## Critical dynamics: Langevin and Fokker–Planck equations

In the framework of critical phenomena, time evolution is generally described by a phenomenological Langevin equation and its RG properties then govern critical dynamics. However, to a given equilibrium distribution can be associated an infinite number of Langevin equations and, thus, there are many more dynamic than static universality classes.

We study here only purely dissipative dynamics (which thus satisfies detailed balance) and, to simplify, without conservation laws. In the classification of the review of Halperin and Hohenberg, we investigate model A.

We consider the example of an one-component scalar field $\varphi(t,x)$, $t$ being the time and $x\in\mathbb{R}^d$. The dissipative Langevin equation is a first order in time stochastic differential equation of the general form

$$\dot\varphi(t,x)=-\tfrac{1}{2}\Omega\frac{\delta\mathcal{H}}{\delta\varphi(t,x)}+\nu(t,x), \tag{57}$$

where the constant $\Omega^{-1}$ provides a time scale.

The functional $\mathcal{H}(\varphi)$ is a time-independent, local euclidean Hamiltonian. An example is

$$\mathcal{H}(\varphi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla_x \varphi(x)\bigr)^2 + V\bigl(\varphi(x)\bigr)\right].$$

The noise field $\nu(t,x)$ is a stochastic field which we assume to have the Gaussian local distribution (Gaussian white noise)

$$[\mathrm{d}\rho(\nu)] = [\mathrm{d}\nu] \exp\left[-\int \mathrm{d}t\, \mathrm{d}^d x\, \nu^2(t,x)/2\Omega\right]. \qquad (58)$$

It can also be characterized by its one- and two-point functions,

$$\langle \nu(t,x)\rangle = 0\,, \quad \langle \nu(t,x)\nu(t',x')\rangle = \Omega\,\delta(t-t')\delta^{(d)}(x-x').$$

*The Fokker–Planck equation.*

Given the noise (58), and some initial distribution for the field $\varphi(t,x)$, the Langevin equation generates the time-dependent field distribution

$$P\big(t,\varphi(x)\big) = \langle \delta\left(\varphi(t,x) - \varphi(x)\right)\rangle_{\nu} .$$

From the Langevin equation one derives the evolution equation

$$\dot{P}(\varphi,t) = -\Omega \mathbf{H}_{\mathrm{FP}} P(\varphi,t) ,$$

where the operator $\mathbf{H}_{\mathrm{FP}}$, the Fokker–Planck Hamiltonian, is given by

$$\mathbf{H}_{\mathrm{FP}}\left(\varphi, \frac{\delta}{\delta\varphi}\right) = -\frac{1}{2}\int \mathrm{d}^d x \frac{\delta}{\delta\varphi(x)}\left[\frac{\delta}{\delta\varphi(x)} + \frac{\delta\mathcal{H}}{\delta\varphi(x)}\right].$$

The change $P = \mathrm{e}^{-\mathcal{H}/2}\,\tilde{P}$ transforms $\mathbf{H}_{\mathrm{FP}}$ into the Hermitian positive Hamiltonian

$$\widetilde{\mathbf{H}}_{\mathrm{FP}}\left(\varphi, \frac{\delta}{\delta\varphi}\right) = \frac{1}{2}\mathbf{A}^{\dagger}\mathbf{A} \text{ with } \mathbf{A} = \frac{\delta}{\delta\varphi(x)} + \frac{1}{2}\frac{\delta\mathcal{H}}{\delta\varphi(x)} .$$

From the Fokker–Planck equation, one infers that the Langevin equation (57) together with the noise distribution (58) generates a dynamics that converges toward the distribution $\mathrm{e}^{-\mathcal{H}(\varphi)}$ provided it is normalizable. It is then the equilibrium distribution.

Moreover, one also proves that the purely dissipative Langevin equation is associated to a time evolution with detailed balance.

*Application to critical dynamics*

Within the theory of phase transitions, the important question is whether and how universal properties generalize to the dynamics. The question can be completely answered in the case of the purely dissipative Langevin equation by constructing an RG also for the dynamics. For this purpose, one has first to understand how the Langevin equation for fields renormalizes.

To discuss renormalization, one has to set up a formalism more directly amenable to the ordinary methods of quantum field theory. This can be done by constructing a field integral representation of the time-dependent $\varphi$-field correlation functions in terms of an associated local action, which, in this framework, it is natural to call dynamic action. When the static Hamiltonian $\mathcal{H}(\varphi)$ is renormalizable, one finds that the renormalizations of the static theory together with a time scale renormalization, render the Langevin equation finite.

## Time-dependent correlation functions and dynamic action

The generating functional $\mathcal{Z}(J)$ of dynamic correlation functions is given by the noise expectation value

$$\begin{aligned}\mathcal{Z}(J) &= \left\langle \exp\left[\int \mathrm{d}^d x\,\mathrm{d}t\, J(t,x)\varphi(t,x)\right]\right\rangle_\nu ,\\ &= \int [\mathrm{d}\nu] \exp\left[-\int \mathrm{d}^d x\,\mathrm{d}t \left(\frac{1}{2\Omega}\nu^2(t,x) - J(t,x)\varphi(t,x)\right)\right],\end{aligned}$$

where $\varphi(t,x)$ is a solution of the Langevin equation (57) with fixed noise $\nu$.

