Note that the first singular contribution depends only on the coefficient of σ^4 in the expansion of $B(\sigma)$ and on the asymptotic form of the propagator (the Gaussian two-point function) at large distance or small momenta. The short distance modification has only ensured large momentum convergence.

A systematic study then confirms that the most singular terms in each order of the perturbative expansion can be reproduced, in the critical limit, by a statistical field theory with an interaction of σ^4 type, in continuum Euclidean space.

Therefore, if the sum of the most divergent terms to all orders is the leading contribution, then the existence of a continuum limit and some universal properties follow, since then the corresponding field theory depends only on a small number of parameters.

Finally, the consistency of the analysis can again be verified by evaluating the leading corrections.

Exercises moved to the end of next section.

From Gaussian models to Renormalization Group

We have studied Ising type models (but the study can be easily extended to ferromagnetic models with $O(N)$ symmetry) with short range interactions and determined the behaviour of the thermodynamic functions near a continuous phase transition, within the framework of the quasi-Gaussian or mean field approximations.

We have shown that these approximations predict a set of universal properties, that is, properties independent of the detailed structure of interactions or microscopic Hamiltonians, including dimension of space or symmetries.

However, many experimental observations as well as numerical and analytical results coming from model systems show that such results cannot be quantitatively correct, at least in dimensions 2 or 3. For example, the exact solution of the Ising model in two dimensions yields exponents like $\beta = 1/8$, $\eta = 1/4$ or $\nu = 1$, clearly different from the predictions of the quasi-Gaussian approximation.

By examining the leading corrections to the Gaussian approximation, we have identified the origin of the difficulty. Above dimension 4 these corrections do not affect universal quantities; by contrast, below four dimensions, the corrections diverge at the critical temperature and, thus, invalidate the assumptions that are at the basis of the quasi-Gaussian approximation.

The analysis also indicates that the coupling of degrees of freedom corresponding to very different length scales plays an essential role: it is impossible to consider only effective macroscopic degrees of freedom. One could then fear that physics in dimension $d \leq 4$, even at large distance, is sensitive to the detailed microscopic structure of systems. However, surprisingly, some universal properties survive, though different from those of the quasi-Gaussian approximation. But these properties are less universal: statistical systems that have the same properties in the quasi-Gaussian approximation, divide into universality classes characterized by the dimension of space, symmetries and some other qualitative features.

To explain this somewhat paradoxical situation, a completely new tool, initially suggested by Kadanoff (1966), has been developed by Wilson (1971), Wegner..., and then many other physicists, the Renormalization Group (RG) (different in spirit and more general than the earlier RG of quantum field theory). In this approach, the RG is generated by integrating successively over the degrees of freedom corresponding to the shortest scales. One then obtains a sequence of models which all describe the same large distance physics but in which details of the short distance structure are gradually eliminated. If this sequence has a limit, which implies that the RG transformations admit fixed points, then universality properties are explained: all statistical models which, after these repeated transformations, converge toward the same fixed point, belong to the same universality class.

Stated in this general form, the general ideas of RG are extremely suggestive but somewhat vague. The main issue becomes the implementation and this is a non trivial issue.

First, one has to define precisely the way one integrates over short-distance degrees of freedom. Then this general RG remains difficult to set-up because it acts on the infinite dimensional space of possible statistical models. In the simplest implementation, in the continuum, it leads to quadratic functional equations.

Only Gaussian models can be discussed systematically. One can identify the simplest fixed point, the Gaussian fixed point, which belongs to the class of Gaussian models discussed previously.

Moreover, a complete local stability analysis of the fixed point is possible. It allows classifying all perturbations as relevant, that is, which become increasingly important at large distance, irrelevant in the opposite case and marginal in the limiting situation.

More generally, in this framework, the RG of quantum field theory appears as an asymptotic form of the general renormalization when one applies it to the neighbourhood of a Gaussian fixed point.

In the specific context of quantum field theory, the assumptions at the basis of the RG have been clarified. The analysis has confirmed the major relation, initially recognized by Wilson, between the quantum field theory describing the fundamental physics at the microscopic scale and the theory of the macroscopic critical phenomena.

Quantum field theory techniques combined with RG could then be used to discuss phase transitions and to calculate universal quantities, like critical exponents.

The renormalization group: General idea

To construct a RG flow in continuum space, the basic idea is to integrate in the field integral recursively over short distance degrees of freedom. This procedure generates a sequence of effective Hamiltonians $\mathcal{H}_\lambda$ function of a scale parameter $\lambda > 0$ (such that $\mathcal{H}_1 = \mathcal{H}$), related by a transformation $\mathcal{T}$ acting in the space of Hamiltonians such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}_\lambda = \mathcal{T}\left[\mathcal{H}_\lambda\right], \tag{40}$$

a flow equation called RG equation (RGE). The appearance of the derivative $\lambda \mathrm{d}/\mathrm{d}\lambda = \mathrm{d}/\mathrm{d}\ln\lambda$ reflects the multiplicative character of dilatations. The RGE thus defines a dynamical process in the 'time' $\ln\lambda$. The denomination renormalization group (RG) refers to the property that $\ln\lambda$ belongs to the additive group of real numbers.

RG equation: General structure, fixed points

One looks for a RG flow that defines a stationary Markov process, that is, $\mathcal{T}[\mathcal{H}_\lambda]$ depends on $\mathcal{H}_\lambda$ but not on the trajectory that has led from $\mathcal{H}_{\lambda=1}$ to $\mathcal{H}_\lambda$, and depends on λ only through $\mathcal{H}_\lambda$.

Universality is then related to the existence of fixed points, solution of

$$\mathcal{T}(\mathcal{H}^*) = 0\,.$$

One also assumes that the mapping $\mathcal{T}$ is differentiable so that, near a fixed point, the RG flow can be linearized,

$$\mathcal{T}(\mathcal{H}^* + \Delta\mathcal{H}_\lambda) \sim L^* \Delta\mathcal{H}_\lambda\,,$$

and is governed by the eigenvalues and eigenvectors of the linear operator L^*. Formally, the local solution of the linearized equations can be written as

$$\mathcal{H}_\lambda = \mathcal{H}^* + \lambda^{L^*}\left(\mathcal{H}_{\lambda=1} - \mathcal{H}^*\right).$$

Eigenvalues and local stability

Global stability cannot be studied in general, but local stability near the fixed point can be studied by determining the spectrum of L^*. We consider only the situation where the spectrum of L^* is real, a property true for the simple systems we consider here but not been proved in general.

