Taking into account translation invariance, one also defines the Fourier transforms

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \tilde{W}^{(n)}(p_1, \ldots, p_n)$$
$$= \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1, \ldots, x_n) \exp\left(i \sum_{j=1}^{n} x_j p_j\right),$$

where, in analogy with quantum mechanics, the Fourier variables p_i are called momenta.

Thermodynamic potential. One also introduces a generalized thermodynamic potential $\Gamma(M)$, Legendre transform of $\mathcal{W}(H)$ (*cf.* classical Hamiltonian and Lagrangian):

$$\mathcal{W}(H) + \Gamma(M) = \int \mathrm{d}^d x\, H(x) M(x), \quad M(x) = \frac{\delta \mathcal{W}(H)}{\delta H(x)},$$

where $M(x)$ is the local magnetization. Its expansion in powers of M,

$$\Gamma(M) = \sum_n \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n\, M(x_1) \ldots M(x_n) \Gamma^{(n)}(x_1, \ldots, x_n),$$

defines vertex functions $\Gamma^{(n)}$. In Fourier representation, the relations between connected and vertex functions are algebraic. In particular, since

$$\int \mathrm{d}^d x' \Gamma^{(2)}(x, x') W^{(2)}(x', x'') = \delta^{(d)}(x - x'').$$

one finds

$$\tilde{\Gamma}^{(2)}(p) \tilde{W}^{(2)}(p) = 1\,.$$

The Gaussian field theory

In the spirit of the central limit theorem of probabilities, one could expect that phase transitions on large scales can be described by Gaussian or weakly perturbed Gaussian measures, since they result from an averaging over many degrees of freedom.

However, the argument assumes that the degrees of freedom are independent.

This is plausible in the infinite volume limit when the correlation length is finite and the initial microscopic degrees of freedom can be replaced by effective degrees of freedom, local averages over regions of a linear size of the order of the correlation length.

The argument no longer applies at the critical temperature because the correlation length then diverges, and the problem requires a more detailed analysis.

The Gaussian field theory. Let $\sigma(x)$ be a field in d-dimensional continuum space $\mathbb{R}^d$, representing an average local spin, and $H(x)$ an arbitrary local magnetic field.

We consider the Gaussian field integral, or functional integral,

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)] \exp\left[-\mathcal{H}_{\mathrm{G}}(\sigma) + \int \mathrm{d}^d x\, \sigma(x) H(x)\right], \tag{13}$$

where $\mathcal{H}_{\mathrm{G}}(\sigma)$ is the quadratic Hamiltonian

$$\mathcal{H}_{\mathrm{G}}(\sigma) = \frac{1}{2} \int \mathrm{d}^d x \left[\left(\nabla_x \sigma(x)\right)^2 + r\sigma^2(x)\right], \; r \geq 0\,. \tag{14}$$

Comparing with the Gaussian lattice model, one verifies that $r \propto T - T_c$ and one notes that the Gaussian model can describe only the high temperature phase $T \geq T_c$.

In the case of a Gaussian measure, all correlation functions can be expressed in terms of the two-point function with the help of Wick's theorem.

In the framework of quantum field theories that describe the fundamental interactions at microscopic scale, the Gaussian case corresponds to a free field theory. The form (14), quadratic in the fields, is then called Euclidean action, or action in imaginary time, and the parameter

$$m = \sqrt{r}$$

is the mass of the quantum particle associated to the field σ.

As in lattice models, when this seems necessary, we define the infinite volume or thermodynamic limit as the limit of a cube with periodic boundary conditions.

Maximum of the integrand and two-point function

The calculation of a Gaussian field integral is a simple generalization of the calculation of the Gaussian path integral. One first looks for a maximum of the integrand and thus the minimum of

$$\mathcal{H}_{\mathrm{G}}(\sigma, H) = \mathcal{H}_{\mathrm{G}}(\sigma) - \int \mathrm{d}^d x \, \sigma(x) H(x). \tag{15}$$

One sets

$$\sigma(x) = \sigma_c(x) + \varepsilon(x) \tag{16}$$

and expands in ε. The field $\sigma_c(x)$ at the minimum, is determined by the condition that the term linear in ε vanishes:

$$-\int \mathrm{d}^d x \left[\nabla_x \sigma_c(x) \cdot \nabla_x \varepsilon(x) + m^2 \sigma_c(x) \varepsilon(x)\right] + \int \mathrm{d}^d x \, \varepsilon(x) H(x) = 0 \,.$$

One integrates the term linear in $\nabla_x \varepsilon$ by parts. Then, because the integrated terms cancel due to periodic boundary conditions,

$$\int \mathrm{d}^d x \, \nabla_x \sigma_c(x) \cdot \nabla_x \varepsilon(x) = -\int \mathrm{d}^d x \, \varepsilon(x) \nabla_x^2 \sigma_c(x).$$

One finds the equation

$$(-\nabla_x^2 + m^2)\sigma_c(x) = H(x),$$

where ∇_x^2 is the Laplacian in d dimensions. The solution can be written as

$$\sigma_c(x) = \int \mathrm{d}^d x \, \Delta(x - y) H(y),$$

where Δ satisfies ($\delta^{(d)}$ is the Dirac distribution in d dimensions)

$$(-\nabla_x^2 + m^2)\Delta(x) = \delta^{(d)}(x),$$

as one verifies by acting with $-\nabla_x^2 + m^2$ on σ_c.

