$

Differentiating with respect to $\Lambda$ at renormalized parameters fixed, we obtain the RG equations:

$$\left[ \Lambda \frac{\partial}{\partial \Lambda} + \beta(t) \frac{\partial}{\partial t} - \frac{n}{2} \zeta(t) + \rho(t) h \frac{\partial}{\partial h} \right] \Gamma^{(n)}(p_i; t, h, \Lambda) = 0\,, \tag{3.10}$$

where the RG functions are defined by:

$$\begin{aligned} \beta(t) &= \Lambda \frac{\partial}{\partial \Lambda} \bigg|_{\text{ren. fixed}} t\,, \\ \zeta(t) &= \Lambda \frac{\partial}{\partial \Lambda} \bigg|_{\text{ren. fixed}} (-\ln Z)\,, \\ \rho(t) &= \Lambda \frac{\partial}{\partial \Lambda} \bigg|_{\text{ren. fixed}} \ln h\,. \end{aligned} \tag{3.11}$$

The coefficient of $\partial/\partial h$ can be derived from equation (3.8) which implies (taking the logarithm of both members):

$$0 = h^{-1}\Lambda\frac{\partial}{\partial\Lambda}h + d - 2 - \frac{1}{2}\zeta(t) - \frac{\beta(t)}{t}, \tag{3.12}$$

and therefore:

$$\rho(t) = 2 - d + \frac{1}{2}\zeta(t) + \frac{\beta(t)}{t}. \tag{3.13}$$

To be able to discuss correlation functions involving the $\sigma$-field, we also need the RG equations satisfied by connected correlation functions $W^{(n)}$:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(t)\frac{\partial}{\partial t} + \frac{n}{2}\zeta(t) + \left(\frac{1}{2}\zeta(t) + \frac{\beta(t)}{t} - \varepsilon\right)h\frac{\partial}{\partial h}\right]W^{(n)} = 0\,, \tag{3.14}$$

in which we now have set:

$$d = 2 + \varepsilon\,. \tag{3.15}$$

The two RG functions can be obtained at one-loop order from a calculation of the 2-point function $\Gamma^{(2)}$:

$$\Gamma^{(2)}(p) = \frac{\Lambda^{\varepsilon}}{t}\left(p^2 + h\right) + \left[p^2 + \tfrac{1}{2}(N-1)h\right]\frac{1}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d q}{q^2 + h} + O(t)\,. \tag{3.16}$$

Applying the RG equation (3.10) to $\Gamma^{(2)}$ and identifying the coefficients of $p^2$ and $h$, we derive two equations which determine $\beta(t)$ and $\zeta(t)$ at one-loop order

$$\beta(t) = \varepsilon t - \frac{(N-2)}{2\pi}t^2 + O\left(t^3, t^2\varepsilon\right), \tag{3.17a}$$

$$\zeta(t) = \frac{(N-1)}{2\pi}t + O\left(t^2, t\varepsilon\right). \tag{3.17b}$$

*3.3 Discussion of the RG flow*

From the expression of $\beta(t)$ in equation (3.17$a$) we immediately conclude:

For $d \le 2\,(\varepsilon \le 0)$, $t = 0$ is an unstable IR fixed point, the IR instability being induced by the vanishing mass of the would-be Goldstone bosons. The spectrum of the theory thus is not given by perturbation theory and the perturbative assumption of spontaneous symmetry breaking at low temperature is inconsistent. As mentioned before, this result agrees with rigorous arguments. Note that since the model depends only on one coupling constant, $t = 0$ is also a UV stable fixed point (the property of large momentum asymptotic freedom). Section 3.5 contains a short discussion of the physics in two dimensions for $N > 2$. The abelian case $N = 2$ is special and has to be discussed separately.

For $d > 2$, i.e. $\varepsilon > 0$, $t = 0$ is a stable IR fixed point, the $O(N)$ symmetry is spontaneously broken at low temperature in zero field. The effective coupling constant, which determines the large distance behaviour, approaches the origin for all temperatures $t < t_c$, $t_c$ being the first non-trivial zero of $\beta(t)$. Therefore the large distance properties of the model can be obtained from low temperature expansion and renormalization group, replacing the perturbative parameters by effective parameters obtained by solving the RG equations.

*The critical temperature.* Finally we observe that, at least for $\varepsilon$ positive and small, and $N > 2$, the RG function $\beta(t)$ has a non-trivial zero $t_c$:

$$t_c = \frac{2\pi\varepsilon}{N-2} + O\left(\varepsilon^2\right) \Rightarrow \beta\left(t_c\right) = 0\,, \text{ and } \quad \beta'\left(t_c\right) = -\varepsilon + O\left(\varepsilon^2\right). \tag{3.18}$$

Since $t_c$ is an unstable IR fixed point, it is by definition a critical temperature. Consequences of this property are studied below. Let us only immediately note that $t_c$ is also a UV fixed point, i.e. it governs the large momentum behaviour of the renormalized theory. The large momentum behaviour of correlation functions is not given by perturbation theory but by the fixed point. As a consequence the perturbative result that the theory cannot be rendered finite for $d > 2$ with a finite number of renormalization constants, cannot be trusted.

We now discuss more precisely the solutions of the RG equations.

### *3.4 Integration of the RG equations*

We first examine the implications of the RG equations for the large distance behaviour of correlation functions for $d > 2$ where $t = 0$ is an IR fixed point. Equation (3.10) can be solved as usual by the method of characteristics, i.e. by introducing a scaling parameter $\lambda$ and running parameters. It is here convenient to proceed somewhat differently by looking for a solution of the form

$$\Gamma^{(n)}\left(p_i; t, h, \Lambda\right) = \xi^{-d}(t) M_0^{-n}(t) F^{(n)}\left(p_i \xi(t), h/h_0(t)\right). \tag{3.19}$$

The ansatz (3.19) solves the RG equations provided the unknown functions $M_0(t)$, $\xi(t)$ and $h_0(t)$ are chosen to be

$$M_0(t) = \exp\left[-\frac{1}{2}\int_0^t \frac{\zeta\left(t'\right)}{\beta\left(t'\right)} \mathrm{d}t'\right], \tag{3.20}$$

$$\xi(t) = \Lambda^{-1} t^{1/\varepsilon} \exp\left[\int_0^t \left(\frac{1}{\beta\left(t'\right)} - \frac{1}{\varepsilon t'}\right) \mathrm{d}t'\right], \tag{3.21}$$

with then

$$h_0(t) = t M_0^{-1}(t) \xi^{-d}(t) \Lambda^{2-d}. \tag{3.22}$$

Note that the function $\xi(t)$ has in zero field the nature of a correlation length.

For the connected correlation functions the same analysis leads to:

$$W^{(n)}(p_i;t,H,\Lambda)=\xi^{d(n-1)}(t)M_0^n(t)G^{(n)}(p_i\xi(t),h/h_0(t)). \tag{3.23}$$

It is convenient to also introduce the function $K(t)$

$$K(t)=M_0(t)\left[\Lambda\xi(t)\right]^{d-2}/t=1+O(t). \tag{3.24}$$

Combining equation (3.19) with dimensional analysis we can rewrite the scaling relation in an equivalent form

$$\Gamma^{(n)}(p_i,t,h,\Lambda)\sim M_0^{-n}(t)[K(t)h]^{d/2} \\ \times\Gamma^{(n)}\left(\frac{p_i}{[K(t)h]^{1/2}},\frac{t\left[K(t)\right]^{d/2}}{M_0(t)}\left(\frac{h}{\Lambda^2}\right)^{(d-2)/2},1,1\right) \tag{3.25}$$

Let us apply this result to the determination of the singularities near the coexistence curve, i.e. at $t$ fixed below the critical temperature when the magnetic field $h$ goes to zero.

*The coexistence curve.* The magnetization is given by

$$M(t,h,\Lambda)\equiv\langle\sigma(x)\rangle=\Lambda^{-\varepsilon}t\frac{\partial\Gamma^{(0)}}{\partial h}, \tag{3.26}$$

($\Gamma^{(0)}$ is the magnetic field dependent free energy). At one-loop order in a field one finds

$$M=1-\frac{N-1}{2}\Lambda^{-\varepsilon}t\frac{1}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^dq}{q^2+h}+O\left(t^2\right).$$

Thus from relation (3.25) follows

$$M(t,h,\Lambda=1)=M_0(t)-\frac{N-1}{2}t\left[K(t)\right]^{d/2}h^{(d-2)/2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}+O\left(h,h^{d-2}\right).$$

This result shows that $M_0(t)$ is the spontaneous magnetization and displays the singularity of the scaling equation of state (section 2.6) on the coexistence curve ($H=0$, $T<T_c$) for $N>1$, and in all dimensions $d>2$.

*The equation of state in the critical domain.* Let now instead use the scaling form (3.19)

$$M=\Lambda^{2-d}t\frac{\partial\Gamma^{(0)}}{\partial h}=M_0(t)F^{(0)}(h/h_0(t)). \tag{3.27}$$

Inversion of this relation yields the scaling form of the equation of state:

$$h=h_0(t)f\left(\frac{M}{M_0(t)}\right), \tag{3.28}$$

and the 1PI correlation functions can thus be written in terms of the magnetization as:

$$\Gamma^{(n)}\left(p_i,t,M,\Lambda\right)=\xi^{-d}\left(t\right)M_0^{-n}(t)F^{(n)}\left(p_i\xi\left(t\right),\frac{M}{M_0(t)}\right). \tag{3.29}$$

The equations (3.28,3.29) are consistent with the equations (2.45,2.50): the appearance of two different functions $\xi(t)$ and $M_0(t)$ corresponds to the existence of two independent critical exponents $\nu,\beta$ in the $(\phi^2)^2$ field theory. They extend, in the large distance limit, the scaling form of correlation functions, valid in the critical region, to all temperatures below $t_c$. There is however one important difference between the RG equations of the $\left(\phi^2\right)^2$ theory and of the $\sigma$-model: the $\left(\phi^2\right)^2$ theory depends on two coupling constants, the coefficient of $\phi^2$ which plays the role of the temperature, and the coefficient of $\left(\phi^2\right)^2$ which has no equivalent here. The correlation functions of the continuum $\left(\phi^2\right)^2$ theory have the exact scaling form (3.29) only at the IR fixed point. In contrast, in the case of the $\sigma$-model, it has been possible to eliminate all corrections to scaling corresponding to irrelevant operators order by order in perturbation theory. We are therefore led to a remarkable conclusion: the correlation functions of the $O(N)$ non-linear model are identical to the correlation functions of the $\left(\phi^2\right)^2$ field theory at the IR fixed point. This conclusion is supported by the analysis of the scaling behaviour performed within the $1/N$ expansion (see equation (4.53)).

*Critical exponents.* Let us now study more precisely what happens when $t$ approaches $t_c$ (for $N>2$). The function $\xi(t)$ diverges as:

$$\xi(t)\sim\Lambda^{-1}\left(t_c-t\right)^{1/\beta'(t_c)}. \tag{3.30}$$

We conclude that the correlation length exponent $\nu$ is given by

$$\nu=-\frac{1}{\beta'\left(t_c\right)}. \tag{3.31}$$

For $d$ close to 2 the exponent $\nu$ thus behaves like:

$$\nu\sim 1/\varepsilon\,. \tag{3.32}$$

The function $M_0(t)$ vanishes at $t_c$:

$$\ln M_0(t)=-\frac{1}{2}\frac{\zeta\left(t_c\right)}{\beta'\left(t_c\right)}\ln\left(t_c-t\right)+\ \text{const.}\,. \tag{3.33}$$

This yields the exponent $\beta$ and thus also $\eta$ through the scaling relation $\beta=\frac{1}{2}\nu(d-2+\eta)$:

$$\eta=\zeta\left(t_c\right)-\varepsilon\,. \tag{3.34}$$

A leading order we find:

$$\eta = \frac{\varepsilon}{N-2} + O\left(\varepsilon^2\right). \tag{3.35}$$

We finally note that the singularity of $\Gamma^{(n)}$ coming from the prefactor $\xi^{-d} M_0^{-n}$ indeed agrees near $t_c$ with the result of equation (2.37).

Consideration of operators with four derivatives allows also to calculate the exponent $\omega$ which characterizes leading corrections to scaling. One finds

$$\omega = 4 - d - 2\varepsilon/(N-2) + O(\varepsilon^2).$$

*The nature of the correlation length* $\xi(t)$. The length scale $\xi(t)$ is a cross-over scale between two different behaviours of correlation functions. For distances large compared to $\xi(t)$, the behaviour of correlation functions is governed by the Goldstone modes (spin wave excitations) and can thus be deduced from the perturbative low temperature expansion. However when $t$ approaches $t_c$, $\xi(t)$ becomes large. There then exist distances large with respect to the microscopic scale but small with respect to $\xi(t)$ in which correlation functions have a critical behaviour. In this situation we can construct continuum correlation functions consistent on all scales, the critical behaviour being also the large momentum behaviour of the renormalized field theory.

*General comment.* From the consideration of the low temperature expansion we have been able to describe, for theories with a continuous symmetry, not only the complete structure of the low temperature phase, and this was expected, but also, in the non-abelian case, the critical behaviour near two dimensions .

This result is somewhat surprising: Indeed perturbation theory is only sensitive to the local structure of the sphere $\mathbf{S}^2 = 1$ while the restoration of symmetry involves the sphere globally. This explains the peculiarity of the abelian case $N = 2$ because locally a circle cannot be distinguished from a non-compact straight line. For $N > 2$ the sphere has instead a local characteristic curvature. Still different regular compact manifolds may have the same local metric, and therefore the same perturbation theory. They all have the same low temperature physics. However the previous results concerning the critical behaviour are physically relevant only if they are still valid when $\varepsilon$ is not infinitesimal and $t$ approaches $t_c$, a condition which cannot be checked directly. In particular the low temperature expansion misses in general terms decreasing like $\exp(\text{const.}/t)$ which may in some cases be essential for the physics. Therefore in section 4.6 we will establish a direct relation between the $O(N)$ $\sigma$ model and the $\left(\phi^2\right)^2$ field theory to all orders in the large $N$ expansion. This gives us some confidence that the previous considerations are valid for the $N$-vector model at least for $N$ sufficiently large. On the other hand the physics of $N = 2$ is not well reproduced. Cardy and Hamber have speculated about the RG flow for $N$ close to 2 and dimension $d$ close to 2, incorporating phenomenologically the Kosterlitz–Thouless transition in their analysis.

*3.5 The dimension two*

Dimension two is of special interest from the particle physics point of view. The RG function $\beta(t)$ is then:

$$\beta(t) = -\frac{(N-2)}{2\pi} t^2 + O\left(t^3\right). \tag{3.36}$$

The non-linear $\sigma$-model for $N > 2$ is the simplest example of a so-called asymptotically free field theory (UV free) since the first coefficient of the $\beta$-function is negative, in contrast with the $\phi^4$ field theory. Therefore the large momentum behaviour of correlation functions is entirely calculable from perturbation theory and RG arguments. There is, however a counterpart, the theory is IR unstable and thus, in zero field $h$, the spectrum of the theory is not perturbative. Contrary to perturbative indications, it consists of $N$ massive degenerate states since the $O(N)$ symmetry is not broken. Asymptotic freedom and the non-perturbative character of the spectrum are also properties of QCD, the theory of strong interactions, in four dimensions, .

If we now define a function $\xi(t)$ by:

$$\xi(t) = \mu^{-1} \exp\left[\int^t \frac{\mathrm{d}t'}{\beta\left(t'\right)}\right], \tag{3.37}$$

we can again integrate the RG equations and we find that $\xi(t)$ is the correlation length in zero field. In addition we can use the explicit expression of the $\beta$-function to calculate the correlation length or the physical mass for small $t$:

$$\xi^{-1}(t) = m(t) = K\mu t^{-1/(N-2)} \,\mathrm{e}^{-2\pi/[(N-2)t]} \left(1 + O(t)\right). \tag{3.38}$$

However the exact value of the integration constant $K$, which gives the physical mass in the RG scale, can only be calculated by non-perturbative techniques.

Finally the scaling forms (3.19,3.23) imply that the perturbative expansion at fixed magnetic field is valid, at low momenta or large distances, and for $h/h_0(t)$ large.

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## 4 $\left(\phi^2\right)^2$ Field Theory and Non-Linear $\sigma$ Model in the Large $N$ Limit

In the preceding sections we have derived universal properties of critical systems within the frameworks of the formal $\varepsilon = 4 - d$ and $\varepsilon = d - 2$ expansions (at least for $N > 2$). It is therefore reassuring to verify, at least in some limiting case, that the results obtained in this way remain valid even when $\varepsilon$ is no longer infinitesimal. We show in this section that, in the case of the $O(N)$ symmetric $\left(\phi^2\right)^2$ field theory, the same universal properties can also be derived at fixed dimension in the large $N$ limit, and more generally order by order in the large $N$-expansion. We then examine the non-linear $\sigma$-model in the same limit.

### *4.1 Introduction*

We again consider the partition function:

$$Z = \int [\mathrm{d}\phi(x)] \exp\left[-S(\phi)\right], \tag{4.1}$$

where $S(\phi)$ is the $O(N)$ symmetric action (2.8) ($u = \Lambda^{4-d} g$):

$$S(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{u}{4!} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x\,. \tag{4.2}$$

A cut-off $\Lambda$, consistent with the symmetry, is implied.

The solution of the model in the large $N$ limit is based on a idea of mean field theory type: it can be expected that for $N$ large the $O(N)$ invariant quantities self-average and therefore have small fluctuations. Thus for example

$$\left\langle \phi^2(x) \phi^2(y) \right\rangle \underset{N\to\infty}{\sim} \left\langle \phi^2(x) \right\rangle \left\langle \phi^2(y) \right\rangle .$$

This suggests to take $\phi^2(x)$ as a dynamical variable. Technically, in the case of the $\left(\phi^2\right)^2$ theory, this can be achieved by using an identity similar to the Hubbard transformation:

$$\exp\left[\frac{1}{2} r \phi^2 + \frac{u}{4!} \left(\phi^2\right)^2\right] \propto \int \mathrm{d}\lambda \exp\left(\frac{3}{2u} \lambda^2 - \frac{3r}{u} \lambda - \frac{1}{2} \lambda \phi^2\right), \tag{4.3}$$

where the integration contour is parallel to the imaginary axis. By introducing a field $\lambda(x)$ the identity can be used for each point $x$ inside the functional integral (4.1). The new functional integral is then gaussian in $\phi$ and the integral over the field $\phi$ can be performed. The dependence on $N$ of the partition function becomes explicit. Actually it is convenient to separate the components of $\phi$ into one component $\sigma$, and $N-1$ components $\pi$, and integrate only over $\pi$ (for $T < T_c$ it may even be convenient to integrate over only $N-2$ components). For $N$ large the difference is negligible. To generate $\sigma$ correlation functions we also add a source $H(x)$ to the action

$$Z(H)=\int[\mathrm{d}\lambda(x)]\,[\mathrm{d}\sigma(x)]\exp\left[-S_N(\lambda,\sigma)+\int\mathrm{d}^d x\,H(x)\sigma(x)\right], \tag{4.4}$$

with:

$$S_N(\lambda,\sigma)=\int\left[\frac{1}{2}(\partial_\mu\sigma)^2-\frac{3}{2u}\lambda^2(x)+\frac{3r}{u}\lambda(x)+\frac{1}{2}\lambda(x)\sigma^2(x)\right]\mathrm{d}^d x + \frac{(N-1)}{2}\operatorname{tr}\ln\left[-\Delta+\lambda(\cdot)\right]. \tag{4.5}$$

*$\lambda$-field correlation functions.* In this formalism it is natural to also calculate correlation functions involving the $\lambda$-field. These have a simple interpretation in the initial $\phi$-field formalism. Indeed let us add a source $j_\lambda$ for $\lambda$ in the action (4.5). Then reintroducing the $\phi$-field and integrating over $\lambda$ we recover instead of action (4.2),

$$S(\phi)-(u/6)\phi^2 j_\lambda+(u/6)j_\lambda^2-rj_\lambda\,. \tag{4.6}$$

Therefore $j_\lambda$ generates the $\phi^2$ correlation functions, up to a multiplicative factor and a translation of the connected 2-point function.

*4.2 Large N limit: the critical domain*

We now take the large $N$ limit at $Nu$ fixed. With this condition $S_N$ is of order $N$ and the functional integral can be calculated for $N$ large by steepest descent. We expect $\sigma=O(N^{1/2})$, $\lambda=O(1)$. We look for a uniform saddle point ($\sigma(x),\lambda(x)$ space-independent),

$$\sigma(x)=\sigma\,,\qquad \lambda(x)=\lambda\,.$$

Differentiating then action (4.5) with respect to $\sigma$ and $\lambda$ we obtain the saddle point equations:

$$\lambda\sigma=0\,, \tag{4.7a}$$

$$\frac{\sigma^2}{N}-\frac{6}{Nu}(\lambda-r)+\frac{1}{(2\pi)^d}\int^\Lambda\frac{\mathrm{d}^d p}{p^2+\lambda}=0\,. \tag{4.7b}$$

*Remark.* In the large $N$ limit the leading perturbative contributions come from chains of "bubble" diagrams of the form displayed in figure 1. These diagrams form asymptotically a geometrical series which is summed by the algebraic techniques explained above.

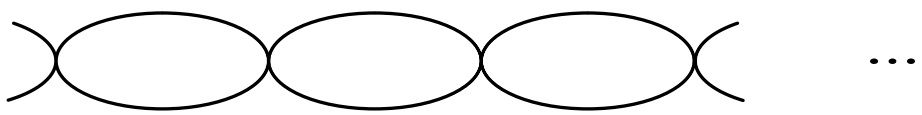

Fig. 1 Leading diagrams in the limit $N\to\infty$.

*The low temperature phase.* Equation (4.7*a*) implies either $\sigma = 0$ or $\lambda = 0$. In the low temperature phase $\sigma$, the average value of the field, does not vanish. Equation (4.7*b*) then yields:

$$\frac{\sigma^2}{N} = -\frac{6}{Nu} r - \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d p}{p^2} . \tag{4.8}$$

Note that this equation has solutions only for $d > 2$. This is a manifestation of the Mermin–Wagner–Coleman theorem: in a system with only short range forces a continuous symmetry cannot be broken for $d \le 2$, in the sense that the average $\sigma$ of the order parameter necessarily vanishes. Physically the would-be Goldstone modes are responsible for this property: being massless, as we know from general arguments and as the propagator in the r.h.s. of (4.8) confirms, they induce an IR instability for $d \le 2$.