In the example of the random walk, we have already introduced the RG terminology:

(i) Positive eigenvalues correspond to directions of instability and the corresponding eigenoperators are called relevant.

(ii) Vanishing eigenvalues correspond to marginal operators. Then, the linearized flow is in general no longer sufficient to determine whether the perturbation corresponds to a marginally stable or unstable situation. An expansion of the RG flow to second order in the perturbation to the fixed point is required and this leads in general to a logarithmic behaviour.

A special case of vanishing eigenvalues corresponds to redundant operators, which merely correspond to a simple change of parametrization.

(iii) Negative eigenvalues correspond to irrelevant operators and to directions of stability.

When only a small number of relevant operators are present, universal properties can be proved.

The Gaussian fixed point

A RG can be constructed that has the Gaussian model as a fixed point. The Hamiltonian flow can be implemented by the simple scaling

$$\sigma(x) \mapsto \lambda^{(2-d)/2} \sigma(x/\lambda). \tag{41}$$

After the change of variables $x' = x/\lambda$, one verifies that the Hamiltonian

$$\mathcal{H}^*_{\rm G}(\sigma) = \frac{1}{2} \int {\rm d}^d x \big(\nabla_x \sigma(x)\big)^2, \tag{42}$$

corresponding to the critical Gaussian model, is invariant. The RG has $\mathcal{H}^*_{\rm G}$ as a fixed point. The Hamiltonian flow (41) corresponds in fact to the linear approximation of the general RG near the Gaussian fixed point.

The linearized RG flow

The transformation (41) generates the linearized RG flow at the Gaussian fixed point. $\mathbb{Z}_2$ symmetric local eigenvectors of the flow are monomials in σ of the form

$$\mathcal{O}_{n,k}(\sigma) = \int \mathrm{d}^d x\, O_{n,k}(\sigma, x) \text{ with } \mathcal{O}_{n,k}(\theta\sigma) = \theta^{2n}\mathcal{O}_{n,k}(\sigma),$$

$O_{n,k}(\sigma, x)$ being a product of powers of the field and its derivatives at point x with $2k$ powers of ∂_μ.

One then defines the dimension of x as -1 of ∂_μ thus as $+1$ and the (Gaussian) dimension of the field as $[\sigma] = (d-2)/2$. The dimension $[\mathcal{O}_{n,k}]$ of $\mathcal{O}_{n,k}$ is

$$[\mathcal{O}_{n,k}] = -d + n(d-2) + 2k\,. \tag{43}$$

Its RG behaviour under the transformation (41) is then given by a simple dimensional analysis. It can be verified that $\mathcal{O}_{n,k}$ scales like $\lambda^{-[\mathcal{O}_{n,k}]}$, and the corresponding eigenvalue of L^* thus is $\ell_{n,k} = -[\mathcal{O}_{n,k}]$.

Discussion

$\ell_{1,0} = 2$: the corresponding eigenvector $\int \mathrm{d}^d x \, \sigma^2(x)$ is relevant: it induces a deviation from the critical temperature and thus a finite correlation length.

$\ell_{1,1} = 0$: the corresponding perturbation $\int \mathrm{d}^d x \, [\nabla_x \sigma(x)]^2$ is simply a redundant eigenvector since it changes only the normalization of $\sigma(x)$.

$\ell_{2,0} = 4 - d$. For $d > 4$, the corresponding eigenvector $\int \mathrm{d}^d x \, \sigma^4(x)$ is irrelevant and one verifies that no other perturbation is relevant: the Gaussian fixed point is stable on the critical surface ($\xi = \infty$).

At $d = 4$, the eigenvector becomes marginal and below dimension 4 it becomes relevant. In dimension $d = 4 - \varepsilon$, $\varepsilon > 0$ small (a notion we define later), it is the only relevant eigenvector and we will study the RG properties of a Gaussian theory to which this unique term is added.

$\ell_{3,0} = 6 - 2d$: the corresponding operator $\int \mathrm{d}^d x \, \sigma^6(x)$ becomes marginal in $d = 3$ dimension.

Above dimension 2 all other eigenoperators are irrelevant.

To summarize, for systems with a $\mathbb{Z}_2$ or, more generally, with an $O(N)$ symmetry, one concludes that

(i) the Gaussian fixed point is stable above space dimension 4;

(ii) by expanding beyond the linearized local flow, one shows that it is marginally stable in dimension 4;

(iii) it is unstable below dimension 4.

Finally, in dimension 3, if by adjusting some parameter the most relevant operator $\int \mathrm{d}^d x\, \sigma^4(x)$ is suppressed, the flow is governed by the marginal operator $\int \mathrm{d}^d x\, \sigma^6(x)$ and this then corresponds to a tricritical behaviour.

Rescaling and Gaussian renormalization

We now assume that the initial Hamiltonian is very close to the Hamiltonian of the Gaussian fixed point. The RG flow is then initially very close to the local linear flow. Therefore, we first perform the corresponding RG transformation. We introduce a parameter $\Lambda \gg 1$ and substitute

$$\sigma(x) \mapsto \Lambda^{(2-d)/2}\sigma(x/\Lambda).$$

In quantum field theory, this could be called a Gaussian renormalization. After the change of variables $x' = x/\Lambda$, a monomial $\mathcal{O}_{n,k}(\sigma)$ is multiplied by $\Lambda^{-[\mathcal{O}_{n,k}]}$, where $-[\mathcal{O}_{n,k}]$ is the associated Gaussian eigenvalue. The Gaussian renormalization can now be inferred from the dimensions given by Λ: coordinates x have dimension Λ^{-1}, derivatives and momenta dimension Λ and the field dimension $\Lambda^{(d-2)/2}$. The Hamiltonian is dimensionless.

In the context of quantum field theory, since the regularization has then the effect, in the Fourier representation, to suppress field contributions with momenta $|p| \gg \Lambda$ in the perturbative expansion, Λ is also called the cut-off.

Statistical field theory: Perturbative expansion

The Gaussian model in the critical domain

After rescaling, the Hamiltonian of the Gaussian model takes the form

$$\mathcal{H}_{\mathrm{G}}(\sigma) = \frac{1}{2}\int \mathrm{d}^d x \left[\left(\nabla_x \sigma(x)\right)^2 + \alpha_0 \Lambda^2 \sigma^2(x) + \sum_{k=2} \alpha_k \Lambda^{2-2k} \sigma(x) \nabla_x^{2k} \sigma(x) \right],$$

where α_0 is the amplitude of the only relevant term. For $\alpha_0 = 0$, except for the two-point function at coinciding points, one can take the $\Lambda \to \infty$ limit. However, for $\alpha_0 \neq 0$, to obtain a non-trivial universal large distance behaviour, it is also necessary to compensate the effect of the RG flow by choosing α_0 infinitesimal, that is, by taking the $\Lambda \to \infty$ limit at $r = \alpha_0 \Lambda^2$ fixed (a Gaussian mass renormalization in quantum field theory language). This defines the critical domain.