The equation can be solved by Fourier transformation. In the infinite volume limit, one finds

$$\Delta(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot x}}{k^2 + m^2}\,,$$

as one verifies by acting with $-\nabla_x^2 + m^2$ on $\Delta(x)$ since

$$\int \mathrm{d}^d k \; \mathrm{e}^{-ik\cdot x} = (2\pi)^d \delta^{(d)}(x).$$

After an integration by parts, the Hamiltonian for $\sigma = \sigma_c$ then becomes

$$\begin{aligned} \mathcal{H}_{\mathrm{G}}(\sigma_c, H) &= \int \mathrm{d}^d x \, \sigma_c(x) \left[-\tfrac{1}{2}\nabla_x^2 + \tfrac{1}{2} m^2 - H(x) \right] \sigma_c(x) \\ &= -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, H(x) \Delta(x-y) H(y). \end{aligned} \tag{17}$$

Gaussian integration

One now performs the change of variables $\sigma(x) \mapsto \varepsilon(x) = \sigma(x) - \sigma_c(x)$. The initial Hamiltonian becomes

$$\mathcal{H}_{\mathrm{G}}(\sigma, H) = \mathcal{H}_{\mathrm{G}}(\sigma_c, H) + \mathcal{H}_{\mathrm{G}}(\varepsilon).$$

Thus,

$$\mathcal{Z}(H) = \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\sigma_c, H)} \int [\mathrm{d}\varepsilon(x)]\, \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\varepsilon)}\,.$$

The remaining Gaussian integration over $\varepsilon(x)$ yields a normalization,

$$\mathcal{Z}(0) = \int [\mathrm{d}\varepsilon(x)]\, \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\varepsilon)},$$

independent of H, and that can be explicitly evaluated only by replacing the continuum by a lattice.

The generating functional of connected correlation functions thus is

$$\mathcal{W}(H) = \ln \mathcal{Z}(H) = \mathcal{W}(0) + \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, H(x)\Delta(x-y)H(y).$$

The two-point function is the only connected correlation function.

In a uniform field, the free energy density becomes

$$\begin{aligned} W(H) &= \left(\mathcal{W}(H) - W(0)\right)/\text{volume} \\ &= \tfrac{1}{2}H^2 \int \mathrm{d}^d x\, \Delta(x) = \tfrac{1}{2}H^2 \tilde{\Delta}(0) = \tfrac{1}{2}H^2/m^2. \end{aligned}$$

The equation of state, relation between magnetic field H, magnetization M and temperature is given by

$$M = \frac{\partial W}{\partial H} = \frac{H}{r} \propto \frac{H}{T - T_c}\,.$$

The magnetic susceptibility follows

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{r} \propto \frac{1}{T - T_c}\,.$$

In general, one defines an exponent γ that characterizes the divergence of χ at T_c. Here $\gamma = 1$.

The thermodynamic potential density, Legendre transform of $W(H)$ then is

$$\mathcal{G}(M) = \tfrac{1}{2} m^2 M^2 .$$

Differentiating $\mathcal{W}(H)$ twice with respect to $H(x)$, one verifies that $\Delta(x-y)$ is the Gaussian two-point function:

$$\langle \sigma(x)\sigma(y) \rangle \equiv W^{(2)}(x,y) = \Delta(x-y).$$

It has an Ornstein–Zernike or free field form.

Explicit calculation of the two-point function

First, at T_c,

$$W^{(2)}(x) = \Delta(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot x}}{k^2} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-ik\cdot x} \int_0^\infty \mathrm{d}t \; \mathrm{e}^{-tk^2} \quad .$$

Performing the Gaussian integral over k, one finds

$$\Delta(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t}{t^{d/2}} \mathrm{e}^{-x^2/4t} \, .$$

After the change of variables $u = x^2/4t$, the integration over u yields

$$\Delta(x) = \frac{2^{d-2}}{(4\pi)^{d/2}} \Gamma(d/2 - 1) \frac{1}{x^{d-2}} \, . \tag{18}$$

At T_c, the two-point function is not defined for $d = 2$. For $d > 2$, , one finds an algebraic decay of the two-point function at large distance.

In general one defines

$$W^{(2)}(x) \underset{x\to\infty}{\propto} \frac{1}{x^{d-2+\eta}}.$$

In the Gaussian model the exponent η, which cannot be negative, vanishes.

For the function $1/(k^2+m^2)$ the strategy is the same. One then finds

$$\Delta(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t}{t^{d/2}}\, \mathrm{e}^{-x^2/4t-m^2t} = \frac{2}{(4\pi)^{d/2}} \left(\frac{2m}{x}\right)^{d/2-1} K_{1-d/2}(mx),$$

where $K_\nu(z)$ is a Bessel function of third kind. For $z \to +\infty$, $K_\nu(z)$ can be evaluated by the steepest descent method. One infers

$$\Delta(x) \underset{x\to\infty}{\sim} \frac{1}{2m} \left(\frac{m}{2\pi}\right)^{(d-1)/2} \frac{\mathrm{e}^{-mx}}{x^{(d-1)/2}}\,.$$

The correlation length, which diverges as $\xi = 1/m \propto (T-T_c)^{-\nu}$ with exponent $\nu = \frac{1}{2}$, characterizes the exponential decay of the two-point function.

Class of fields contributing to the field integral. To get an idea of the class of typical fields that contribute to the field integral, one can evaluate the two-point function in the limit of coinciding points:

$$\langle\sigma(\mathbf{x})\sigma(\mathbf{y})\rangle \underset{|\mathbf{x}-\mathbf{y}|\to 0}{\sim} \Delta(\mathbf{x}-\mathbf{y}, m=0) = \frac{2^{d-2}}{(4\pi)^{d/2}}\Gamma(d/2-1)\frac{1}{|\mathbf{x}-\mathbf{y}|^{d-2}}\,.$$

One notices that the fields $\sigma(\mathbf{x})$ contributing to the field integrals are so singular that the expectation value of $\sigma^2(\mathbf{x})$ diverges for $d>1$, and with a rate that increases with the dimension of space d. This singularity of the Gaussian measure is the source of new difficulties.