To impose the Langevin equation, we insert into the field integral the identity

$$\int [\mathrm{d}\varphi] \det \mathbf{M} \prod_{t,x} \delta \left[\dot{\varphi}(t,x) + (\Omega/2)\delta\mathcal{H}/\delta\varphi - \nu(t,x)\right] = 1\,,$$

where $\mathbf{M}$ is the differential operator,

$$\mathbf{M} = \frac{\delta \,\text{Langevin equation}}{\delta\varphi(t',x')} = \frac{\partial}{\partial t}\delta(t-t')\delta(x-x') + \frac{\Omega}{2}\frac{\delta^2\mathcal{H}}{\delta\varphi(t',x')\delta\varphi(t,x)}.$$

The $\delta$-function can immediately be used to integrate over the noise $\nu$:

$$\mathcal{Z}(J) = \int [\mathrm{d}\varphi] \det \mathbf{M} \exp\left[-\int \mathrm{d}^d x\, \mathrm{d}t \left(\frac{1}{2\Omega}\left(\dot{\varphi} + \tfrac{1}{2}\Omega\frac{\delta\mathcal{H}(\varphi)}{\delta\varphi}\right)^2 - J\varphi\right)\right].$$

For a system with a discrete set of degrees of freedom (a $d = 0$ dimensional or a lattice regularized field theory), the determinant can be calculated, using the identity

$$\det \mathbf{M} \propto \exp \operatorname{tr} \ln \left[ 1 + \left( \frac{\partial}{\partial t} \right)^{-1} \frac{\Omega}{2} \frac{\delta \mathcal{H}}{\delta \varphi \delta \varphi} \right].$$

As a consequence of the causality of the Langevin equation, the inverse of the operator $(\partial/\partial t)\delta(t - t')$ is the kernel $\theta(t - t')$ ($\theta(t)$ is the Heaviside step function). In an expansion in powers of $\Omega$, all terms thus vanish when one takes the trace, but the first one that yields

$$\det M \propto \exp \left\{ \theta(0) \frac{\Omega}{2} \int \mathrm{d}t \, \mathrm{d}^d x \, \left. \frac{\delta^2 \mathcal{H}}{\delta \varphi(t, x) \delta \varphi(t, x')} \right|_{x' = x} \right\}.$$

For the undefined quantity $\theta(0)$ we choose $\theta(0) = 1/2$, a choice symmetric in time. The final expression can then formally be written as

$$\mathcal{Z}(J) = \int [\mathrm{d}\varphi] \exp \left[ -\mathcal{S}(\varphi) + \int \mathrm{d}^d x \, \mathrm{d}t \, J(t, x) \varphi(t, x) \right]$$

with

$$\mathcal{S}(\varphi) = \frac{1}{2\Omega}\int \mathrm{d}^d x\,\mathrm{d}t \left[\left(\dot\varphi(t,x)\right)^2 + \tfrac{1}{4}\Omega^2\left(\frac{\delta\mathcal{H}(\varphi)}{\delta\varphi(t,x)}\right)^2\right]$$
$$-\frac{\Omega}{4}\int \mathrm{d}t\,\mathrm{d}^d x\ \frac{\delta^2\mathcal{H}(\varphi)}{\delta\varphi(x,t)\delta\varphi(x',t)}\bigg|_{x'=x}.$$

where we have expanded the square and integrated the cross term:

$$\int_{t'}^{t''}\mathrm{d}t\int \mathrm{d}^d x\,\dot\varphi(t,x)\frac{\delta\mathcal{H}(\varphi)}{\delta\varphi(t,x)} = \mathcal{H}(\varphi(t'')) - \mathcal{H}(\varphi(t')),$$

an equation valid inside the field integral only for $\theta(0) = 1/2$.

*The problem of the determinant*

In dimension $d > 0$, the dynamic action is undefined when $\mathcal{H}(\varphi)$ is a local functional because the contribution of the determinant is formally proportional to $\delta^{(d)}(0)$:

$$\ln \det \mathbf{M} \propto \int \mathrm{d}t\, \mathrm{d}^d x\, \left.\frac{\delta^2 \mathcal{H}}{\delta\varphi(t,x)\delta\varphi(x',t)}\right|_{x'=x} \propto \delta^{(d)}(0).$$

The determinant has thus to be regularized. We have two options:

(i) With dimensional regularization, terms like $\delta^{(d)}(0)$ vanish and, therefore, the determinant can be completely omitted.

(ii) However, it is useful to keep this divergent term in some regularized form in order to preserve the algebraic structure of the dynamic action.Indeed, this algebraic structure determines the form of the renormalization. This can be achieved with lattice regularization.

## The purely dissipative Langevin equation and supersymmetry

We have explained how to associate to the Langevin or Fokker–Planck equations a dynamic action. Quite generally, the dynamic action has a BRS symmetry. This symmetry and its consequences in the form of WT identities can be used to prove that under some general conditions the structure of the Langevin equation is stable under renormalization.

In the particular example of purely dissipative equations with Gaussian noise, the dynamic action has an additional Grassmann symmetry which, combined with the first one, provides the simplest example of supersymmetry: quantum mechanics supersymmetry. To exhibit the supersymmetry, it is convenient to introduce an alternative formalism based on Grassmann coordinates and superfields.

*Grassmann coordinates and algebraic properties*

To discuss supersymmetry, we add to time and space two Grassmann coordinates, $\theta$ and $\bar{\theta}$, generators of a Grassmann algebra $\mathfrak{A}$:

$$\theta^2 = \bar{\theta}^2 = 0\,, \quad \theta\bar{\theta} = -\bar{\theta}\theta\,.$$

*Grassmann derivatives.* We also define two linear operations acting on $\mathfrak{A}$:

$$\frac{\partial}{\partial\theta}\,, \quad \frac{\partial}{\partial\bar{\theta}}\,,$$

such that (with $\theta \mapsto \theta_1, \bar{\theta} \mapsto \theta_2$) with the $\theta_i$ they form a representation of fermion creation and annihilation operators:

$$\begin{aligned} \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} &= 0 \\ \frac{\partial}{\partial\theta_i}\theta_j + \theta_j\frac{\partial}{\partial\theta_i} &= \delta_{ij} \\ \theta_i\theta_j + \theta_i\theta_j &= 0\,. \end{aligned}$$

We then define definite integrals on $\mathfrak{A}$ by

$$\int \mathrm{d}\theta \equiv \frac{\partial}{\partial\theta}\,, \quad \int \mathrm{d}\bar{\theta} \equiv \frac{\partial}{\partial\bar{\theta}}\,.$$

Then, for example,

$$\int \mathrm{d}\theta \int \mathrm{d}\bar{\theta}\; \mathrm{e}^{\mu\bar{\theta}\theta} = \int \mathrm{d}\theta \int \mathrm{d}\bar{\theta}\,(1+\mu\bar{\theta}\theta) = \mu\,.$$

*Grassmann parity.* One defines an algebra automorphism on $\mathfrak{A}$ by

$$P(\theta) = -\theta\,, \quad P(\bar{\theta}) = -\bar{\theta}\,.$$

Finally, when the number of generators is even, one can define a 'complex' conjugation in $\mathfrak{A}$ (with the properties of a hermitian conjugation) and a scalar product.