The weakly perturbed or quasi-Gaussian model

To allow for spontaneous $\mathbb{Z}_2$ symmetry breaking and, thus, to be able to describe physics below T_c, terms have necessarily to be added to the Gaussian Hamiltonian to generate a double-well potential for constant fields. The minimal addition, and the leading term near the Gaussian fixed point from the RG viewpoint, is

$$\mathcal{H}_{\mathrm{G}} \mapsto \mathcal{H}(\sigma) = \mathcal{H}_{\mathrm{G}}(\sigma) + \frac{g}{4!}\Lambda^{4-d}\int \mathrm{d}^d x\, \sigma^4(x), \quad g \geq 0\,.$$

The σ^4 term generates a shift of the critical temperature. To recover a critical theory ($T = T_c$), it is necessary to adjust the coefficient of the σ^2 term: $\alpha_0 = (\alpha_0)_c(g)$, a mass renormalization in the quantum field theory terminology, and this defines the critical Hamiltonian $\mathcal{H}_c$.

Dimensions $d > 4$

For $d > 4$, the σ^4 term, as we have shown, is an irrelevant perturbation, as the power of Λ^{4-d} confirms, which does not invalidate the universal predictions of the Gaussian model. Leading corrections to the Gaussian model are obtained by expanding in powers of the coefficient g of the σ^4 term.

In terms of $u = g\Lambda^{4-d}$, the partition function, for example, is given by

$$\mathcal{Z} = \sum_{k=0}^{\infty} \frac{(-u)^k}{(4!)^k k!} \left\langle \left(\int \mathrm{d}^d x \, \sigma^4(x) \right)^k \right\rangle_{\mathrm{G}} .$$

The Gaussian expectations values $\langle \bullet \rangle_{\mathrm{G}}$ can then be evaluated in terms of the Gaussian two-point function with the help of Wick's theorem (Feynman graph expansion).

Dimensions $d < 4$

By contrast, for any $d < 4$, the σ^4 contribution is relevant: the Gaussian fixed point is unstable and no longer governs the large distance behaviour. This reflects in the behaviour of the perturbative expansion of the critical theory ($T = T_c$) in powers of u: it contains so-called infra-red, that is, long distance, or zero momentum in the Fourier representation, divergences.

Renormalization group in dimension $d = 4 - \varepsilon$

For $d < 4$ fixed, the determination of the large distance behaviour of correlation functions requires the construction of a general renormalization group: this leads to functional equations (Wegner, Wilson) that we do not describe here, but which, in general, unfortunately cannot be solved analytically.

However, a trick has been discovered to extend the definition of all terms of the perturbative expansion to arbitrary complex values of the dimension d in the form of meromorphic functions.

This allows s allows studying the neighbourhood of dimension 4, replacing, in dimension $d = 4 - \varepsilon$ and in the framework of a double series expansion in g and ε, the general renormalization group by a much simpler asymptotic form, valid when a non-trivial fixed point is close to the Gaussian fixed point, and studying the model analytically. (Though a numerical method has been developed, based on the field theory RG in the form of Callan–Symanzik equations, that circumvents the problem of the ε-expansion but requires the additional, non-perturbative, assumption that the hierarchy of eigenoperators has not changed.)

Dimensional continuation and regularization

Dimensional continuation. To define dimensional continuation, one introduces the Fourier representation of the two-point function (or propagator) $\Delta(x)$, corresponding to the Hamiltonian of the Gaussian model,

$$\Delta(x) \equiv \langle \sigma(x)\sigma(0)\rangle_{\mathrm{G}} = \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, \mathrm{e}^{-ipx}\tilde{\Delta}(p).$$

A representation of $\tilde{\Delta}(p)$ useful for dimensional continuation then is the Laplace representation (here written for the critical propagator)

$$\tilde{\Delta}(p) = \int_0^\infty \mathrm{d}s\, \rho(s\Lambda^2)\mathrm{e}^{-sp^2}, \tag{44}$$

where $\rho(s) \to 1$ when $s \to \infty$. Moreover, to reduce the field integration to continuous fields and, thus, to render the perturbative expansion finite, one needs for $s \to 0$ at least $\rho(s) = O(s^q)$ with $q > (d-2)/2$.

If, in addition, one wants the expectation values of all local polynomials to be defined, one must impose to $\rho(s)$ to converge to zero faster than any power for $s \to 0$.

A contribution to perturbation theory (represented graphically by a Feynman diagram) takes, in Fourier representation, the form of a product of propagators integrated over a subset of momenta. With the Laplace representation, all momentum integrations become Gaussian and can be performed, resulting in explicit analytic meromorphic functions of the dimension parameter d. This can be illustrated by two simple but useful examples.

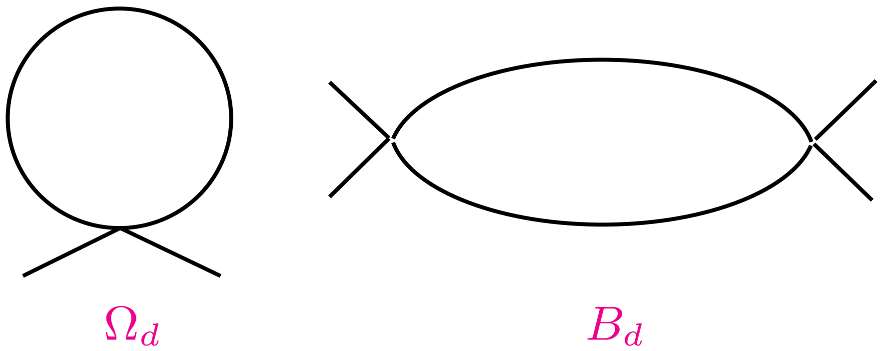

Fig. 4 Two one-loop diagrams.

The contribution of order g to the two-point function (Fig. 4) is proportional to

$$\Omega_d = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, \tilde{\Delta}(k) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \int_0^\infty \mathrm{d}s \, \rho(s\Lambda^2) \mathrm{e}^{-sk^2}$$
$$= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \mathrm{d}s \, s^{-d/2} \rho(s\Lambda^2),$$

which, in the latter form, is holomorphic for $2 < \mathrm{Re}\, d < 2(1+q)$.