For $d=2$, the short distance behaviour takes the form

$$\langle\sigma(\mathbf{x})\sigma(\mathbf{y})\rangle \underset{|\mathbf{x}-\mathbf{y}|\to 0}{\sim} -\frac{1}{2\pi}\ln(m|\mathbf{x}-\mathbf{y}|).$$

Quasi-Gaussian or classical approximation

Below the transition point, the Gaussian model is clearly no longer valid since the quadratic form in the Hamiltonian is not positive and thus the Gaussian integral is not defined.

However, even in the framework of the central limit theorem, the Gaussian distribution is only asymptotic. The analysis of the Gaussian model shows that below the transition point, corrections to the Gaussian distribution, that is, terms of higher degree in the effective field distribution, even if their amplitude is small, can no longer be neglected.

Quasi-Gaussian approximation. Since the field integral then is no longer Gaussian, it cannot be calculated exactly. But since the Hamiltonian remains formally analytic, the integral over the fields can be evaluated by the steepest descent method.

Moreover, if one assumes that the fluctuations around the saddle point vary slowly in H, one can neglect the contributions coming from integrating out the fluctuations around the saddle point and approximate the free energy by value of the Hamiltonian at the saddle point, an approximation which one can call quasi-Gaussian or classical.

Such an assumption implies, in particular, that the fields $\sigma(x)$ are the sum of an average value $M(x)$ and a weakly correlated fluctuating part. This assumption goes beyond an idea of central limit theorem in the sense that the average value $M(x)$ is no longer related only to the distribution in each site but also results from the interactions.

One can show that the quasi-Gaussian approximation reproduces, at leading order, the universal results of the lattice model in infinite dimension. However, unlike the model in infinite dimension, it allows also studying the behaviour of correlation functions at the transition.

Effective model. To go beyond the Gaussian model, we thus consider a more general one-site local distribution. In the continuum limit, this corresponds to adding to the Hamiltonian $\mathcal{H}_{\mathrm{G}}$ a local function of the form

$$\int \mathrm{d}^d x \, B\big(\sigma(x)\big)$$

where we choose a function $B(\sigma)$ having the form of the thermodynamic potential of the one-site model and thus convex. Moreover, with our general assumptions, such a function is analytic and we parametrize its expansion at $\sigma = 0$ in the form

$$B(\sigma) = \sum_{p=1} \frac{b_{2p}}{2p!} \, \sigma^{2p} \,, \quad b_2 > 0 \,. \tag{19}$$

We also assume $b_4 > 0$ since we want to study continuous transitions.

The generating function of correlation functions can then be written as

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)] \exp\left[-\mathcal{H}(\sigma) + \int \mathrm{d}^d x\, H(x)\sigma(x)\right], \tag{20}$$

where the Hamiltonian $\mathcal{H}(\sigma)$ takes the form

$$\mathcal{H}(\sigma) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla_x \sigma(x)\bigr)^2 + \tfrac{1}{2} r \sigma^2(x) + B\bigl(\sigma(x)\bigr)\right]. \tag{21}$$

Steepest descent method. The maximum of the integrand in the integral (20) is given by a solution of the saddle point equation

$$\frac{\delta \mathcal{H}}{\delta \sigma(x)} = H(x) \tag{22}$$

and, at leading order,

$$\mathcal{W}(H) = -\mathcal{H}(\sigma) + \int \mathrm{d}^d x \, \sigma(x) H(x),$$

where σ is a function of H through (22).

Together, these equations show that $\mathcal{W}(H)$ is the Legendre transform of $\mathcal{H}(\sigma)$. As a consequence, the thermodynamic potential $\Gamma(M)$, Legendre transform of $\mathcal{W}(H)$, is simply

$$\Gamma(M) = \mathcal{H}(M). \tag{23}$$

In the case of the models invariant under space translations that we study, the magnetization in a uniform field is uniform. The thermodynamic potential density is then

$$\mathcal{G}(M) = \Omega^{-1}\Gamma(M) = \tfrac{1}{2}rM^2 + B(M). \tag{24}$$

The equation of state follows:

$$H = \frac{\partial \mathcal{G}}{\partial M} = rM + B'(M) = M(r + b_2) + \tfrac{1}{6}b_4 M^3 + O(M^5). \tag{25}$$

In zero field, the magnetization is solution of $rM + B'(M) = 0$. The minimum of the thermodynamic potential corresponds to the symmetric solution $M = 0$ for $r > r_c = -b_2$ and a non-vanishing value, the spontaneous magnetization

$$M \sim \pm\sqrt{6(r_c - r)/b_4} \tag{26}$$

for $0 < r_c - r \ll 1$. If one defines in general $M \propto (T_c - T)^{\beta}$, one finds $\beta = \frac{1}{2}$.