*Superfields and covariant derivatives*

We introduce a superfield notation

$$\phi(t,x;\bar{\theta},\theta) = \varphi(t,x) + \theta\bar{c}(t,x) + c(t,x)\bar{\theta} + \theta\bar{\theta}\bar{\varphi}(t,x)\,,$$

where $\varphi(t,x)$ and $\bar{\varphi}(t,x)$ are (scalar) boson fields, $\bar{c}(t,x)$ and $c(t,x)$ are (spinless) fermion fields.

We define also two Grassmann-type derivatives,

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar{\theta}}, \quad \mathrm{D} = \frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t}. \tag{59}$$

$\bar{\mathrm{D}}$ and $\mathrm{D}$ satisfy the anticommutation relations

$$\mathrm{D}^2 = \bar{\mathrm{D}}^2 = 0\,, \qquad \mathrm{D}\bar{\mathrm{D}} + \bar{\mathrm{D}}\mathrm{D} = -\frac{\partial}{\partial t}\,. \tag{60}$$

*Supersymmetry*

The dynamic action associated to the Langevin equation

$$\dot{\varphi}(t,x) = -\tfrac{1}{2}\Omega \frac{\delta \mathcal{H}}{\delta \varphi(t,x)} + \nu(t,x),$$

with the noise Gaussian distribution

$$[\mathrm{d}\rho(\nu)] = [\mathrm{d}\nu] \exp\left[-\int \mathrm{d}t\, \mathrm{d}^d x\, \nu^2(t,x)/2\Omega\right],$$

can then be rewritten in supersymmetric form as

$$\mathcal{S}(\phi) = \int \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, \mathrm{d}t \left[\frac{2}{\Omega}\int \mathrm{d}^d x\, \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}(\phi)\right]. \tag{61}$$

For convenience, we have rescaled the Langevin equation by a factor $2/\Omega$.

We then introduce the two other Grassmann-type differential operators

$$\mathrm{Q} = \frac{\partial}{\partial\theta}, \qquad \bar{\mathrm{Q}} = \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t}. \tag{62}$$

Both anticommute with $\mathrm{D}$ and $\bar{\mathrm{D}}$ and satisfy

$$\mathrm{Q}^2 = \bar{\mathrm{Q}}^2 = 0\,, \quad \mathrm{Q}\bar{\mathrm{Q}} + \bar{\mathrm{Q}}\mathrm{Q} = \frac{\partial}{\partial t}\,. \tag{63}$$

The two pairs $\mathrm{D}, \bar{\mathrm{D}}$ and $\mathrm{Q}, \bar{\mathrm{Q}}$ provide the simplest examples of generators of supersymmetry. Moreover, $\mathrm{Q}$ and $\bar{\mathrm{Q}}$ are generators of symmetries of the dynamic action, as one verifies by performing variations of $\phi$ of the form

$$\delta\phi(t,\theta,\bar{\theta}) = \varepsilon Q\phi(t,\theta,\bar{\theta}), \quad \delta\phi(t,\theta,\bar{\theta}) = \bar{\varepsilon}\bar{\mathrm{Q}}\phi(t,\theta,\bar{\theta}), \tag{64}$$

where $\varepsilon$ and $\bar{\varepsilon}$ are anticommuting constants. The variation of the action density is then a total time derivative. The action is thus supersymmetric.

This confirms that the operators $\mathrm{D}$ and $\bar{\mathrm{D}}$ are covariant derivatives from the point of view of supersymmetry.

This supersymmetry is directly related to the property that the corresponding Fokker–Planck Hamiltonian is equivalent to a positive Hamiltonian of the form $\mathbf{A}^\dagger\mathbf{A}$.

*Remarks.*

(i) The anticommutator of $\bar{\mathrm{Q}}$ and $\mathrm{Q}$ generates time translations. Supersymmetry implies time translation invariance.

(ii) It is possible to emphasize the symmetric role played by $\bar\theta$ and $\theta$ by performing the substitution $t \mapsto t + \frac{1}{2}\theta\bar\theta$.

(iii) Considering the fermions as real dynamic variables, one can associate to the supersymmetric action a Hamiltonian in boson–fermion space. It corresponds then both to the Langevin equation and its time-reversed form.

(iv) Supersymmetry provides a proof of detailed balance, alternative to the proof based on the Fokker–Planck equation.

*Ward–Takahashi (WT) identities*

One symmetry simply implies that correlation functions are invariant under a translation of the coordinate $\theta$. The second transformation has a slightly more complicated form. It implies that the generating functional $\mathcal{W}(J)$ of connected correlation functions satisfies,

$$\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\bar{\mathrm{Q}}J(t,x,\theta,\bar{\theta})\frac{\delta\mathcal{W}}{\delta J(t,x,\theta,\bar{\theta})}=0\,.$$

Connected correlation functions $W^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i)$ and vertex functions $\Gamma^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i)$ thus satisfy the WT identities:

$$\bar{\mathrm{Q}}\,W^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i)=0\,,\qquad \bar{\mathrm{Q}}\,\Gamma^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i)=0$$

with

$$\bar{\mathrm{Q}}\equiv\sum_{k=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_k}+\theta_k\frac{\partial}{\partial t_k}\right).$$