Setting

$$r_c = -\frac{Nu}{6} \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d p}{p^2} , \tag{4.9}$$

$$r = r_c + (u/6)\, \tau , \tag{4.10}$$

we can rewrite equation (4.8):

$$\sigma^2 = -\tau = (-\tau)^{2\beta} \quad \text{with } \beta = \tfrac{1}{2} \cdot \tag{4.11}$$

The expectation value of the field vanishes for $r = r_c$, which therefore corresponds to the critical temperature. Moreover we find that for $N$ large the exponent $\beta$ remains classical, i.e. mean-field like, in all dimensions.

*The high temperature phase.* Above $T_c$, $\sigma$ vanishes. In expression (4.5) we see that the $\sigma$-propagator then becomes

$$\Delta_\sigma = \frac{1}{p^2 + \lambda} . \tag{4.12}$$

Therefore $\lambda^{1/2}$ is at this order the physical mass, i.e. the inverse correlation length $\xi^{-1}$ of the field $\sigma$

$$m = \xi^{-1} = \lambda^{1/2} . \tag{4.13}$$

From equation (4.7*b*) we can verify that $\partial r / \partial \lambda$ is positive. The minimum value of $r$, obtained for $\lambda = 0$, is $r_c$. Using equations (4.9,4.10) in equation (4.7*b*) we then find:

$$\frac{6}{u} + \frac{N}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d p}{p^2 \left(p^2 + m^2\right)} = \frac{\tau}{m^2} . \tag{4.14}$$

(i) For $d > 4$ the integral in (4.14) has a limit for $m = 0$ and therefore at leading order:

$$m^2 = \xi^{-2} \sim \tau \quad \text{and thus} \quad \nu = \tfrac{1}{2}\,, \tag{4.15}$$

which is the mean field result.

(ii) For $2 < d < 4$ instead, the integral behaves for $m$ small like (setting $d = 4 - \varepsilon$):

$$D_1(m^2) \equiv \frac{1}{(2\pi)^d} \int^\Lambda \frac{\mathrm{d}^d p}{p^2\,(p^2 + m^2)} = C(d) m^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + O\left(m^{2-\varepsilon}\Lambda^{-2}\right), \tag{4.16}$$

with

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \tag{4.17a}$$

$$C(d) = -\frac{\pi}{2\sin(\pi d/2)} N_d\,, \tag{4.17b}$$

where we have introduced for convenience the usual loop factor $N_d$. The constant $a(d)$ which characterizes the leading correction in equation (4.16), depends explicitly on the regularization, i.e. the way large momenta are cut.

The leading contribution, for $m \to 0$, to the l.h.s. of equation (4.14) now comes from the integral. Keeping only the leading term in (4.16) we obtain:

$$m = \xi^{-1} \sim \tau^{1/(2-\varepsilon)}, \tag{4.18}$$

which shows that the exponent $\nu$ is not classical:

$$\nu = \frac{1}{2 - \varepsilon} = \frac{1}{d-2}\,. \tag{4.19}$$

(iii) For $d = 4$ the l.h.s. is still dominated by the integral:

$$D_1(m^2) = \frac{1}{(2\pi)^4} \int^\Lambda \frac{\mathrm{d}^4 p}{p^2\,(p^2 + m^2)} \underset{m \to 0}{\sim} \frac{1}{8\pi^2} \ln(\Lambda/m).$$

The correlation length no longer has a power law behaviour but instead a mean-field behaviour modified by a logarithm. This is typical of a situation where the gaussian fixed point is stable, in the presence of a marginal operator.

(iv) Examining equation (4.7$b$) for $\sigma = 0$ and $d = 2$ we find that the correlation length becomes large only for $r \to -\infty$. This peculiar situation will be discussed in the framework of the non-linear $\sigma$-model.

Finally, in the critical limit $\tau = 0$, $\lambda$ vanishes and thus from the form (4.12) of the $\sigma$-propagator we find that the critical exponent $\eta$ remains classical for all $d$

$$\eta = 0 \;\Rightarrow\; d_\phi = \tfrac{1}{2}(d-2)\,. \tag{4.20}$$

We verify that the exponents $\beta, \nu, \eta$ satisfy the scaling relation proven within the framework of the $\varepsilon$-expansion

$$\beta = \nu d_\phi \,.$$

*Singular free energy and scaling equation of state.* In a constant magnetic field $H$ in the $\sigma$ direction, the free energy $W(H)/\Omega$ per unit volume is given by

$$W(H)/\Omega = \ln Z/\Omega = \frac{3}{2u}\lambda^2 - \frac{3r}{u}\lambda - \frac{1}{2}\lambda\sigma^2 + H\sigma - \frac{N}{2}\,\mathrm{tr}(-\Delta+\lambda),$$

where $\Omega$ is the total space volume and $\lambda, \sigma$ the saddle point values are given by equation (4.7$b$) and the modified saddle point equation (4.7$a$):

$$\lambda\sigma = H \,. \tag{4.21}$$

The thermodynamical potential $\Gamma(M)$ is the Legendre transform of $W(H)$. First

$$M = \Omega^{-1}\frac{\partial W}{\partial H} = \sigma \,,$$

because partial derivatives of $W$ with respect to $\lambda, \sigma$ vanish as a consequence of the saddle point equations. It follows

$$V(M) \equiv \Gamma(M)/\Omega = HM - W(H)/\Omega = -\frac{3}{2u}\lambda^2 + \frac{3r}{u}\lambda + \frac{1}{2}\lambda M^2 + \frac{N}{2}\,\mathrm{tr}(-\Delta+\lambda).$$

As a property of the Legendre transform, the saddle point equation for $\lambda$ is now obtained by writing that the derivative of $\Gamma$ vanishes.

The term $\mathrm{tr}\ln$ can be evaluated for large $\Lambda$ in terms of $r_c$ and the quantities defined in (4.16). One finds

$$\begin{aligned}\mathrm{tr}\ln[(\Delta-\lambda)\Delta^{-1}] &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, \ln[(p^2+\lambda)/p^2] \\ &= -2\frac{C(d)}{d}\lambda^{d/2} - \frac{6r_c}{Nu}\lambda + \frac{a(d)}{2}\lambda^2\Lambda^{4-d} + O(\lambda^{1+d/2}\Lambda^{-2}).\end{aligned}$$

The thermodynamical potential becomes

$$V(M) = \frac{3}{2}\left(\frac{1}{u^*} - \frac{1}{u}\right)\lambda^2 + \frac{3(r-r_c)}{u}\lambda + \frac{1}{2}\lambda M^2 - \frac{NC(d)}{d}\lambda^{d/2}, \tag{4.22}$$

where we have defined

$$u^* = \frac{6}{Na(d)}\Lambda^{\varepsilon}. \tag{4.23}$$

Note that for $\lambda$ small the term proportional to $\lambda^2$ is negligible with respect to the singular term $\lambda^{d/2}$ for $d<4$. At leading order in the critical domain

$$V(M)=\frac{1}{2}\tau\lambda+\frac{1}{2}\lambda M^2-\frac{NC(d)}{d}\lambda^{d/2}, \qquad (4.24)$$

where $\tau$ has been defined in (4.10).

The saddle point equation for $\lambda$ takes the simple form

$$\tau+M^2-NC(d)\lambda^{d/2-1}=0,$$

and thus

$$\lambda=\left[\frac{1}{NC(d)}\left(\tau+M^2\right)\right]^{2/(d-2)}.$$

It follows that the leading contribution, in the critical domain, to the thermodynamical potential is given by

$$V(M)\sim\frac{(d-2)}{2d}\frac{1}{\bigl(NC(d)\bigr)^{2/(d-2)}}(\tau+M^2)^{d/(d-2)}. \qquad (4.25)$$

Various quantities can be derived from $V(M)$, for example the equation of state by differentiating with respect to $M$. The resulting scaling equation of state is

$$H=\frac{\partial V}{\partial M}=h_0M^\delta f\left(\tau/M^2\right), \qquad (4.26)$$

in which $h_0$ is a normalization constant, The exponent $\delta$ is given by:

$$\delta=\frac{d+2}{d-2}, \qquad (4.27)$$

in agreement with the general scaling relation relation $\delta=d/d_\phi-1$, and the function $f(x)$ by:

$$f(x)=(1+x)^{2/(d-2)}. \qquad (4.28)$$

The asymptotic form of $f(x)$ for $x$ large implies $\gamma=2/(d-2)$ again in agreement with the scaling relation $\gamma=\nu(2-\eta)$. Taking into account the values of the critical exponents $\gamma$ and $\beta$ it is then easy to verify that the function $f$ satisfies all required properties like for example Griffith's analyticity (see section 2.6). In particular the equation of state can be cast into the parametric form:

$$\begin{aligned}\sigma&=R^{1/2}\theta\,,\\ \tau&=3R\left(1-\theta^2\right),\\ H&=h_0R^{\delta/2}\theta\left(3-2\theta^2\right)^{2/(d-2)}.\end{aligned}$$

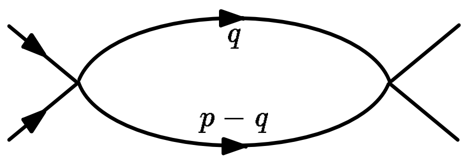


Fig. 2 The "bubble" diagram $B_\Lambda(p,m)$.

*Leading corrections to scaling.* The $\lambda^2$ term yields the leading corrections to scaling. It is subleading by a power of $\tau$

$$\lambda^2/\lambda^{d/2} = O(\tau^{(4-d)/(d-2)}).$$

We conclude

$$\omega\nu = (4-d)/(d-2) \ \Rightarrow\ \omega = 4-d\,. \tag{4.29}$$

We have identified the exponent $\omega$ which governs the leading corrections to scaling. Note that for the special value $u = u^*$ this correction vanishes.

*Specific heat exponent. Amplitude ratios.* Differentiating twice $V(M)$ with respect to $\tau$ we obtain the specific heat at fixed magnetization

$$C_H = \frac{1}{(d-2)} \frac{1}{\left(NC(d)\right)^{2/(d-2)}} (\tau + M^2)^{(4-d)/(d-2)}. \tag{4.30}$$

For $M = 0$ we identify the specific exponent $\alpha$

$$\alpha = \frac{4-d}{d-2}, \tag{4.31}$$

which indeed is equal to $2 - d\nu$, as predicted by scaling laws. Among the ratio of amplitudes one can calculate for example $R_\xi^+$ and $R_c$ (for definitions see chapter 28 of main reference)

$$(R_\xi^+)^d = \frac{4N}{(d-2)^3} \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}}, \quad R_c = \frac{4-d}{(d-2)^2}. \tag{4.32}$$

*The $\lambda$ and $(\phi)^2$ two-point functions.* Differentiating twice the action (4.5) with respect to $\lambda(x)$, then replacing the field $\lambda(x)$ by its expectation value $m^2$, we find the $\lambda$-propagator $\Delta_\lambda(p)$ above $T_c$

$$\Delta_\lambda(p) = -\frac{2}{N} \left[\frac{6}{Nu} + B_\Lambda(p,m)\right]^{-1}, \tag{4.33}$$

where $B_\Lambda(p,m)$ is the bubble diagram of figure 2:

$$B_\Lambda(p,m) = \frac{1}{(2\pi)^d} \int^\Lambda \frac{\mathrm{d}^d q}{(q^2+m^2)\left[(p-q)^2+m^2\right]}. \tag{4.34}$$

The $\lambda$-propagator is negative because the $\lambda$-field is imaginary. As noted in 4.1, it is simply related to the $\phi^2$ 2-point function

$$\left\langle \phi^2\phi^2 \right\rangle = \frac{B_\Lambda(p,m)}{1+(Nu/6)B_\Lambda(p,m)}. \tag{4.35}$$

At zero momentum we recover the specific heat. The small $m$ expansion of $B_\Lambda(0,m)$ can be derived from the expansion (4.16). One finds

$$B_\Lambda(0,m) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{(q^2+m^2)^2}$$
$$= \frac{\partial}{\partial m^2}\left(m^2 D_1(m^2)\right) \underset{m\ll\Lambda}{=} (d/2-1)C(d)m^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + \cdots \,. \quad (4.36)$$

The singular part of the specific heat thus vanishes as $m^\varepsilon$, in agreement with equation (4.30) for $M=0$.

In the critical theory ($m=0$ at this order) for $2\le d\le 4$ the denominator is also dominated at low momentum by the integral

$$B_\Lambda(p,0) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2(p-q)^2} \underset{2<d<4}{=} b(\varepsilon)p^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + O\left(\Lambda^{-2}p^{2-\varepsilon}\right), \quad (4.37)$$

where

$$b(\varepsilon) = -\frac{\pi}{\sin(\pi d/2)}\frac{\Gamma^2(d/2)}{\Gamma(d-1)}N_d\,, \quad (4.38)$$

and thus:

$$\Delta_\lambda(p) \underset{p\to 0}{\sim} -\frac{2}{Nb(\varepsilon)}p^\varepsilon. \quad (4.39)$$

We again verify consistency with scaling relations. In particular we note that in the large $N$ limit the *dimension* $[\lambda]$ *of the field* $\lambda$ is

$$[\lambda] = \tfrac{1}{2}(d+\varepsilon) = 2\,, \quad (4.40)$$

a result important for the $1/N$ perturbation theory.

*Remarks.*

(i) For $d=4$ the behaviour of the propagator is still dominated by the integral which has a logarithmic behaviour $\Delta_\lambda \propto 1/\ln(\Lambda/p)$.

(ii) Note therefore that for $d\le 4$ the contributions generated by the term proportional to $\lambda^2(x)$ in (4.5) always are negligible in the critical domain.

### *4.3 RG functions and leading corrections to scaling*

*The RG functions.* For a more detailed verification of the consistency of the large $N$ results with the RG framework, we now calculate RG functions at leading order. One first easily verifies that, at leading order for $\Lambda$ large, $m$ solution of equation (4.14) satisfies

$$\Lambda\frac{\partial m}{\partial\Lambda} + N\varepsilon a(d)\Lambda^{-\varepsilon}\frac{u^2}{6}\frac{\partial m}{\partial u} = 0\,,$$

where the constant $a(\varepsilon)$ has been defined in (4.16). It depends on the cut-off procedure but for $\varepsilon = 4 - d$ small satisfies

$$a(\varepsilon) \sim 1/(8\pi^2 \varepsilon). \tag{4.41}$$

We then set (equation (4.23)):

$$u = g\Lambda^{\varepsilon}, \quad g^* = u^* \Lambda^{-\varepsilon} = 6/(Na)\,. \tag{4.42}$$

In the new variables $\Lambda, g, \tau$ we obtain an equation which expresses that $m$ is RG invariant

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta\left(g\right) \frac{\partial}{\partial g} - \eta_2(g)\tau \frac{\partial}{\partial \tau}\right) m(\tau, g, \Lambda) = 0\,, \tag{4.43}$$

with

$$\beta(g) = -\varepsilon g(1 - g/g^*), \tag{4.44}$$

$$\nu^{-1}(g) = 2 + \eta_2(g) = 2 - \varepsilon g/g^*. \tag{4.45}$$

When $a(d)$ is positive (but this not true for all regularizations, see the discussion below), one finds an IR fixed point $g^*$, as well as exponents $\omega = \varepsilon$, and $\nu^{-1} = d - 2$, in agreement with equations (4.29,4.19). In the framework of the $\varepsilon$-expansion $\omega$ is associated with the leading corrections to scaling. In the large $N$ limit $\omega$ remains smaller than two for $\varepsilon < 2$, and this extends the property to all dimensions $2 \leq d \leq 4$.

Finally, applying the RG equations to the propagator (4.12), we find

$$\eta(g) = 0\ , \tag{4.46}$$

a result consistent with the value (4.20) found for $\eta$.

*Leading corrections to scaling.* From the general RG analysis we expect the leading corrections to scaling to vanish for $u = u^*$. This property has already been verified for the free energy. Let us now consider the correlation length or mass $m$ given by equation (4.14). If we keep the leading correction to the integral for $m$ small (equation (4.16)) we find

$$\frac{6}{u} - \frac{6}{u^*} + NC(d)m^{-\varepsilon} + O\left(m^{2-\varepsilon}\Lambda^{-2}\right) = \frac{\tau}{m^2}\,, \tag{4.47}$$

where equation (4.42) has been used. We see that the leading correction again vanishes for $u = u^*$. Actually all correction terms suppressed by powers of order $\varepsilon$ for $d \to 4$ vanish simultaneously as expected from the RG analysis of the $\phi^4$ field theory. Moreover one verifies that the leading correction is proportional to $(u - u^*)\tau^{\varepsilon/(2-\varepsilon)}$, which leads to $\omega\nu = \varepsilon/(2 - \varepsilon)$, in agreement with equations (4.29,4.19).

In the same way if we keep the leading correction to the $\lambda$-propagator in the critical theory (equation (4.37)) we find:

$$\Delta_\lambda(p) = -\frac{2}{N}\left[\frac{6}{Nu} - \frac{6}{Nu^*} + b(\varepsilon)p^{-\varepsilon}\right]^{-1}, \tag{4.48}$$

where terms of order $\Lambda^{-2}$ and $1/N$ have been neglected. The leading corrections to scaling again exactly cancel for $u = u^*$ as expected.

*Discussion.*

(i) One can show that a perturbation due to irrelevant operators is equivalent, at leading order in the critical region, to a modification of the $(\phi^2)^2$ coupling. This can be explicitly verified here. The amplitude of the leading correction to scaling has been found to be proportional to $6/Nu - a(d)\Lambda^{-\varepsilon}$ where the value of $a(d)$ depends on the cut-off procedure and thus of contributions of irrelevant operators. Let us call $u'$ the $(\phi^2)^2$ coupling constant in another scheme where $a$ is replaced by $a'$. Identifying the leading correction to scaling we find the relation:

$$\frac{6\Lambda^\varepsilon}{Nu} - a(d) = \frac{6\Lambda^\varepsilon}{Nu'} - a'(d),$$

homographic relation which is consistent with the special form (4.44) of the $\beta$-function.

(ii) *The sign of* $a(d)$. It is generally assumed that $a(d) > 0$. This is indeed what one finds in the simplest regularization schemes, like the simplest Pauli–Villars's regularization where $a(d)$ is positive in all dimensions $2 < d < 4$. Moreover $a(d)$ is always positive near four dimensions where it diverges like

$$a(d) \underset{d\to 4}{\sim} \frac{1}{8\pi^2\varepsilon}.$$

Then there exists an IR fixed point, non-trivial zero of the $\beta$-function. For this value $u^*$ the leading corrections to scaling vanish.

However for $d$ fixed, $d < 4$, this is not a universal feature. For example in the case of simple lattice regularizations it has been shown that in $d = 3$ the sign is arbitrary.

However, if $a(d)$ is negative, the RG method for large $N$ (at least in the perturbative framework) is confronted with a serious difficulty. Indeed the coupling flows in the IR limit to large values where the large $N$ expansion is no longer reliable. It is not known whether this signals a real physical problem, or is just an artifact of the large $N$ limit.

Another way of stating the problem is to examine directly the relation between bare and renormalized coupling constant. Calling $g_r m^{4-d}$ the renormalized 4-point function at zero momentum, we find

$$m^{4-d} g_r = \frac{\Lambda^{4-d} g}{1 + \Lambda^{4-d} g N B_\Lambda(0,m)/6}. \tag{4.49}$$

In the limit $m \ll \Lambda$ the relation can be written

$$\frac{1}{g_{\rm r}} = \frac{(d-2)NC(d)}{12} + \left(\frac{m}{\Lambda}\right)^{4-d} \left(\frac{1}{g} - \frac{Na(d)}{6}\right). \tag{4.50}$$

We see that when $a(d) < 0$ the renormalized IR fixed point value cannot be reached by varying $g > 0$ for any finite value of $m/\Lambda$. In the same way leading corrections to scaling can no longer be cancelled.

*4.4 Small coupling constant and large momentum expansions for $d < 4$*

Section 4.6 is devoted to a systematic discussion of the $1/N$ expansion. However the $1/N$ correction to the two-point function will help us to immediately understand the problem of the massless field theory for $d < 4$.

We have seen that, in the framework at the $1/N$ expansion, we can calculate at fixed dimension $d < 4$ in the critical limit $(T = T_c, m^2 = 0)$. This implies that the terms of the $1/N$ expansion cannot be expanded in a power series of the coupling constant, at least with integer powers. Note that since the gaussian fixed point is an UV fixed point, the small coupling expansion is also a large momentum expansion. To understand the phenomenon we consider the $\langle\sigma\sigma\rangle$ correlation function at order $1/N$. At this order only one diagram contributes (figure 3), containing two $\lambda^2\sigma$ vertices. After mass renormalization and in the large cut-off limit we find:

$$\Gamma^{(2)}_{\sigma\sigma}(p) = p^2 + \frac{2}{N(2\pi)^d} \int \frac{{\rm d}^d q}{(6/Nu) + b(\varepsilon) q^{-\varepsilon}} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2}\right) + O\left(\frac{1}{N^2}\right). \tag{4.51}$$

An analytic study of the integral reveals that it has an expansion of the form

$$\sum_{k\geq 1} \alpha_k u^k p^{2-k\varepsilon} + \beta_k u^{(2+2k)/\varepsilon} p^{-2k}. \tag{4.52}$$

The coefficients $\alpha_k, \beta_k$ can be obtained by performing a Mellin transformation over $u$ on the integral. Indeed if a function $f(u)$ behaves like $u^t$ for $u$ small, then the Mellin transform $M(s)$

$$M(s) = \int_0^\infty {\rm d}u\, u^{-1-s} f(u),$$

has a pole at $s = t$. Applying the transformation to the integral, and inverting $q$ and $u$ integrations we have to calculate the integral

$$\int_0^\infty {\rm d}u \frac{u^{-1-s}}{(6/Nu) + b(\varepsilon) q^{-\varepsilon}} = \frac{N}{6} \left(\frac{N b(\varepsilon) q^{-\varepsilon}}{6}\right)^{1-s} \frac{\pi}{\sin \pi s}.$$

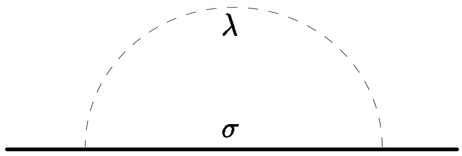


Fig. 3 The diagram contributing to $\Gamma^{(2)}_{\sigma\sigma}$ at order $1/N$.

