The situation before the Higgs discovery at LHC: An RG upper-bound on the Higgs mass

In the Standard Model, the Higgs field through its various couplings gives masses to all fields. The observed masses determine the corresponding couplings. Before the LHC discovery, only the Higgs mass and thus the Higgs self-coupling were unknown parameters. However, it was likely that the renormalized $(\phi^2)^2$ self-coupling g would be small enough so that perturbation theory remained at least semi-quantitatively applicable. Otherwise, the successes of the Standard Model would have been difficult to understand.

In the perturbative regime, the Higgs mass increases with g. To obtain an upper-bound on the Higgs mass one has to examine what happens when g increases. For g large enough, the Higgs mass is mostly determined by the Higgs self-coupling. In the pure $(\phi^2)^2$ field theory and in the perturbative regime, simple RG arguments are applicable.

Any quantum field theory requires eventually a cut-off to cure the unavoidable problem of infinities. Here, the cut-off Λ corresponds to the onset of some new physics beyond the Standard Model.

The renormalized coupling constant g is the effective coupling constant $g_0(\mu/\Lambda)$ at the renormalization scale μ, $\mu \ll \Lambda$. Then,

$$g \sim g_0(\mu/\Lambda), \quad \int_{g_0}^{g_0(\mu/\Lambda)} \frac{\mathrm{d}g'}{\beta(g')} = \ln(\mu/\Lambda).$$

For g_0 small, the perturbative expansion of the β-function is

$$\beta(g_0) = \beta_2 g_0^2 + \beta_3 g_0^3 + O\left(g_0^4\right), \quad 8\pi^2 \beta_2 = 2\,, \beta_3/\beta_2^2 = -13/24\,.$$

For g small, one infers

$$\ln(\Lambda/\mu) = \frac{1}{\beta_2 g} + \frac{\beta_3}{\beta_2^2} \ln g + K(g_0) + O(g),$$

where $K(g_0) = O(1)$ can only be determined by non-perturbative methods.

For g small, perturbation theory to relate the Higgs field expectation value, which is known from the Z mass ($\langle H \rangle \sim 250\,\text{GeV}$), and the Higgs mass. At leading order, one finds

$$m_H^2 = \tfrac{1}{3} g \langle H \rangle^2 + O\left(g^2\right).$$

To minimize higher order corrections, one chooses for g the renormalized coupling constant at scale $\langle \phi \rangle$. One can then eliminate g and finds

$$\ln\left(\frac{\Lambda}{\langle H \rangle}\right) = \frac{1}{3\beta_2} \frac{\langle H \rangle^2}{m_H^2} + \frac{2\beta_3}{\beta_2^2} \ln\left(\frac{m_H}{\langle H \rangle}\right) + \tilde{K}(g_0) + O(g).$$

If one can neglect in the r.h.s. all terms but the two first ones, one obtains a relation between the two ratios $\Lambda / \langle \phi \rangle$ and $m_H / \langle \phi \rangle$. Moreover, if the Higgs is really associated to a physical particle, its mass must be smaller than the cut-off. Taking for the two coefficients of the β-function the values for $O(4)$, $8\pi^2 \beta_2 = 2$, $\beta_3/\beta_2^2 = -13/24$, one obtains the upper-bound

$$m_H < 2.6 \langle \phi \rangle \;\Rightarrow m_H < 640\ \text{GeV}.$$

The value could be compared with computer simulation values, which vary in the range 670–700 GeV. Moreover, the corresponding value of g is such that perturbation theory at leading order should still be semi-quantitatively correct.

Conversely, from the value physical coupling constant at scale μ, one can infer from the equation an upper-bound on the cut-off or scale of new physics. Clearly, this bound is very sensitive to small corrections since the equation determines $\ln(\Lambda/\mu)$. Moreover, for smaller values of m_H the coupling to the quark top and vector bosons should be taken into account.

The conclusion at the time where the construction of a new collider had to be decided was that by exploring physics in the TeV range one would either find the Higgs particle or discover some new physics, or both.

Later, the study of radiative corrections at LEP, even though they varied only like $\ln m_H$, actually suggested that the Higgs mass should be expected to lie between 114 and 200 GeV.