*Renormalization of the dissipative Langevin equation*

To be more specific, we now assume that $\mathcal{H}(\varphi)$ has the form

$$\mathcal{H}(\varphi) = \tfrac{1}{2}\int \mathrm{d}^d x (\nabla_x \varphi)^2 + \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = \tfrac{1}{2} m^2 \varphi^2 + O(\varphi^3).$$

Then the propagator in the dynamic theory, in the Fourier representation, reads $(\delta(\theta) = \theta)$

$$\tilde{\Delta}(\omega, \mathbf{k}, \boldsymbol{\theta}', \boldsymbol{\theta}) = \frac{\Omega \left[1 + \frac{1}{2} i\omega \left(\theta' - \theta\right) \left(\bar{\theta} + \bar{\theta}'\right) + \frac{1}{4}\Omega \left(k^2 + m^2\right) \delta^2(\bar{\boldsymbol{\theta}}' - \bar{\boldsymbol{\theta}})\right]}{\omega^2 + \frac{\Omega^2}{4} \left(k^2 + m^2\right)^2}.$$

A power counting follows

$$[k] = 1\,, \ \ [\omega] = 2\,, \ \ [\theta] = [\bar{\theta}] = -1\,.$$

Similarly, then (since integration and differentiation over anticommuting variables are identical operations, the dimension of $\mathrm{d}\theta$ is $-[\theta]$)

$$[x] = -1\,, \ \ [t] = -2\,, \ \ [\mathrm{d}\theta] = [\mathrm{d}\bar{\theta}] = 1 \ \ \Rightarrow [\mathrm{d}t] + [\mathrm{d}\theta] + [\mathrm{d}\bar{\theta}] = 0\,.$$

Therefore, the term proportional to $\mathcal{H}(\phi)$ in the action has the same canonical dimension as in the static case: the static and the dynamic theories have the same power counting and both theories are renormalizable in the same space dimension $d$.

The supersymmetry transformation is linearly represented on the fields and, therefore, the renormalized action remains supersymmetric. We then write in superfield notation the most general form of the renormalized action $\mathcal{S}_{\mathrm{r}}$ consistent with power counting and supersymmetry:

$$\mathcal{S}_{\mathrm{r}}(\phi) = \int \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, \mathrm{d}t \left[ \frac{2Z_\omega}{\Omega} \int \mathrm{d}^d x\, \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}_{\mathrm{r}}(\phi) \right]. \tag{65}$$

Only one new renormalization constant, $Z_\omega$, is generated. After renormalization, the drift force in the Langevin equation is thus proportional to the functional derivative of the renormalized Hamiltonian.

## The dissipative Langevin equation: RG equations in $d = 4 - \varepsilon$ dimension

*The $N$-vector model near four dimensions.* We consider the dissipative dynamics for the $N$-component field $\boldsymbol{\varphi}$ that satisfies,

$$\dot{\boldsymbol{\varphi}}(t,x) = -\frac{\Omega_0}{2}\frac{\delta\mathcal{H}(\boldsymbol{\varphi})}{\delta\boldsymbol{\varphi}(t,x)} + \boldsymbol{\nu}(t,x) \tag{66}$$

with

$$\mathcal{H}(\boldsymbol{\varphi}) = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\nabla_x\boldsymbol{\varphi}\right)^2 + \frac{1}{2}r_0\boldsymbol{\varphi}^2 + g_0\frac{\Lambda^\varepsilon}{4!}\left(\boldsymbol{\varphi}^2\right)^2\right].$$

The noise field has the Gaussian distribution

$$[\mathrm{d}\rho(\boldsymbol{\nu})] = [\mathrm{d}\nu]\exp\left[-\int \mathrm{d}t\,\mathrm{d}^d x\,\boldsymbol{\nu}^2(t,x)/2\Omega_0\right],$$

in such way that $\mathrm{e}^{-\mathcal{H}}$ is the equilibrium distribution.

In terms of the superfield

$$\boldsymbol{\phi} = \boldsymbol{\varphi} + \theta\bar{\mathbf{c}} + \mathbf{c}\bar{\theta} + \theta\bar{\theta}\bar{\boldsymbol{\varphi}}\,,$$

the corresponding dynamic action $\mathcal{S}(\boldsymbol{\phi})$ takes the supersymmetric form

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\left[\int \mathrm{d}^d x\,\frac{2}{\Omega_0}\bar{\mathrm{D}}\phi\mathrm{D}\phi + \mathcal{H}(\boldsymbol{\phi})\right]$$

with

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar{\theta}}, \quad \mathrm{D} = \frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t}.$$

*RG equations at and above $T_c$*

We have shown that static and supersymmetric dynamic theories have the same upper-critical dimension. Therefore, fluctuations are only relevant for dimensions $d \le 4$. We have also shown that the renormalized dynamic action $\mathcal{S}_{\rm r}(\boldsymbol{\phi})$ then takes the form

$$\mathcal{S}_{\rm r}(\boldsymbol{\phi}) = \int \mathrm{d}\bar\theta\,\mathrm{d}\theta\,\mathrm{d}t \left[\int \mathrm{d}^d x \frac{2}{\Omega} Z_\omega \bar{\mathrm{D}}\phi \mathrm{D}\phi + \mathcal{H}_{\rm r}(\boldsymbol{\phi})\right],$$

in which $\boldsymbol{\phi}$ is now the renormalized field and $\mathcal{H}_{\rm r}(\boldsymbol{\phi})$ is the static renormalized Hamiltonian.

To renormalize the dynamic action, we need, in addition to the static renormalization constants, a renormalization of the parameter $\Omega$:

$$\Omega_0 = \Omega Z / Z_\omega \,,$$

where $Z$ is the field renormalization constant.

The derivation of RG equations for the dynamics is standard. We write below RG equations for the renormalized theory, using subscript 0 for the initial parameters.