In the same way, the contribution of order g^2 to the four-point function (Fig. 4), is proportional to

$$\begin{aligned} B_d(p) &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, \tilde{\Delta}(k)\tilde{\Delta}(p-k) \\ &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d k \int_0^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \rho(s_1\Lambda^2)\rho(s_2\Lambda^2)\mathrm{e}^{-s_1k^2-s_2(p-k)^2} \\ &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty \frac{\mathrm{d}s_1\, \mathrm{d}s_2}{(s_1+s_2)^{d/2}}\rho(s_1\Lambda^2)\rho(s_2\Lambda^2)\, \mathrm{e}^{-p^2 s_1 s_2/(s_1+s_2)}, \end{aligned}$$

which, in the latter form, is holomorphic for $2 < \mathrm{Re}\, d < 4(1+q)$.

For the theory of critical phenomena, dimensional continuation is sufficient since it allows exploring the neighbourhood of dimension four, determining fixed points and calculating universal quantities as $\varepsilon = (4-d)$-expansions.

Dimensional regularization

However, for practical calculations, but then restricted to the leading large distance behaviour, an additional step is extremely useful. It can be verified that if one decreases $\operatorname{Re} d$ enough, so that by naive power counting all momentum integrals are convergent, one can, after explicit dimensional continuation, take the infinite Λ limit. The resulting perturbative contributions become meromorphic functions with poles at dimensions at which large momentum, and low momentum in the critical theory, divergences appear. This method of regularizing large momentum divergences is called dimensional regularization and is extensively used in quantum field theory. In the theory of critical phenomena, it has also been used to calculate universal quantities like critical exponents, as ε-expansions. For example,

$$B_d(p) = -\frac{2\pi\Gamma(d/2)}{(4\pi)^{d/2}\sin(\pi d/2)\Gamma(d-1)}p^{d-4} = \frac{1}{8\pi^2\varepsilon}(1-\varepsilon\ln p) + O(\varepsilon).$$

Perturbative renormalization group: The critical theory

The perturbative renormalization group, as it has been developed in the framework of the perturbative expansion of quantum field theory, relies on the so-called renormalization theory. For the σ^4 field theory it has been first formulated in space dimension $d = 4$. For critical phenomena, a minor extension is required that involves an additional expansion in powers of $\varepsilon = 4 - d$, after dimensional continuation.

We first consider the critical theory ($T = T_c$) corresponding to the Hamiltonian $\mathcal{H}_c(\sigma)$.

To formulate the renormalization theorem, one introduces a momentum scale $\mu \ll \Lambda$, called the renormalization scale, and a parameter g_r characterizing the effective σ^4 coefficient at scale μ, called the renormalized coupling constant.

The renormalization theorem

One can find two dimensionless functions $Z(\Lambda/\mu, g)$ and $Z_g(\Lambda/\mu, g)$ that satisfy (g and Λ/μ are the only two dimensionless combinations)

$$\Lambda^{4-d} g = \mu^{4-d} Z_g(\Lambda/\mu, g) g_{\mathrm{r}} = \mu^{4-d} g_{\mathrm{r}} + O(g^2),\ \ Z(\Lambda/\mu, g) = 1 + O(g),$$

calculable order by order in a double series expansion in powers of g and ε, such that all connected correlations functions

$$\tilde{W}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = Z^{-n/2}(g, \Lambda/\mu) \tilde{W}^{(n)}(p_i; g, \Lambda),$$

called renormalized functions, have, order by order in g_{r} (and ε), finite limits $\tilde{W}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu)$ when $\Lambda \to \infty$ at p_i, μ, g_{r} fixed.

The renormalization constant $Z^{1/2}(\Lambda/\mu, g)$ is the ratio between the full field renormalization in presence of the σ^4 interaction and the Gaussian field renormalization $\Lambda^{(d-2)/2}$.

Universality: a first essential step

There is some arbitrariness in the choice of the renormalization constants Z and Z_g since they can be multiplied by arbitrary functions of $g_{\rm r}$. The renormalization constants can be completely determined by imposing two renormalization conditions to the renormalized correlation functions, which are then independent of the specific form of the short distance regularization.

This leads to a first very important result: since initial and renormalized correlation functions are proportional, they have the same large distance behaviour. This behaviour is thus to a large extent universal since it can depend at most on only one parameter, the σ^4 coefficient g.

Perturbative limit

In addition to the limit $\tilde{W}_{\rm r}^{(n)}(p_i; g_{\rm r}, \mu)$, one defines asymptotic functions $\tilde{W}_{\rm as.}^{(n)}(p_i; g, \Lambda)$ and $Z_{\rm as.}(g, \Lambda/\mu)$ by expanding the perturbative contributions to the functions $\tilde{W}^{(n)}(p_i; g, \Lambda)$ and $Z(g, \Lambda/\mu)$, respectively, for $\Lambda \to \infty$ and keeping, order by order in g and ε, only the terms that do not go to zero.

Critical RG equations

From the relation between initial and renormalized functions and the existence of a limit $\Lambda \to \infty$, a new equation follows, obtained by differentiating the equation with respect to Λ at $\mu, g_{\rm r}$ fixed:

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\rm r},\mu \text{ fixed}} Z^{-n/2}(g,\Lambda/\mu)\tilde{W}^{(n)}(p_i;g,\Lambda) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\rm r},\mu \text{ fixed}} \tilde{W}^{(n)}_{\rm r}(p_i;g_{\rm r},\mu,\Lambda) \to 0\,.$$

Then, introducing the asymptotic functions,

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\rm r},\mu \text{ fixed}} Z^{-n/2}_{\rm as.}(g,\Lambda/\mu)\tilde{W}^{(n)}_{\rm as.}(p_i;g,\Lambda) = 0\,.$$

Using the chain rule, one infers

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g,\Lambda/\mu)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g,\Lambda/\mu)\right]\tilde{W}^{(n)}_{\rm as.}(p_i;g,\Lambda) = 0\,.$$

The functions β and η are defined by

$$\beta(g,\Lambda/\mu)=\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\mathrm{r}},\mu} g\,,\quad \eta(g,\Lambda/\mu)=-\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\mathrm{r}},\mu}\ln Z_{\mathrm{as.}}(g,\Lambda/\mu).$$

Since the functions $\tilde{W}^{(n)}_{\mathrm{as.}}$ do not depend on μ, the functions β and η cannot depend on Λ/μ, and one finally obtains the RG equations (Zinn-Justin 1973):

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\frac{n}{2}\eta(g)\right)\tilde{W}^{(n)}_{\mathrm{as.}}(p_i;g,\Lambda)=0\,. \tag{45}$$

From the relation between g and g_{r}, one immediately infers that $\beta(g)=-\varepsilon g+O(g^2)$.