The Gross–Neveu–Yukawa model (GNY): a toy Higgs-top model

Since indications were that the Higgs would not be on the high side, it became necessary to include the couplings to the top quark and vector bosons. Neglecting vector bosons, a semi-quantitative picture can be obtained from a model that represents only the Higgs–top physics, the GNY model.

The model is renormalizable in four dimensions with a simple fermion-boson interaction. In the GNY model, particles still receive masses by spontaneous chiral symmetry breaking. No Goldstone boson is generated because the chiral symmetry is discrete.

In the model, the ratio of fermion and boson masses can then be predicted as a simple consequence of IR freedom and the *natural* assumption that coupling constants have generic values at the cut-off scale.

More generally, the renormalization group flow can be studied as a function of the physical masses when the physical ratio differs from the prediction.

The Gross–Neveu–Yukawa model

The GNY model involves a set of N massless fermions $\{\psi^i, \bar{\psi}^i\}$ and a scalar field H.

It has a discrete chiral $\mathbb{Z}_2$ symmetry under which the fields transform like

$$\boldsymbol{\psi} \mapsto \gamma_5 \boldsymbol{\psi}, \quad \bar{\boldsymbol{\psi}} \mapsto -\bar{\boldsymbol{\psi}}\gamma_5, \quad H \mapsto -H\,, \tag{47}$$

which prevents the addition of a fermion mass term to the action.

The $U(N)$ symmetry is implemented by the transformation

$$\psi \mapsto U\psi\,, \quad \bar{\psi} \mapsto U^\dagger \bar{\psi}\,.$$

The model illustrates the physics of spontaneous chiral symmetry breaking and fermion mass generation.

A renormalizable action then reads (a cut-off Λ, consistent with the symmetries, is implied)

$$\mathcal{S}\left(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, H\right) = \int \mathrm{d}^4 x \left[-\bar{\boldsymbol{\psi}} \cdot \left(\not{\partial} + g_0 H\right) \boldsymbol{\psi} + \tfrac{1}{2}\left(\nabla_x H\right)^2 + \tfrac{1}{2} m_0^2 H^2 + \frac{\lambda_0}{4!} H^4\right].$$

The renormalized action. Calling μ the renormalization scale and g, λ the renormalized couplings, one can write the renormalized action as

$$\mathcal{S}_{\mathrm{r}}(H, \boldsymbol{\psi}, \bar{\boldsymbol{\psi}}) = \int \mathrm{d}^4 x \left\{ -Z_\psi \left[\bar{\boldsymbol{\psi}}(x) \cdot \left(\not{\partial} + g Z_g Z_H^{1/2} H(x) \right) \boldsymbol{\psi}(x) \right] \right.$$
$$\left. + \frac{1}{2} Z_H \left[\left(\nabla_x H(x) \right)^2 + m_0^2 H^2(x) \right] + Z_\lambda \frac{\lambda}{4!} Z_H^2 H^4(x) \right\}. \quad (48)$$

where $Z_\psi, Z_g, Z_H, Z_m Z_\lambda$ are renormalization constants. In what follows we set

$$Z_H m_0^2 = Z_H m_{0c}^2 + Z_m \tau \,,$$

where m_{0c}^2 is defined by the property that for $m_0^2 = m_{0c}^2$ the physical masses of ψ and H vanish. Then, Z_m is a renormalization constant and the new parameter τ, in the language of phase transitions, plays the role of the deviation from the critical temperature.

The phase transition in the tree approximation. In the tree approximation,

$$S_{\text{tree}}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, H) = \int \mathrm{d}^4 x \left[-\bar{\boldsymbol{\psi}} \cdot \not{\partial} \boldsymbol{\psi} - g\, H \bar{\boldsymbol{\psi}} \cdot \boldsymbol{\psi} + \tfrac{1}{2} (\nabla_x H)^2 + \tfrac{1}{2} \tau H^2 + \frac{\lambda}{4!} H^4 \right],$$

a phase transition occurs, for $\tau = 0$, between a fermion massless symmetric phase for $\tau > 0$ and a phase for $\tau < 0$ where the chiral symmetry is spontaneously broken and a fermion mass is generated. The H expectation value

$$\langle H \rangle = \pm \sqrt{-6\tau/\lambda}\,,$$

gives a mass to the fermions by a mechanism reminiscent of the Standard Model of weak-electromagnetic interactions. The fermion and boson masses are then

$$m_\psi = g \langle H \rangle\,, \quad m_H = \sqrt{\frac{\lambda}{3}} \langle H \rangle \quad \Rightarrow \quad \frac{m_H}{m_\psi} = \frac{1}{g} \sqrt{\frac{\lambda}{3}}\,.$$