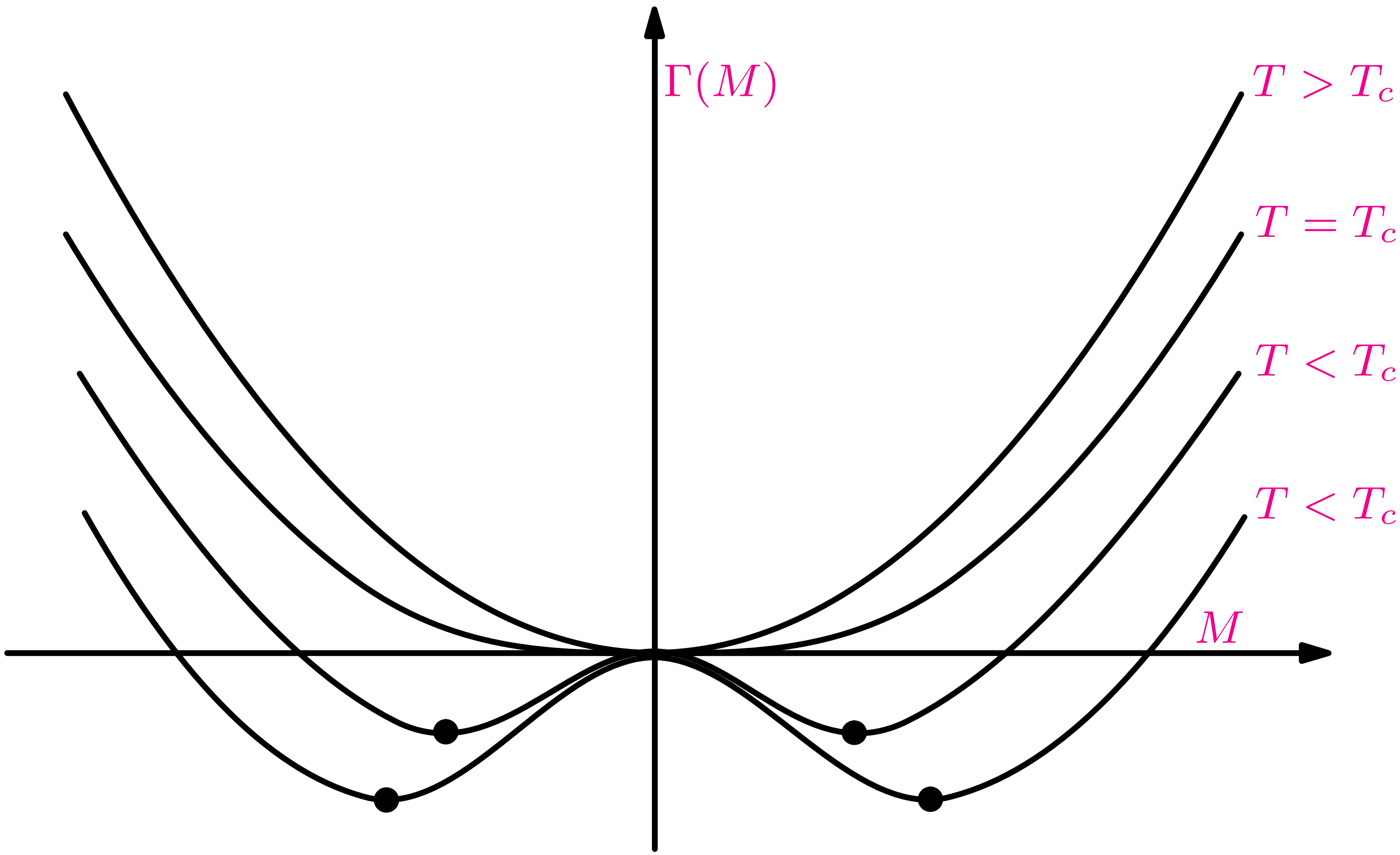

Fig. 2 Thermodynamic potential: second order phase transition.

Quasi-Gaussian approximation: The two-point function

Divergence of the correlation length and continuous transition. A continuous transition is characterized by the property

$$\left.\frac{\partial^2 \mathcal{G}}{(\partial M)^2}\right|_{M=0} = 0 \tag{27}$$

and, thus, by the divergence of the magnetic susceptibility $\chi = \partial^2 W/(\partial H)^2$ in zero field. Moreover,

$$\frac{\partial W(H)}{\partial H} = \left.\frac{\delta \mathcal{W}}{\delta H(x)}\right|_{H(x)=H} .$$

The second derivative is, thus, related to the connected two-point function:

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d y \left.\frac{\delta^2 \mathcal{W}}{\delta H(x)\delta H(y)}\right|_{H(x)=H} = \int \mathrm{d}^d y\, W^{(2)}(x,y).$$

Translation invariance in a uniform field implies

$$W^{(2)}(x,y) = W^{(2)}(x-y).$$

Thus,

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d x\, W^{(2)}(x).$$

We now introduce the Fourier transforms of the connected and vertex functions

$$\widetilde{W}^{(2)}(k) = \int \mathrm{d}^d x\ \mathrm{e}^{ik\cdot x}\, W^{(2)}(x),$$

$$\tilde{\Gamma}^{(2)}(k) = \int \mathrm{d}^d x\ \mathrm{e}^{ik\cdot x}\, \Gamma^{(2)}(x).$$

Then,

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d x\, W^{(2)}(x) = \widetilde{W}^{(2)}(k=0) = 1/\tilde{\Gamma}^{(2)}(k=0).$$

The integral $\int \mathrm{d}^d x\, W^{(2)}(\mathbf{x})$ diverges only if the correlation length diverges. The condition of continuous transition thus implies the divergence of the correlation length for vanishing magnetization.

Other universal properties

Two-point function at fixed magnetization. More generally, from $\Gamma(M) = \mathcal{H}(M)$ one infers the relation between local magnetic field and magnetization

$$H(x) = \frac{\delta \Gamma}{\delta M(x)} = -\nabla_x^2 M(x) + rM(x) + B'\big(M(x)\big). \tag{28}$$

By differentiating again, one obtains the two-point vertex function at fixed magnetization

$$\Gamma^{(2)}(x-y) \equiv \left.\frac{\delta^2 \Gamma}{\delta M(x) \delta M(x)}\right|_{M(x)=M} = [-\nabla_x^2 + r + B''(M)]\delta^{(d)}(x-y).$$

Its Fourier transform is given by

$$\tilde{\Gamma}^{(2)}(k) = k^2 + r + B''(M). \tag{29}$$

The Fourier transform of the connected two-point function follows:

$$\widetilde{W}^{(2)}(k) = 1/\tilde{\Gamma}^{(2)}(k) = \left(k^2 + r + B''(M)\right)^{-1}.$$

In zero field, above T_c, the magnetization vanishes and one recovers the form of the Gaussian model

$$\widetilde{W}^{(2)}(k) = \left(r - r_c + k^2\right)^{-1},$$

where $r_c = -b_2$. If the transition is second order, the expression remains valid up to $r = r_c$ ($T = T_c$) where the correlation length diverges. In particular, one recovers the Gaussian or classical values of the exponents $\eta = 0$, $\nu = 1/2$.