The RG differential operator then takes the form ($\mu$ is the renormalization scale)

$$\mathrm{D}_{\mathrm{RG}} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \eta_\omega(g) \Omega \frac{\partial}{\partial \Omega} - \frac{n}{2} \eta(g),$$

where the new independent RG function $\eta_\omega(g)$ is given by

$$\eta_\omega(g) = \mu \left. \frac{\mathrm{d}}{\mathrm{d}\mu} \right|_{g_0, \Omega_0} \ln \Omega \,. \tag{67}$$

The RG equations for the critical theory, in Fourier space and for the vertex functions, then read ($\boldsymbol{\theta} \equiv (\bar{\theta}, \theta)$)

$$\mathrm{D}_{\mathrm{RG}} \tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \mu, \Omega, g) = 0 \ .$$

At an IR fixed point $g^*$, the RG equations reduce to

$$\left(\mu\frac{\partial}{\partial\mu}+\eta_\omega(g^*)\Omega\frac{\partial}{\partial\Omega}-\frac{n}{2}\eta(g^*)\right)\tilde{\Gamma}^{(n)}(p_i,\omega_i,\boldsymbol{\theta},\mu,\Omega)=0\,.$$

We then set $z=2+\eta_\omega(g^*)$.

From dimensional analysis, we obtain

$$\tilde{\Gamma}^{(n)}\left(\lambda p_i,\rho\omega_i,\frac{\boldsymbol{\theta}}{\sqrt{\rho}},\lambda\mu,\frac{\rho\Omega}{\lambda^2}\right)=\lambda^{d-n(d-2)/2}\rho^{1-n}\tilde{\Gamma}^{(n)}(p_i,\omega_i,\boldsymbol{\theta},\mu,\Omega),$$

In the dimensional equation, we choose $\rho=\Omega\lambda^z\mu^{-\eta_\omega}$. Then, combining the solution of the RG equation with dimensional analysis, we find the dynamic scaling relations

$$\tilde{\Gamma}^{(n)}(\lambda p_i,\omega_i,\boldsymbol{\theta},\mu=\Omega=1)=\lambda^{d-nd_\varphi-z(n-1)}F^{(n)}(p_i,\lambda^{-z}\omega_i,\boldsymbol{\theta}\lambda^{z/2}),$$

where $d_\varphi=\frac{1}{2}(d-2+\eta)$ is the field dimension.

A few algebraic manipulations yield the corresponding relation for connected correlation functions:

$$\tilde{W}^{(n)}(\lambda p_i, \omega_i, \boldsymbol{\theta}, \mu = \Omega = 1) = \lambda^{(d+z)(1-n)+nd_\varphi} G^{(n)}(p_i, \omega_i \lambda^{-z}, \boldsymbol{\theta}\lambda^{z/2}).$$

The $\boldsymbol{\varphi}$-field two-point correlation function is obtained for $n = 2$ and $\boldsymbol{\theta} = 0$:

$$\tilde{W}^{(2)}(p, \omega, \boldsymbol{\theta} = 0) \sim p^{-2+\eta-z} G^{(2)}(\omega/p^z).$$

The equal-time correlation function is obtained by integrating over $\omega$. One verifies consistency with static scaling.

The dynamic critical two-point function thus depends on a frequency scale that vanishes at small momentum like $p^z$ or a time scale that diverges like $p^{-z}$.

The RG function $\eta_\omega$ at one-loop order for this model is

$$\eta_\omega(g) = N_d^2 \frac{N+2}{72} \left[6 \ln(4/3) - 1\right] g^2 + O\left(\tilde{g}^3\right), \quad N_d = \frac{2}{\Gamma(d/2)(4\pi)^{d/2}}. \tag{68}$$

The dynamic critical exponent $z$ follows:

$$z = 2 + \frac{N+2}{2(N+8)^2}\left[6\ln(4/3) - 1\right]\varepsilon^2 + 0\left(\varepsilon^3\right).$$

*Correlation functions above $T_c$ in the critical domain.* We only write the RG equations at the IR fixed point:

$$\left(\mu\frac{\partial}{\partial\mu} + \eta_\omega\Omega\frac{\partial}{\partial\Omega} - \eta_2\sigma\frac{\partial}{\partial\sigma} - \frac{n}{2}\eta\right)\tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \sigma, \mu, \Omega) = 0\,, \tag{69}$$

in which $\sigma$ is a measure of the deviation from the critical temperature:

$$\sigma \propto T - T_c\,.$$

Dimensional analysis yields

$$\tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \sigma, \mu, \Omega) = \lambda^{d-n(d-2)/2}\rho^{1-n}\tilde{\Gamma}^{(n)}\left(\frac{p_i}{\lambda}, \frac{\omega_i}{\rho}, \boldsymbol{\theta}\sqrt{\rho}, \frac{\sigma}{\lambda^2}, \frac{\mu}{\lambda}, \frac{\Omega\lambda^2}{\rho}\right).$$

Finally, combining this equation with the RG equation (69) and choosing

$$\lambda = \sigma^{\nu} \mu^{\nu\eta 2} \sim \xi^{-1}, \qquad \rho = \Omega \mu^{-\eta\omega} \lambda^{z} \sim \xi^{-z},$$

in which $\xi$ is the correlation length, one obtains

$$\tilde{\Gamma}^{(n)}(p_i, \omega_i, \boldsymbol{\theta}, \sigma, \mu = 1, \Omega = 1) \sim \xi^{-d+n(d-2+\eta)/2+z(n-1)} \\ \times F^{(n)}(p_i\xi, \omega_i\xi^{z}, \boldsymbol{\theta}\xi^{-z/2}).$$

Only the combination $\omega_i \xi^z$ appears in the r.h.s. and this implies, after Fourier transformation, that all times are measured in units of a correlation time $\tau$ that diverges at the critical temperature as

$$\tau \propto \xi^{z}, \tag{70}$$

an effect called critical slowing down.

## General stochastic equations and BRS symmetry

A number of topics share a common feature: they involve an equation like a stochastic field equation or the gauge fixing equation of gauge theories that can be considered as a constraint equation and which, in the framework of some perturbation theory, leads to problems of divergences and renormalization.

Since such questions can best be dealt with by quantum field theory methods, it is convenient to associate to these equations a kind of quantum action. As we show below, this can be achieved by a set of formal identities based on field integration.