RG equations: β-functions

Beyond the tree approximation, the model can be discussed, like the $(\phi^2)^2$ theory, by RG techniques. For $|\tau| \ll \Lambda^2$ (this is the usual fine tuning problem), the corresponding renormalized (1PI) vertex functions of l fermion fields and n H-fields satisfy the RG equations

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_{g^2}\frac{\partial}{\partial g^2} + \beta_\lambda\frac{\partial}{\partial\lambda} - \tfrac{1}{2}l\eta_\psi - \tfrac{1}{2}n\eta_H - \eta_m\tau\frac{\partial}{\partial\tau}\right]\Gamma^{(l,n)} = 0\,. \qquad (49)$$

At one-loop order, the RG β-functions have the form

$$\beta_\lambda = \frac{1}{8\pi^2}\left(a\lambda^2 + b\lambda g^2 + cg^4\right), \quad \beta_{g^2} = \frac{d}{8\pi^2}g^4$$

with

$$a = \tfrac{3}{2}, \quad b = 4N, \quad c = -24N, \quad d = 2N+3\,.$$

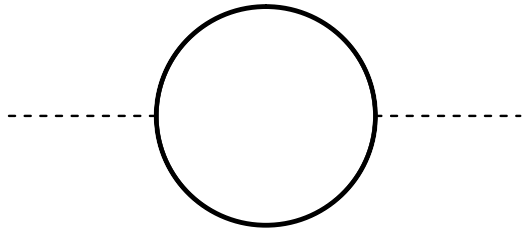

Fig. 6 Boson two-point function: contribution from the fermion loop (the fermions and bosons correspond to continuous and dotted lines, respectively).

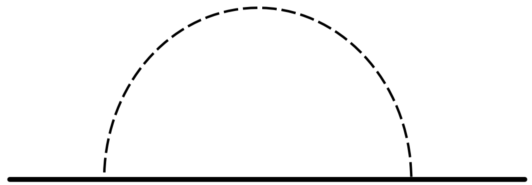

Fig. 7 Fermion two-point function at one-loop.

IR freedom and mass ratio

One easily verifies that the origin $\lambda = g^2 = 0$ is IR stable. The model GNY is thus trivial or IR free, that is, Gaussian up to logarithmic corrections vanishing for large cut-off.

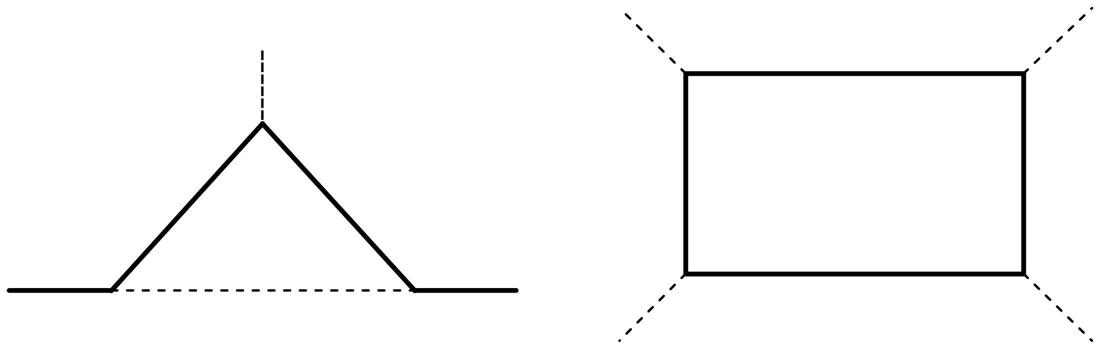

Fig. 8 Three and four-point functions: other divergent one-loop diagrams.

