RENORMALIZATION GROUP: AN INTRODUCTION

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The renormalization group has played a crucial role in 20th century physics in two apparently unrelated domains: the theory of fundamental interactions at the microscopic scale and the theory of continuous macroscopic phase transitions.

In the former framework, it emerged as a consequence of the necessity of renormalization to cancel infinities that appear in a straightforward interpretation of quantum field theory, and of the freedom of then defining the parameters of the renormalized theory at different momentum scales.

In the statistical physics of phase transitions, a more general renormalization group, based on a recursive averaging over short distance degrees of freedom, was later introduced to explain the universal properties of continuous phase transitions.

The renormalization group of quantum field theory can now be understood as the asymptotic form of the general renormalization group in some neighbourhood of the Gaussian fixed point.

In these lectures, we first illustrate the notion of universality with the elementary example of random walk, directly linked to the central limit theorem of probabilities. We revisit the problem with renormalization group (RG) inspired methods, introducing in this way the RG terminology. We recover that the large scale behaviour is universal and defines a continuum limit that can be described by a path integral.

Then, we argue that similarly large distance properties of statistical models near a continuous phase transition can be described by statistical field theories. We explain the perturbative renormalization group. We review a few important applications like the proof of scaling laws and the determination of singularities of thermodynamic functions at the transition.

We apply an RG analysis to a toy model of top quark-Higgs boson interaction. We introduce the RG equations relevant to critical dynamics.

Finally, if time is left, we briefly describe the application of RG ideas to properties of random matrices in the large size limit.

For an elementary introduction to the renormalization group in the spirit of these lectures, *cf.*, for example,

J. Zinn-Justin, *Phase transitions and renormalization group*, Oxford Univ. Press (Oxford 2007),

initially published in French *Transitions de phase et groupe de renormalisation.* EDP Sciences/CNRS Editions, Les Ulis 2005,

including the functional renormalization group in Chapter 16.

In www.scholarpedia.org, see

Jean Zinn-Justin (2010), *Critical Phenomena: field theoretical approach*, Scholarpedia, 5(5):8346.

More advanced material can be found in

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002),

including critical dynamics in Chapter 36.

Universality and continuum limit: Elementary example

The universality of the large scale behaviour and, correspondingly, the existence of a macroscopic continuum limit, emerge as a collective property of systems involving a large number of random variables, provided the probability of large deviations with respect to the average decreases fast enough.

These properties, as well as the importance of Gaussian distributions, are illustrated here by the simple example of the random walk.

The renormalization group viewpoint. Inspired by renormalization group (RG) ideas, we introduce transformations, acting on the transition probability, which decrease the number of random steps. We show that Gaussian distributions are attractive fixed points for these transformations. Universal scaling properties and a continuum limit follow.

The properties of the continuum limit can then be described by a path integral.

Translation invariant local random walk

We consider a stochastic process, a random walk, in discrete times, first on the real axis and then, briefly, on the lattice of points with integer coordinates.

The random walk is specified by a probability distribution $P_0(q)$ (q being a position) at initial time $n = 0$ and a density of transition probability $\rho(q, q') \geq 0$ from the point q' to the point q, which we assume independent of the (integer) time n.

These conditions define a Markov chain, a Markovian process, in the sense that the displacement at time n depends only on the position at time n, but not on the positions at prior times, homogeneous or stationary, that is, invariant under time translation, up to the boundary condition.

Walk in continuum space

The probability distribution $P_n(q)$ for a walker to be at point q at time n satisfies the evolution equation

$$P_{n+1}(q) = \int \mathrm{d}q' \, \rho(q, q') P_n(q') \, .$$

Probability conservation implies

$$\int \mathrm{d}q \, \rho(q, q') = 1 \, . \tag{1}$$

To slightly simplify the analysis, we take as initial distribution

$$P_0(q) = \delta(q).$$

where δ is Dirac's distribution.

An iteration of the flow equation yields

$$P_n(q) = \int \mathrm{d}q' \mathrm{d}q_1 \, \mathrm{d}q_2 \, \ldots \mathrm{d}q_{n-1} \, \rho(q, q_{n-1}) \ldots \rho(q_2, q_1) \rho(q_1, q') P_0(q').$$

Translation symmetry. We have already assumed ρ independent of n and, thus, the random walk transition probability is invariant under time translation. In addition, we now assume that the transition probability is also invariant under space translation and, thus,

$$\rho(q, q') \equiv \rho(q - q').$$

As a consequence, the evolution equation takes the form of the convolution equation,

$$P_{n+1}(q) = \int \mathrm{d}q' \, \rho(q - q') P_n(q'),$$

which also appears in the discussion of the central limit theorem of probabilities.

Locality. We consider only transition functions piecewise differentiable and with bounded variation, and satisfying a property of locality in the form of an exponential decay: qualitatively, large displacements have a very small probability. More precisely, we assume that the transition probabilities $\rho(q)$ satisfy a bound of the exponential type,

$$\rho(q) \le M\, \mathrm{e}^{-A|q|}, \quad M, A > 0\,.$$

Fourier representation

The evolution equation is an equation that simplifies after Fourier transformation. We thus introduce

$$\tilde{P}_n(k) = \int \mathrm{d}q\; \mathrm{e}^{-ikq}\, P_n(q)\,,$$

which is a generating function of the moments of the distribution $P_n(q)$.