The quantum action then is automatically invariant under transformations that depend on anticommuting parameters. This symmetry has no geometric origin but is merely a consequence of the construction though it has been first discovered in the context of quantized gauge theories by Becchi, Rouet and Stora and is called now BRS symmetry (or BRST).

The generator of the BRS symmetry has a vanishing square and generalizes exterior differentiation. It can be conveniently expressed in compact form by introducing Grassmann coordinates.

For simplicity, we discuss this mathematical structure in the case of a finite number of variables but the generalization to field equations is simple. We show that the BRS symmetry is stable against a number of algebraic transformations.

Langevin equations are stochastic field equations that have been proposed to describe the dynamics of critical phenomena. In the perturbative solution of these equations, divergences appear and it is necessary to understand how these equations renormalize. The BRS symmetry of the associated (dynamic) action allows proving that under some general conditions the structure of the Langevin equation is stable under renormalization.

In the case of purely dissipative Langevin equations, BRS symmetry becomes part of a larger symmetry called supersymmetry.

## BRS symmetry and constraint equations

Let $\varphi^\alpha$ be a set of dynamical variables satisfying a system of equations,

$$E_\alpha(\varphi) = 0\,, \tag{71}$$

where the $E_\alpha(\varphi)$ are smooth functions and $E_\alpha = E_\alpha(\varphi)$ is a one-to-one mapping in some neighbourhood of $E_\alpha = 0$ which can be inverted in $\varphi^\alpha = \varphi^\alpha(E)$. This implies in particular that the equations have a unique solution $\varphi^\alpha_{\mathrm{s}} \equiv \varphi^\alpha(0)$. In the neighbourhood of $\varphi_{\mathrm{s}}$, the determinant $\det \mathbf{E}$ of the matrix $\mathbf{E}$ with elements $E_{\alpha\beta}$,

$$E_{\alpha\beta} \equiv \partial_\beta E_\alpha\,,$$

does not vanish and thus we choose $E_\alpha(\varphi)$ such that it is positive.

Note that it will be convenient throughout the section to use the notation $\partial/\partial\varphi^\alpha \mapsto \partial_\alpha$.

For any function $F(\varphi)$, we now derive a formal expression for $F(\varphi_{\mathrm{s}})$ that does not involve solving equations (71) explicitly. We start from the trivial identity

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}E^{\alpha}\, \delta(E_{\alpha}) \right\} F\big(\varphi(E)\big),$$

where $\delta(E)$ is Dirac's $\delta$-function. We then change variables $E \mapsto \varphi$. This generates the Jacobian $\mathcal{J}(\varphi) = \det \mathbf{E} > 0$. Thus,

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}\varphi^{\alpha}\, \delta\left[E_{\alpha}(\varphi)\right] \right\} \mathcal{J}(\varphi)\, F(\varphi). \tag{72}$$

We then replace the $\delta$-function by its Fourier representation:

$$\prod_{\alpha} \delta\left[E_{\alpha}(\varphi)\right] = \int_{-i\infty}^{+i\infty} \prod_{\alpha} \frac{\mathrm{d}\bar{\varphi}^{\alpha}}{2i\pi} \exp\left[\sum_{\alpha} -\bar{\varphi}^{\alpha} E_{\alpha}(\varphi)\right],$$

where the $\bar{\varphi}$ integration runs along the imaginary axis.

Moreover, a determinant can be written as an integral over Grassmann variables $\bar{c}^{\alpha}$ and $c^{\alpha}$:

$$\det \mathbf{E} = \int \prod_{\alpha} (\mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha}) \exp \left( \sum_{\alpha,\beta} c^{\alpha} E_{\alpha\beta} \bar{c}^{\beta} \right) .$$

The expression (72) then becomes

$$F(\varphi_{\mathrm{s}}) = \mathcal{N} \int \prod_{\alpha} (\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\varphi}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha})\, F(\varphi) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c})\right] , \tag{73}$$

in which $S(\varphi, \bar{\varphi}, c, \bar{c})$ is the function (and element of the Grassmann algebra)

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \sum_{\alpha} \bar{\varphi}^{\alpha} E_{\alpha}(\varphi) - \sum_{\alpha,\beta} c^{\alpha} E_{\alpha\beta}(\varphi) \bar{c}^{\beta} \tag{74}$$

and $\mathcal{N}$ the normalization constant determined by setting $F = 1$:

$$\mathcal{N}^{-1} = \int \prod_{\alpha} (\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\varphi}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha}) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c})\right] .$$

*BRS symmetry.* Somewhat surprisingly, the function $S$ has a symmetry, BRS symmetry, first discovered in quantized gauge theories by Becchi, Rouet and Stora. It is a Grassmann symmetry in the sense that the parameter $\varepsilon$ of the transformation is an anticommuting constant, an additional generator of the Grassmann algebra. The variations of the various dynamic variables are

$$\delta\varphi^\alpha = \varepsilon\bar{c}^\alpha\,, \qquad \delta\bar{c}^\alpha = 0\,, \tag{75$a$}$$

$$\delta c^\alpha = \varepsilon\bar{\varphi}^\alpha\,, \qquad \delta\bar{\varphi}^\alpha = 0 \tag{75$b$}$$

with

$$\varepsilon^2 = 0\,, \qquad \varepsilon\bar{c}^\alpha + \bar{c}^\alpha\varepsilon = 0\,, \qquad \varepsilon c^\alpha + \varepsilon c^\alpha = 0\,.$$

The transformation is obviously nilpotent of vanishing square: $\delta^2 = 0$.