We assume that the dimensionless couplings $\lambda(\Lambda)$ and $g(\Lambda)$ are generic ($i.e.$, of order 1, numerically $8\pi^2$, which is the loop factor) at the cut-off scale Λ. Solving the RG equations, one infers that the coupling constants at a scale $\mu \ll \Lambda$ decrease like

$$g^2(\mu) \sim \frac{8\pi^2}{(2N+3)\ln(\Lambda/\mu)}, \quad \lambda(\mu) \sim \frac{8\pi^2 R_*(N)}{(2N+3)\ln(\Lambda/\mu)}$$

with

$$R_*(N) = \tfrac{1}{3}\left[-(2N-3) + \sqrt{4N^2+132N+9}\right].$$

In particular, choosing $\mu \sim \langle H \rangle$ and if the mass scale $\langle H \rangle \ll \Lambda$, one concludes that the ratio of H and fermion masses goes to the limit

$$\frac{m_H^2}{m_\psi^2} = \frac{\lambda(\langle H \rangle)}{3g^2(\langle H \rangle)} = \tfrac{1}{3} R_*(N) = \frac{1}{9}\left(-(2N-3) + \sqrt{4N^2 + 132N + 9}\right).$$

As a function of N, when N varies from 1 to ∞, the ratio m_H/m_ψ varies from about 1.20 to 2, which correspond to the $\bar{\psi}\psi$ threshold and the large N limit.

IR freedom of the theory and the assumption that the couplings are generic at the cut-off scale imply a fixed ratio between the masses of the top and Higgs particles.

The general renormalization group flow at one-loop

Identifying the boson with the Higgs field and the fermion with the top field, we can put numbers on the vacuum expectation value and couplings:

$$\langle H \rangle = 246.\,\text{Gev}\,, \quad m_\psi = 173.2\,\text{Gev}\,, m_H = 125.\,\text{Gev}\,.$$

Then, $\lambda = 0.775\,, \quad g^2 = 0.496$. The main neglected contributions correspond to Higgs couplings to vector bosons and, therefore, the picture we find can only be semi-quantitative but the analysis is here much simplified and, thus, more transparent.

We now analyse the general RG flow. Two-dimensional flows can be easily studied because, quite generally, RG trajectories can only meet at fixed points, here $g = \lambda = 0$.

One verifies immediately that the lines $g = 0$ and $\lambda = R_* g^2$ are fixed trajectories and thus cannot be crossed. By contrast, the line $\lambda = 0$ can be formally crossed and the RG trajectories then enter an unphysical region.

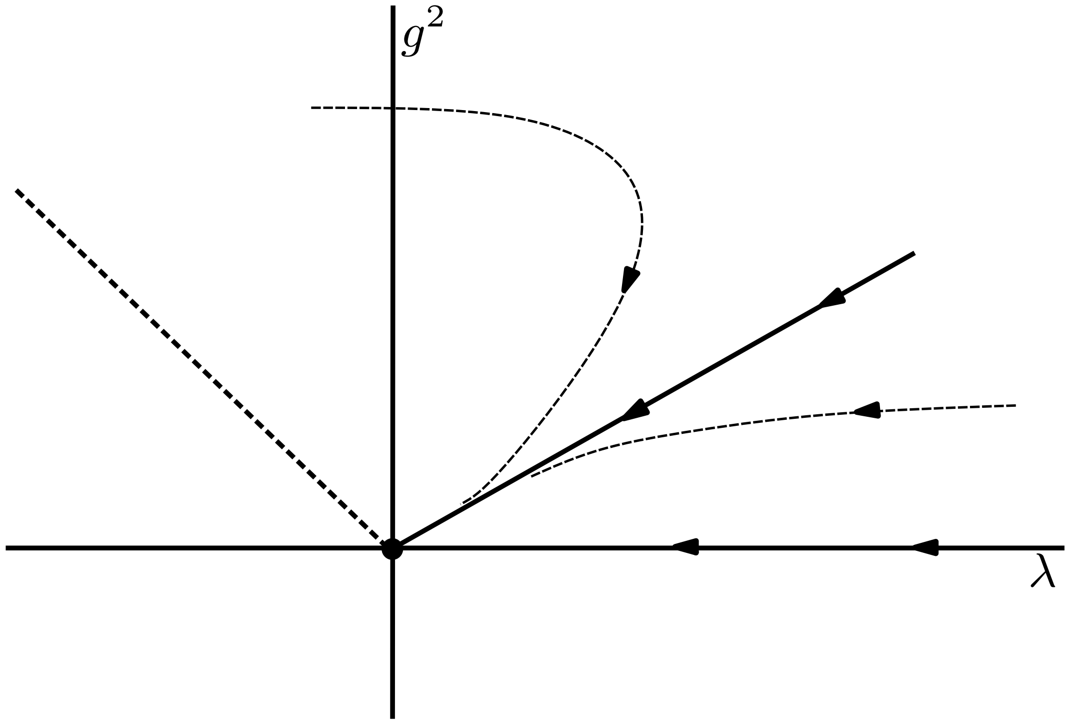

Fig. 9 RG flow: the dotted line on the left is an unphysical fixed line.