Then the value of the remaining $q$ integral follows from the generic result (4.78).

The terms with integer powers of $u$ correspond to the formal perturbative expansion where each integral is calculated for $\varepsilon$ small enough. $\alpha_k$ has poles at $\varepsilon = (2l+2)/k$ for which the corresponding power of $p^2$ is $-l$, i.e. an integer. One verifies that $\beta_l$ has a pole at the same value of $\varepsilon$ and that the singular contributions cancel in the sum. For these dimensions logarithms of $u$ appear in the expansion.

### *4.5 The non-linear σ-model in the large N limit*

We have noticed that the term proportional to $\int \mathrm{d}^d x\, \lambda^2(x)$, which has dimension $4-d$ for large $N$ in all dimensions, is irrelevant in the critical domain for $d<4$ and can thus be omitted at leading order (this also applies to $d=4$ where it is marginal but yields only logarithmic corrections). Actually the constant part in the inverse propagator as written in equation (4.48) plays the role of a large momentum cut-off. Let us thus consider the action (4.67) without the $\lambda^2$ term. If we then work backwards, reintroduce the initial field $\phi$ and integrate over $\lambda(x)$ we find

$$Z = \int [\mathrm{d}\phi(x)]\, \delta\left[\phi^2(x) - \frac{6}{u}\left(m^2 - r\right)\right] \exp\left[-\int \frac{1}{2}\left(\partial_\mu \phi(x)\right)^2 \mathrm{d}^d x\right]. \qquad (4.53)$$

Under this form we recognize the partition function of the $O(N)$ symmetric non-linear $\sigma$-model in an unconventional normalization. We have therefore discovered a remarkable correspondence: to all orders in an $1/N$ expansion the renormalized non-linear $\sigma$-model is identical to the renormalized $\left(\phi^2\right)^2$ field theory at the IR fixed point.

*The large N limit.* In order to more explicitly show the correspondence between the set of parameters used in the two models, let us directly solve the $\sigma$-model in the large $N$ limit. We rewrite the partition function:

$$Z = \int [\mathrm{d}\phi(x)\mathrm{d}\lambda(x)] \exp\left[-S(\phi,\lambda)\right], \qquad (4.54)$$

with:

$$S(\phi,\lambda) = \frac{1}{2t}\int \mathrm{d}^d x \left[\left(\partial_\mu \phi\right)^2 + \lambda\left(\phi^2 - 1\right)\right]. \qquad (4.55)$$

Integrating, as we did in section 4.1, over $N-1$ components of $\phi$ and calling $\sigma$ the remaining component, we obtain:

$$Z=\int[\mathrm{d}\sigma(x)\mathrm{d}\lambda(x)]\exp\left[-S_N(\sigma,\lambda)\right], \tag{4.56}$$

with:

$$S_N(\sigma,\lambda)=\frac{1}{2t}\int\left[(\partial_\mu\sigma)^2+\left(\sigma^2(x)-1\right)\lambda(x)\right]\mathrm{d}^d x+\frac{1}{2}(N-1)\operatorname{tr}\ln\left[-\Delta+\lambda(\cdot)\right]. \tag{4.57}$$

The large $N$ limit is here taken at $tN$ fixed. The saddle point equations, analogous to equations (4.7), are:

$$m^2\sigma=0\,, \tag{4.58a}$$

$$\sigma^2=1-\frac{(N-1)t}{(2\pi)^d}\int^\Lambda\frac{\mathrm{d}^d p}{p^2+m^2}\,, \tag{4.58b}$$

where we have set $\langle\lambda(x)\rangle=m^2$. At low temperature $\sigma$ is different from zero and thus $m$, which is the mass of the $\pi$-field, vanishes. Equation (4.58$b$) gives the spontaneous magnetization:

$$\sigma^2=1-\frac{(N-1)t}{(2\pi)^d}\int^\Lambda\frac{\mathrm{d}^d p}{p^2}. \tag{4.59}$$

Setting

$$\frac{1}{t_c}=\frac{(N-1)}{(2\pi)^d}\int^\Lambda\frac{\mathrm{d}^d p}{p^2}, \tag{4.60}$$

we can write equation (4.59):

$$\sigma^2=1-t/t_c\,. \tag{4.61}$$

Thus $t_c$ is the critical temperature where $\sigma$ vanishes.

Above $t_c$, $\sigma$ instead vanishes and $m$, which is now the common mass of the $\pi$- and $\sigma$-field, is for $2<d<4$ given by:

$$\frac{1}{t_c}-\frac{1}{t}=m^{d-2}\frac{(N-1)}{(2\pi)^d}\int\frac{\mathrm{d}^d p}{p^2\left(p^2+1\right)}+O\left(m^2\Lambda^{d-4}\right). \tag{4.62}$$

We recover the scaling form of the correlation length $\xi=1/m$. From the equations (4.61,4.62), we can also derive the RG functions at leading order for $N$ large:

$$\beta(t)=\varepsilon t-\frac{N}{2\pi}t^2\,,\qquad \zeta(t)=\frac{N}{2\pi}t\,. \tag{4.63}$$

It is also easy to calculate the thermodynamical potential, Legendre transform of $W(H) = t \ln Z(H)$:

$$V(M) = \Gamma(M)/\Omega = \frac{d-2}{2d} \frac{1}{\left(NC(d)\right)^{2/(d-2)}} (M^2 - 1 + t/t_c)^{d/(d-2)}, \qquad (4.64)$$

a result which extends equation (4.25) to all temperatures below $t_c$. The calculation of other physical quantities and the expansion in $1/N$ follow from the considerations of previous sections and section 4.6.

*Two dimensions and the question of Borel summability.* For $d = 2$ the critical temperature vanishes and the parameter $m$ has the form:

$$m \sim \Lambda \, \mathrm{e}^{-2\pi/(Nt)}, \qquad (4.65)$$

in agreement with the RG predictions. Note that the field 2-point function takes in the large $N$-limit the form:

$$\Gamma^{(2)}_{\sigma\sigma}(p) = p^2 + m^2 \,. \qquad (4.66)$$

The mass term vanishes to all orders in the expansion in powers of the coupling constant $t$, preventing any perturbative calculation of the mass of the field. The perturbation series is trivially not Borel summable. Most likely this property is also true for the model at finite $N$. On the other hand if we break the $O(N)$ symmetry by a magnetic field, adding a term $h\sigma$ to the action, the physical mass becomes calculable in perturbation theory.

*Corrections to scaling and the dimension four.* In equation (4.62) we have neglected corrections to scaling. If we take into account the leading correction we get instead:

$$m^2 \left(C(d) m^{d-4} - a(d) \Lambda^{d-4}\right) \propto t - t_c \,,$$

where $a(d)$, as we have already explained, is a constant which explicitly depends on the cut-off procedure and can thus be varied by changing contributions of irrelevant operators. By comparing with the results of section 4.3, we discover that, although the non-linear $\sigma$-model superficially depends on one parameter less than the corresponding $\phi^4$ field theory, actually this parameter is hidden in the cut-off function. This remark becomes important in the four dimensional limit where most leading contributions come from the leading corrections to scaling. For example for $d = 4$ equation (4.62) takes a different form, the dominant term in the r.h.s. is proportional to $m^2 \ln m$. We recognize in the factor $\ln m$ the effective $\phi^4$ coupling at mass scale $m$. Beyond the $1/N$ expansion, to describe with perturbation theory and renormalization group the physics of the non-linear $\sigma$ model it is necessary to introduce the operator $\int \mathrm{d}^d x \, \lambda^2(x)$, which irrelevant for $d < 4$, becomes marginal, and to return to the $\phi^4$ field theory.

*4.6 The $1/N$-expansion: an alternative field theory*

*Preliminary remarks. Power counting.* Higher order terms in the steepest descent calculation of the functional integral (4.4) generate a systematic $1/N$ expansion. Let us first slightly rewrite action (4.5). We shift the field $\lambda(x)$ by its expectation value $m^2$ (equation (4.13)), $\lambda(x) \mapsto m^2 + \lambda(x)$:

$$S_N(\sigma,\lambda) = \frac{1}{2}\int \mathrm{d}^d x \left[(\partial_\mu \sigma)^2 + m^2\sigma^2 + \lambda(x)\sigma^2(x) - \frac{3}{u}\lambda^2(x) - \frac{6}{u}\left(m^2 - r\right)\lambda(x)\right] + \frac{(N-1)}{2}\operatorname{tr}\ln\left[-\Delta + m^2 + \lambda(\cdot)\right]. \qquad (4.67)$$

We now analyze the terms in the action (4.67) from the point of view of large $N$ power counting. The dimension of the field $\sigma(x)$ is $(d-2)/2$. From the critical behaviour (4.39) of the $\lambda$-propagator we have deduced the canonical dimension $[\lambda]$ of the field $\lambda(x)$:

$$2[\lambda] - \varepsilon = d \qquad \text{i.e.} \qquad [\lambda] = 2\,.$$

As noted above, $\lambda^2$ has dimension $4 > d$ and is thus irrelevant. The interaction term $\int \lambda(x)\sigma^2(x)\mathrm{d}^d x$ has dimension zero. It is easy to verify that the non-local interactions involving the $\lambda$-field, coming from the expansion of the $\operatorname{tr}\ln$, have all also the canonical dimension zero:

$$\left[\operatorname{tr}\left[\lambda(x)\left(-\Delta + m^2\right)^{-1}\right]^k\right] = k[\lambda] - 2k = 0\,.$$

This power counting property has the following implication: In contrast with usual perturbation theory, the $1/N$ expansion generates only logarithmic corrections to the leading long distance behaviour for any fixed dimension $d$, $2 < d \le 4$. The situation is thus similar to the situation one encounters for the $\varepsilon$-expansion (at the IR fixed point) and one expects to be able to calculate universal quantities like critical exponents for example as power series in $1/N$. However, because the interactions are non-local, the results of renormalization theory do not immediately apply. We now construct an alternative quasi-local field theory, for which the standard RG analysis is valid, and which reduces to the large $N$ field theory in some limit.

*An alternative field theory.* To be able to use the standard results of renormalization theory we reformulate the critical theory to deal with the non-local interactions. Neglecting corrections to scaling we start from the non-linear $\sigma$-model in the form (4.55):

$$Z = \int [\mathrm{d}\lambda(x)]\,[\mathrm{d}\phi(x)]\exp\left[-S(\phi,\lambda)\right], \qquad (4.68)$$

$$S(\phi,\lambda) = \frac{1}{2t}\int \mathrm{d}^d x\left[(\partial_\mu\phi)^2 + \lambda\left(\phi^2 - 1\right)\right]. \qquad (4.69)$$

The difficulty arises from the $\lambda$-propagator, absent in the perturbative formulation, and generated by the large $N$ summation. We thus add to the action (4.69) a term quadratic in $\lambda$ which at tree level of standard perturbation theory generates a $\lambda$-propagator of the form (4.39). The modified action $S_v$ then is

$$S_v(\phi,\lambda)=\frac{1}{2}\int \mathrm{d}^d x\left\{\frac{1}{t}\left[(\partial_\mu\phi)^2+\lambda\left(\phi^2-1\right)\right]-\frac{1}{v^2}\lambda(-\partial^2)^{-\varepsilon/2}\lambda\right\}. \quad (4.70)$$

In the limit where the parameter $v$ goes to infinity the coefficient of the additional term vanishes, and the initial action is recovered.

We below consider only the critical theory. This means that the couplings of all relevant interactions will be set to their critical values. These interactions contain a term linear in $\lambda$ and a polynomial in $\phi^2$ of degree depending on the dimension. Note that in some discrete dimensions some monomials become just renormalizable. We therefore work in generic dimensions. The quantities we shall calculate are regular in the dimension. The field theory with the action (4.70) can be studied with standard field theory methods. The peculiar form of the $\lambda$ quadratic term, which is not strictly local, does not create a problem. Similar terms are encountered in statistical systems with long range forces. The simple consequence is that the $\lambda$-field is not be renormalized because counterterms are always local.

It is convenient to rescale $\phi\mapsto\phi\sqrt{t}$, $\lambda\mapsto v\lambda$:

$$S_v(\phi,\lambda)=\frac{1}{2}\int \mathrm{d}^d x\left[(\partial_\mu\phi)^2+v\lambda\phi^2-\lambda(-\partial^2)^{-\varepsilon/2}\lambda+\text{relevant terms}\right].$$

The renormalized critical action then reads:

$$[S_v]_{\mathrm{ren}}=\frac{1}{2}\int \mathrm{d}^d x\left[Z_\phi(\partial_\mu\phi)^2+v_{\mathrm{r}}Z_v\lambda\phi^2-\lambda(-\partial^2)^{-\varepsilon/2}\lambda+\text{relevant terms}\right]. \quad (4.71)$$

It follows that the RG equations for 1PI correlation functions of $l$ $\lambda$ fields and $n$ $\phi$ fields in the critical theory take the form:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta_{v^2}(v)\frac{\partial}{\partial v^2}-\frac{n}{2}\eta(v)\right]\Gamma^{(l,n)}=0\,. \quad (4.72)$$

We can then calculate the RG functions as power series in $1/N$. It is easy to verify that $v^2$ has to be taken of order $1/N$. Therefore to generate a $1/N$ expansion one first has to sum the multiple insertions of the one-loop $\lambda$ two-point function, contributions which form a geometrical series. The $\lambda$ propagator then becomes

$$\Delta_\lambda(p)=-\frac{2p^{4-d}}{b(\varepsilon)D(v)}\,, \quad (4.73)$$

where we have defined

$$D(v) = 2/b(\varepsilon) + Nv^2 .$$

The solution to the RG equations (4.72) can be written:

$$\Gamma^{(l,n)}(\tau p, v, \Lambda) = Z^{-n/2}(\tau)\tau^{d-2l-n(d-2)/2}\Gamma^{(l,n)}(p, v(\tau), \Lambda), \tag{4.74}$$

with the usual definitions

$$\tau\frac{\mathrm{d}v^2}{\mathrm{d}\tau} = \beta(v(\tau)), \quad \tau\frac{\mathrm{d}\ln Z}{\mathrm{d}\tau} = \eta(v(\tau)).$$

We are interested in the neighbourhood of the fixed point $v^2 = \infty$. One verifies that the RG function $\eta(v)$ approaches the exponent $\eta$ obtained by direct calculation, and the RG $\beta$-function behaves like $v^2$. The flow equation for the coupling constant becomes:

$$\tau\frac{\mathrm{d}v^2}{\mathrm{d}\tau} = \rho v^2, \Rightarrow v^2(\tau) \sim \tau^{\rho}. \tag{4.75}$$

We then note that to each power of the $\lambda$ field corresponds a power of $v$. It follows

$$\begin{aligned}\Gamma^{(l,n)}(\tau p, v, \Lambda) &\propto v^l(\tau)\tau^{d-2l-n(d-2+\eta)} \\ &\propto \tau^{d-(2-\rho/2)l-n(d-2+\eta)}.\end{aligned} \tag{4.76}$$

To compare with the result (2.30) obtained from the perturbative renormalization group one has still to take into account that the functions $\Gamma^{(l,n)}$ defined here are obtained by an additional Legendre transformation with respect to the source of $\phi^2$. Therefore

$$2 - \rho/2 = d_{\phi^2} = d - 1/\nu . \tag{4.77}$$

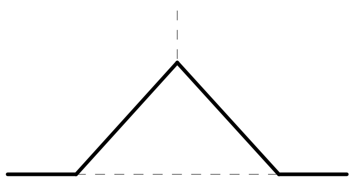

Fig. 4 Diagram contributing to $\Gamma^{(3)}_{\sigma\sigma\lambda}$ at order $1/N$.

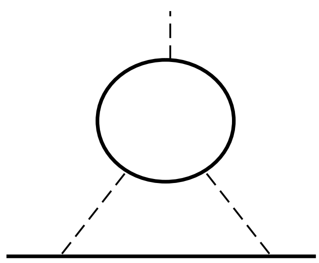

Fig. 5 Diagram contributing to $\Gamma^{(3)}_{\sigma\sigma\lambda}$ at order $1/N$.

*RG functions at order* $1/N$. Most calculations at order $1/N$ rely on the evaluation of the generic integral

$$\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q}{(p+q)^{2\mu}q^{2\nu}}=p^{d-2\mu-2\nu}\frac{\Gamma(\mu+\nu-d/2)\Gamma(d/2-\mu)\Gamma(d/2-\nu)}{(4\pi)^{d/2}\Gamma(\mu)\Gamma(\nu)\Gamma(d-\mu-\nu)}\,. \tag{4.78}$$

For later purpose it is convenient to set:

$$X_1=\frac{2N_d}{b(\varepsilon)}=\frac{4\Gamma(d-2)}{\Gamma(d/2)\Gamma(2-d/2)\Gamma^2(d/2-1)}=\frac{4\sin(\pi\varepsilon/2)\Gamma(2-\varepsilon)}{\pi\Gamma(1-\varepsilon/2)\Gamma(2-\varepsilon/2)}. \tag{4.79}$$

To compare with fixed dimension results note $X_1\sim 2(4-d)$ for $d\to 4$ and $X_1\sim(d-2)$ for $d\to 2$.

The calculation of the $\langle\phi\phi\rangle$ correlation function at order $1/N$ involves the evaluation of the diagram of figure 3. We want to determine the coefficient of $p^2\ln\Lambda/p$. Since we work at one-loop order we can instead replace the $\lambda$ propagator $q^{-\varepsilon}$ by $q^{2\nu}$ and send the cut-off to infinity. We then use the result (4.78) with $\mu=1$. In the limit $2\nu\to-\varepsilon$ the integral has a pole. The residue of the pole yields the coefficient of $p^2\ln\Lambda$ and the finite part contains the $p^2\ln p$ contribution

$$\Gamma^{(2)}_{\sigma\sigma}(p)=p^2+\frac{\varepsilon}{4-\varepsilon}\frac{2N_d}{b(\varepsilon)D(v)}v^2p^2\ln(\Lambda/p).$$

Expressing that the function satisfies the RG equation we obtain the function $\eta(v)$.

The second RG function can be deduced from the divergent parts of the $\langle\phi\phi\lambda\rangle$ function

$$\Gamma^{(3)}_{\sigma\sigma\lambda}=v+A_1v^3D^{-1}(v)\ln\Lambda+A_2v^5D^{-2}(v)\ln\Lambda+\text{ finite },$$

with

$$\begin{aligned}A_1&=-\frac{2}{b(\varepsilon)}N_d=-X_1\\ A_2&=-\frac{4N}{b^2(\varepsilon)}(d-3)b(\varepsilon)N_d=-2N(d-3)X_1\,,\end{aligned}$$

where $A_1$ and $A_2$ correspond to the diagrams of figures 4 and 5 respectively.

Applying the RG equation one finds the relation at order $1/N$

$$\beta_{v^2}(v)=2v^2\eta(v)-2A_1v^4D^{-1}(v)-2A_2v^6D^{-2}(v). \tag{4.80}$$

We thus obtain

$$\eta(v)=\frac{\varepsilon v^2}{4-\varepsilon}X_1D^{-1}(v), \tag{4.81}$$

$$\beta_{v^2}(v)=\frac{8v^4}{4-\varepsilon}X_1D^{-1}(v)+4N(1-\varepsilon)v^6X_1D^{-2}(v), \tag{4.82}$$

where the first term in $\beta_{v^2}$ comes from $A_1$ and $\eta$ and the second from $A_2$.

Extracting the large $v^2$ behaviour we find

$$\eta = \frac{\varepsilon}{N(4-\varepsilon)} X_1 + O(1/N^2).,\tag{4.83}$$

$$\rho = \frac{4(3-\varepsilon)(2-\varepsilon)}{N(4-\varepsilon)} X_1 > 0\,,$$

and thus

$$\frac{1}{\nu} = d - 2 + \frac{2(3-\varepsilon)(2-\varepsilon)}{N(4-\varepsilon)} X_1 + O(1/N^2).\tag{4.84}$$

*4.7 Additional results*

The calculations beyond the order $1/N$ are rather technical. The reason is easy to understand: Because the effective field theory is renormalizable in all dimensions $2 \le d \le 4$, the dimensional regularization, which is so useful in perturbative calculations, no longer works. Therefore either one keeps a true cut-off or one introduces more sophisticated regularization schemes. For details the reader is referred to the literature.

*Generic dimensions.* The exponents $\gamma$ and $\eta$ are known up to order $1/N^2$ and $1/N^3$ respectively in arbitrary dimensions but the expressions are too complicated to be reproduced here. The expansion of $\gamma$ up to order $1/N$ can be directly deduced from the results of the preceding sections:

$$\gamma = \frac{1}{1-\varepsilon/2}\left(1 - \frac{3}{2N} X_1\right) + O\left(\frac{1}{N^2}\right).\tag{4.85}$$

The exponents $\omega$ and $\theta = \omega\nu$, governing the leading corrections to scaling, can also be calculated for example from the $\langle \lambda^2 \lambda\lambda \rangle$ function:

$$\omega = \varepsilon\left(1 - \frac{2(3-\varepsilon)^2}{(4-\varepsilon)N} X_1\right) + O\left(\frac{1}{N^2}\right),\tag{4.86}$$

$$\theta = \omega\nu = \frac{\varepsilon}{2-\varepsilon}\left(1 - \frac{2(3-\varepsilon)}{N} X_1\right) + O\left(\frac{1}{N^2}\right).\tag{4.87}$$

Note that the exponents are regular functions of $\varepsilon$ up to $\varepsilon = 2$ and free of renormalon singularities at $\varepsilon = 0$.

The equation of state and the spin–spin correlation function in zero field are also known at order $1/N$, but since the expressions are complicated we refer the reader to the literature for details.

*Three dimensional results.* Let us give the expansion of $\eta$ in three dimensions at the order presently available:

$$\eta = \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \frac{\eta_3}{N^3} + O\left(\frac{1}{N^4}\right),$$

with

$$\eta_1 = \frac{8}{3\pi^2}\,, \quad \eta_2 = -\tfrac{8}{3}\eta_1^2\,, \quad \eta_3 = \eta_1^3\left[-\tfrac{797}{18} - \tfrac{61}{24}\pi^2 + \tfrac{27}{8}\psi''(1/2) + \tfrac{9}{2}\pi^2 \ln 2\right],$$

$\psi(x)$ being the logarithmic derivative of the $\Gamma$ function.