The BRS transformation can be represented by a Grassmann differential operator $\mathcal{D}$, when acting on functions of $\{\varphi, \bar{\varphi}, c, \bar{c}\}$:

$$\mathcal{D} = \sum_{\alpha} \left( \bar{c}^{\alpha} \frac{\partial}{\partial \varphi^{\alpha}} + \bar{\varphi}^{\alpha} \frac{\partial}{\partial c^{\alpha}} \right). \tag{76}$$

The nilpotency of the BRS transformation is then expressed by the identity

$$\mathcal{D}^2 = 0\,. \tag{77}$$

The differential operator $\mathcal{D}$ is a cohomology operator, generalization of the exterior differentiation of differential forms. In particular, the first term $\sum_{\alpha} \bar{c}^{\alpha} \partial_{\alpha}$ in the BRS operator is identical to the differentiation of forms in a formalism in which the Grassmann variables $\bar{c}^{\alpha}$ are introduced as external variables to exhibit the antisymmetry of the corresponding tensors.

Equation (77) implies that all quantities of the form $\mathcal{D}Q(\varphi, \bar{\varphi}, c, \bar{c})$, quantities we call BRS exact, are BRS invariant. We immediately verify that the function $S$ defined by equation (74) is BRS exact:

$$S = \mathcal{D}\left[\sum_{\alpha} c^{\alpha} E_{\alpha}(\varphi)\right]. \tag{78}$$

It follows that $S$ is BRS invariant,

$$\mathcal{D}S = 0\,. \tag{79}$$

The reciprocal property, the meaning and implications of the BRS symmetry will be discussed below.

These properties play an important role, in particular, in the discussion of the renormalization of gauge theories.

**Grassmann coordinates, gradient equations**

A more compact representation of BRS transformations is obtained by introducing a Grassmann coordinate $\theta$ and two functions of $\theta$:

$$\phi^\alpha(\theta) = \varphi^\alpha + \theta\bar{c}^\alpha, \qquad C^\alpha(\theta) = c^\alpha + \theta\bar{\varphi}^\alpha\,. \tag{80}$$

With this notation, the BRS transformations (75) simply become a translation of $\theta$:

$$\begin{aligned} \delta\phi^\alpha(\theta) &= \varepsilon\frac{\partial\phi^\alpha}{\partial\theta} = \phi^\alpha(\theta+\varepsilon) - \phi^\alpha(\theta), \\ \delta C^\alpha(\theta) &= \varepsilon\frac{\partial C^\alpha}{\partial\theta} = C^\alpha(\theta+\varepsilon) - C^\alpha(\theta). \end{aligned} \tag{81}$$

In particular, the BRS operator $\mathcal{D}$ is represented by $\partial/\partial\theta$:

$$\mathcal{D} \mapsto \frac{\partial}{\partial\theta}\,.$$

From the expansion

$$\sum_\alpha C^\alpha(\theta)E_\alpha(\phi(\theta)) = \sum_\alpha c^\alpha E_\alpha(\varphi) + \theta\left[\sum_\alpha \bar{\varphi}^\alpha E_\alpha(\varphi) - \sum_{\alpha,\beta} c^\alpha \frac{\partial E_\alpha}{\partial \varphi^\beta}\bar{c}^\beta\right],$$

we recover equation (78) in a different notation:

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \frac{\partial}{\partial\theta}\left[C^\alpha(\theta)E_\alpha(\phi(\theta))\right]. \tag{82}$$

In the case of Grassmann variables integration and differentiation are identical operations. Therefore, the equation can be rewritten as

$$S(\varphi, \bar{\varphi}, c, \bar{c}) = \int \mathrm{d}\theta\, C^\alpha(\theta)E_\alpha(\phi(\theta)). \tag{83}$$

In this expression the BRS symmetry is manifest: the integrand does not depend on $\theta$ explicitly.

Note that since the function $S$ involves only a Grassmann combination of the form $c\bar{c}$, in a representation in terms of the functions (80), as in equation (83), each integration over $\theta$ is associated with a factor $C^\alpha(\theta)$.

*Gradient equations.* In general, the two Grassmann variables $\bar{c}^\alpha$ and $c^\alpha$ play different roles. However, there is one special situation in which a symmetry is established between them—when the matrix $E_{\alpha\beta}$ is symmetric:

$$E_{\alpha\beta} = E_{\beta\alpha} \iff \partial_\beta E_\alpha = \partial_\alpha E_\beta \,.$$

Hence, in the absence of topological obstructions, there exists a function $A(\varphi)$ such that

$$E_\alpha(\varphi) = \partial_\alpha A(\varphi). \tag{84}$$

The symmetry between $c$ and $\bar{c}$ generates an additional independent BRS symmetry of generator

$$\bar{\mathcal{D}} = \sum_\alpha \left( c^\alpha \frac{\partial}{\partial \varphi^\alpha} + \bar{\varphi}^\alpha \frac{\partial}{\partial \bar{c}^\alpha} \right).$$

Introducing now two Grassmann coordinates $\bar{\theta}$ and $\theta$ (and then $\bar{\mathcal{D}} \mapsto \partial/\partial\bar{\theta}$), and a function

$$\phi^\alpha(\bar{\theta}, \theta) = \varphi^\alpha + \theta \bar{c}^\alpha + c^\alpha \bar{\theta} + \theta\bar{\theta}\bar{\varphi}^\alpha \,. \tag{85}$$

one verifies that the function $S(\phi)$ then takes the remarkable form

$$S(\phi) = \int \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, A\left[\phi(\bar{\theta}, \theta)\right] = \bar{\mathcal{D}}\mathcal{D}A(\varphi).$$

The two symmetries, which correspond to independent translations of $\theta$ and $\bar{\theta}$, are here explicit.

## Stochastic equations

We now assume that equation (71) depends on a set of stochastic variables $\nu_a$, the "noise",

$$E_\alpha(\varphi, \nu) = 0 \tag{86}$$

with normalized probability distribution $\mathrm{d}\rho(\nu)$.