The correlation length above and below T_c. More generally, for $|r - r_c|$, $|k|$, $M \ll 1$ (which also implies a weak magnetic field) one finds

$$\widetilde{W}^{(2)}(k) \sim \left(r - r_c + \tfrac{1}{2} b_4 M^2 + k^2\right)^{-1}. \tag{30}$$

The correlation function keeps an Ornstein–Zernike or free field form. The correlation length for $M \neq 0$ follows:

$$\xi^{-2} = r - r_c + \tfrac{1}{2} b_4 M^2 . \tag{31}$$

In zero magnetic field, using below T_c the expression (26) of the spontaneous magnetization, one finds

$$\xi_+^{-2} = r - r_c \ \text{for} \quad T > T_c \,, \quad \xi_-^{-2} = 2\,(r_c - r) \ \text{for} \quad T < T_c \,. \tag{32}$$

Introducing also quite generally a correlation length exponent ν' for $T \to T_{c-}$, and defining the critical amplitudes $f_\pm$ for $|T - T_c| \to 0$ by

$$\xi_+ \sim f_+ \,(T - T_c)^{-\nu} \,, \quad \xi_- \sim f_- \,(T_c - T)^{-\nu'} \,,$$

one infers the quasi-Gaussian value of the exponent $\nu' = \frac{1}{2}$ and the universal amplitude ratio

$$f_+ / f_- = \sqrt{2} \,.$$

Notice that sometimes the correlation length is defined in terms of the second moment ξ_1^2 of $W^{(2)}(x)$ which is proportional to ξ^2, and thus has the same universal properties

$$\tilde{\Gamma}^{(2)}(k) = \left[\widetilde{W}^{(2)}(k)\right]^{-1} \sim \tilde{\Gamma}^{(2)}(0)\left(1 + k^2\xi_1^2 + O(k^4)\right). \qquad (33)$$

Another universal amplitude. If for $r = r_c$, $H \to 0$, one sets

$$\chi \sim C^c/H^{2/3}, \ \Rightarrow \ 3C^c = (6/b_4)^{1/3}$$

and, in zero field,

$$M \sim M_0(r - r_c)^{1/2} \ \Rightarrow \ M_0^2 = 12/b_4\,.$$

Then, the combination

$$R_\chi = C^+ M_0^2 (3C^c)^{-3} = 1\,,$$

is universal.

Quasi-Gaussian approximation and Landau's theory

The universal results that we have obtained within the framework of the quasi-Gaussian approximation also follow from Landau's theory, which we recall here. Landau's theory is based on general assumptions concerning the properties of systems with short range interactions, of which we have used some to justify the quasi-Gaussian approximation.

We suppose that in zero field the physical system is invariant under space translations. Landau's theory then takes the form of several regularity conditions of the thermodynamic potential as a function of temperature and local magnetization (more generally of a local order parameter):

(i) The thermodynamic potential $\Gamma(M)$, function of the local magnetization $M(\mathbf{x})$ (generated by an inhomogeneous magnetic field), which is also the generating function of vertex functions, is expandable in powers of M at $M = 0$.

(ii) Introducing the Fourier representation of the magnetization field,

$$M(\mathbf{x}) = \int \mathrm{d}^d k \ \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \, \tilde{M}(\mathbf{k}),$$

we expand the thermodynamic potential $\Gamma(M)$ in powers of $\tilde{M}(\mathbf{k})$:

$$\Gamma(M) = \int \mathrm{d}^d x_n \frac{1}{n!} \int \mathrm{d}^d k_1 \dots \mathrm{d}^d k_n \, \tilde{M}(\mathbf{k}_1) \dots \tilde{M}(\mathbf{k}_n)$$
$$\times (2\pi)^d \delta^{(d)} \left(\int \mathrm{d}^d x_i \mathbf{k}_i \right) \tilde{\Gamma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n),$$

where the Dirac $\delta^{(d)}$ functions are the direct consequence of translation invariance which implies that the sum of Fourier variables must vanish.

Then, the vertex functions $\tilde{\Gamma}^{(n)}$, that appear in this expansion, are regular at $\mathbf{k}_i = 0$.

(iii) The coefficients of the expansion are regular functions of the temperature for T near T_c, the temperature at which the coefficient of $\tilde{\Gamma}^{(2)}(\mathbf{k} = 0)$ vanishes.

Finally, the positivity of $\tilde{\Gamma}^{(4)}(0,0,0,0)$ is a necessary condition for the transition to be second order.

These conditions are motivated by some general assumptions: the effective spins are microscopic averages of weakly coupled variables whose fluctuations can be treated perturbatively. They rely also on a decoupling of the various scales of physics, and leads to the conclusion that critical phenomena can be described, at leading order, in terms of a finite number of effective macroscopic variables, as in the mean field approximation.

These remarks render even more puzzling the empirical observation that the universal results of the quasi-Gaussian or mean field approximations are in quantitative disagreement (and sometimes even qualitative) with experimental results and with results, exact or numerical, coming from lattice models. An examination of the leading corrections to the Gaussian theory indicates the origin of this difficulty.

Corrections to the quasi-Gaussian approximation

To describe the low temperature phase, it is necessary to go beyond the Gaussian model. But the quasi-Gaussian approximation is justified only if the steepest descent method is justified. Formally, this condition seems to be satisfied if all the coefficients b_{2p} of the expansion of $B(\sigma)$, except the coefficient b_2 of the quadratic term, are in some sense small.

However, it is also necessary that the unavoidable corrections to the leading order result change only the coefficients of the expansion of the thermodynamic potential, without affecting its regularity properties.