The exponent $\gamma$ is known only up to order $1/N^2$:

$$\gamma = 2 - \frac{24}{N\pi^2} + \frac{64}{N^2\pi^4}\left(\tfrac{44}{9} - \pi^2\right) + O\left(\tfrac{1}{N^3}\right).$$

Note that the $1/N$ expansion seems to be rapidly divergent and certainly a direct summation of these terms does not provide very good estimates of critical exponents in 3 dimensions for useful values of $N$.

*4.8 Dimension four: triviality, renormalons, Higgs mass*

A number of issues concerning the physics of the $(\phi^2)^2$ theory in four dimensions can be addressed within the framework of the large $N$ expansion. For simplicity reasons we consider here only the critical (i.e massless) theory.

*Triviality and UV renormalons.* It is easy to verify that the renormalized coupling constant $g_{\mathrm{r}}$, defined as the value of the vertex $\langle\sigma\sigma\sigma\sigma\rangle$ at momenta of order $\mu \ll \Lambda$, is given by:

$$g_{\mathrm{r}} = \frac{g}{1 + \frac{1}{6}NgB_\Lambda(\mu)}\,, \tag{4.88}$$

where $B_\Lambda(p)$ corresponds to the bubble diagram (figure 2)

$$B_\Lambda(p) \underset{p\ll\Lambda}{\sim} \frac{1}{8\pi^2}\ln(\Lambda/p) + \text{ const.}. \tag{4.89}$$

We see that when the ratio $\mu/\Lambda$ goes to zero, the renormalized coupling constant vanishes, for that all $g$. This is the so-called *triviality* property. In the standard treatment of quantum field field, one usually insists in taking the infinite cut-off $\Lambda$ limit. Here one then finds only a free field theory. Another way of formulating the problem is the following: it is impossible to construct in four dimensions a $\phi^4$ field theory consistent (in the sense of satisfying all usual physical requirements) on all scales for non zero coupling. Of course in the logic of *effective* field theories this is no longer an issue. The triviality property just implies that the renormalized or effective charge is logarithmically small as indicated by equations (4.88,4.89). Note that if $g$ is generic (not too small) and $\Lambda/\mu$ large, $g_r$ is essentially independent of the initial coupling constant. Only if the bare coupling is small is the renormalized coupling an adjustable, but bounded, quantity.

Let us now imagine that we work formally and, ignoring the problem, we express the leading contribution to the four-point function in terms of the renormalized constant:

$$\frac{g}{1 + \frac{N}{48\pi^2}g\ln(\Lambda/p)} = \frac{g_{\mathrm{r}}}{1 + \frac{N}{48\pi^2}g_{\mathrm{r}}\ln(\mu/p)}\,.$$

We then find that the function has a pole for

$$p = \mu\, \mathrm{e}^{48\pi^2/(Ng_{\mathrm{r}})} .$$

This pole corresponds to the Landau ghost for this theory which has $g = 0$ as an IR fixed point. If we calculate contributions of higher orders, for example to the two-point function, this pole makes the loop integrals diverge. In an expansion in powers of $g_{\mathrm{r}}$, each term is instead calculable but one finds, after renormalization, UV contributions of the type

$$\int^{\infty} \frac{\mathrm{d}^4 q}{q^6} \left( -\frac{Ng_{\mathrm{r}}}{48\pi^2} \ln(\mu/q) \right)^k \underset{k\to\infty}{\propto} \left( \frac{Ng_{\mathrm{r}}}{96\pi^2} \right)^k k! .$$

The perturbative manifestation of the Landau ghost is the appearance of contributions to the perturbation series which are not Borel summable. By contrast the contributions due to the finite momentum region, which can be evaluated by a semiclassical analysis, are Borel summable, but invisible for $N$ large. This effect is called UV renormalon effect. Note finally that this UV problem is independent of the mass of the field $\phi$, that we have taken zero for simplicity reasons.

*IR renormalons.* We now illustrate the problem of IR renormalons with the same example of the massless $(\phi^2)^2$ theory (but now zero mass is essential), in four dimensions, in the large $N$ limit. We calculate the contribution of the small momentum region to the mass renormalization, at cut-off $\Lambda$ fixed. In the large $N$ limit the mass renormalization is then proportional to (see equation (4.51))

$$\int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2 \left(1 + \frac{1}{6} N g B_{\Lambda}(q)\right)} \sim \int \frac{\mathrm{d}^4 q}{q^2 \left(1 + \frac{N}{48\pi^2} g \ln(\Lambda/q)\right)} .$$

It is easy to expand this expression in powers of the coupling constant $g$. The term of order $k$ in the limit $k \to \infty$ behaves as $(-1)^k (N/96\pi^2)^k k!$. This contribution has the alternating sign of the semiclassical contribution. Note that more generally for $N$ finite on finds $(-\beta_2/2)^k k!$. IR singularities are responsible for additional, Borel summable, contributions to the large order behaviour.

In a theory asymptotically free for large momentum, clearly the roles of IR and UV singularities are interchanged.

*The mass of the $\sigma$ field in the phase of broken symmetry.* The $\phi^4$ theory is a piece of the Standard Model, and the field $\sigma$ then represents the Higgs field. With some reasonable assumptions it is possible to establish for finite $N$ a semi-quantitative bound on the Higgs mass. Let us examine here what happens for $N$ large.

In the phase of broken symmetry the action, after translation of average values, includes a term proportional to $\sigma\lambda$ and thus the propagators of the fields $\sigma$ and $\lambda$ are elements of a $2 \times 2$ matrix $\mathbf{M}$:

$$\mathbf{M}^{-1}(p) = \begin{pmatrix} p^2 & \sigma \\ \sigma & -3/u - \frac{1}{2} N B_{\Lambda}(p) \end{pmatrix} ,$$

where $\sigma = \langle \sigma(x) \rangle$. In four dimensions $B_\Lambda$ is given by equation (4.89). It is convenient to introduce a mass scale $M$, RG invariant, such that

$$\frac{48\pi^2}{Nu} + 8\pi^2 B_\Lambda(p) \sim \ln(M/p),$$

and thus

$$M \propto e^{48\pi^2/Nu} \Lambda .$$

The mass of the field $\sigma$ at this order is a solution to the equation $\det \mathbf{M} = 0$. One finds

$$p^2 \ln(M/p) = -(16\pi^2/N)\sigma^2 \ \Rightarrow \ m_\sigma^2 \ln(iM/m_\sigma) = (16\pi^2/N)\sigma^2 .$$

The mass $m_\sigma$ solution to the equation is complex, because the particle $\sigma$ can decay into massless Goldstone bosons. At $\sigma$ fixed, the mass decreases when the cut-off increases or when the coupling constant goes to zero. Expressing that the mass must be smaller than the cut-off, one obtains an upper-bound on $m_\sigma$ (but which slightly depends on the chosen regularization).

### *4.9 Finite size effects*

Another question can be studied in the large $N$ limit, finite size effects. It is difficult to discuss all possible finite size effects because the results depend both on the geometry of the system and on the boundary conditions. In particular one must discuss separately boundary conditions depending whether they break or not translation invariance. In the first case new effects appear which are surface effects, and that we do not examine here. Note that the periodic conditions are not the only ones which preserve translation invariance. For systems which have a symmetry one can glue the boundaries after having made a group transformation. Thus here one could also choose antiperiodic conditions or more generally fields differing by a transformation of the $O(N)$ group.

Moreover if we are interested only in qualitative aspects we can limit ourselves to a simple geometry, in each direction the system having the same finite size $L$, all other sizes being infinite (but we thus exclude some questions concerning crossover regimes). Even so the number of different possible situations remains large, and we limit ourselves here to two examples.

We consider the example of periodic boundary conditions in two cases: finite volume (the geometry of the hypercube or rather hypertorus) in this section, and QFT at finite temperature in next section.

From the point of view of renormalization group, finite size effects, which only affect the IR domain, do not change UV divergences. The RG equations remain the same, only the solutions are modified by the appearance of new dimensional quantities. Thus if finite sizes are characterized by only one length $L$, solutions will be functions of an additional argument $L/\xi$ where $\xi$ is the correlation length.

A property characteristic of a system of finite size is the quantification of momenta in Fourier space. For periodic conditions, if we call $L$ the size du system in each direction, we have

$$p_\mu = 2\pi n_\mu / L\,, \quad n_\mu \in \mathbb{Z}\,.$$

In particular, in a massless theory the zero mode $\mathbf{p} = 0$ now corresponds to an isolated pole of the propagator. This automatically leads to IR divergences in all dimensions. Therefore in equations (4.7) the solution $\sigma \neq 0$ no longer exists. This is not surprising: there are no phase transitions in a finite volume. Neglecting corrections to scaling laws we can then write equation (4.58$b$):

$$1 = (N-1) t L^{-d} \sum_{n_\mu} \frac{1}{m^2 + (2\pi \mathbf{n}/L)^2}\,, \tag{4.90}$$

where the sums are cut by a cut-off $\Lambda$.

To discuss the equation it is convenient to introduce the function $A(s)$ (related to Jacobi's elliptic functions)

$$A(s) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-sn^2}\,. \tag{4.91}$$

Using Poisson's transformation it is easy to show

$$A(s) = (\pi/s)^{1/2} A\left(\pi^2/s\right). \tag{4.92}$$

Using this definition, and introducing the critical temperature $t_c$, one can write equation (4.90) (for $2 < d < 4$)

$$\frac{1}{t} - \frac{1}{t_c} = (N-1) L^{-d} \int_0^\infty \mathrm{d}s \left(\mathrm{e}^{-sm^2} A^d(4\pi^2 s/L^2) - L^d (4\pi s)^{-d/2}\right). \tag{4.93}$$

Setting $s \mapsto L^2 s$ and introducing the function $F$:

$$F(z) = \int_0^\infty \mathrm{d}s \left(\mathrm{e}^{-sz^2} A^d(4\pi^2 s) - (4\pi s)^{-d/2}\right), \tag{4.94}$$

we can rewrite the relation

$$\frac{1}{t} - \frac{1}{t_c} = (N-1) L^{2-d} F(mL). \tag{4.95}$$

For $|t - t_c| \ll \Lambda^{d-2}$ we find a scaling form which is in agreement with the RG result, which predicts ($1/\nu = d - 2 + O(1/N)$):

$$Lm(t, L) = L/\xi(t, L) = f\big((t - t_c) L^{1/\nu}\big).$$

Here the length $\xi$ has the meaning of a correlation length only for $\xi < L$. Since $\eta = 0$, the magnetic susceptibility $\chi$ in zero field instead is always given by $\chi = t/m^2$.

One verifies that for $t > t_c$ fixed, $L \to \infty$ and thus $mL \to \infty$ one recovers the infinite volume limit. On the contrary in the low temperature phase for $t < t_c$ fixed, $L \to \infty$, $mL$ goes to zero. Thus the contribution of the zero mode dominates in the r.h.s. of equation (4.90). Using the relation (4.92) one then finds

$$F(z) = \frac{1}{z^2} + K(d) + O\left(z^2\right),$$
$$K(d) = \int_0^\infty \mathrm{d}s \left[A^d(4\pi^2 s) - 1 - (4\pi s)^{-d/2}\right],$$

and thus

$$\chi(L,t) = \frac{t}{m^2} = \frac{1}{N-1}(1 - t/t_c)L^d - tL^2 K(d) + O\left(L^{4-d}/(t-t_c)\right). \qquad (4.96)$$

We see that the susceptibility diverges with the volume, an indication of the existence of a broken symmetry phase.

Note finally that it is instructive to make a similar analysis for different boundary conditions which have no zero mode.

For $d = 2$ the regime where finite size effects are observables corresponds to $t \ln(L\Lambda) = O(1)$, i.e. to a regime of low temperature. The zero mode dominates for $t \ln(L\Lambda) \ll 1$, and the susceptibility is then given by

$$\chi(t,L) \sim \tfrac{1}{N} L^2 \left[1 + O(t \ln(L\Lambda))\right].$$

*4.10 Field theory at finite temperature*

Quantum field theory at finite temperature can be considered as a system which has a finite size in one direction. Indeed the partition function is given by $\operatorname{tr} \mathrm{e}^{-LH}$, where $H$ is the hamiltonian and $L^{-1}$ the temperature. For a scalar field theory with euclidean lagrangian density $\mathcal{L}(\phi)$ this leads to the functional integral

$$Z = \int [\mathrm{d}\phi] \exp\left[-\int_0^L \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, \mathcal{L}(\phi)\right],$$

where the field $\phi$ satisfies periodic boundary conditions only in one direction

$$\phi(\tau = 0, x) = \phi(\tau = L, x).$$

Let us again consider, as an example, the non-linear $\sigma$ model. We find a finite size system, but the interpretation of parameters is different. The variable $t$ now represents the coupling constant of the QFT. Since $L$ is the inverse temperature, the limit $L \to \infty$ corresponds to the limit of vanishing temperature.

The saddle point equation (4.58$b$), in the symmetric phase $\sigma = 0$, becomes

$$1 = (N-1)t \frac{1}{(2\pi)^{d-1} L} \int \mathrm{d}^{d-1}k \sum_n \frac{1}{m^2 + k^2 + (2\pi n/L)^2} \,. \tag{4.97}$$

On immediately verifies that the IR problem induced by the zero mode has the following consequences: since one integrates only over $d-1$ dimensions, a phase transition is only possible for $d > 3$. Qualitatively at large distance the condition of finite temperature leads to a property of *dimensional reduction* $d \mapsto d-1$. The large $N$ expansion is thus particularly well suited to the study of this problem which exhibits a crossover between two different dimensions.

Again using Schwinger's representation of the propagator, integrating over $k$ and introducing the function (4.91) we can rewrite equation (4.97):

$$\frac{1}{t} - \frac{1}{t_c} = \frac{N-1}{(4\pi)^{(d-1)/2}} L^{2-d} G(mL) \tag{4.98}$$

$$G(z) = \int_0^\infty \mathrm{d}s\, s^{-(d-1)/2} \left[ \mathrm{e}^{-z^2 s} A(4\pi^2 s) - (4\pi s)^{-1/2} \right] . \tag{4.99}$$

Here $\xi_L = m^{-1}$ has really the meaning of a correlation length.

This equation has a scaling form for $d < 4$. The behaviour of the system then depends on the ratio between $L$ and the correlation length $\xi_\infty$ of the system at zero temperature. For $t > t_c$ fixed and $L$ large (with respect to $1/\Lambda$) we recover the zero temperature limit. For $t - t_c$ small we find a crossover between a regime of small and high temperature. In the regime $t < t_c$ fixed and $L$ large, we have to examine the behaviour of $G(z)$ for $z$ small.

At $d = 3$:

$$G(z) = -2 \ln z + \text{ const.} \,.$$

Hence

$$\frac{1}{m^2} \propto \chi(L,t) \propto L^2 \exp\left[ \frac{4\pi L}{N} \left( \frac{1}{t} - \frac{1}{t_c} \right) \right] . \tag{4.100}$$

One finds that $\xi_L$ remains finite below $t_c$ for all non vanishing temperatures, and has when the coupling constant $t$ goes to zero or $L \to \infty$ the exponential behaviour characteristic of the dimension two.

For $d = 4$ the situation is different because a transition is possible in dimension $d - 1 = 3$. This is consistent with the existence of the quantity $G(0) > 0$ which appears in the relation between coupling constant and temperature at the critical point:

$$\frac{1}{t} - \frac{1}{t_c} = \frac{(N-1)G(0)}{(4\pi)^{3/2}} \frac{1}{L^2} \,. \tag{4.101}$$

For a coupling constant $t$ which corresponds to a phase of broken symmetry at zero temperature ($t < t_c$), one now finds a transition temperature $L^{-1} \propto \sqrt{t_c - t}$. Studying more generally the saddle point equations one can derive all other properties of this system.

### *4.11 Other methods. General vector field theories*

The large $N$ limit can be obtained by several other algebraic methods. Without being exhaustive, let us list a few. Schwinger–Dyson equations for $N$ large lead to a self-consistent equation for the two-point function. From the point of view of stochastic quantization or critical dynamics the Langevin equation also becomes linear and self-consistent for $N$ large. One replaces $\phi^2(x,t)$ by $\langle \phi^2(x,t) \rangle$ ($\langle \cdot \rangle$ means noise average) at leading order. Finally a version of the Hartree–Fock approximation also yields the large $N$ result.

*General vector field theories.* We now briefly explain how the algebraic method presented in section 4.1 can be generalized to actions which have a more complicated dependence in one or several vector fields. Again in a general $O(N)$ symmetric field theory the composite fields with small fluctuations are the scalars constructed from all vectors. The strategy is then to introduce pairs of fields and Lagrange multipliers for all independent $O(N)$ invariant scalar products constructed from the many-component fields.

Let us first take the example of one field $\phi$ and assume that the interaction is an arbitrary function of the only invariant $\phi^2(x)$

$$S(\phi) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[ \partial_\mu \phi(x) \right]^2 + V\left(\phi^2\right) \right\}. \tag{4.102}$$

We then introduce two fields $\rho(x)$ and $\lambda(x)$ and use the identity:

$$\exp\left[ -\int \mathrm{d}^d x \, V(\phi^2) \right] \propto \int [\mathrm{d}\rho(x)\, \mathrm{d}\lambda(x)] \exp\left\{ -\int \mathrm{d}^d x \left[ \tfrac{1}{2}\lambda\left(\phi^2 - \rho\right) + V(\rho) \right] \right\}. \tag{4.103}$$

In the special case in which $V(\rho)$ is a quadratic function, the integral over $\rho$ can be performed. In all cases, however, the identity (4.103) transforms the action into a quadratic form in $\phi$ and therefore the integration over $\phi$ can be performed and the dependence in $N$ becomes explicit. This method will be applied in section 7 to the study of multi-critical points and double scaling limit.

If the action is an $O(N)$ invariant function of two fields $\phi_1$ and $\phi_2$ the potential depends on the three scalar products $\phi_1 \cdot \phi_2$, $\phi_1^2$ and $\phi_2^2$. Then three pairs of fields are required.

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## 5 Gross–Neveu and Gross–Neveu–Yukawa Models

To illustrate the techniques developed in sections 3, 4, we now discuss models with fermions exhibiting the phenomenon of chiral phase transition. Again we consider two different field theory models with the same symmetries, the Gross–Neveu (GN) and the Gross–Neveu–Yukawa (GNY) models. The GN model is renormalizable in two dimensions, and describes in perturbation theory only one phase, the symmetric phase. The GNY model is renormalizable in four dimensions and instead allows a perturbative analysis of the chiral phase transition. We now show that the physics of these models can indeed be studied by the same techniques as ferromagnetic systems, that is RG equations near two and four dimensions, and large $N$ expansion.

### *5.1 The Gross–Neveu model*

The GN model is described in terms of a $U(N)$ symmetric action for a set of $N$ massless Dirac fermions $\{\psi^i, \bar\psi^i\}$:

$$S\left(\bar\psi, \psi\right) = -\int \mathrm{d}^d x \left[\bar\psi \cdot \not\partial \psi + \tfrac{1}{2} G \left(\bar\psi \cdot \psi\right)^2\right].$$

The GN model has in even dimensions a discrete chiral symmetry:

$$\psi \mapsto \gamma_S \psi, \quad \bar\psi \mapsto -\bar\psi \gamma_S\,, \tag{5.1}$$

which prevents the addition of a fermion mass term while in odd dimensions a mass term breaks space parity. Actually the two symmetry operations can be written in a form

$$\mathbf{x} = \{x_1, x_2, \dots, x_d\} \mapsto \tilde{\mathbf{x}} = \{-x_1, x_2, \dots, x_d\}, \quad \begin{cases} \psi(x) \mapsto \gamma_1 \psi(\tilde x), \\ \bar\psi(x) \mapsto -\bar\psi(\tilde x)\gamma_1 \end{cases},$$

valid in all dimensions.

This model illustrates the physics of spontaneous fermion mass generation and, in even dimensions, chiral symmetry breaking. It is renormalizable and asymptotically free in two dimensions. However, as in the case of the non-linear $\sigma$ model, the perturbative GN model describes only one phase. The main difference is that the role of the spontaneously broken and the explicitly symmetric phase are interchanged. Indeed it is always the massless phase which is unstable in low dimensions.

Since the symmetry breaking mechanism is non-perturbative it will eventually be instructive to compare the GN model with a different model with the same symmetries: the Gross–Neveu–Yukawa model.

*RG equations near and in two dimensions.* The GN model is renormalizable in two dimensions, and in perturbation theory describes only the massless symmetric phase. Perturbative calculations in two dimensions can be made with an IR cut-off of the form of a mass term $M\bar{\psi}\psi$, which breaks softly the chiral symmetry. It is possible to use dimensional regularization in practical calculations. Note that in two dimensions the symmetry group is really $O(2N)$, as one verifies after some relabelling of the fields. Therefore the $(\bar{\psi}\psi)^2$ interaction is multiplicatively renormalized. It is convenient to introduce here a dimensionless coupling constant

$$u = G\Lambda^{2-d}. \tag{5.2}$$

As a function of the cut-off $\Lambda$ the bare correlation functions satisfy the RG equations:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} - \frac{n}{2}\eta_\psi(u) - \eta_M(u)M\frac{\partial}{\partial M}\right]\Gamma^{(n)}(p_i; u, M, \Lambda) = 0\,. \tag{5.3}$$

A direct calculation of the $\beta$-function in $d = 2 + \varepsilon$ dimension yields:

$$\beta(u) = \varepsilon u - (N'-2)\frac{u^2}{2\pi} + (N'-2)\frac{u^3}{4\pi^2} + \frac{(N'-2)(N'-7)}{32\pi^3}u^4 + O\left(u^5\right), \tag{5.4}$$

Note that for $d = 2$ $N' = 2N$.

The special case $N' = 2$, for which the $\beta$-function vanishes identically in two dimensions, corresponds to the Thirring model (because for $N' = 2$ $(\bar{\psi}\gamma_\mu\psi)^2 = -2(\bar{\psi}\psi)^2$). The latter model is to the equivalent the sine-Gordon or the $O(2)$ vector model.