RG equations in the critical domain above T_c

Correlation functions also exhibit universal properties near T_c when the correlation length ξ is large in the microscopic scale, here, $\xi\Lambda \gg 1$. To describe universal properties in the critical domain above T_c, one adds to the critical Hamiltonian the σ^2 relevant term:

$$\mathcal{H}_\tau(\sigma) = \mathcal{H}_c(\sigma) + \frac{\tau}{2}\int \mathrm{d}^d x\, \sigma^2(x),$$

where $\tau \propto T - T_c \ll \Lambda^2$ characterizes the deviation from the critical temperature. The extended renormalization theorem involves a new renormalization factor $Z_2(\Lambda/\mu, g)$, ratio between the full renormalization of $\int \mathrm{d}^d x\, \sigma^2(x)$ and the Gaussian renormalization. One then derives a more general RGE of the form (Zinn-Justin 1973)

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g) - \eta_2(g)\tau\frac{\partial}{\partial\tau}\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;\tau,g,\Lambda) = 0\,,$$

where a new RG function $\eta_2(g)$, related to $Z_2(\Lambda/\mu, g)$, appears.

These equations can be further generalized to deal with an external field (a magnetic field for magnetic systems) and the corresponding induced field expectation value (magnetization for magnetic systems).

Renormalized RG equations

For $d = 4 - \varepsilon$, if one is only interested in the leading scaling behaviour (and the first correction), it is technically simpler to use dimensional regularization and the renormalized theory in the so-called minimal (or modified minimal) subtraction scheme. The relation between initial and renormalized correlation functions is asymptotically symmetric. One thus derives also (for the critical theory)

$$\left(\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} + \frac{n}{2}\tilde{\eta}(g_{\mathrm{r}})\right)\tilde{W}_{\mathrm{r}}^{(n)}(p_i, g_{\mathrm{r}}, \mu) = 0$$

with the definitions

$$\tilde{\beta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g g_{\mathrm{r}}\,, \quad \tilde{\eta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g \ln Z(g_{\mathrm{r}}, \varepsilon)\,.$$

In this scheme, the renormalization constants are obtained by continuation to low dimensions where the infinite Λ limit, at $g_{\rm r}$ fixed, can be taken. For example,

$$\lim_{\Lambda\to\infty} Z(\Lambda/\mu, g)\big|_{g_{\rm r}\ \text{fixed}} = Z(g_{\rm r}, \varepsilon).$$

Then, order by order in powers of $g_{\rm r}$, they have a Laurent expansion in powers of ε. In the minimal subtraction scheme, the freedom in the choice of renormalization constants is used to reduce the Laurent expansion to the singular terms. For example, $Z(g_{\rm r}, \varepsilon)$ takes the form

$$Z(g_{\rm r}, \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{\sigma_n(g_{\rm r})}{\varepsilon^n} \text{ with } \sigma_n(g_{\rm r}) = O(g_{\rm r}^{n+1}).$$

A remarkable consequence is that the RG functions $\tilde{\eta}(g_{\rm r})$, and $\tilde{\eta}_2(g_{\rm r})$ when a σ^2 term is added, become independent of ε and $\tilde{\beta}(g_{\rm r})$ has the simple dependence $\tilde{\beta}(g_{\rm r}) = -\varepsilon g_{\rm r} + \tilde{\beta}_2(g_{\rm r})$, where $\tilde{\beta}_2(g_{\rm r}) = O(g_{\rm r}^2)$ is also independent of ε.

Solution of the RG equations: The epsilon-expansion

RG equations can be solved by the method of characteristics. In the simplest example of the critical theory, one introduces a scale parameter λ and two functions of $g(\lambda)$ and $\zeta(\lambda)$ defined by

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\big(g(\lambda)\big),\ g(1) = g\,, \quad \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \zeta(\lambda) = -\eta\big(g(\lambda)\big),\ \zeta(1) = 1\,.$$

The function $g(\lambda)$ is the effective amplitude of the σ^4 term at the scale λ. One verifies that the differential RG equation then implies

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda/\lambda\big)\right] = 0\,,$$

from which one infers ($\Lambda \mapsto \lambda\Lambda$)

$$\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g, \lambda\Lambda\big) = \zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda\big).$$

From its definition, one verifies that $\tilde{W}^{(n)}_{\text{as.}}$ has dimension $(d-(d+2)n/2)$. Therefore,

$$\begin{aligned}\tilde{W}^{(n)}_{\text{as.}}(p_i/\lambda; g, \Lambda) &= \lambda^{(d+2)n/2-d}\tilde{W}^{(n)}_{\text{as.}}(p_i; g, \lambda\Lambda) \\ &= \lambda^{(d+2)n/2-d}\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}(p_i; g(\lambda), \Lambda).\end{aligned}$$

These equations show that the general Hamiltonian flow reduces here to the flow of $g(\lambda)$ and, thus, the zeros of the function $\beta(g)$ determine the fixed points that govern the large distance behaviour.

Since

$$\beta(g) = -\varepsilon g + O(g^2),$$

when $\lambda \to \infty$, if $g > 0$ is initially very small, $g(\lambda)$ moves away from the unstable Gaussian fixed point, in agreement with the general RG analysis of the Gaussian fixed point.

If one assumes the existence of another zero $g^* > 0$ with then $\beta'(g^*) > 0$, $g(\lambda)$ will converge toward g^*. If $\eta(g^*) \equiv \eta$ is finite, one finds the universal behaviour

$$\tilde{W}^{(n)}_{\text{as.}}\bigl(p_i/\lambda; g, \Lambda\bigr) \underset{\lambda\to\infty}{\propto} \lambda^{(d+2-\eta)n/2-d} \tilde{W}^{(n)}_{\text{as.}}\bigl(p_i; g^*, \Lambda\bigr).$$

For the connected correlation functions in position space, this result translates into

$$W^{(n)}_{\text{as.}}\bigl(\lambda x_i; g, \Lambda\bigr) \underset{\lambda\to\infty}{\propto} \lambda^{-n(d-2+\eta)/2} W^{(n)}_{\text{as.}}\bigl(x_i; g^*, \Lambda\bigr),$$

for all x_i distinct.