First, the coupling constants at physical scale (or renormalized) are small. This justifies using perturbation theory and indicates that IR freedom is relevant since it predicts small renormalized couplings when the initial couplings at a large momentum cut-off scale are of order 1.

By contrast, the ratio $R = \frac{\lambda}{3g^2} \approx 0.52$ is smaller than what is predicted by the model. However, more realistic calculations, including vector bosons, have been performed and seem to indicate that the physical Higgs mass is very close to a fixed line.

Depending on the precise top mass, deviations appear but at a very high energy scale, at least 10^{10} GeV.

This RG result is puzzling since it points toward the possibility that the Standard Model could be valid up to such a high scale. However, then the problem of the fine-tuning of the Higgs mass, which is of the order of $(\Lambda/m_H)^2$ (Λ is the scale of new physics), which we have disregarded up to now, becomes extremely severe.

Exercises

Note that in these lectures we use a Euclidean (or imaginary time) notation, in particular for fermions, $\not\partial = \gamma_\mu \partial_\mu$, ($\mathbf{1}$ is the unit element)

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu} \mathbf{1}\,,$$
$$\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4 \;\Rightarrow\; \gamma_5^2 = \mathbf{1}\,.$$

The modified minimal subtraction scheme ($\overline{\mathrm{MS}}$). In the calculation of low order Feynman diagrams, a factor

$$N_d = \frac{\text{area of the sphere } S_{d-1}}{(2\pi)^d} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}, \tag{50}$$

L being the number of loops of the diagram, is generated naturally. To avoid expanding N_d in $\varepsilon = 4 - d$, it is convenient to rescale the loop expansion parameter to suppress this factor, for instance, by multiplying each Feynman diagram by a factor $(N_4/N_d)^L$, where L is the number of loops.

Exercise 9

Calculation of RG β-functions of the GNY model. Determine the RG functions of the GNY model, in particular, verifying the expressions for the two β-functions, using dimensional regularization and working in the $\overline{MS}$ scheme. This involves a determination of the divergent part at one-loop of various two, three and four point functions.

We give below some elements of the calculation of the divergent parts.

The boson diagrams: One-loop divergences

Figure 4 displays the two one-loop divergent diagrams generated by the σ^4 interaction.

In the massless theory, the contribution (a) to the σ two-point function,

$$\Omega_0 = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{q^2},$$

vanishes in dimensional regularization.

The diagram (b) contributes to the σ four-point function. In the massless limit,

$$B_d(p) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{q^2 (p-q)^2}$$
$$= -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} N_d\, p^{4-d} \equiv N_d\, b(d) p^{-\varepsilon} \tag{51}$$

with

$$b(d) = -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} = \frac{1}{\varepsilon} \left(1 + \tfrac{1}{2}\varepsilon + O\left(\varepsilon^2\right)\right).$$

Then,

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle_{1\mathrm{PI}} \big|_{g=0} = \lambda - \tfrac{1}{2}\lambda^2 \left[B_d(p_{12}) + B_d(p_{13}) + B_d(p_{14})\right] + O(\lambda^3),$$

where $\sigma_i \equiv \sigma(p_i)$ and $p_{ij} \equiv p_i + p_j$. Expanding for $\varepsilon \to 0$, one finds the divergent contribution

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle_{1\mathrm{PI,div.}} \Big|_{g=0} = -\frac{3N_d}{2\varepsilon} \lambda^2. \tag{52}$$

Diagrams involving fermion propagators: One-loop divergences

We evaluate below only the one-loop divergent parts of the additional diagrams involving fermion propagators.