Finally the field and mass RG functions are

$$\begin{aligned}
\eta_\psi(u) &= \frac{N'-1}{8\pi^2}u^2 - \frac{(N'-1)(N'-2)}{32\pi^3}u^3 + \frac{(N'-1)(N'-2)(N'-5)}{128\pi^4}u^4, \\
\eta_M(u) &= \frac{N'-1}{2\pi}u - \frac{N'-1}{8\pi^2}u^2 - \frac{(2N'-3)(N'-1)}{32\pi^3}u^3.
\end{aligned} \tag{5.5}$$

As in the case of the non-linear $\sigma$ model, the solution of the RG equations (5.3) involves a length scale $\xi$ of the type of a correlation length which is a RG invariant

$$\xi^{-1}(u) \equiv \Lambda(u) \propto \Lambda \exp\left[-\int^u \frac{\mathrm{d}u'}{\beta(u')}\right]. \tag{5.6}$$

*Two dimensions.* For $d = 2$ the model is asymptotically free. In the chiral theory ($M = 0$) the spectrum, then, is non-perturbative, and many arguments lead to the conclusion that the chiral symmetry is always broken and a fermion mass generated. From the statistical point of view this corresponds to the existence of a gap in the spectrum of fermion excitation (as in a superfluid or superconductor). All masses are proportional to the mass parameter $\Lambda(u)$ which is a RG invariant. Its dependence in the coupling constant is given by equation (5.6):

$$\Lambda(u) \propto \Lambda u^{1/(N'-2)}\,\mathrm{e}^{-2\pi/(N'-2)u}\left(1 + O(u)\right). \tag{5.7}$$

We see that the continuum limit, which is reached when the masses are small compared to the cut-off, corresponds to $u \to 0$.

$S$-matrix considerations have then led to the conjecture that, for $N$ finite, the spectrum is:

$$m_n = \Lambda(u) \frac{2(N-1)}{\pi} \sin\left(\frac{n\pi}{2(N-1)}\right), \quad n = 1, 2 \ldots < N\,,\ N > 2\,,$$

To each mass value corresponds a representation of the $O(2N)$ group. The nature of the representation leads to the conclusion that $n$ odd corresponds to fermions and $n$ even to bosons.

This result is consistent with the spectrum for $N$ large evaluated by semiclassical methods. In particular the ratio of the masses of the fundamental fermion and the lowest lying boson is:

$$\frac{m_\sigma}{m_\psi} = 2\cos\left(\frac{\pi}{2(N-1)}\right) = 2 + O(1/N^2). \tag{5.8}$$

The large $N$ limit will be recovered in section 5.4.

Note that the two first values of $N$ are special, the model $N = 2$ is conjectured to be equivalent to two decoupled sine-Gordon models.

*Dimension* $d = 2 + \varepsilon$. As in the case of the $\sigma$-model, asymptotic freedom implies the existence of a non-trivial UV fixed point $u_c$, in $2 + \varepsilon$ dimension

$$u_c = \frac{2\pi}{N' - 2}\varepsilon\left(1 - \frac{\varepsilon}{N' - 2}\right) + O\left(\varepsilon^3\right).$$

$u_c$ is also the critical coupling constant for the transition between a phase in which the chiral symmetry is spontaneously broken and a massless small $u$ phase.

At the fixed point one finds the correlation length exponent $\nu$:

$$\nu^{-1} = -\beta'(u_c) = \varepsilon - \frac{\varepsilon^2}{N' - 2} + O\left(\varepsilon^3\right). \tag{5.9}$$

The fermion field dimension $[\psi]$ is:

$$2[\psi] = d - 1 + \eta_\psi(u_c) = 1 + \varepsilon + \frac{N' - 1}{2(N' - 2)^2}\varepsilon^2 + O\left(\varepsilon^3\right). \tag{5.10}$$

The dimension of the composite field $\sigma = \bar\psi\psi$ is given by

$$[\sigma] = d - 1 - \eta_M(u_c) = 1 - \frac{\varepsilon}{N' - 2}.$$

As for the $\sigma$-model the existence of a non-trivial UV fixed point implies that large momentum behaviour is not given by perturbation theory above two dimensions, and this explains why the perturbative result that the model cannot be defined in higher dimensions cannot be trusted. However, to investigate whether the $\varepsilon$ expansion makes sense beyond an infinitesimal neighbourhood of dimension two other methods are required, like the $1/N$ expansion which will be considered in section 5.4.

*5.2 The Gross–Neveu–Yukawa model*

The Gross–Neveu–Yukawa (GNY) model has the same chiral and $U(N)$ symmetries as the GN model. The action is ($\varepsilon = 4 - d$):

$$S\left(\bar{\psi},\psi,\sigma\right) = \int \mathrm{d}^d x \left[ -\bar{\psi}\cdot\left(\not{\partial} + g\Lambda^{\varepsilon/2}\sigma\right)\psi + \tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 + \tfrac{1}{2}m^2\sigma^2 + \frac{\lambda}{4!}\Lambda^{\varepsilon}\sigma^4 \right], \tag{5.11}$$

where $\sigma$ is an additional scalar field, $\Lambda$ the momentum cut-off, and $g, \lambda$ dimensionless "bare" i.e. effective coupling constants at large momentum scale $\Lambda$.

The action still has a reflection symmetry, $\sigma$ transforming into $-\sigma$ when the fermions transform by (5.1). In contrast with the GN model, however, the chiral transition can here be discussed by perturbative methods. An analogous situation has already been encountered when comparing the $(\phi^2)^2$ field theory with the non-linear $\sigma$ model. Even more, the GN model is renormalizable in dimension two and the GNY model in dimension four.

*The phase transition.* Examining the action (5.11) we see that in the tree approximation when $m^2$ is negative the chiral symmetry is spontaneously broken. The $\sigma$ expectation value gives a mass to the fermions, a mechanism reminiscent of the Standard Model of weak-electromagnetic interactions:

$$m_\psi = g\left\langle\sigma\right\rangle, \tag{5.12}$$

while the $\sigma$ mass then is:

$$m_\sigma^2 = \frac{\lambda}{3g^2}m_\psi^2 . \tag{5.13}$$

As a result of interactions the transition value $m_c^2$ of the parameter $m^2$ will be modified. In what follows we set

$$m^2 = m_c^2 + t\,, \tag{5.14}$$

where the new parameter $t$, in the language of phase transitions, plays the role of the deviation from the critical temperature.

To study the model beyond the tree approximation we now discuss RG equations near four dimensions.

*5.3 RG equations near four dimensions*

The model (5.11) is trivial above four dimensions, renormalizable in four dimensions and can thus be studied near dimension 4 by RG techniques. Five renormalization constants are required, corresponding to the two field renormalizations, the $\sigma$ mass, and the two coupling constants. The RG equations thus involve five RG functions. The 1PI correlation functions $\Gamma^{(l,n)}$, for $l$ $\psi$ and $n$ $\sigma$ fields, then satisfy

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_{g^2}\frac{\partial}{\partial g^2} + \beta_\lambda\frac{\partial}{\partial\lambda} - \tfrac{1}{2}l\eta_\psi - \tfrac{1}{2}n\eta_\sigma - \eta_m t\frac{\partial}{\partial t}\right)\Gamma^{(l,n)} = 0\,. \tag{5.15}$$

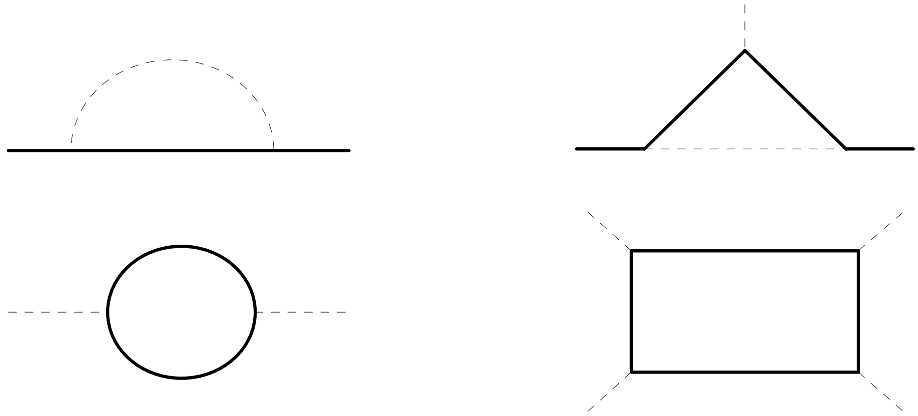

Fig. 6 One-loop diagrams: fermions are represented by solid lines.

*The RG functions.* The RG functions at one-loop order involve the calculation of the diagrams of figure 6. One finds:

$$\beta_\lambda = -\varepsilon\lambda + \frac{1}{8\pi^2}\left(\frac{3}{2}\lambda^2 + 4N\lambda g^2 - 24Ng^4\right), \tag{5.16}$$

$$\beta_{g^2} = -\varepsilon g^2 + \frac{2N+3}{8\pi^2}g^4. \tag{5.17}$$

Note that in these expressions for convenience we have set in the algebra of $\gamma$ matrices $\operatorname{tr}\mathbf{1} = 4$ as in four dimensions. To extrapolate the results to other dimensions one has to replace everywhere $N$ by $N'/4$, where $N' = N\operatorname{tr}\mathbf{1}$ is the total number of fermion degrees of freedom.

*Dimension four.* In four dimensions the origin $\lambda = g^2 = 0$ is IR stable. Indeed the second equation implies that $g$ goes to zero, and the first then that $\lambda$ also goes to zero. As a consequence if the bare coupling constants are generic, i.e. if the effective couplings at cut-off scale are of order 1, the effective couplings at scale $\mu \ll \Lambda$ go to zero and in a way asymptotically independent from the bare couplings. One finds

$$g^2(\mu) \sim \frac{8\pi^2}{(2N+3)\ln(\Lambda/\mu)}, \quad \lambda(\mu) \sim \frac{8\pi^2\tilde{\lambda}^*}{\ln(\Lambda/\mu)},$$

where we have defined

$$\tilde{\lambda}_* = \frac{48N}{(2N+3)\left[(2N-3) + \sqrt{4N^2+132N+9}\right]}. \tag{5.18}$$

This result allows to use renormalized perturbation theory to calculation physical observables. For example we can evaluate the ratio between the masses of the scalar and fermion fields. It is then optimal to take for $\mu$ a value of order $\langle\sigma\rangle$. A remarkable consequence follows: the ratio (5.13) of scalar and fermion masses is fixed

$$\frac{m_\sigma^2}{m_\psi^2}=\frac{\lambda_*}{3g_*^2}=\frac{16N}{(2N-3)+\sqrt{4N^2+132N+9}}, \tag{5.19}$$

while in the classical limit it seems arbitrary.

Of course if the bare couplings are "unnaturally" small the same will apply to the renormalized couplings at scale $\mu$ and the ratio will be modified.

*Dimension* $d=4-\varepsilon$. One then finds a non-trivial IR fixed point (we recall $N'=N\,\mathrm{tr}\,\mathbf{1}$):

$$g_*^2=\frac{16\pi^2\varepsilon}{N'+6}, \quad \lambda_*=8\pi^2\varepsilon\tilde{\lambda}_*\,. \tag{5.20}$$

The matrix of derivatives of the $\beta$-functions has two eigenvalues $\omega,\omega'$,

$$\omega_1=\varepsilon, \quad \omega_2=\varepsilon\sqrt{N'^2+132N'+36}/(N'+6), \tag{5.21}$$

and thus the fixed point is IR stable. The first eigenvalue is always the smallest.

The field renormalization RG functions are at the same order:

$$\eta_\sigma=\frac{N'}{16\pi^2}g^2, \qquad \eta_\psi=\frac{1}{16\pi^2}g^2. \tag{5.22}$$

At the fixed point one finds

$$\eta_\sigma=\frac{N'\varepsilon}{N'+6}, \quad \eta_\psi=\frac{\varepsilon}{(N'+6)}, \tag{5.23}$$

and thus the dimensions $d_\psi$ and $d_\sigma$ of the fields

$$d_\psi=\frac{3}{2}-\frac{N'+4}{2(N'+6)}\varepsilon\,, \qquad d_\sigma=1-\frac{3}{N'+6}\varepsilon\,. \tag{5.24}$$

The RG function $\eta_m$ corresponding to the mass operator is at one-loop order:

$$\eta_m=-\frac{\lambda}{16\pi^2}-\eta_\sigma\,,$$

and thus the exponent $\nu$:

$$\frac{1}{\nu}=2+\eta_m=2-\frac{\varepsilon}{2}\tilde{\lambda}_*-\frac{N'\varepsilon}{N'+6}=2-\varepsilon\frac{5N'+6+\sqrt{N'^2+132N'+36}}{6(N'+6)}. \tag{5.25}$$

Finally we can evaluate the ratio of masses (5.13) at the fixed point:

$$\frac{m_\sigma^2}{m_\psi^2}=\frac{\lambda_*}{3g_*^2}=\frac{8N'}{(N'-6)+\sqrt{N'^2+132N'+36}}.$$

In $d=4$ and $d=4-\varepsilon$ the existence of an IR fixed point has the same consequence: If we assume that the $\sigma$ expectation value is much smaller than the cut-off and that the coupling constants are generic at the cut-off scale, then *the ratio of fermion and scalar masses is fixed.*

*5.4 GNY and GN models in the large N limit*

We now show that the GN model plays with respect to the GNY model (5.11) the role the non-linear $\sigma$-model plays with respect to the $\phi^4$ field theory. For this purpose we start from the action (5.11) of the GNY model and integrate over $N-1$ fermion fields. We also rescale for convenience $\Lambda^{(4-d)/2}g\sigma$ into $\sigma$, and then get the large $N$ action:

$$\begin{aligned} S_N\left(\bar\psi,\psi,\sigma\right) = \int \mathrm{d}^d x \left\{ -\bar\psi\left(\not\partial+\sigma\right)\psi + \Lambda^{d-4}\left[\frac{1}{2g^2}\left(\partial_\mu\sigma\right)^2 + \frac{m^2}{2g^2}\sigma^2 + \frac{\lambda}{4!g^4}\sigma^4\right]\right\} \\ -\left(N-1\right)\operatorname{tr}\ln\left(\not\partial+\sigma\right). \end{aligned} \tag{5.26}$$

To take the large $N$ limit we assume $\sigma$ finite and $g^2,\lambda = O(1/N)$.

Let us call $V(\sigma)$ the action per unit volume for constant field $\sigma(x)$ and vanishing fermion fields

$$\begin{aligned} V(\sigma) &= \Lambda^{d-4}\left(\frac{m^2}{2g^2}\sigma^2+\frac{\lambda}{4!g^4}\sigma^4\right) - N\operatorname{tr}\ln\left(\not\partial+\sigma\right) \\ &= \Lambda^{d-4}\left(\frac{m^2}{2g^2}\sigma^2+\frac{\lambda}{4!g^4}\sigma^4\right) - \frac{N'}{2}\int^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d}\ln(q^2+\sigma^2). \end{aligned} \tag{5.27}$$

The expectation value of $\sigma$ for $N$ large is given by a *gap* equation:

$$V'(\sigma)\Lambda^{4-d} = \frac{m^2}{g^2}\sigma + \frac{\lambda}{6g^4}\sigma^3 - N'\Lambda^{4-d}\frac{\sigma}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2+\sigma^2} = 0\,. \tag{5.28}$$

It is also useful to calculate the second derivative to check stability of the extrema

$$V''(\sigma)\Lambda^{4-d} = \frac{m^2}{g^2} + \frac{\lambda}{2g^4}\sigma^2 + N'\Lambda^{4-d}\int^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\sigma^2-q^2}{(q^2+\sigma^2)^2}\,.$$

The solution $\sigma=0$ is stable provided

$$V''(0)>0 \;\Leftrightarrow\; \frac{m^2}{g^2} > N'\Lambda^{4-d}\frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2}.$$

Instead the non-trivial solution to the gap equation exists only for

$$\frac{m^2}{g^2} > N'\Lambda^{4-d}\frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2},$$

but then it is stable. We conclude that the critical temperature or critical bare mass is given by:

$$\frac{m_c^2}{g^2} = N'\Lambda^{4-d}\frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2}, \tag{5.29}$$

which shows that the fermions favour the chiral transition. In particular when $d$ approaches 2 we observe that $m_c^2 \to +\infty$ which implies that the chiral symmetry is always broken in 2 dimensions. Using equation (5.29) and setting

$$t = \Lambda^{d-4}(m^2 - m_c^2)/g^2, \tag{5.30}$$

we can write the equation for the non-trivial solution

$$t + \Lambda^{d-4}\frac{\lambda}{6g^4}\sigma^2 + N'\frac{\sigma^2}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d q}{q^2(q^2+\sigma^2)} = 0\,.$$

We now expand the integral for $\sigma$ small (equation (4.16))

$$D_1(\sigma^2) = \frac{1}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d q}{q^2\,(q^2+\sigma^2)} = C(d)\sigma^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + O\left(\frac{\sigma^{2-\varepsilon}}{\Lambda^2}\right). \tag{5.31}$$

Keeping only the leading terms for $t \to 0$ we obtain for $d < 4$ the scaling behaviour

$$\sigma \sim (-t/N'C)^{1/(d-2)}. \tag{5.32}$$

Since, at leading order, the fermion mass $m_\psi = \sigma$, it immediately follows that the exponent $\nu$ is also given by:

$$\nu \sim \beta \sim 1/(d-2) \;\Rightarrow\; \eta_\sigma = 4 - d\,. \tag{5.33}$$

At leading order, for $N \to \infty$, $\nu$ has the same value as in the non-linear $\sigma$-model.

At leading order in the scaling limit the thermodynamical (or effective) potential $V(\sigma)$ then becomes

$$V(\sigma) = \tfrac{1}{2}t\sigma^2 + (N'/d)C(d)|\sigma|^d. \tag{5.34}$$

We note that, although in terms of the $\sigma$-field the model has a simple Ising-like symmetry, the scaling equation of state for large $N$ is quite different.

We read from the large $N$ action that at this order $\eta_\psi = 0$.

Finally from the large $N$ action we can calculate the $\sigma$-propagator at leading order. Quite generally, using the saddle point equation, one finds for the inverse $\sigma$-propagator in the massive phase:

$$\Delta_\sigma^{-1}(p) = \Lambda^{d-4}\left(\frac{p^2}{g^2} + \frac{\lambda}{3g^4}\sigma^2\right) + \frac{N'}{2(2\pi)^d}\left(p^2 + 4\sigma^2\right)\int^{\Lambda}\frac{\mathrm{d}^d q}{(q^2+\sigma^2)\left[(p+q)^2+\sigma^2\right]}. \tag{5.35}$$

We see that in the scaling limit $p, \sigma \to 0$, the integral yields the leading contribution. Neglecting corrections to scaling we find that the propagator vanishes for $p^2 = -4\sigma^2$ which is just the $\bar\psi\psi$ threshold. Thus, in this limit, $m_\sigma = 2m_\psi$ in all dimensions, a result consistent with $d = 2$ exact value.

At the transition the propagator reduces to

$$\Delta_\sigma \sim \frac{2}{N' b(\varepsilon) p^{d-2}}, \tag{5.36}$$

with (equation (4.38))

$$b(\varepsilon) = -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} N_d \,. \tag{5.37}$$

The result is consistent with the value of $\eta_\sigma$ found above.

Let us finally note that the behaviour of the propagator at the critical point, $\Delta_\sigma(p) \propto p^{2-d}$, implies for the field $\sigma$ the canonical dimension $[\sigma]$ in the large $N$ expansion, for $2 \le d \le 4$:

$$[\sigma] = 1 \,. \tag{5.38}$$

*Corrections to scaling and the IR fixed point.* The IR fixed point is determined by demanding the cancellation of the leading corrections to scaling. Let us thus consider the effective potential $V(\sigma)$. The leading correction to scaling is proportional to

$$\left(\frac{\lambda}{4! g^4} - \frac{N' a(d)}{4}\right) \sigma^4 ,$$

($a(\varepsilon) \sim 1/8\pi^2\varepsilon$). Demanding the cancellation of the coefficient of $\sigma^2$, we obtain a relation between $\lambda$ and $g^2$

$$g_*^4 = \frac{\lambda_*}{6N' a(d)} = \frac{4\lambda_* \varepsilon \pi^2}{3N'} + O\left(\varepsilon^2\right),$$

a result consistent with the results of the $\varepsilon$-expansion.

In the same way it is possible to calculate the leading correction to the $\sigma$-propagator (5.35). Demanding the cancellation of the leading correction we obtain

$$\frac{p^2}{g_*^2} + \frac{\lambda_*}{3g_*^4}\sigma^2 - \tfrac{1}{2} N' \left(p^2 + 4\sigma^2\right) a(d) = 0 \,.$$

The coefficient of $\sigma^2$ cancels from the previous relation and the cancellation the coefficient of $p^2$ yields

$$g_*^2 = \frac{2}{N' a(d)} = \frac{16\pi^2 \varepsilon}{N'} + O\left(\varepsilon^2\right),$$

in agreement with the $\varepsilon$-expansion for $N$ large.

*The relation to the GN model for dimensions* $2 \le d \le 4$. We have seen that the terms $(\partial_\mu \sigma)^2$ and $\sigma^4$ of the large $N$ action which have a canonical dimension 4, are irrelevant in the IR critical region for $d \le 4$. We recognize a situation already encountered in the $(\phi^2)^2$ field theory in the large $N$ limit. In the scaling region it is possible to omit them and one then finds the action:

$$S_N\left(\bar\psi,\psi,\sigma\right) = \int \mathrm{d}^d x \left[-\bar\psi\cdot\left(\not\partial+\sigma\right)\psi + \Lambda^{d-4}\frac{m^2}{2g^2}\sigma^2\right]. \tag{5.39}$$

The integral over the $\sigma$ field can explicitly be performed and yields the action of the GN model:

$$S_N\left(\bar\psi,\psi\right) = -\int \mathrm{d}^d x \left[\bar\psi\cdot\not\partial\psi + \frac{\Lambda^{4-d}}{2m^2}g^2\left(\bar\psi\cdot\psi\right)^2\right].$$

The GN and GNY models are thus equivalent for the large distance physics. In the GN model, in the large $N$ limit, the $\sigma$ particle appears as a $\bar\psi\psi$ boundstate at threshold.