The exponent $d_\sigma = (d-2+\eta)/2$ is the dimension of the field σ, from the point of view of large distance properties.

Explicit calculations: the RG functions at one-loop

The inverse or vertex two-point function. At one-loop order,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \tfrac{1}{2} g \Omega_d + O(g^2),$$

where Ω_d is a constant given by the first diagram of Fig. 4. The critical theory is defined by $\tilde{\Gamma}^{(2)}(0) = 0$ and this determines the critical value of the parameter r at order g:

$$r = r_c(g) \equiv -\tfrac{1}{2} g \Omega_d + O(g^2) \ \Rightarrow\ \tilde{\Gamma}^{(2)}(p) = p^2 + O(g^2).$$

Since β is of order g and $\tilde{\Gamma}^{(2)}$ satisfies the RG equation

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta(g)\right) \tilde{\Gamma}^{(2)}(p; g, \Lambda) = 0 \ \Rightarrow\ \eta(g) = O(g^2)\,.$$

The four-point vertex (or 1PI) function. At one-loop order,

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = \Lambda^{\varepsilon} g - \tfrac{1}{2} g^2 \Lambda^{2\varepsilon} \left[B_d(p_1+p_2) + B_d(p_1+p_3)\right] + B_d(p_1+p_4) + O(g^3),$$

where B_d is the second diagram of Fig. 4:

$$B_d(p) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q\, \tilde{\Delta}(q) \tilde{\Delta}(p-q) \underset{\Lambda\to\infty}{\sim} \frac{1}{(2\pi)^d} \int_{1<|q|<\Lambda} \frac{\mathrm{d}^d q}{q^4} \underset{\Lambda\to\infty}{\sim} \frac{1}{8\pi^2} [\ln\Lambda + O(1)] + O(\varepsilon).$$

Thus, $\quad \tilde{\Gamma}^{(4)} = g + g\varepsilon \ln\Lambda - \dfrac{3g^2}{16\pi^2} \ln\Lambda + O(g^2)\times 1 + O(g^3, g^2\varepsilon).$

The four-point vertex satisfies

$$\left(\Lambda \frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - 2\eta(g)\right) \tilde{\Gamma}^{(4)}(p_i; g, \Lambda) = 0 \;\Rightarrow\; \beta = -\varepsilon g + \frac{3}{16\pi^2} g^2 + \cdots \; .$$

The RG β-function and the IR fixed point. Using the perturbative calculation of the two- and four-point functions at one-loop order, one has thus derived

$$\beta(g) = -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3, \varepsilon g^2).$$

In the sense of an ε-expansion, $\beta(g)$ has a zero g^* with a positive slope (Wilson–Fisher 1972),

$$g^* = \frac{16\pi^2 \varepsilon}{3} + O(\varepsilon^2), \quad \omega = \beta'(g^*) = \varepsilon + O(\varepsilon^2),$$

which governs the large momentum behaviour of correlation functions. Moreover, the exponent ω governs the leading correction to the critical behaviour.

Inserting the expansion of $g^*(\varepsilon)$ in the perturbative expansions of other RG functions, one infers the ε-expansion of other critical exponents or universal functions.

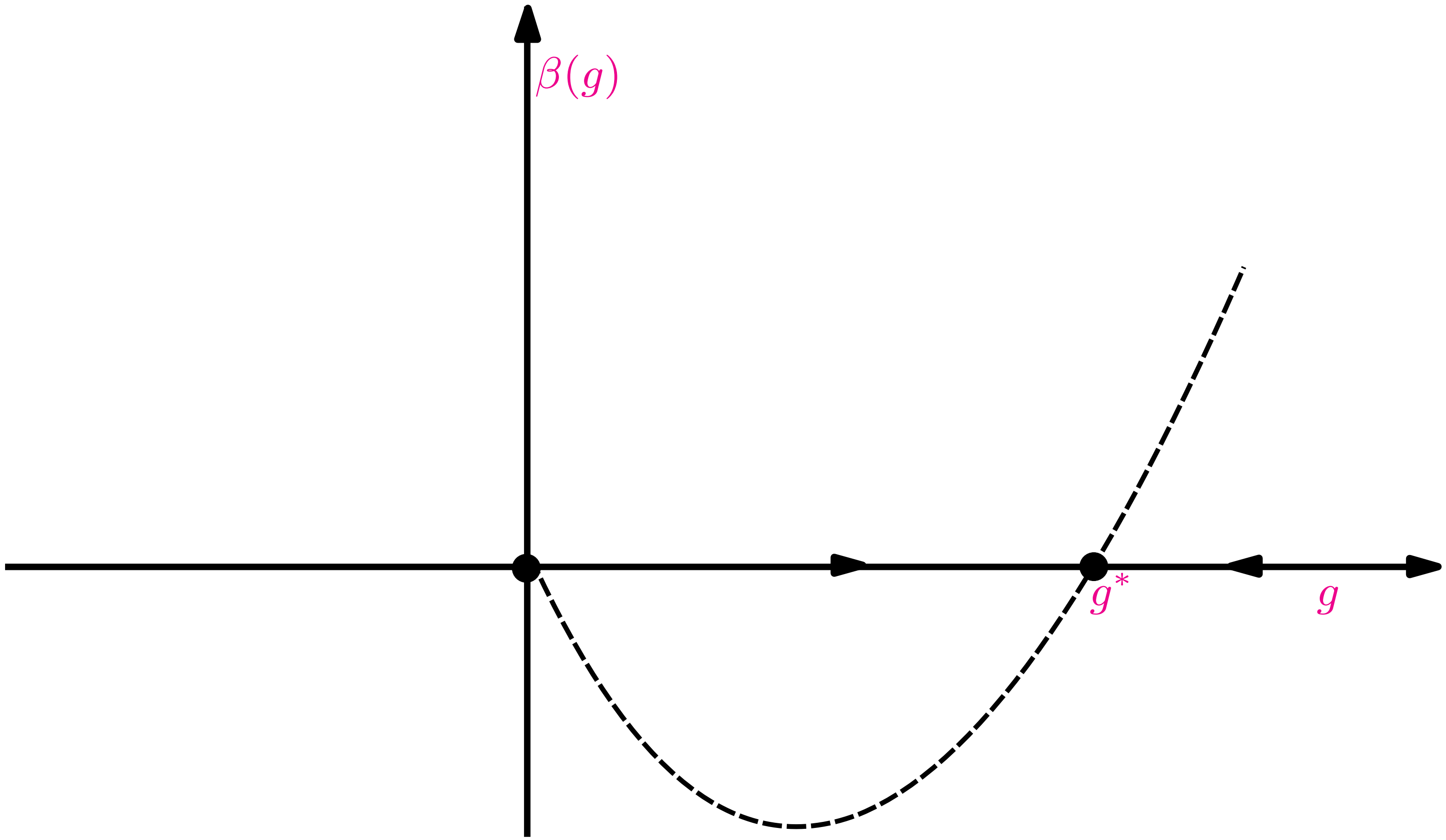

Fig. 5 The RG β-function and RG flow in the $(\boldsymbol{\sigma}^2)^2$ field theory for $d < 4$.