*Boson two-point function.*The diagram of figure 6 has a factor Ng^2, the sign coming from the Legendre transformation cancelling the sign coming from the fermion loop. It is then multiplied by

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d q \frac{\mathrm{tr}\, i\not{q} i\left(\not{q}+\not{p}\right)}{q^2(p+q)^2} = 2p^2 B_d(p) = N_d \frac{2}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon + O\left(\varepsilon^2\right)\right) p^{2-\varepsilon}, \tag{53}$$

where we have used the identity

$$2(q^2+p\cdot q) = (p+q)^2 + q^2 - p^2 .$$

The contributions of the two first terms then vanishes in dimensional regularization. Therefore, the divergent part is

$$\langle \sigma\sigma \rangle_{1\mathrm{PI},1\,\mathrm{loop},\,\mathrm{div.}} = N_d \frac{2N}{\varepsilon} g^2 .$$

Fermion two-point function. Taking into account the sign coming from the Legendre transformation, the diagram of figure 7 has a factor $-g^2$. Moreover, the diagram is proportional to $\not{p}$:

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(p-q)^2}\,\frac{i\not{q}}{q^2} = i\not{p}X(p)$$

and thus multiplying both sides with $\not{p}$ and taking the trace, one infers

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(p-q)^2}\,\frac{i\not{q}}{q^2} = \frac{i\not{p}}{p^2}\,\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q\, p\cdot q}{(p-q)^2 q^2} = \tfrac{1}{2} i\not{p} B_d(p)\,, \tag{54}$$

where the identity $2p\cdot q = p^2 + q^2 - (p-q)^2$ has been used. The divergent part is

$$\langle \bar{\psi}\psi\rangle_{\text{1PI,1 loop, div.}} = -g^2 N_d \frac{i\not{p}}{2\varepsilon}\,.$$

The $\langle\bar\psi\psi\sigma\rangle$ vertex function. Figure 8 displays the remaining two Feynman diagrams. The diagram on the left has a factor $-g^2$ multiplied by

$$\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q}{(p_1-q)^2}\frac{i\not{q}\, i\,(\not{q}-\not{p}_1-\not{p}_2)}{q^2(q-p_1-p_2)^2}.$$

Evaluated at $p_1=-p_2=p$ (zero boson momentum), the diagram reduces to

$$\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q(i\not{q})^2}{q^4(p-q)^2}=-B_d(p)=-\frac{N_d}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon+O\left(\varepsilon^2\right)\right)p^{-\varepsilon}. \qquad (55)$$

Its divergent part is

$$\left\langle\bar\psi\psi\sigma\right\rangle_{1\mathrm{PI},1\,\mathrm{loop},\,\mathrm{div.}}=g^3\frac{N_d}{\varepsilon}\,.$$

The $\langle\sigma\sigma\sigma\sigma\rangle$ vertex function. The diagram in the right of figure 8 has a factor Ng^4, the sign of the Legendre transformation cancelling the sign of the fermion loop. It is multiplied by

$$SQ(p_1,p_2,p_3,p_4)=\frac{1}{(2\pi)^d}\int \mathrm{d}^d q \frac{\operatorname{tr}\left[i\not{q}\, i\left(\not{q}+\not{p}_1\right) i\left(\not{q}+\not{p}_1+\not{p}_2\right) i\left(\not{q}-\not{p}_4\right)\right]}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}$$

to which five diagrams corresponding to permutations of $\{p_2,p_3,p_4\}$ have to be added.

Evaluating the diagram for vanishing opposite momenta, for instance for $p_2=p_4=0$, $p_1=-p_3=p$, one simply finds the contribution

$$\begin{aligned} SQ(p,0,-p,0) &= \frac{1}{(2\pi)^d}\operatorname{tr}\int \mathrm{d}^d q \left(\frac{i\not{q}}{q^2}\right)^2\left(\frac{i(\not{q}+\not{p})}{(p+q)^2}\right)^2 \\ &= 4B_d(p)=\frac{4N_d}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon+O\left(\varepsilon^2\right)\right)p^{-\varepsilon}. \end{aligned} \qquad (56)$$

The total one-loop divergence coming from the six diagrams, to which the contribution (52) has to be added, is then

$$\langle \sigma\sigma\sigma\sigma \rangle_{\text{1PI,1 loop, div.}} = \left(-\tfrac{3}{2}\lambda^2 + 24Ng^4\right)\frac{N_d}{\varepsilon}.$$

Exercise 10

RG equations. Determine the RG flow for the GNY model explicitly by solving the RG equations at one-loop order and discuss the solution. It will be convenient to parametrize the scale parameter as e^t.