The solution $\varphi^\alpha$ of the equation becomes a stochastic variable. Quantities of interest are expectation values of functions of $\varphi$:

$$\begin{aligned}\langle F(\varphi)\rangle_\nu &= \int \mathrm{d}\rho(\nu) \prod_\alpha \mathrm{d}\varphi^\alpha\, \delta\left[E_\alpha(\varphi, \nu)\right] \det \mathbf{E}\, F(\varphi) \\ &\propto \int \mathrm{d}\rho(\nu) \prod_\alpha (\mathrm{d}\varphi^\alpha \mathrm{d}\bar{\varphi}^\alpha \mathrm{d}\bar{c}^\alpha \mathrm{d}c^\alpha)\, F(\varphi) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c}, \nu)\right]\end{aligned}$$

with $S$ given by equation (74):

$$S = \sum_\alpha \bar{\varphi}^\alpha E_\alpha(\varphi, \nu) - \sum_{\alpha,\beta} c^\alpha E_{\alpha\beta}(\varphi, \nu) \bar{c}^\beta\,. \tag{87}$$

Let us introduce the function $\Sigma(\varphi, \bar{\varphi}, c, \bar{c})$ obtained after noise averaging:

$$\langle F(\varphi)\rangle \propto \int \prod_{\alpha} (\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\varphi}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha})\, F(\varphi) \exp\left[-\Sigma(\varphi, \bar{\varphi}, c, \bar{c})\right]$$

with

$$\exp\left[-\Sigma(\varphi, \bar{\varphi}, c, \bar{c})\right] = \int \mathrm{d}\rho(\nu) \exp\left[-S(\varphi, \bar{\varphi}, c, \bar{c}, \nu)\right]. \tag{88}$$

We have shown that $S$ has a BRS symmetry. Applying the BRS operator (76) on both sides of equation (88), we conclude that $\Sigma(\varphi, \bar{\varphi}, c, \bar{c})$ is still BRS symmetric,

$$\mathcal{D}\Sigma = 0\,, \tag{89}$$

although it no longer has the simple form (87), that is, a function linear in $\bar{\varphi}$ and $c\bar{c}$. Moreover, because $S$ is BRS exact, the function $\Sigma$ is also BRS exact, as can be shown by simple algebraic manipulations based on the identity

$$f(\mathcal{D}X) = f(0) + \mathcal{D}\left[X g(\mathcal{D}X)\right] \quad \text{with} \quad g(x) = \frac{f(x) - f(0)}{x}\,.$$

*A simple example: Stochastic equations linear in the noise*

Stochastic equations of the simple algebraic form

$$E_\alpha(\nu, \varphi) \equiv E_\alpha(\varphi) - \nu_\alpha \,, \tag{90}$$

are often met. Introducing the Laplace transform of the measure $\mathrm{d}\rho(\nu)$,

$$\mathrm{e}^{w(\bar\varphi)} = \int \mathrm{d}\rho(\nu) \exp\left[\sum_\alpha \bar\varphi^\alpha \nu_\alpha\right],$$

we obtain for the function $\Sigma$ defined by equation (88),

$$\Sigma(\varphi, \bar\varphi, c, \bar c) = -w(\bar\varphi) + \sum_\alpha \bar\varphi^\alpha E_\alpha(\varphi) - \sum_{\alpha,\beta} c^\alpha E_{\alpha\beta} \bar c^\beta \tag{91a}$$

$$= \mathcal{D}\tilde\Sigma \,, \quad \tilde\Sigma = \sum_\alpha c^\alpha \left[E_\alpha(\varphi) - \frac{\partial}{\partial\bar\varphi^\alpha} \int_0^1 \mathrm{d}s \, w(s\bar\varphi)\right]. \tag{91b}$$

*Remarks.*

(i) After integration over the noise, the expression of the function $\Sigma$ in the notation of Grassmann coordinates is, in general, rather complicated. However, in the case of equation (90) with Gaussian noise the additional term $w(\bar{\varphi}) = \frac{1}{2}\sum_{\alpha,\beta} w_{\alpha\beta}\bar{\varphi}^{\alpha}\bar{\varphi}^{\beta}$ is represented in the notation (85) by

$$\frac{1}{2}\sum_{\alpha,\beta} w_{\alpha\beta}\bar{\varphi}^{\alpha}\bar{\varphi}^{\beta} = \int \mathrm{d}\bar{\theta}\mathrm{d}\theta\, \frac{1}{2}\sum_{\alpha,\beta} w_{\alpha\beta}\frac{\partial\phi^{\alpha}}{\partial\bar{\theta}}\frac{\partial\phi^{\beta}}{\partial\theta}.$$

(ii) In the latter case, it is also possible to integrate explicitly over the $\bar{\varphi}$ variables. The resulting integrand corresponds to

$$\Sigma(\varphi, c, \bar{c}) = \tfrac{1}{2}\sum_{\alpha,\beta}\left[E_{\alpha}(\varphi)(w^{-1})^{\alpha\beta}E_{\beta}(\varphi) - c^{\alpha}E_{\alpha\beta}(\varphi)\bar{c}^{\beta}\right].$$

The BRS transformation of $c$ is now non-linear:

$$\delta_{\mathrm{BRS}} c^{\alpha} = \varepsilon \sum_{\beta} [w^{-1}]^{\alpha\beta} E_{\beta}(\varphi)\,.$$

We note that in this form the BRS transformation has a vanishing square only when $\varphi$ is a solution of the equation $E(\varphi) = 0$. We conclude that the property $\mathcal{D}^2 = 0$ of BRS transformations is not true in all formulations and may be satisfied only after the introduction of some auxiliary variables.

*BRS cohomology.* We have shown that for a general stochastic equation linear in a Gaussian noise, the weight function $\Sigma$ obtained after noise averaging is BRS exact and quadratic in $\{\bar{\varphi}, c\bar{c}\}$. Conversely, one may look for the most general function $\Sigma$ quadratic in $\{\bar{\varphi}, c\bar{c}\}$ and BRS symmetric. BRS cohomology techniques allow showing that in the case of simply connected manifolds, any BRS symmetric function is BRS exact, up to a constant:

$$\mathcal{D}\Sigma = 0 \ \Rightarrow \ \Sigma = \mathcal{D}\widetilde{\Sigma} + \text{const.}\ .$$