The critical domain. Calculating with a small deviation from criticality, $r = r_c + \tau$, one finds

$$\tilde{\Gamma}^{(2)}(p=0) = \tau + \frac{g\Lambda^{\varepsilon}}{2(2\pi)^d} \int \mathrm{d}^d k \left[\tilde{\Delta}(k,\tau) - \tilde{\Delta}(k,0)\right] + O(g^2).$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial\tau}\tilde{\Gamma}^{(2)}(p=0) &= 1 - \frac{g\Lambda^{\varepsilon}}{2(2\pi)^d} \int \mathrm{d}^d k \, \tilde{\Delta}^2(k,\tau) + O(g^2) \\ &= 1 - \frac{g}{16\pi^2}[\ln(\Lambda/\sqrt{\tau}) + O(1)] + O(g^2, g\varepsilon). \end{aligned}$$

Using an RG equation in the critical domain, which at leading order reduces to

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \eta_2(g)\tau\frac{\partial}{\partial\tau} - \eta_2(g)\right]\frac{\partial}{\partial\tau}\tilde{\Gamma}^{(2)}(p=0) = O(g^2, g\varepsilon)\,,$$

we conclude

$$\eta_2(g) = -\frac{g}{16\pi^2} + O(g^2).$$

Generalization: $O(N)$ symmetric models

The results obtained for models with a $\mathbb{Z}_2$ reflection symmetry can easily be generalized to N-vector models with $O(N)$ orthogonal symmetry, but, in contrast with the Gaussian model prediction, different values of N correspond to different universality classes.

Their universal properties can then be derived from a field theory with an N-component field $\boldsymbol{\sigma}(x)$ and an $O(N)$ symmetric Hamiltonian of the form

$$\mathcal{H}(\boldsymbol{\sigma}) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \boldsymbol{\sigma}(x)\right)^2 + \tfrac{1}{2} r \boldsymbol{\sigma}^2(x) + \frac{g}{4!}\left(\boldsymbol{\sigma}^2(x)\right)^2\right] + \text{higher derivatives.}$$

The RG β-function becomes

$$\beta(g) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O(g^3, \varepsilon g^2) \ \Rightarrow\ g^* = \frac{48\pi^2}{N+8}\varepsilon + O(\varepsilon^2).$$

Moreover, one finds

$$\eta(g) = O(g^2),\ \eta_2(g) = -\frac{(N+2)g}{48\pi^2} + O(g^2).$$

Finally, correlation functions of the $O(N)$ model can be evaluated in the large N limit explicitly and the general predictions of the ε-expansion can then be verified in this limit for all dimensions.

Further generalizations involve theories with N-component fields but with symmetry groups subgroup of $O(N)$, such that only one quadratic invariant but several independent quartic σ^4 terms are allowed. The structure of fixed points may then be more complicate as the discussion in a later section illustrates.

Epsilon-expansion: A few general results

First, from the mere existence of a fixed point and of the corresponding ε-expansion, universal properties of an important class of critical phenomena can be proved to all orders in ε: this includes relations between critical exponents (only two are independent), scaling behaviour of correlation functions or of the equation of state.

The scaling equation of state

The scaling properties of the equation of state of magnetic systems, that is, the relation between applied magnetic field H, magnetization M and temperature T, provide an example of the general results that can be obtained. In the relevant limit $|H| \ll 1$, $|T-T_c| \ll 1$, using RG arguments, one proves Widom's conjectured scaling form

$$H = M^{\delta} f\bigl((T-T_c)/M^{1/\beta}\bigr),$$

where $f(z)$ is a universal function (up to z and f normalizations).

Moreover, the exponents satisfy the relations ($\eta = \eta(g^*)$)

$$\delta = \frac{d+2-\eta}{d-2+\eta}, \quad \beta = \tfrac{1}{2}\nu(d-2+\eta) = \nu d_\sigma\,,$$

where ν, the correlation length exponent, given by $\nu = 1/\big(\eta_2(g^*)+2\big)$, characterizes the divergence ξ of the correlation length at T_c:

$$\xi \propto |T-T_c|^{-\nu}.$$

Other relations can be derived, involving the magnetic susceptibility exponent γ characterizing the divergence of the two-point correlation function at zero momentum at T_c, or the exponent α characterizing the behaviour of the specific heat:

$$\gamma = \nu(2-\eta), \quad \alpha = 2-\nu d\,.$$

Note that the relations involving the dimension d explicitly are not valid for the Gaussian fixed point.

Critical exponents as ε-expansions

Universal quantities can be calculated as ε-expansions. As an illustration, we give here two terms of the expansion of the exponents η, γ and ω for the $O(N)$ symmetric $(\boldsymbol{\sigma}^2)^2$ theory, although the RG functions of the field theory are known to five-loop order and, thus, critical exponents are known up to order ε^5.

In terms of the variable $v = N_d g$ where $N_d = 2/(4\pi)^{d/2}\Gamma(d/2)$ is the usual loop factor, the RG functions $\beta(v)$ and $\eta_2(v)$ at two-loop order, $\eta(v)$ at three-loop order, are

$$\beta(v) = -\varepsilon v + \frac{(N+8)}{6}v^2 - \frac{(3N+14)}{12}v^3 + O(v^4),$$

$$\eta(v) = \frac{(N+2)}{72}v^2\left(1 - \frac{(N+8)}{24}v\right) + O(v^4),$$

$$\eta_2(v) = -\frac{(N+2)}{6}v\left(1 - \frac{5}{12}v\right) + O(v^3).$$

The fixed point value solution of $\beta(v^*) = 0$ is then

$$v^*(\varepsilon) = \frac{6\varepsilon}{(N+8)} \left[1 + \frac{3(3N+14)}{(N+8)^2}\varepsilon\right] + O(\varepsilon^3).$$

The values of the critical exponents

$$\eta = \eta(v^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(v^*)}, \quad \omega = \beta'(v^*),$$

follow

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2} \left[1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon\right] + O(\varepsilon^4),$$

$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + O(\varepsilon^3),$$

$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + O(\varepsilon^3).$$

Though this may not be obvious on these few terms, the ε-expansion is divergent for any $\varepsilon > 0$, as estimates of the large order behaviour of perturbation series based on instanton calculus have demonstrated. For example, adding simply the known successive terms for $\varepsilon = 1$ and $N = 1$ yields

$$\gamma = 1.000\ldots,\ 1.1666\ldots,\ 1.2438\ldots, 1.1948\ldots,\ 1.3384\ldots,\ 0.8918\ldots,$$

the best estimate being provided by summing only up to order ε^2 since the best field theory estimate is $\gamma = 1.2396 \pm 0.0013$.