Conversely, it would seem that the GN model depends on a smaller number of parameters than its renormalizable extension. Again this problem is only interesting in four dimensions where corrections to scaling, i.e. to free field theory, are important. However, if we examine the divergences of the term $\operatorname{tr}\ln\left(\not\partial+\sigma\right)$ in the effective action (5.26) relevant for the large $N$ limit, we find a local polynomial in $\sigma$ of the form:

$$\int \mathrm{d}^4 x \left[A\sigma^2(x) + B\left(\partial_\mu\sigma\right)^2 + C\sigma^4(x)\right].$$

Therefore the value of the determinant can be modified by a local polynomial of this form by changing the way the cut-off is implemented: additional parameters, as in the case of the non-linear $\sigma$-model, are hidden in the cut-off procedure. Near two dimensions these operators can be identified with $(\bar\psi\psi)^2, [\partial_\mu(\bar\psi\psi)]^2, (\bar\psi\psi)^4$. It is clear that by changing the cut-off procedure we change the amplitude of higher dimension operators. These bare operators in the IR limit have a component on all lower dimensional renormalized operators.

Note finally that we could have added to the GNY model an explicit breaking term linear in the $\sigma$ field, which becomes a fermion mass term in the GN model, and which would have played the role of the magnetic field of the ferromagnets.

### 5.5 *The large N expansion*

Using the large $N$ dimension of fields and power counting arguments one can then prove that the $1/N$ expansion is renormalizable with arguments quite similar to those presented in section 4.6.

*Alternative theory.* To prove that the large $N$ expansion is renormalizable one proceeds as in the case of the scalar theory in section 4.6. One starts from a critical action with an additional term quadratic in $\sigma$ which generates the large $N$ $\sigma$-propagator already in perturbation theory

$$S(\psi,\bar{\psi},\sigma)=\int \mathrm{d}^d x\left[-\bar{\psi}(\not\partial+\sigma)\psi+\frac{1}{2v^2}\sigma(-\partial^2)^{d/2-1}\sigma\right]. \tag{5.40}$$

The initial theory is recovered in the limit $v\to\infty$. One then rescales $\sigma$ in $v\sigma$. The model is renormalizable without $\sigma$ field renormalization because divergences generate only local counter-terms

$$S_{\mathrm{r}}(\psi,\bar{\psi},\sigma)=\int \mathrm{d}^d x\left[-Z_\psi\bar{\psi}(\not\partial+v_{\mathrm{r}}Z_v\sigma)\psi+\frac{1}{2}\sigma(-\partial^2)^{d/2-1}\sigma\right]. \tag{5.41}$$

RG equations follow

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta_{v^2}(v)\frac{\partial}{\partial v^2}-\frac{n}{2}\eta_\psi(v)\right]\Gamma^{(l,n)}=0\,. \tag{5.42}$$

Again the large $N$ expansion is obtained by first summing the bubble contributions to the $\sigma$-propagator. We define

$$D(v)=\frac{2}{b(\varepsilon)}+N'v^2.$$

Then the large $N$ $\sigma$ propagator reads

$$\langle\sigma\sigma\rangle=\frac{2}{b(\varepsilon)D(v)p^{d-2}}. \tag{5.43}$$

The solution to the RG equations can be written:

$$\Gamma^{(l,n)}(\tau p,v,\Lambda)=Z^{-n/2}(\tau)\tau^{d-l-n(d-2)/2}\Gamma^{(l,n)}(p,v(\tau),\Lambda), \tag{5.44}$$

with the usual definitions

$$\tau\frac{\mathrm{d}v^2}{\mathrm{d}\tau}=\beta(v(\tau))\,,\quad \tau\frac{\mathrm{d}\ln Z}{\mathrm{d}\tau}=\eta_\psi(v(\tau))\,.$$

We are interested in the neighbourhood of the fixed point $v^2=\infty$. Then the RG function $\eta(v)$ approaches the exponent $\eta$. The flow equation for the coupling constant becomes:

$$\tau\frac{\mathrm{d}v^2}{\mathrm{d}\tau}=\rho v^2,\ \Rightarrow\ v^2(\tau)\sim\tau^\rho.$$

We again note that a correlation function with $l$ $\sigma$ fields becomes proportional to $v^l$. Therefore

$$\Gamma^{(l,n)}(\tau p,v,\Lambda)\propto\tau^{d-(1-\rho/2)l-n(d-2+\eta_\psi)/2}. \tag{5.45}$$

We conclude

$$d_\sigma = \tfrac{1}{2}(d-2+\eta_\sigma) = 1-\tfrac{1}{2}\rho \ \Leftrightarrow\ \eta_\sigma = 4-d-\rho\,. \tag{5.46}$$

*RG functions at order* $1/N$. A new generic integral is useful here

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q(\not p+\not q)}{(p+q)^{2\mu}q^{2\nu}} = \not p p^{d-2\mu-2\nu}\frac{\Gamma(\mu+\nu-d/2)\Gamma(d/2-\mu+1)\Gamma(d/2-\nu)}{(4\pi)^{d/2}\Gamma(\mu)\Gamma(\nu)\Gamma(d-\mu-\nu+1)}\,. \tag{5.47}$$

We first calculate the $1/N$ contribution to the fermion two-point function at the critical point (from a diagram similar to diagram 3)

$$\Gamma^{(2)}_{\bar\psi\psi}(p) = i\not p + \frac{2iv^2}{b(\varepsilon)D(v)(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q(\not p+\not q)}{q^{d-2}(p+q)^2}.$$

We need the coefficient of $\not p \ln\Lambda/p$. Since we work only at one-loop order we again replace the $\sigma$ propagator $1/q^{d-2}$ by $1/q^{2\nu}$, and send the cut-off to infinity. The residue of the pole at $2\nu = d-2$ gives the coefficient of the term $\not p\ln\Lambda$ and the finite part the $\not p\ln p$ contribution. We find

$$\Gamma^{(2)}_{\bar\psi\psi}(p) = i\not p + \frac{2iv^2}{b(\varepsilon)D(v)}N_d\left(\frac{d-2}{d}\right)\not p\ln(\Lambda/p)\,, \tag{5.48}$$

where $N_d$ is the loop factor (4.17$a$). Expressing that the $\langle\bar\psi\psi\rangle$ function satisfies RG equations we immediately obtain the RG function $\eta_\psi(v)$

$$\eta_\psi(v) = \frac{v^2}{D(v)}\frac{(d-2)}{d}X_1\,, \tag{5.49}$$

where $X_1$ is given by equation (4.79). We then calculate the function $\langle\sigma\bar\psi\psi\rangle$ at order $1/N$

$$\Gamma^{(3)}_{\sigma\bar\psi\psi}(p) = v + A_1 D^{-1}(v)v^3\ln\Lambda\,,$$

with

$$A_1 = -\frac{2}{b(\varepsilon)}N_d = -X_1\,,$$

where $A_1$ corresponds to the diagram of figure 4. The diagram of figure 5 vanishes because the $\sigma$ 3-point function vanishes for symmetry reasons.

The $\beta$-function follows

$$\beta_{v^2}(v) = \frac{4(d-1)v^4}{d}X_1 D^{-1}(v), \tag{5.50}$$

and thus

$$\rho = \frac{8(d-1)N_d}{db(\varepsilon)N'} = \frac{4(d-1)}{dN'}X_1 \,.$$

The exponents $\eta_\psi$ and $\eta_\sigma$ at order $1/N$, and thus the corresponding dimensions of fields $d_\psi, d_\sigma$ follow

$$\eta_\psi = \frac{(d-2)}{d}\frac{X_1}{N'} = \frac{(d-2)^2}{d}\frac{\Gamma(d-1)}{\Gamma^3(d/2)\Gamma(2-d/2)N'}. \qquad (5.51)$$

$$2d_\psi = d-1-\frac{2(d-2)}{d}\frac{X_1}{N'}. \qquad (5.52)$$

For $d=4-\varepsilon$ we find $\eta_\psi \sim \varepsilon/N'$, result consistent with (5.23) for $N$ large. For $d=2+\varepsilon$ instead one finds $\eta_\psi \sim \varepsilon^2/2N'$, consistent with (5.10). The dimension $d_\sigma$ of the field $\sigma$ is

$$d_\sigma = \tfrac{1}{2}(d-2+\eta_\sigma) = 1-\frac{2(d-1)}{dN'}X_1 + O(1/N'^2). \qquad (5.53)$$

A similar evaluation of the $\langle \sigma^2\sigma\sigma \rangle$ function allows to determine the exponent $\nu$ to order $1/N$

$$\frac{1}{\nu} = d-2-\frac{2(d-1)(d-2)}{dN'}X_1 \,. \qquad (5.54)$$

Actually all exponents are known to order $1/N^2$ except $\eta_\psi$ which is known to order $1/N^3$.

## 6 Other models with chiral fermions

Let us for completeness shortly examine two other models with chiral fermions, in which large $N$ techniques can be applied, massless QED and the $U(N)$ massless Thirring model.

### *6.1 Massless electrodynamics*

Let us give another example with a structure different from a Yukawa-type theory. We now consider a model of $N$ charged massless fermion fields $\psi, \bar\psi$, coupled through an abelian gauge field $A_\mu$ (massless QED):

$$S\left(\bar\psi,\psi,A_\mu\right) = \int \mathrm{d}^d x \left[\tfrac{1}{4e^2}F_{\mu\nu}^2(x) - \bar\psi(x)\cdot\left(\not\partial + i\not A\right)\psi(x)\right]. \qquad (6.1)$$

This model possesses, in addition to the $U(1)$ gauge invariance, a chiral $U(N)\times U(N)$ symmetry because the fermions are massless. Again the interesting question is whether the model exhibits in some dimension $2 \le d \le 4$ a spontaneous breaking of the chiral symmetry.

*Dimension* $d = 4 - \varepsilon$. In terms of the standard coupling constant $\alpha$:

$$\alpha \equiv \frac{e^2}{4\pi}, \tag{6.2}$$

the RG $\beta$ function reads (taking $\operatorname{tr} \mathbf{1} = 4$ in the space of $\gamma$ matrices):

$$\begin{aligned}\beta(\alpha) &= -\varepsilon\alpha + \frac{2N}{3\pi}\alpha^2 + + \frac{N}{2\pi^2}\alpha^3 - \frac{N(22N+9)}{144\pi^3}\alpha^4 \\ &- \frac{1}{64\pi^4} N \left[\tfrac{616}{243}N^2 + \left(\tfrac{416}{9}\zeta(3) - \tfrac{380}{27}\right) N + 23\right] \alpha^5 + O\left(\alpha^6\right). \end{aligned}\tag{6.3}$$

The model is free at low momentum in four dimensions. Therefore no phase transition is expected, at least for $e^2$ small enough. A hypothetical phase transition would rely on the existence on non-trivial fixed points outside of the perturbative regime.

In the perturbative framework the model provides an example of the famous triviality problem. For a generic effective coupling constant at cut-off scale (i.e. bare coupling), the effective coupling constant at scale $\mu \ll \Lambda$ is given by

$$\alpha(\mu) \equiv \frac{e^2(\mu)}{4\pi} \sim \frac{3\pi}{2N \ln(\Lambda/\mu)}.$$

This result can be used to bound $N$.

In $4 - \varepsilon$ dimension, one instead finds a non-trivial IR fixed point corresponding to a coupling constant:

$$e_*^2 = 24\pi^2 \varepsilon \Lambda^\varepsilon / N',$$

($N' = N \operatorname{tr} \mathbf{1}$) and correlation functions have a scaling behaviour at long distance. As we have discussed in the case of the $\phi^4$ field theory, the effective coupling constant at large distance becomes close to the IR fixed point, except when the initial coupling constant is very small.

The RG function associated with the field renormalization is also known at order $\alpha^3$ but this is a non-physical quantity since gauge dependent

$$\eta_\psi = \xi \frac{\alpha}{2\pi} - \frac{4N+3}{16\pi^2}\alpha^2 + \frac{40N^2 + 54N + 27}{576\pi^3}\alpha^3 + O\left(\alpha^4\right),$$

where the gauge is specified by a term $(\partial_\mu A_\mu)^2/2\xi$.

### *6.2 The large N limit*

To evaluate correlation functions for $N$ large, one first integrates over the fermion fields and one obtains the effective action:

$$S\left(\bar\psi, \psi, A_\mu\right) = \int \mathrm{d}^d x \left[\tfrac{1}{4e^2} F_{\mu\nu}^2(x) - N \operatorname{tr} \ln\left(\not\partial + i\not{A}\right)\right]. \tag{6.4}$$

The large $N$ limit is taken with $e^2N$ fixed. Therefore, at leading order, only $S_2(A_\mu)$, the quadratic term in $A_\mu$ in the expansion of the fermion determinant, contributes. A short calculation yields

$$\begin{aligned} S_2(A_\mu) &= -N' \int \mathrm{d}^d k \, A_\mu(k) \left[k^2 \delta_{\mu\nu} - k_\mu k_\nu\right] A_\nu(-k) K(k), \\ \text{with} \quad K(k) &= \frac{d-2}{4(d-1)} \left[b(\varepsilon) k^{d-4} - a(d) \Lambda^{d-4}\right] + O\left(\Lambda^{-2}\right), \end{aligned} \tag{6.5}$$

where $a(d)$ is a regularization-dependent constant.

For $d < 4$ the leading term in the IR region comes from the integral. The behaviour at small momentum of the vector field is modified, which confirms the existence of a non-trivial IR fixed point. The fixed point is found by demanding cancellation of the leading corrections to scaling coming from $F^2_{\mu\nu}$ and the divergent part of the loop integral,

$$e_*^2 = \frac{2(d-1)}{(d-2)a(d)} \frac{\Lambda^{4-d}}{N'}.$$

However there is still no indication of chiral symmetry breaking. Power counting within the $1/N$ expansion confirms that the IR singularities have been eliminated, because the large $N$ vector propagator is less singular than in perturbation theory. Of course this result is valid only for $N$ large. Since the long range forces generated by the gauge coupling have not been totally eliminated the problem remains open for $d$ not close to four, or for $e^2$ not very small and $N$ finite. Some numerical simulations indeed suggest a chiral phase transition for $d = 4$ and $d = 3$, $N \le N_c \sim 3$.

The exponents corresponding to the IR fixed point have been calculated up to order $1/N^2$. At order $1/N$ ($X_1$ is defined by equation (4.79))

$$\begin{aligned} \eta_\psi &= -\frac{(d-1)^2(4-d)}{d(d-2)} \frac{X_1}{4N} + O\left(1/N^2\right) \\ \eta_m &= -\frac{(d-1)^2}{d(d-2)} \frac{X_1}{N} + O\left(1/N^2\right) \\ \beta'(\alpha^*) &= 4 - d - \frac{(d-3)(d-6)(d-1)^2(4-d)}{d(d-2)} \frac{X_1}{4N} + O\left(1/N^2\right) . \end{aligned}$$

Finally note that in the $d = 2$ limit, the integral generates a contribution $Ne^2/\pi k^2$ times the propagator of the free gauge field

$$N' K(k) \underset{d \to 2}{\sim} \frac{N}{2\pi} \frac{1}{k^2}.$$

As a direct analysis of the $d = 2$ case confirms, this corresponds to a massive bound state, of mass squared $Ne^2/\pi$. However, for generic values of the coupling constant, this mass is of the order of the cut-off $\Lambda$. Only when $e$ is small with respect to the microscopic scale, as one assumes in conventional renormalized perturbation theory, does this mass correspond in the continuum limit to a propagating particle.

*Two dimensions.* As stated above we now assume that the dimensional quantity $e^2$ is small in the microscopic scale. The model is then a simple extension of the Schwinger model and can be exactly solved in the same way. For $N=1$ the model exhibits the simplest example of a chiral anomaly, illustrates the property of confinement and spontaneous chiral symmetry breaking. For $N>1$ the situation is more subtle. The neutral $\bar\psi\psi$ two-point function decays algebraically

$$\left\langle \bar\psi(x)\cdot\psi(x)\bar\psi(0)\cdot\psi(0)\right\rangle \propto x^{2/N-2},$$

indicating the presence of a massless mode and $\left\langle\bar\psi\psi\right\rangle=0$. Instead if we calculate the two-point function of the composite operator $\mathcal{O}_N(x)$

$$\mathcal{O}_N(x)=\prod_{i=1}^{N}\bar\psi_i(x)\psi_i(x),$$

we find

$$\langle\mathcal{O}_N(x)\mathcal{O}_N(0)\rangle \propto \text{ const.}\,.$$

We have thus identified an operator which has a non-zero expectation value. As a consequence of the fermion antisymmetry, if we perform a transformation under the group $U(N)\times U(N)$ corresponding to matrices $U_+,U_-$, the operator is multiplied by $\det U_+/\det U_-$. The operator thus is invariant under the group $SU(N)\times SU(N)\times U(1)$. Its non-vanishing expectation value is the sign of the spontaneous breaking of the remaining $U(1)$ chiral group.

### *6.3 The U(N) Thirring model*

We now consider the model

$$S(\bar\psi,\psi)=-\int \mathrm{d}^d x\left[\bar\psi\left(\not\partial+m_0\right)\psi-\tfrac{1}{2}gJ_\mu J_\mu\right], \tag{6.6}$$

where

$$J_\mu=\bar\psi\gamma_\mu\cdot\psi\,. \tag{6.7}$$

The special case $N=1$ corresponds to the simple Thirring model. In two dimensions it is then equivalent to a free massless boson field theory (with mass term for fermions one obtains the sine–Gordon model). Both to bozonize the model in $d=2$ and to study that large $N$ properties one introduces a abelian gauge field $A_\mu$ coupled to the current $J_\mu$

$$\tfrac{1}{2}gJ_\mu J_\mu \longmapsto A_\mu^2/2g+iA_\mu J_\mu\,. \tag{6.8}$$

One then finds massive QED without the $F_{\mu\nu}^2$ term

$$S(A_\mu, \bar{\psi}, \psi) = -\int \mathrm{d}^2 x \left[\bar{\psi}\left(\not\partial + i\not A + m_0\right)\psi - A_\mu^2/2g\right]. \qquad (6.9)$$

If we integrate over the fermions, the fermion determinant generates a kinetic term for the gauge field. For $m_0 = 0$ we are thus in situation very similar to massless QED, except that the gauge field is massive.

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L. Del Dubbio, S.J. Hands, J.C. Mehegan, *Nucl. Phys.* B502 (1997) 269, hep-lat/9701016.

Finally these analytic techniques have also been applied to the supersymmetric extension of the non-linear $\sigma$ model and other models, see for example

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## 7 The $O(N)$ vector model in the large $N$ limit: multi-critical points and double scaling limit

We now discuss the large $N$ limit of the general $N$-vector models with one scalar field. To illustrate the method we study multi-critical points. Of particular interest are the subtleties involved in the stability of the phase structure at critical dimensions.

Another issue involves the so-called *double scaling limit.* Statistical mechanical properties of random surfaces as well as randomly branched polymers can be analyzed within the framework of large $N$ expansion. In the same manner in which matrix models in their double scaling limit provide representations of dynamically triangulated random surfaces summed on different topologies, $O(N)$ symmetric vector models represent discretized branched polymers in this limit, where $N \to \infty$ and the coupling constant $g \to g_c$ in a correlated manner. The surfaces in the case of matrix models, and the randomly branched polymers in the case of vector models are classified by the different topologies of their Feynman graphs and thus by powers of $1/N$. Though matrix theories attract most attention, a detailed understanding of these theories exists only for dimensions $d \leq 1$. On the other hand, in many cases, the $O(N)$ vector models can be successfully studied also in dimensions $d > 1$, and thus, provide us with intuition for the search for a possible description of quantum field theory in terms of extended objects in four dimensions, which is a long lasting problem in elementary particle theory.

The double scaling limit in $O(N)$ vector quantum field theories reveals an interesting phase structure beyond $N \to \infty$ limit. In particular, though the $N \to \infty$ multicritical structure of these models is generaly well understood, there are certain cases where it is still unclear which of the features survives at finite $N$, and to what extent. One such problem is the multicritical behavior of $O(N)$ models *at critical dimensions.* Here, one finds that in the $N \to \infty$ limit, there exists a non-trivial UV fixed point, scale invariance is spontaneously broken, and the one parameter family of ground states contains a massive vector and a massless bound state, a Goldstone boson-dilaton. However, since it is unclear whether this structure is likely to survive for finite $N$ one would like to know whether it is possible to construct a local field theory of a massless dilaton via the double scaling limit, where all orders in $1/N$ contribute. The double scaling limit is viewed as the limit at which the attraction between the $O(N)$ vector quanta reaches a value at $g \to g_c$, at which a massless bound state is formed in the $N \to \infty$ limit, while the mass of the vector particle stays finite. In this limit, powers of $1/N$ are compensated by IR singularities and thus all orders in $1/N$ contribute.

In section 7.1 the double scaling limit for simple integrals and quantum mechanics is explained, introducing a formalism which will be useful for field theory examples.

In section 7.2 the special case of field theory in dimension two is discussed.

In higher dimensions a new phenomenon arises: the possibility of a spontaneous breaking of the $O(N)$ symmetry of the model, associated to the Goldstone phenomenon.

Before discussing a possible double scaling limit, the critical and multicritical points of the $O(N)$ vector model are examined in section 7.3. In particular, a certain sign ambiguity that appears in the expansion of the gap equation is noted, and related to the existence of the IR fixed point in dimensions $2 < d < 4$ discussed in section 4.3. In section 7.4 we discuss the subtleties and conditions for the existence of an $O(N)$ singlet massless bound state along with a small mass $O(N)$ vector particle excitation. It is pointed out that the correct massless effective field theory is obtained after the massive $O(N)$ scalar is integrated out. Section 7.5 is devoted to the double scaling limit with a particular emphasis on this limit in theories at their critical dimensions. In section 7.6 the main conclusions are summarized.

### *7.1 Double scaling limit: simple integrals and quantum mechanics*

We first discuss $d = 0$ and $d = 1$ dimensions, dimensions in which the matrix models has equally been solved. We however introduce a general method, not required here, but useful in the general field theory examples.