This can be verified by calculating the first correction to the second derivative of the thermodynamic potential density, $\mathcal{G}''(0) = \chi^{-1}$, in the disordered phase above T_c $(r > r_c)$ and in zero field for $r \to r_c$.

The calculation involves two steps: first a determination of the value of r for which $\mathcal{G}''(0)$ vanishes, which yields a non-universal correction to r_c and thus T_c, then a calculation of the leading contribution to $\mathcal{G}''(0)$ for $r \to r_c$.

Perturbative expansion and regularization

To describe physics in the ordered phase below T_c, one needs to perturbe the quadratic Hamiltonian by adding higher power terms to the quadratic potential. Near the transition, the expectation value of the field is small and thus we can make a small field expansion:

$$\mathcal{H}(\sigma) = \mathcal{H}_{\mathrm{G}}(\sigma) + \tfrac{1}{2} b_2 \int \mathrm{d}^d x\, \sigma^2(x) + \frac{b_4}{4!} \int \mathrm{d}^d x\, \sigma^4(x) + \cdots \ .$$

Thermodynamic quantities can then be calculated by expanding in powers of $b_4, b_6 \ldots$.

However, the Gaussian two-point function generated by the Hamiltonian $\mathcal{H}_{\mathrm{G}}$ leads to a first unphysical problem: for $d > 1$ too singular, in particular nowhere continuous, fields contribute to the field integral in such a way that correlation functions at coinciding points are not defined. For example,

$\left\langle \sigma^2(x) \right\rangle = W^{(2)}(0,0) = \dfrac{1}{(2\pi)^d} \displaystyle\int \frac{\mathrm{d}^d p}{p^2 + r}$, diverges for large momenta (reflecting a short distance singularity) in all space dimensions $d \geq 2$.

Regularization. The problem of 'UV' divergences is absent in lattice models due to the lattice structure, as it is absent as well for other statistical systems due to their intrinsic short distance structure. It appears here because we insist on making no reference to a microscopic scale.

Therefore, it is necessary to introduce an artificial short distance structure in the continuum field integral by modifying the Gaussian measure to restrict the field integration to more regular fields, continuous to define expectation values of powers of the field at the same point, satisfying differentiability conditions to define expectation values of the field and its derivatives taken at the same point. This procedure is called regularization. After Fourier transformation, this modification has the effect of decreasing the contribution of field components corresponding to momenta $|p| \gg 1$.

This impossibility to construct a model describing the long distance properties without reference to the short distance structure, is a first evidence of non scale-decoupling.

Regularization can be achieved by adding to $\mathcal{H}_{\mathrm{G}}(\sigma)$ enough terms with more derivatives (this preserves locality):

$$\mathcal{H}_{\mathrm{G}}(\sigma) = \frac{1}{2}\int \mathrm{d}^d x \left[\nabla_x \sigma(x))^2 + r\sigma^2(x)\right] + \frac{1}{2}\sum_{\ell=2}^{\ell_{\max}} \alpha_\ell \int \mathrm{d}^d x\, \sigma(x) \left(-\nabla_x^2\right)^\ell \sigma(x).$$

For example, simple continuity requires $2\ell_{\max} > d$.

The two-point function is then given by

$$\Delta(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{-ipx}\, \tilde{\Delta}(p)$$

with (taking into account the leading term in the expansion of $B(\sigma)$)

$$\tilde{\Delta}^{-1}(p) = r + b_2 + K(p^2) \text{ and } K(p^2) = p^2 + p^4 \sum_{\ell=2}^{\ell_{\max}} \alpha_\ell p^{2\ell-4}.$$

Renormalization group arguments will then be required to prove regularization independence in non-Gaussian theories.

Calculation of the leading correction

In the disordered phase $r > r_c$, in zero field, the magnetization $M = \langle \sigma \rangle$ vanishes and the leading saddle point is simply $\sigma = 0$. The first correction to the steepest descent method then is also the first correction to the Gaussian model.

The corrections to the Gaussian result are obtained by expanding expression (20), separating in the Hamiltonian $\mathcal{H}(\sigma)$ a quadratic part from a remainder called perturbation:

$$\mathcal{H}(\sigma) = \mathcal{H}_{\mathrm{G}}(\sigma) + \int \mathrm{d}^d x \, B\bigl(\sigma(x)\bigr) = \mathcal{H}_0(\sigma) + \int \mathrm{d}^d x \bigl(B(\sigma(x)) - \tfrac{1}{2} b_2 \sigma^2(x)\bigr)$$

with

$$\mathcal{H}_0(\sigma) = \mathcal{H}_{\mathrm{G}}(\sigma) + \tfrac{1}{2} b_2 \int \mathrm{d}^d x \, \sigma^2(x) \, .$$

The second derivative of the thermodynamic potential is also the inverse of the Fourier transform of the connected two-point function, at vanishing argument.

The first correction to the Gaussian form of the two-point function is then given by the contribution of order b_4 (Fig. 3) generated by the quartic term in $B(\sigma)$:

$$B(\sigma) - \tfrac{1}{2} b_2 \sigma^2 = \frac{1}{4!} b_4 \sigma^4 + O(\sigma^6).$$

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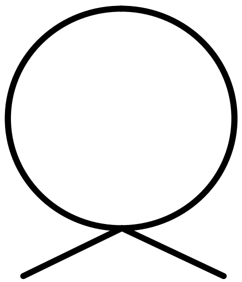

Fig. 3 One-loop contribution to the two-point function.