Divergent series do not define a unique analytic function in general. Extracting more precise estimates from the known terms of the series thus requires an assumption concerning its Borel summability and a practical summation method.

Numerical estimates of exponents from a summation of the ε-expansion

We display below (Table 1) the results for the critical exponents γ, ν, η, β and the correction exponent ω of the $O(N)$ model obtained from a Borel summation of the ε-expansion (Guida and Zinn-Justin 1998), assuming Borel summability. Due to scaling relations like $\gamma = \nu(2-\eta)$, $\gamma + 2\beta = \nu d$, only two among the first four are independent, but the series have been summed independently to check consistency and precision.

We recall that $N = 0$ corresponds to statistical properties of polymers (mathematically the self-avoiding random walk), $N = 1$ to the Ising universality class, which includes liquid-vapour, binary mixtures or anisotropic magnet phase transitions. $N = 2$ describes the superfluid Helium transition, while $N = 3$ corresponds to isotropic ferromagnets.

Table 1

Critical exponents of the $O(N)$ model for $d = 3$, obtained from the ε-expansion.

N	0	1	2	3
γ	1.1571 ± 0.0030	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
ν	0.5878 ± 0.0011	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
η	0.0315 ± 0.0035	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
β	0.3032 ± 0.0014	0.3265 ± 0.0015	0.3465 ± 0.0035	0.3655 ± 0.0035
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018

As a comparison, we also display (Table 2) the best available field theory results obtained from a Borel summation of $d = 3$ renormalized perturbative series (Le Guillou and Zinn-Justin 1980, Guida and Zinn-Justin 1998) based on the Callan–Symanzik (CS) formalism, following an initial suggestion of Parisi (Borel summability has been proven). In the CS formalism, the renormalized vertex functions are defined by the conditions

$$\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p) = m^2 + p^2 + O(p^4),$$
$$\tilde{\Gamma}_{\mathrm{r}}^{(4)}(0,0,0,0) = m^{4-d} g_{\mathrm{r}}\,.$$

The IR stable zero g_{r}^* of the β-function of the renormalized theory then corresponds to the limit $g\Lambda^{4-d} \to \infty$ in the initial theory. Beyond the ε-expansion, at d fixed, the zero has to be determined numerically.

Table 2

Critical exponents of the $O(N)$ model for $d = 3$, obtained from the $(\boldsymbol{\sigma}^2)^2_3$ field theory.

N	0	1	2	3
g_r^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013

Exercises

Critical phenomena in the large N limit. One considers a model involving an N-component field $\boldsymbol{\phi}$ and an $O(N)$ symmetric Hamiltonian of the form

$$\mathcal{S}(\boldsymbol{\phi}, \lambda) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[(\nabla_x \boldsymbol{\phi}(x))^2 + \lambda(x) \boldsymbol{\phi}^2(x) \right] - \frac{3N}{2g} \left(\lambda(x) - r \right)^2 \right\}, \quad (46)$$

where $\lambda(x)$ is an auxiliary scalar field and the integration runs along the imaginary axis, r and $g > 0$ are parameters.

Eventually, the action has to be regularized by introducing a momentum cut-off.

Exercise 5

λ-integration. Eliminate the auxiliary field λ, by performing explicitly the Gaussian integration over the λ, and determine the corresponding $\boldsymbol{\phi}$-field action.

Exercise 6

***ϕ**-integration.* Alternatively, integrate over the N-component field $\boldsymbol{\phi}$ and show that the corresponding effective action has the form (using the general identity $\ln\det = \operatorname{tr}\ln$)

$$\mathcal{S}(\lambda) = N\left\{\tfrac{1}{2}\operatorname{tr}\ln\left[-\nabla_x^2 + \lambda(x)\right] - \frac{3}{2g}\int \mathrm{d}^d x\,(\lambda(x) - r)^2\right\}.$$

Exercise 7

The steepest descent method for large N. In the latter form, it is clear that, for $N \to \infty$, the partition function can be calculated by the steepest descent method. It can be justified that the saddle value of the field $\lambda(x)$ is a constant $\bar{\lambda} = \langle\lambda\rangle = m^2$, where m from the action (46) is the $\boldsymbol{\phi}$-field mass. To determine $\bar{\lambda}$, one only needs the action density for constant field.

Justify the expression

$$\frac{\mathcal{S}(\bar{\lambda})}{N \times \text{volume}} = \frac{1}{2(2\pi)^d} \int \mathrm{d}^d p \, \ln(p^2 + \bar{\lambda}) - \frac{3}{2g}(\bar{\lambda} - r)^2.$$

Differentiate with respect to $\bar{\lambda}$ to obtain the gap equation, which determines the $\boldsymbol{\phi}$-field mass or, correspondingly, the correlation length $\xi = 1/m$. Note that the momentum integral has to be regularized by replacing p^2 by a polynomial $\Lambda^2 K(p^2/\Lambda^2) = p^2 + O(p^4)$.

Discuss the solution as a function of the parameter r and the space dimension d. First, identify the transition temperature r_c. Verify that the solutions of the saddle point equations describe only the domain $T \geq T_c$.

Exercise 8

Partial integration. Repeat the exercise by integrating only over $(N-1)$ components of the field $\boldsymbol{\phi}$. First verify that indeed above T_c, $\bar{\lambda} = m^2$ where m is the physical mass of the N field components.

Below T_c, the remaining field component has a non-zero expectation value. Derive the saddle point equations obtained now by varying both $\bar{\lambda}$ and the expectation value of $\boldsymbol{\phi}$. Calculate the different $\boldsymbol{\phi}$ two-point functions. Discuss.