*The zero dimensional example.* Let us first consider the zero dimensional example. The partition function $Z$ is given by

$$\mathrm{e}^{Z} = \int \mathrm{d}^{N}\phi \exp\left[-NV\left(\phi^{2}\right)\right].$$

The simplest method for discussing the large $N$ limit is of course to integrate over angular variables. Instead we introduce two new variables $\lambda, \rho$ and use the identity

$$\exp\left[-NV(\phi^{2})\right] \propto \int \mathrm{d}\rho\, \mathrm{d}\lambda \exp\left\{-N\left[\tfrac{1}{2}\lambda\left(\phi^{2}-\rho\right)+V(\rho)\right]\right\}. \qquad (7.1)$$

The integral over $\lambda$ is really a Fourier representation of a $\delta$-function and thus the contour of integration runs parallel to the imaginary axis. The identity (7.1) transforms the action into a quadratic form in $\phi$. Hence the integration over $\phi$ can be performed and the dependence in $N$ becomes explicit

$$\mathrm{e}^{Z} \propto \int \mathrm{d}\rho\, \mathrm{d}\lambda \exp\left\{-N\left[-\tfrac{1}{2}\lambda\rho+V(\rho)+\tfrac{1}{2}\ln\lambda\right]\right\}.$$

The large $N$ limit is obtained by steepest descent. The saddle point is given by

$$V'(\rho)=\tfrac{1}{2}\lambda, \qquad \rho = 1/\lambda\,.$$

The leading contribution to $Z$ is proportional to $N$ and obtained by replacing $\lambda,\rho$ by the saddle point value. The leading correction is obtained by expanding $\lambda,\rho$ around the saddle point and performing the gaussian integration. It involves the determinant $D$ of the matrix $\mathbf{M}$ of second derivatives

$$\mathbf{M}=\begin{pmatrix}-\frac{1}{2}\lambda^{-2} & -\frac{1}{2}\\ -\frac{1}{2} & V''(\rho)\end{pmatrix},\quad D=\det\mathbf{M}=-\tfrac{1}{2}\left(V''(\rho)/\lambda^2+\tfrac{1}{2}\right).$$

In the generic situation the resulting contribution to $Z$ is $-\frac{1}{2}\ln D$. However if the determinant $D$ vanishes the leading order integral is no longer gaussian, at least for the degree of freedom which corresponds to the eigenvector with vanishing eigenvalue. The condition of vanishing of the determinant also implies that two solutions of the saddle point equation coincide and thus corresponds to a surface in the space of the coefficients of the potential $V$ where the partition function is singular.

To examine the corrections to the leading large $N$ behaviour it remains however possible to integrate over one of the variables by steepest descent. At leading order this corresponds to solving the saddle point equation for one of the variables, the other being fixed. Here it is convenient to eliminate $\lambda$ by the equation $\lambda=1/\rho$. One finds

$$\mathrm{e}^Z\propto\int\mathrm{d}\rho\,\exp\left[-N\bigl(V(\rho)-\tfrac{1}{2}\ln\rho\bigr)+O(1)\right].$$

In the leading term we obviously recover the result of the angular integration with $\rho=\phi^2$. For $N$ large the leading contribution arises from the leading term in the expansion of $W(\rho)=V(\rho)-\frac{1}{2}\ln\rho$ near the saddle point:

$$W(\rho)-W(\rho_s)\sim\tfrac{1}{n!}W^{(n)}(\rho_s)(\rho-\rho_s)^n.$$

The integer $n$ characterizes the nature of the critical point. Adding relevant perturbations $\delta_k V$ of parameters $v_k$ to the critical potential

$$\delta_k V=v_k(\rho-\rho_s)^k,\quad 1\le k\le n-2$$

(the term $k=n-1$ can always be eliminated by a shift of $\rho$) we find the partition function at leading order for $N$ large in the scaling region:

$$\mathrm{e}^{Z(\{u_k\})}\propto\int\mathrm{d}z\,\exp\left(-z^n-\sum_{k=1}^{n-2}u_k z^k\right),$$

where $z\propto N^{1/n}(\rho-\rho_s)$ and

$$u_k\propto N^{1-k/n}v_k$$

is held fixed

*Quantum mechanics.* The method we have used above immediately generalizes to quantum mechanics, although a simpler method involves solving the radial Schrödinger equation. We consider the euclidean action

$$S(\phi) = N \int \mathrm{d}t \,\left[\tfrac{1}{2}\left(\mathrm{d}_t\phi(t)\right)^2 + V(\phi^2)\right]. \tag{7.2}$$

Note the unusual field normalization, the factor $N$ in front of the action simplifying all expressions in the large $N$ limit.

To explore the large $N$ limit one has to take the scalar function $\phi^2$, which self-averages, as a dynamical variable. At each time $t$ we thus perform the transformation (7.1). One introduces two paths $\rho(t), \lambda(t)$ and writes

$$\begin{aligned}&\exp\left[-N\int \mathrm{d}t\, V(\phi^2)\right]\\ &\qquad \propto \int [\mathrm{d}\rho(t)\,\mathrm{d}\lambda(t)] \exp\left\{-N\int \mathrm{d}t\,\left[\tfrac{1}{2}\lambda\left(\phi^2-\rho\right)+V(\rho)\right]\right\}.\end{aligned} \tag{7.3}$$

The integral over the path $\phi(t)$ is then gaussian and can be performed. One finds

$$e^Z = \int [\mathrm{d}\rho(t)\mathrm{d}\lambda(t)]\, \exp\left[-S_N(\lambda,\rho)\right] \tag{7.4}$$

with

$$S_N = N\int \mathrm{d}t\, \left[-\tfrac{1}{2}\lambda\rho + V(\rho)\right] + \tfrac{1}{2}\operatorname{tr}\ln\left(-\mathrm{d}_t^2 + \lambda(\cdot)\right). \tag{7.5}$$

Again, in the large $N$ limit the path integral can be calculated by steepest descent. The saddle points are constant paths solution of

$$V'(\rho) = \tfrac{1}{2}\lambda\,, \qquad \rho = \frac{1}{2\pi}\int \frac{\mathrm{d}\omega}{\omega^2+\lambda} = \frac{1}{2\sqrt{\lambda}}\,, \tag{7.6}$$

where $\omega$ is the Fourier energy variable conjugated to $t$. Again a critical point is defined by the property that at least two solutions to the saddle point equations coalesce. This happens when the determinant of the matrix of first derivatives of the equations vanishes:

$$\det\begin{pmatrix} V''(\rho) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8\lambda^{3/2}}\end{pmatrix} = -\frac{1}{8\lambda^{3/2}}V''(\rho) - \frac{1}{4} = 0\,. \tag{7.7}$$

The leading correction to the saddle point contribution is given by a gaussian integration. The result involves the determinant of the operator second derivative of $S_N$. By Fourier transforming time the operator becomes a tensor product of $2 \times 2$ matrices with determinant $D(\omega)$

$$D(\omega) = \det \begin{pmatrix} V''(\rho) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}B(\omega) \end{pmatrix} \text{ with } B(\omega) = \frac{1}{2\pi} \int \frac{\mathrm{d}\omega'}{(\omega'^2 + \lambda)[(\omega - \omega')^2 + \lambda]} .$$

Thus, the criticality condition is equivalent to $D(0) = 0$. When the criticality condition is satisfied, the leading correction is no longer given by steepest descent. Again, since at most one mode can be critical, we can integrate over one of the path by steepest descent, which means solving the saddle point equation for one function, the other being fixed. While the $\rho$ equation remains local, the $\lambda$ is now non-local, involving the diagonal matrix element of the inverse of the differential operator $-\mathrm{d}_t^2 + \lambda(t)$. We shall see in next section how this problem can be overcome in general. A special feature of quantum mechanics, however, is that the determinant can be calculated, after a simple change of variables. We set

$$\lambda(t) = \dot{s}(t) + s^2(t), \tag{7.8}$$

in such a way that the second order differential operator factorizes

$$-\mathrm{d}_t^2 + \lambda(t) = -\left(\mathrm{d}_t + s(t)\right)\left(\mathrm{d}_t - s(t)\right). \tag{7.9}$$

The determinant of a first order differential operator can be calculated by expanding formally in $s$. Only the first term survives but the coefficient is ambiguous

$$\operatorname{tr}\ln\left(1 - \mathrm{d}_t^{-1} s(\cdot)\right) = -\theta(0) \int \mathrm{d}t\, s(t).$$

A more refined analysis, which involves boundary conditions, is required to determine the ambiguous value $\theta(0)$ of step function. Here one finds

$$\ln\det\left(-\mathrm{d}_t^2 + \lambda(\cdot)\right) = \operatorname{tr}\ln\left(-\mathrm{d}_t^2 + \lambda(\cdot)\right) = \int \mathrm{d}t\, s(t). \tag{7.10}$$

The jacobian of the transformation (7.8) contributes at higher order in $1/N$ and can be neglected. Therefore the large $N$ action becomes

$$\begin{aligned} S_N &= N \int \mathrm{d}t\ \left[-\tfrac{1}{2}(\dot{s} + s^2)\rho + V(\rho) + \tfrac{1}{2}s\right] \\ &= N \int \mathrm{d}t\ \left[-\tfrac{1}{2}\rho s^2 + \tfrac{1}{2}s(\dot{\rho} + 1) + V(\rho)\right] . \end{aligned}$$

We can now replace $s$ by the solution of a local saddle point equation (or perform the gaussian integration, but neglect the determinant which is of higher order):

$$\frac{\delta S_N}{\delta s(t)} = 0 \ \Leftrightarrow\ -s\rho + \tfrac{1}{2}(\dot{\rho} + 1) = 0 ,$$

and find

$$S_N = N \int \mathrm{d}t \left[ \frac{\dot{\rho}^2}{8\rho} + \frac{1}{8\rho} + V(\rho) \right] . \qquad (7.11)$$

We recognize the action for the large $N$ potential at zero angular momentum in the radial coordinate $\rho(t) = \phi^2(t)$. Critical points then are characterized by the behaviour of the potential $W(\rho)$

$$W(\rho) = V(\rho) + \frac{1}{8\rho} ,$$

near the saddle point $\rho_s$

$$W(\rho) - W(\rho_s) \sim W^{(n)}(\rho_s) \frac{(\rho - \rho_s)^n}{n!} .$$

At critical points the ground state energy, after subtraction of the classical term which is linear in $N$, has a non-analytic contribution. To eliminate $N$ from the action we set

$$t \mapsto t N^{(n-2)/(n+2)}, \qquad \rho(t) - \rho_s \mapsto N^{-2/(n+2)} z(t).$$

We conclude that the leading correction to the energy levels is proportional to $N^{-(n-2)/(n+2)}$. Note also that the scaling of time implies that higher order time derivatives would be irrelevant, an observation which can be used more directly to expand the determinant in local terms, and will be important in next section.

If we add relevant corrections to the potential

$$\delta_k V = v_k (\rho - \rho_s)^k, \quad 1 \le k \le n-2 ,$$

the coefficients $v_k$ must scale like

$$v_k \propto N^{2(k-n)/(n+2)}.$$

### *7.2 The 2D $V(\phi^2)$ field theory in the double scaling limit*

In the first part we study the $O(N)$ symmetric $V(\phi^2)$ field theory, where $\phi$ is $N$-component field, in the large $N$ limit in dimension two because phase transitions occur in higher dimensions, a problem which has to be considered separately. The action is:

$$S(\phi) = N \int \mathrm{d}^2 x \left\{ \tfrac{1}{2} \left[ \partial_\mu \phi(x) \right]^2 + V\left(\phi^2\right) \right\} , \qquad (7.12)$$

where an implicit cut-off $\Lambda$ is always assumed below. Whenever the explicit dependence in the cut-off will be relevant we shall assume a Pauli–Villars's type regularization, i.e. the replacement in action (7.12) of $-\phi\partial^2\phi$ by

$$-\phi \partial^2 D(-\partial^2/\Lambda^2) \phi , \qquad (7.13)$$

where $D(z)$ is a positive non-vanishing polynomial with $D(0) = 1$.

As before one introduces two fields $\rho(x)$ and $\lambda(x)$ and uses the identity (7.3). The large $N$ action is then:

$$S_N = N \int \mathrm{d}^2 x \left[V(\rho) - \tfrac{1}{2}\lambda\rho\right] + \tfrac{1}{2} N \operatorname{tr}\ln(-\Delta + \lambda). \tag{7.14}$$

Again for $N$ large we evaluate the integral by steepest descent. Since the saddle point value $\lambda$ is the $\phi$-field mass squared, we set in general $\lambda = m^2$. With this notation the two equations for the saddle point $m^2, \rho_s = \langle \phi^2 \rangle$ are:

$$V'(\rho_s) = \tfrac{1}{2} m^2 \,, \tag{7.15a}$$

$$\rho_s = \frac{1}{(2\pi)^2} \int^{\Lambda} \frac{\mathrm{d}^2 k}{k^2 + m^2} \,, \tag{7.15b}$$

where we have used a short-cut notation

$$\frac{1}{(2\pi)^2} \int^{\Lambda} \frac{\mathrm{d}^2 k}{k^2 + m^2} \equiv \frac{1}{(2\pi)^2} \int \frac{\mathrm{d}^2 k}{D(k^2/\Lambda^2) k^2 + m^2} \equiv B_1(m^2). \tag{7.16}$$

For $m \ll \Lambda$ one finds

$$B_1(m^2) = \frac{1}{2\pi} \ln(\Lambda/m) + \frac{1}{4\pi} \ln(8\pi K) + O(m^2/\Lambda^2),$$

where $K$ is a regularization dependent constant.

As we have discussed in the case of quantum mechanics a critical point is characterized by the vanishing at zero momentum of the determinant of second derivatives of the action at the saddle point. The mass-matrix has then a zero eigenvalue which, in field theory, corresponds to the appearance of a new massless excitation other than $\phi$. In order to obtain the effective action for this scalar massless mode we must integrate over one of the fields. In the field theory case the resulting effective action can no longer be written in local form. To discuss the order of the critical point, however, we only need the action for space independent fields, and thus for example we can eliminate $\lambda$ using the $\lambda$ saddle point equation.

The effective $\rho$ potential $W(\rho)$ then reads

$$W(\rho) = V(\rho) - \tfrac{1}{2} \int^{\lambda(\rho)} \mathrm{d}\lambda' \, \lambda' \frac{\partial}{\partial \lambda'} B_1(\lambda'), \tag{7.17}$$

where at leading order for $\Lambda$ large

$$\lambda(\rho) = 8\pi K \Lambda^2 \, \mathrm{e}^{-4\pi\rho} \,.$$

The expression for the effective action in equation (7.17) is correct for any $d$ and will be used also in section 7.5. Here we have:

$$W(\rho) = V(\rho) + K\Lambda^2 \, \mathrm{e}^{-4\pi\rho} = V(\rho) + \tfrac{1}{8\pi} m^2 \, \mathrm{e}^{-4\pi(\rho-\rho_s)} \,.$$

A multicritical point is defined by the condition

$$W(\rho) - W(\rho_s) = O\left((\rho-\rho_s)^n\right) \qquad (7.18).$$

This yields the conditions:

$$V^{(k)}(\rho_s) = \tfrac{1}{2}(-4\pi)^{k-1} m^2 \quad \text{for } 1 \le k \le n-1 \,.$$

Note that the coefficients $V^{(k)}(\rho_s)$ are the coupling constants renormalized at leading order for $N$ large. If $V(\rho)$ is a polynomial of degree $n-1$ (the minimal polynomial model) the multicritical condition in equation (7.18) determines the critical values of renormalized coupling constants as well as $\rho_s$

When the fields are space-dependent it is simpler to eliminate $\rho$ instead, because the corresponding field equation:

$$V'\big(\rho(x)\big) = \tfrac{1}{2}\lambda(x). \qquad (7.19)$$

is local. This equation can be solved by expanding $\rho(x) - \rho_s$ in a power series in $\lambda(x) - m^2$:

$$\rho(x) - \rho_s = \frac{1}{2V''(\rho_s)}\left(\lambda(x) - m^2\right) + O\left((\lambda - m^2)^2\right). \qquad (7.20)$$

The resulting action for the field $\lambda(x)$ remains non-local but because, as we shall see, adding powers of $\lambda$ as well as adding derivatives make terms less relevant, only the few first terms of a local expansion of the effective action will be important.

If in the local expansion of the determinant we keep only the two first terms we obtain an action containing at leading order a kinetic term proportional to $(\partial_\mu\lambda)^2$ and the interaction $(\lambda(x) - m^2)^n$:

$$S_N(\lambda) \sim N \int \mathrm{d}^2 x \left[\frac{1}{96\pi m^4}(\partial_\mu\lambda)^2 + \tfrac{1}{n!} S_n \left(\lambda(x) - m^2\right)^n\right],$$

where the neglected terms are of order $(\lambda - m^2)^{n+1}$, $\lambda\partial^4\lambda$, and $\lambda^2\partial^2\lambda$ and

$$S_n = W^{(n)}(\rho_s)[2V''(\rho_s)]^{-n} = W^{(n)}(\rho_s)(-4\pi m^2)^{-n}.$$

Moreover we note that together with the cut-off $\Lambda$, $m$ now also acts as a cut-off in the local expansion.

To eliminate the $N$ dependence in the action we have, as in the example of quantum mechanics, to rescale both the field $\lambda - m^2$ and space:

$$\lambda(x) - m^2 = \sqrt{48\pi} m^2 N^{-1/2} \varphi(x) , \quad x \mapsto N^{(n-2)/4} x . \tag{7.21}$$

We find

$$S_N(\varphi) \sim \int \mathrm{d}^2 x \left[ \tfrac{1}{2} (\partial_\mu \varphi)^2 + \tfrac{1}{n!} g_n \varphi^n \right] .$$

In the minimal model, where the polynomial $V(\rho)$ has exactly degree $n-1$, we find $g_n = 6(48\pi)^{(n-2)/2} m^2$.

As anticipated we observe that derivatives and powers of $\varphi$ are affected by negative powers of $N$, justifying a local expansion. However we also note that the cut-offs ($\Lambda$ or the mass $m$) are now also multiplied by $N^{(n-2)/4}$. Therefore the large $N$ limit also becomes a large cut-off limit.

*Double scaling limit.* The existence of a double scaling limit relies on the existence of IR singularities due to the massless or small mass bound state which can compensate the $1/N$ factors appearing in the large $N$ perturbation theory.

We now add to the action relevant perturbations:

$$\delta_k V = v_k (\rho(x) - \rho_s)^k , \quad 1 \le k \le n-2.$$

proportional to $\int \mathrm{d}^2 x (\lambda - m^2)^k$:

$$\delta_k S_N(\lambda) = N S_k \int \mathrm{d}^2 x \, (\lambda - m^2)^k ,$$

where the coefficients $S_k$ are functions of the coefficients $v_k$. After the rescaling (7.21)

$$\delta_k S_N(\varphi) = \tfrac{1}{k!} g_k N^{(n-k)/2} \int \mathrm{d}^2 x \, \varphi^k(x) \quad 1 \le k \le n-2$$

However, unlike quantum mechanics, it is not sufficient to scale the coefficients $g_k$ with the power $N^{(k-n)/2}$ to obtain a finite scaling limit. Indeed perturbation theory is affected by UV divergences, and we have just noticed that the cut-off diverges with $N$. In two dimensions the nature of divergences is very simple: it is entirely due to the self-contractions of the interactions terms and only one divergent integral appears:

$$\left\langle \varphi^2(x) \right\rangle = \frac{1}{4\pi^2} \int \frac{\mathrm{d}^2 q}{q^2 + \mu^2} ,$$

where $\mu$ is the small mass of the bound state, required as an IR cut-off to define perturbatively the double scaling limit. We can then extract the $N$ dependence

$$\left\langle \varphi^2(x) \right\rangle = \frac{1}{8\pi} (n-2) \ln N + O(1).$$

Therefore the coefficients $S_k$ have also to cancel these UV divergences, and thus have a logarithmic dependence in $N$ superposed to the natural power obtained from power counting arguments. In general for any potential $U(\varphi)$

$$U(\varphi) =: U(\varphi) : + \left[\sum_{k=1} \frac{1}{2^k k!} \left\langle \varphi^2 \right\rangle^k \left(\frac{\partial}{\partial \varphi}\right)^{2k}\right] : U(\varphi) : ,$$

where $: U(\varphi) :$ is the potential from which self-contractions have been subtracted (it has been normal-ordered). For example for $n = 3$

$$\varphi^3(x) =: \varphi^3(x) : +3 \left\langle \varphi^2 \right\rangle \varphi(x),$$

and thus the double scaling limit is obtained with the behaviour

$$N g_1 + \frac{1}{16\pi} \ln N g_3 \text{ fixed } .$$

For the example $n = 4$

$$g_1 N^{3/2} \quad \text{and} \quad N g_2 + \frac{g_4}{8\pi} \ln N \text{ fixed } .$$

*7.3 The $V(\phi^2)$ in the large $N$ limit: phase transitions*

In higher dimensions something new happens: the possibility of phase transitions associated with spontaneous breaking the $O(N)$ symmetry. In the first part we thus study the $O(N)$ symmetric $V(\phi^2)$ field theory, in the large $N$ limit to explore the possible phase transitions and identify the corresponding multicritical points. The action is:

$$S(\phi) = N \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + V\left(\phi^2\right) \right\}, \tag{7.22}$$

where, as above (equations (7.12,7.13)), an implicit cut-off $\Lambda$ is always assumed below.

The identity (7.3) transforms the action into a quadratic form in $\phi$ and therefore the integration over $\phi$ can be performed. It is convenient however here to integrate only over $N - 1$ components, to keep a component of the vector field, which we denote $\sigma$, in the action. The large $N$ action is then:

$$S_N = N \int \mathrm{d}^d x \left[ \tfrac{1}{2} \left(\partial_\mu \sigma\right)^2 + V(\rho) + \tfrac{1}{2} \lambda \left(\sigma^2 - \rho\right) \right] + \tfrac{1}{2}(N-1) \operatorname{tr} \ln(-\Delta + \lambda). \tag{7.23}$$

*The saddle point equations: the $O(N)$ critical point.* Let us then write the saddle point equations for a general potential $V$. At high temperature $\sigma = 0$

and $\lambda$ is the $\phi$-field mass squared. We thus set in general $\lambda = m^2$. With this notation the three saddle point equations are:

$$m^2\sigma = 0\,, \tag{7.24a}$$
$$V'(\rho) = \tfrac{1}{2}m^2\,, \tag{7.24b}$$
$$\sigma^2 = \rho - \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d k}{k^2+m^2}\,. \tag{7.24c}$$

In the ordered phase $\sigma \neq 0$ and thus $m$ vanishes. Equation (7.24$c$) has a solution only for $\rho > \rho_c$,

$$\rho_c = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d k}{k^2}\,,\ \Rightarrow\ \sigma = \sqrt{\rho-\rho_c}\,.$$

Equation (7.24$b$) which reduces to $V'(\rho) = 0$ then yields the critical temperature. Setting $V(\rho) = U(\rho) + \frac{1}{2}r\rho$, we find

$$r_c = -2U'(\rho_c).$$

To find the magnetization critical exponent $\beta$ we need the relation between the $r$ and $\rho$ near the critical point.