Moreover, since the magnetization vanishes, the connected two-point function is equal to the complete two-point function. One obtains

$$W^{(2)}(x-y) = \langle \sigma(x)\sigma(y) \rangle = \Delta(x-y) - \frac{b_4}{2} \int \mathrm{d}^d z \Delta(x-z)\Delta(0)\Delta(z-y) + O(b_4^2).$$

The inverse of the connected two-point function (in the sense of kernels) is the vertex function $\Gamma^{(2)}(x-y)$. Here, one finds

$$\Gamma^{(2)}(x-y) = \left[K(-\nabla_x^2) + r + b_2 + \tfrac{1}{2} b_4 \Delta(0)\right] \delta^{(d)}(x-y) + O(b_4^2).$$

In the Fourier representation,

$$\Delta(x=0) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\Delta}(p)$$

with

$$\tilde{\Delta}^{-1}(p) = r + b_2 + K(p^2).$$

The Fourier transform of $\Gamma^{(2)}(x-y)$ is then given by

$$\tilde{\Gamma}^{(2)}(k) = K(k^2) + r + b_2 + \tfrac{1}{2} b_4 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\Delta}(p) + O(b_4^2)\,. \tag{34}$$

We recall that the coefficient of M^2 in the expansion of the thermodynamic potential density $\mathcal{G}(M)$, which is also the inverse of the magnetic susceptibility in zero field, is given by

$$\chi^{-1}(M=0) = \left.\frac{\partial^2 \mathcal{G}}{(\partial M)^2}\right|_{M=0} = \int \mathrm{d}^d x\, \Gamma^{(2)}(x-y) = \tilde{\Gamma}^{(2)}(k=0).$$

The critical behaviour

We infer the expansion

$$\mathcal{G}''(0) = r + b_2 + \tfrac{1}{2} b_4 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\Delta}(p) + O(b_4^2). \tag{35}$$

The criticality condition is now

$$\mathcal{G}''(0) = r + b_2 + \tfrac{1}{2} b_4 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\Delta}(p) + O(b_4^2) = 0\,. \tag{36}$$

The first effect of the correction is to modify the critical value r_c and, thus, the critical temperature. In the term of order b_4, one can replace r_c by $-b_2$, its leading order value, and the equation for r_c becomes

$$0 = r_c + b_2 + \frac{b_4}{2(2\pi)^d} \int \frac{\mathrm{d}^d p}{K(p^2)} + O(b_4^2),$$

Since $K(p^2) \sim p^2$ for $p \to 0$, one verifies again the pathological character of the model in dimension $d = 2$ where the integral diverges at $p = 0$: continuous phase transitions in dimension 2 cannot be described by the Gaussian model and, thus, a perturbed Gaussian model.

We then differentiate $\mathcal{G}''(0)$ with respect to r. At this order we can substitute $b_2 = -r_c$. Thus,

$$\frac{\partial \mathcal{G}''(0)}{\partial r} = 1 - \frac{b_4}{2(2\pi)^d} \int \frac{\mathrm{d}^d p}{[K(p^2) + r - r_c)]^2} + O(b_4^2)\,. \qquad (37)$$

If the integral has a finite limit when $r \to r_c$, the derivative exists at $r = r_c$ and the correction to $\mathcal{G}''(0)$, beyond the Gaussian contribution, remains proportional to $r - r_c \propto T - T_c$:

$$\mathcal{G}''(0) \underset{r \to r_c}{\sim} (r - r_c) \left. \frac{\partial \mathcal{G}''(0)}{\partial r} \right|_{r=r_c} .$$

Then, $\mathcal{G}''(0)$ vanishes linearly at the critical point like $T - T_c$, as in the quasi-Gaussian theory, and only the non-universal coefficient is weakly modified.

One finds

$$\left. \frac{\partial \mathcal{G}''(0)}{\partial r} \right|_{r=r_c} = 1 - \frac{b_4}{2(2\pi)^d} \int \frac{\mathrm{d}^d p}{K^2(p^2)} + O(b_4^2) , \tag{38}$$

where, independently of the regularization,

$$K^2(p^2) \underset{p \to 0}{\sim} p^4 .$$

The role of dimension 4. Since $K^2(p^2)$ for $p \to 0$ behaves like p^4, the integral converges only for $d > 4$. We conclude:

For $d > 4$, the perturbation to the Gaussian theory is small, and modifies only non-universal quantities. The magnetic susceptibility still diverges like $1/(T - T_c)$ and the critical exponent γ keeps its Gaussian value: $\gamma = 1$.

For $2 < d \leq 4$, on the contrary, the integral diverges when $r \to r_c$. Thus, however small the amplitude b_4 of the first correction to the Gaussian distribution is, for $d \leq 4$ when the correlation length ξ diverges the contribution of order b_4 eventually becomes larger than the Gaussian term.

For $d \leq 4$, the perturbative expansion is not valid close to T_c, and the universal predictions of the Gaussian model and the perturbed Gaussian model are inconsistent.

It is instructive to evaluate more precisely the behaviour of the integral when $|r - r_c| \ll 1$ for $d < 4$:

$$\frac{\partial \mathcal{G}''(0)}{\partial r} \sim 1 - \frac{b_4}{2(2\pi)^d} \int \frac{\mathrm{d}^d p}{[r - r_c + K(p^2)]^2} + O(b_4^2)\,.$$

For $d < 4$, the integral converges at infinity. Replacing $K(p^2)$ by p^2 modifies the result only by a negligible constant for $r \to r_c$. After the change of variables $p = p' \sqrt{r - r_c}$, the integral becomes

$$\begin{aligned}\frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{(r - r_c + p^2)^2} &= (r - r_c)^{d/2-2} \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{(1 + p^2)^2} \\ &= \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} (r - r_c)^{d/2-2}.\end{aligned}$$

Integrating over r and introducing the Gaussian correlation length $\xi = 1/\sqrt{r - r_c}$, one infers

$$\mathcal{G}''(0) = \chi^{-1} \underset{\xi \gg 1}{=} (r - r_c) \left[1 + \frac{b_4}{2} \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \xi^{4-d} \right] + O(b_4^2). \tag{39}$$

(Here, the correlation is measured in units of the microscopic scale.) For $d = 4$, the correction has a logarithmic divergence that requires a regularization, but the leading logarithmic term does not depend on the regularization parameter:

$$\mathcal{G}''(0) = \chi^{-1} \underset{\xi \gg 1}{=} (r - r_c) \left(1 - \frac{b_4}{16\pi^2} \ln \xi \right) + O(b_4^2).$$

These expressions relate explicitly the failure of the quasi-Gaussian approximation to the divergence of the correlation length.