In the disordered phase, $\sigma = 0$, equation (7.24$c$) relates $\rho$ to the $\phi$-field mass $m$. For $m \ll \Lambda$, $\rho$ approaches $\rho_c$, and the relation becomes (equation (4.16)):

$$\rho - \rho_c = -C(d)m^{d-2} + a(d)m^2\Lambda^{d-4} + O\left(m^d\Lambda^{-2}\right) + O\left(m^4\Lambda^{d-6}\right). \tag{7.25}$$

For $2 < d < 4$ (the situation we shall assume below except when stated otherwise) the $O\left(m^d\Lambda^{-2}\right)$ from the non-analytic part dominates the corrections to the leading part of this expression. For $d = 4$ instead

$$\rho - \rho_c = \tfrac{1}{8\pi^2}m^2\left(\ln m/\Lambda + \text{ const.}\right),$$

and for $d > 4$ the analytic contribution dominates and

$$\rho - \rho_c \sim a(d)m^2\Lambda^{d-4}.$$

The constant $C(d)$ is universal (equation (4.17$b$)). The constant $a(d)$, which also appears in equation (4.16), instead depends on the cut-off procedure, and is given by

$$a(d) = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{k^4}\left(1 - \frac{1}{D^2(k^2)}\right). \tag{7.26}$$

*Critical point.* In a generic situation $V''(\rho_c) = U''(\rho_c)$ does not vanish. We thus find in the low temperature phase

$$t = r - r_c \sim -2U''(\rho_c)(\rho-\rho_c)\ \Rightarrow\ \beta = \tfrac{1}{2}\,. \tag{7.27}$$

This is the case of an ordinary critical point. Stability implies $V''(\rho_c) > 0$ so that $t < 0$.

At high temperature, in the disordered phase, the $\phi$-field mass $m$ is given by $2U'(\rho) + r = m^2$ and thus, using (7.25), at leading order

$$t \sim 2U''(\rho_c)C(d)m^{d-2},$$

in agreement with the result of the normal critical point. Of course the simplest realization of this situation is to take $V(\rho)$ quadratic, and we recover the $(\phi^2)^2$ field theory.

*The sign of the constant* $a(d)$. A comment concerning the non-universal constant $a(d)$ defined in (7.25) is here in order because, while its absolute value is irrelevant, its sign plays a role in the discussion of multicritical points. Actually the relevance of this sign to the RG properties of the large $N$ limit of the simple $(\phi^2)^2$ field theories has already mentioned (section 4.3). For the simplest Pauli–Villars's type regularization we have $D(z) > 1$ and thus $a(d)$ is finite and positive in dimensions $2 < d < 4$, but this clearly is not a universal feature.

A new situation arises if we can adjust a parameter of the potential in such a way that $U''(\rho_c) = 0$. This can be achieved only if the potential $V$ is at least cubic. We then expect a tricritical behavior. Higher critical points can be obtained when more derivatives vanish. We shall examine the general case though, from the point of view of real phase transitions, higher order critical points are not interesting because $d > 2$ for continuous symmetries and mean-field behavior is then obtained for $d \geq 3$. The analysis will however be useful in the study of double scaling limit.

Assuming that the first non-vanishing derivative is $U^{(n)}(\rho_c)$, we expand further equation (7.24*b*). In the ordered low temperature phase we now find

$$t = -\frac{2}{(n-1)!}U^{(n)}(\rho_c)(\rho-\rho_c)^{n-1}, \Rightarrow \ \sigma \propto (-t)^{\beta}, \quad \beta = \frac{1}{2(n-1)}, \qquad (7.28)$$

which leads to the exponent $\beta$ expected in the mean field approximation for such a multicritical point. We have in addition the condition $U^{(n)}(\rho_c) > 0$.

In the high temperature phase instead

$$m^2 = t + (-1)^{n-1}\frac{2}{(n-1)!}U^{(n)}(\rho_c)C^{n-1}(d)m^{(n-1)(d-2)}. \qquad (7.29)$$

For $d > 2n/(n-1)$ we find a simple mean field behavior, as expected since we are above the upper-critical dimension .

For $d < 2n/(n-1)$ we find a peculiar phenomenon, the term in the r.h.s. is always dominant, but depending on the parity of $n$ the equation has solutions for $t > 0$ or $t < 0$. For $n$ even, $t$ is positive and we find

$$m \propto t^{\nu}, \qquad \nu = \frac{1}{(n-1)(d-2)}, \qquad (7.30)$$

which is a non mean-field behavior below the critical dimension. However for $n$ odd (this includes the tricritical point) $t$ must be negative, in such a way that we have now two competing solutions at low temperature. We have to find out which one is stable. We shall verify below that only the ordered phase is stable, so that the correlation length of the $\phi$-field in the high temperature phase remains always finite. Although these dimensions do not correspond to physical situations because $d < 3$ the result is peculiar and inconsistent with the $\varepsilon$-expansion.

For $d = 2n/(n-1)$ we find a mean field behavior without logarithmic corrections, provided one condition is met:

$$\frac{2}{(n-1)!} U^{(n)}(\rho_c) C^{n-1}\left(2n/(n-1)\right) < 1\,, \qquad C(3) = 1/(4\pi). \tag{7.31}$$

We examine, as an example, in more details the tricritical point below. We will see that the special point

$$\frac{2}{(n-1)!} U^{(n)}(\rho_c) C^{n-1}\left(2n/(n-1)\right) = 1\,, \tag{7.32}$$

has several peculiarities. In what follows we call $\Omega_c$ this special value of $U^{(n)}(\rho_c)$.

*Discussion.* In the mean field approximation the function $U(\rho) \propto \rho^n$ is not bounded from below for $n$ odd, however $\rho = 0$ is the minimum because by definition $\rho \geq 0$. Here instead we are in the situation where $U(\rho) \sim (\rho - \rho_c)^n$ but $\rho_c$ is positive. Thus this extremum of the potential is likely to be unstable for $n$ odd. To check the global stability requires further work. The question is whether such multicritical points can be studied by the large $N$ limit method.

Another point to notice concerns renormalization group: The $n = 2$ example is peculiar in the sense that the large $N$ limit exhibits a non-trivial IR fixed point. For higher values of $n$ no coupling renormalization arises in the large $N$ limit and the IR fixed point remains pseudo-gaussian. We are in a situation quite similar to usual perturbation theory, the $\beta$ function can only be calculated perturbatively in $1/N$ and the IR fixed point is outside the perturbative regime.

*Local stability and the mass matrix.* The matrix of the general second partial derivatives of the effective action is:

$$N \begin{pmatrix} p^2 + m^2 & 0 & \sigma \\ 0 & V''(\rho) & -\frac{1}{2} \\ \sigma & -\frac{1}{2} & -\frac{1}{2} B_\Lambda(p,m) \end{pmatrix}, \tag{7.33}$$

where $B_\Lambda(p,m)$ is defined in (4.34).

We are in position to study the local stability of the critical points. Since the integration contour for $\lambda = m^2$ should be parallel to the imaginary axis, a necessary condition for stability is that the determinant remains negative.

*The disordered phase.* Then $\sigma = 0$ and thus we have only to study the $2 \times 2$ matrix $\mathbf{M}$ of the $\rho, m^2$ subspace. Its determinant must remain negative which implies

$$\det \mathbf{M} < 0 \;\Leftrightarrow\; 2V''(\rho) B_\Lambda(p,m) + 1 > 0\,. \tag{7.34}$$

For Pauli–Villars's type regularization the function $B_\Lambda(p,m)$ is decreasing so that this condition is implied by the condition at zero momentum

$$\det \mathbf{M} < 0 \;\Leftarrow\; 2V''(\rho) B_\Lambda(0,m) + 1 > 0\,.$$

For $m$ small we use equation (4.36) and at leading order the condition becomes:

$$C(d)(d-2) m^{d-4} V''(\rho) + 1 > 0\,.$$

This condition is satisfied by a normal critical point since $V''(\rho_c) > 0$. For a multicritical point, and taking into account equation (7.25) we find:

$$(-1)^n \frac{d-2}{(n-2)!} C^{n-1}(d) m^{n(d-2)-d} V^{(n)}(\rho_c) + 1 > 0\,. \tag{7.35}$$

We obtain a result consistent with our previous analysis: For $n$ even it is always satisfied. For $n$ odd it is always satisfied above the critical dimension and never below. At the upper-critical dimension we find a condition on the value of $V^{(n)}(\rho_c)$ which we recognize to be identical to condition (7.31) because then $2/(n-1) = d-2$.

*The ordered phase.* Now $m^2 = 0$ and the determinant $\Delta$ of the complete matrix is:

$$-\Delta > 0 \;\Leftrightarrow\; 2V''(\rho) B_\Lambda(p,0) p^2 + p^2 + 4V''(\rho)\sigma^2 > 0\,. \tag{7.36}$$

We recognize a sum of positive quantities, and the condition is always satisfied. Therefore in the case where there is a competition with a disordered saddle point only the ordered one can be stable.

### *7.4 The scalar bound state*

In this section we study the limit of stability in the disordered phase ($\sigma = 0$). This is a problem which only arises when $n$ is odd, the first case being provided by the tricritical point.

The mass-matrix has then a zero eigenvalue which corresponds to the appearance of a new massless excitation other than $\phi$. Let us denote by $\mathbf{M}$ the $\rho, m^2$ $2 \times 2$ submatrix. Then

$$\det \mathbf{M} = 0 \;\Leftrightarrow\; 2V''(\rho) B_\Lambda(0,m) + 1 = 0\,.$$

In the two-space the corresponding eigenvector has components $(\frac{1}{2}, V''(\rho))$.

*The small mass m region.* In the small $m$ limit the equation can be rewritten in terms of the constant $C(d)$ defined in (4.16):

$$C(d)(d-2)m^{d-4}V''(\rho)+1=0\,. \tag{7.37}$$

Equation (7.37) tells us that $V''(\rho)$ must be small. We are thus close to a multicritical point. Using the result of the stability analysis we obtain

$$(-1)^{n-1}\frac{d-2}{(n-2)!}C^{n-1}(d)m^{n(d-2)-d}V^{(n)}(\rho_c)=1\,. \tag{7.38}$$

We immediately notice that this equation has solutions only for $n(d-2)=d$, i.e. at the critical dimension. The compatibility then fixes the value of $V^{(n)}(\rho_c)$. We again find the point (7.32), $V^{(n)}(\rho_c)=\Omega_c$. If we take into account the leading correction to the small $m$ behavior we find instead:

$$V^{(n)}(\rho_c)\Omega_c^{-1}-1\sim(2n-3)\frac{a(d)}{C(d)}\left(\frac{m}{\Lambda}\right)^{4-d}. \tag{7.39}$$

This means that when $a(d)>0$ there exists a small region $V^{(n)}(\rho_c)>\Omega_c$ where the vector field is massive with a small mass $m$ and the bound-state massless. The value $\Omega_c$ is a fixed point value.

*The scalar field at small mass.* We want to extend the analysis to a situation where the scalar field has a small but non-vanishing mass $M$ and $m$ is still small. The goal is in particular to explore the neighbourhood of the special point (7.32). Then the vanishing of the determinant of $\mathbf{M}$ implies

$$1+2V''(\rho)B_\Lambda(iM,m)=0\,. \tag{7.40}$$

Because $M$ and $m$ are small, this equation still implies that $\rho$ is close to a point $\rho_c$ where $V''(\rho)$ vanishes. Since reality imposes $M<2m$, it is easy to verify that this equation has also solutions for only the critical dimension. Then

$$V^{(n)}(\rho_c)f(m/M)=\Omega_c\,, \tag{7.41}$$

where we have set:

$$f(z)=\int_0^1 \mathrm{d}x\left[1+(x^2-1)/(4z^2)\right]^{d/2-2},\qquad \tfrac{1}{2}<z\,. \tag{7.42}$$

In three dimensions it reduces to:

$$f(z)=z\ln\left(\frac{2z+1}{2z-1}\right).$$

$f(z)$ is a decreasing function which diverges for $z = \frac{1}{2}$ because $d \le 3$. Thus we find solutions in the whole region $0 < V^{(n)}(\rho_c) < \Omega_c$, i.e. when the multicritical point is locally stable.

Let us calculate the propagator near the pole. We find the matrix $\mathbf{\Delta}$

$$\mathbf{\Delta} = \frac{2}{G^2}\left[N \left.\frac{\mathrm{d}B_\Lambda(p,m)}{\mathrm{d}p^2}\right|_{p^2=-M^2}\right]^{-1} \frac{1}{p^2+M^2}\begin{pmatrix} 1 & G \\ G & G^2 \end{pmatrix}, \tag{7.43}$$

where we have set

$$G = \frac{2(-C)^{n-2}W^{(n)}}{(n-2)!}m^{4-d}.$$

For $m/M$ fixed the residue goes to zero with $m$ as $m^{d-2}$ because the derivative of $B$ is of the order of $m^{d-6}$. Thus the bound-state decouples on the multicritical line.

*The scalar massless excitation: general situation.* Up to now we have explored only the case where both the scalar field and the vector field propagate. Let us now relax the latter condition, and examine what happens when $m$ is no longer small. The condition $M = 0$ then reads

$$2V''(\rho_s)B_\Lambda(0,m) + 1 = 0$$

together with

$$m^2 = 2V'(\rho_s), \qquad \rho_s = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d k}{k^2+m^2}. \tag{7.44}$$

An obvious remark is: there exist solutions only for $V''(\rho_s) < 0$, and therefore the ordinary critical line can never be approached. In terms of the function $F(z)$

$$\Lambda^{d-2}F(m^2/\Lambda^2) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d k}{k^2+m^2} \equiv \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{k^2 D(k^2)+m^2} \tag{7.45}$$

and thus

$$F(z) = N_d \int_0 \frac{k^{d-1}\mathrm{d}k}{k^2 D(k^2)+z}, \tag{7.46}$$

the equations can be rewritten

$$\rho_s = \Lambda^{d-2}F(z), \quad z = 2V'(\rho_s)\Lambda^{-2}, \quad 2\Lambda^{d-4}V''(\rho_s)F'(z) = 1.$$

The function $F(z)$ in Pauli–Villars's regularization is a decreasing function. In the same way $-F'(z)$ is a positive decreasing function.

The third equation is the condition for the two curves corresponding to the two first ones become tangent. For any value of $z$ we can find potentials and thus solutions. Let us call $z_s$ such a value and specialize to cubic potentials. Then

$$\rho_s = \Lambda^{d-2}F(z_s)\ ,$$

$$V(\rho) = V'(\rho_s)(\rho-\rho_s) + \tfrac{1}{2}V''(\rho_s)(\rho-\rho_s)^2 + \tfrac{1}{3!}V^{(3)}(\rho_s)(\rho-\rho_s)^3, \tag{7.47}$$

which yields a two parameter family of solutions. For $z$ small we see that for $d < 4$ the potential becomes proportional to $(\rho-\rho_c)^3$.

*7.5 Stability and double scaling limit*

In order to discuss in more details the stability issue and the double scaling limit we now construct the effective action for the scalar bound state. We consider first only the massless case. We only need the action in the IR limit, and in this limit we can integrate out the vector field and the second massive eigenmode.

*Integration over the massive modes.* As we have already explained in section 7.2 we can integrate over one of the fields, the second being fixed, and we need only the result at leading order. Therefore we replace in the functional integral

$$\mathrm{e}^{Z}=\int[\mathrm{d}\rho\mathrm{d}\lambda]\exp\left[-\tfrac{N}{2}\operatorname{tr}\ln(-\partial^{2}+\lambda)+N\int\mathrm{d}^{d}x\left(-V(\rho)+\tfrac{1}{2}\rho\lambda\right)\right],\qquad(7.48)$$

one of the fields by the solution of the field equation. It is useful to first discuss the effective potential of the massless mode. This requires calculating the action only for constant fields. It is then simpler to eliminate $\lambda$. We assume in this section that $m$ is small (the vector propagates). For $\lambda \ll \Lambda$ the $\lambda$-equation reads ($d<4$)

$$\rho-\rho_{c}=-C(d)\lambda^{(d-2)/2}.\qquad(7.49)$$

It follows that the resulting potential $W(\rho)$, obtained from equation (7.17) is

$$W(\rho)=V(\rho)+\frac{d-2}{2d(C(d))^{2/(d-2)}}(\rho_{c}-\rho)^{d/(d-2)}.\qquad(7.50)$$

In the sense of the double scaling limit the criticality conditions are

$$W(\rho)=O\big((\rho-\rho_{s})^{n}\big).$$

It follows

$$V^{(k)}(\rho_{s})=-\tfrac{1}{2}C^{1-k}(d)\frac{\Gamma\big(k-d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-k(d-2)}\quad 1\leq k\leq n-1\,.$$

For the potential $V$ of minimal degree we find

$$W(\rho)\sim\tfrac{1}{2n!}C^{1-n}(d)\frac{\Gamma\big(n-d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-n(d-2)}(\rho-\rho_{s})^{n}.$$

*The double scaling limit.* We recall here that quite generally one verifies that a non-trivial double scaling limit may exist only if the resulting field theory of the massless mode is super-renormalizable, i.e. below its upper-critical dimension $d=2n/(n-2)$, because perturbation theory has to be IR divergent. Equivalently, to eliminate $N$ from the critical theory, one has to rescale

$$\rho-\rho_{s}\propto N^{-2\theta}\varphi\,,\quad x\mapsto xN^{(n-2)\theta}\quad\text{with }1/\theta=2n-d(n-2),$$

where $\theta$ has to be positive.

We now specialize to dimension three, since $d < 3$ has already been examined, and the expressions above are valid only for $d < 4$. The normal critical point ($n = 3$), which leads to a $\varphi^3$ field theory, and can be obtained for a quadratic potential $V(\rho)$ (the $(\phi^2)^2$) has been discussed elsewhere. We thus concentrate on the next critical point $n = 4$ where the minimal potential has degree three.

*The $d = 3$ tricritical point.* The potential $W(\rho)$ then becomes

$$W(\rho) = V(\rho) + \tfrac{8\pi^2}{3}(\rho_c - \rho)^3. \tag{7.51}$$

If the potential $V(\rho)$ has degree larger than three, we obtain after a local expansion and a rescaling of fields,

$$\rho - \rho_s = (\frac{-1}{32\pi^2 \rho_c})(\lambda - m^2) \propto \varphi/N\,, \quad x \mapsto Nx\,, \tag{7.52}$$

a simple super-renormalizable $\varphi^4(x)$ field theory. If we insist instead that the initial theory should be renormalizable, then we remain with only one candidate, the renormalizable $(\phi^2)^3$ field theory, also relevant for the tricritical phase transition with $O(N)$ symmetry breaking. Inspection of the potential $W(\rho)$ immediately shows a remarkable feature: Because the term added to $V(\rho)$ is itself a polynomial of degree three, the critical conditions lead to a potential $W(\varphi)$ which vanishes identically. This result reflects the property that the two saddle point equation ($\partial S/\partial\rho = 0$ , $\partial S/\partial\lambda = 0$ in equations (7.24)) are proportional and thus have a continuous one-parameter family of solutions. This results in a flat effective potential for $\varphi(x)$. The effective action for $\varphi$ depends only on the derivatives of $\varphi$, like in the $O(2)$ non-linear $\sigma$ model.

We conclude that no non-trivial double scaling limit can be obtained in this way. In three dimensions with a $(\phi^2)^3$ interaction we can generate at most a normal critical point $n = 3$, but then a simple $(\phi^2)^2$ field theory suffices.

The ambiguity of the sign of $a(d)$ discussed in section 7.3 has an interesting appearance in $d = 3$ in the small $m^2$ region. If one keeps the extra term proportional to $a(d)$ in equation (7.50) we have

$$W(\rho) = V(\rho) + \frac{8\pi^2}{3}(\rho_c - \rho)^3 + \frac{a(3)}{\Lambda}4\pi^2(\rho_c - \rho)^4.$$

Using now equation (7.49) and, as mentioned in section 7.4, the fact that in the small $m^2$ region the potential is proportional to $(\rho - \rho_c)^3$ we can solve for $m^2$. Since $m^2 > 0$ the appearance of a phase with small mass depends on the sign of $a(d)$. Clearly this shows a non-commutativity of the limits of $m^2/\Lambda^2 \to 0$ and $N \to \infty$. The small $m^2$ phase can be reached by a special tuning and cannot be reached with an improper sign of $a(d)$. Calculated in this way, $m^2$ can be made proportional to the deviation of the coefficient of $\rho^3$ in $V(\rho)$ from its critical value $16\pi^2$.

*7.6 Conclusions*

This is a study of several subtleties in the phase structure of $O(N)$ vector models around multicritical points of odd and even orders. One of the main topics is the understanding of the multicritical behavior of these models at their critical dimensions and the effective field theory of the $O(N)$-singlet bound state obtained in the $N \to \infty$, $g \to g_c$ correlated limit. It is pointed out that the integration over massive $O(N)$ singlet modes is essential in order to extract the correct effective field theory of the small mass scalar excitation. After performing this integration, it has been established here that the double scaling limit of $(\phi^2)^K$ vector model in its critical dimension $d = 2K/(K-1)$ results in a theory of a free massless $O(N)$ singlet bound state. This fact is a consequence of the existence of flat directions at the scale invariant multicritical point in the effective action. In contrast to the case $d < 2K/(K-1)$ where IR singularities compensate powers of $1/N$ in the double scaling limit, at $d = 2K/(K-1)$ there is no such compensation and only a noninteracting effective field theory of the massless bound state is left.

Another interesting issue in this study is the ambiguity of the sign of $a(d)$. The coefficient of $m^2\Lambda^{d-4}$ denoted by $a(d)$ in the expansion of the gap equation in equations (7.24$c$) and (7.25) seems to have a surprisingly important role in the approach to the continuum limit ($\Lambda^2 \gg m^2$). The existence of an IR fixed point at $g \sim O(N^{-1})$, as seen in the $\beta$ function for the unrenormalized coupling constant (section 4.3), depends on the sign of $a(d)$. Moreover, the existence of a phase with a small mass $m$ for the $O(N)$ vector quanta and a massless $O(N)$ scalar depends also on the sign of $a(d)$. It may very well be that the importance of the sign of $a(d)$ is a mere reflection of the limited coupling constant space used to described the model. This is left here as an open question that deserves a detailed renormalization group or lattice simulation study in the future.

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