To summarize:

i) For dimensions $d > 4$, the correction does not modify the universal predictions of the quasi-Gaussian approximation. One finds some singular corrections but they yield sub-leading contributions.

(ii) For dimensions $d \le 4$, singularities, also called 'infra-red'(IR) a denomination borrowed from quantum field theory, consequences of the large distance behaviour of the Gaussian two-point function, or at vanishing argument of its Fourier transform, imply that the Gaussian predictions cannot be correct in general.

An inspection of higher order corrections confirms these results. For $d \le 4$, the corrections are increasingly singular when the order increases, whereas for $d > 4$ they are less and less singular, which confirms the validity of the first order analysis.

The perturbative terms responsible for this difficulty involve the ratio ξ between the correlation length and the microscopic scale. This gives some indication about the mechanism responsible for the failure of the quasi-Gaussian approximation: physics at the microscopic scale does not decouple from physics at large distance.

Indeed, for $d > 4$, the contribution from arguments $|p| \le \xi^{-1}$ is negligible when ξ diverges, which means that in direct space the degrees of freedom corresponding to distances of the order of the correlation length or larger play a negligible role. On the contrary, for $d \le 4$, at T_c, all scales contribute. This property invalidates ideas based on the central limit theorem, namely that a small number of degrees of freedom with a quasi-Gaussian distribution can replace the infinite number of initial microscopic degrees of freedom.

To solve this problem of coupling between all scales, a new tool has been invented, the renormalization group.

Note that the first singular contribution depends only on the coefficient of σ^4 in the expansion of $B(\sigma)$ and of the asymptotic form of Ornstein–Zernike type of the propagator (the Gaussian two-point function) at large distance or small momenta. The effect of the short distance modification has been limited to ensure large momentum convergence.

A systematic study then confirms that the most singular terms in each order of the perturbative expansion can be reproduced, in the critical limit, by a statistical field theory with an interaction of σ^4 type, in continuum Euclidean space.

As a consequence, if the sum of the most divergent terms to all orders is the leading contribution, then the existence of a continuum limit and some universal properties follow, since then the corresponding field theory depends only on a small number of parameters.

Finally, the consistency of this analysis can again be verified by evaluating the leading corrections.

Exercises

Critical phenomena in the large N limit. One considers a model involving an N-component field $\boldsymbol{\phi}$ and an $O(N)$ symmetric Hamiltonian of the form

$$\mathcal{S}(\boldsymbol{\phi},\lambda)=\int \mathrm{d}^d x\left\{\tfrac{1}{2}\left[(\nabla_x\boldsymbol{\phi}(x))^2+\lambda(x)\boldsymbol{\phi}^2(x)\right]-\frac{3N}{g}\left(\lambda(x)-r\right)^2\right\}, \qquad (40)$$

where $\lambda(x)$ is an auxiliary scalar field, r and $g>0$ parameters.

Eventually, the action has to be regularized by introducing a momentum cut-off.

Exercise 5

λ-integration. Eliminate the auxiliary field λ, by performing explicitly the Gaussian integration over the λ, and determine the corresponding $\boldsymbol{\phi}$ field action.

Exercise 6

ϕ-integration. Alternatively, integrate over the N-component field $\boldsymbol{\phi}$ and show that the corresponding effective action has the form (using the identity $\ln\det = \mathrm{tr}\ln$)

$$\mathcal{S}(\lambda) = N\left\{\tfrac{1}{2}\,\mathrm{tr}\ln\left[-\nabla_x^2 + \lambda(x)\right] - \frac{3}{g}\int \mathrm{d}^d x\,\left(\lambda(x) - r\right)^2\right\}.$$

Exercise 7

The steepest descent method for large N. In the latter form, it is clear that, for $N \to \infty$, the partition function can be calculated by the steepest descent method. It can be justified that the saddle value of the field $\lambda(x)$ is a constant $\bar{\lambda} = \langle\lambda\rangle = m^2$, where m from the action (40) is the $\boldsymbol{\phi}$-field mass. To determine $\bar{\lambda}$, one only needs the action density for constant field.

Justify the expression

$$\frac{\mathcal{S}(\bar{\lambda})}{N \times \text{volume}} = \frac{1}{2(2\pi)^d} \int \mathrm{d}^d p \, \ln(p^2 + \bar{\lambda}) - \frac{3}{g}(\bar{\lambda} - r)^2 .$$

Differentiate with respect to $\bar{\lambda}$ to obtain the gap equation, which determines the $\boldsymbol{\phi}$-field mass or, correspondingly, the correlation length $\xi = 1/m$. Note that the momentum integral has to be regularized by replacing p^2 by a polynomial $K(p^2)$.

Discuss the solution as a function of the parameter r and the space dimension d.

From Gaussian models to Renormalization Group

We have studied Ising type models (but the study can be easily extended to ferromagnetic models with $O(N)$ symmetry) with short range interactions and determined the behaviour of the thermodynamic functions near a continuous phase transition, within the framework of the quasi-Gaussian or mean field approximations.

We have shown that these approximations predict a set of universal properties, that is, properties independent of the detailed structure of interactions or microscopic Hamiltonians, including dimension of space or symmetries.

However, many experimental observations as well as numerical and analytical results coming from model systems show that such results cannot be quantitatively correct, at least in dimensions 2 or 3. For example, the exact solution of the Ising model in two dimensions yields exponents like $\beta = 1/8$, $\eta = 1/4$ or $\nu = 1$, clearly different from the predictions of the quasi-Gaussian approximation.