The reality of $P_n(q)$ and the normalization of the total probability imply

$$\tilde{P}_n^*(k) = \tilde{P}_n(-k), \quad \tilde{P}_n(k=0) = 1\,.$$

Similarly, we introduce

$$\tilde{\rho}(k) = \int \mathrm{d}q \; \mathrm{e}^{-ikq}\,\rho(q),$$

which is a generating function of the moments of the distribution $\rho(q)$. Finally, the exponential decay condition implies that the function $\tilde{\rho}(k)$ is a function analytic in the strip $|\,\mathrm{Im}\,k| < A$.

The evolution equation then becomes

$$\tilde{P}_{n+1}(k) = \tilde{\rho}(k)\tilde{P}_n(k)$$

and since with our choice of initial conditions $\tilde{P}_0(k) = 1$,

$$\tilde{P}_n(k) = \tilde{\rho}^n(k).$$

Generating function of cumulants

We also introduce the function

$$w(k) = \ln \tilde{\rho}(k) \Rightarrow \; w^*(k) = w(-k), \quad w(0) = 0\,, \tag{2}$$

a generating function of the cumulants of $\rho(q)$. The regularity of $\tilde{\rho}(k)$ and the condition $\tilde{\rho}(0) = 1$ imply that $w(k)$ has a regular expansion at $k = 0$ of the form

$$w(k) = -iw_1 k - \frac{1}{2} w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r\,,\; w_2 > 0\,,$$

where w_r is the rth cumulant. Then,

$$\tilde{P}_n(k) = \mathrm{e}^{nw(k)}\,.$$

Random walk: Asymptotic behaviour from a direct calculation

With the hypotheses satisfied by P_0 and ρ, the determination of the asymptotic behaviour for $n \to \infty$ follows from arguments identical to those leading to the central limit theorem of probabilities. One finds the asymptotic behaviour

$$P_n(q) \sim \frac{1}{\sqrt{2\pi n w_2}} \, \mathrm{e}^{-(q-nw_1)^2/2nw_2} \, .$$

At q fixed, the probability converges exponentially to zero for all $w_1 \neq 0$.

By contrast, the random variable $s = q/n$ has the asymptotic distribution

$$R_n(s) = nP_n(ns) \underset{n\to\infty}{\sim} \sqrt{\frac{n}{2\pi w_2}} \, \mathrm{e}^{-n(s-w_1)^2/2w_2} \, . \tag{3}$$

The average value s is thus a random variable that converges with probability 1 toward the expectation value $\langle s \rangle = w_1$ (the mean velocity).

Finally, the random variable that characterizes the deviation with respect to the mean trajectory,

$$X = (s - w_1)\sqrt{n} = \frac{q}{\sqrt{n}} - w_1\sqrt{n}\,, \tag{4}$$

and thus $\langle X \rangle = 0$, has, as limiting distribution, the universal Gaussian distribution

$$L_n(X) = \sqrt{n} P_n(n w_1 + X\sqrt{n}) \sim \frac{1}{\sqrt{2\pi w_2}}\, \mathrm{e}^{-X^2/2w_2}\,.$$

The neglected terms are of two types, multiplicative corrections of order $1/\sqrt{n}$ and additive corrections decreasing exponentially with n.

The result implies that the mean deviation from the mean trajectory increases as the square root of time, a characteristic property of the Brownian motion.

Continuum time limit

The asymptotic Gaussian distribution of the deviation $\bar{q} = q - nw_1$ from the mean trajectory is

$$P_n(\bar{q}) \sim \frac{1}{\sqrt{2\pi n w_2}} \, \mathrm{e}^{-\bar{q}^2/2nw_2} \, .$$

By changing the time scale and by a continuous interpolation, one can define a diffusion process or Brownian motion in continuous time.

Let t and ε be two real positive numbers and n the integer part of t/ε:

$$n = [t/\varepsilon] \, . \tag{5}$$

One then takes the limit $\varepsilon \to 0$ at t fixed and thus $n \to \infty$.

If the time t is measured with a finite precision Δt, as soon as $\Delta t \gg \varepsilon$, time can be considered as a continuous variable for what concerns all expectation values continuous functions of time.

One then performs the change of distance scale

$$\bar{q} = x/\sqrt{\varepsilon}.$$

Since the Gaussian function is continuous, the limiting distribution takes the form

$$P_n(q)/\sqrt{\varepsilon} \underset{\varepsilon\to 0}{\sim} \Pi(t,x) = \frac{1}{\sqrt{2\pi t w_2}}\, \mathrm{e}^{-x^2/2tw_2}\,. \tag{6}$$

(The change of variables $q \mapsto x$ implies a change of normalization of the distribution.) This distribution is a solution of a diffusion or heat equation:

$$\frac{\partial}{\partial t}\Pi(t,\mathbf{x}) = \tfrac{1}{2} w_2 \frac{\partial^2}{(\partial x)^2}\Pi(t,x).$$

In the limit $n \to \infty$ and in suitable macroscopic variables, one thus obtains a diffusion process that can entirely be described in continuum time and space.

The limiting distribution $\Pi(t, x)$ implies a scaling property characteristic of the Brownian motion. The moments of the distribution satisfy

$$\langle x^{2m} \rangle = \int \mathrm{d}x \, x^{2m} \Pi(t, x) \propto t^m. \tag{7}$$

The variable $x/\sqrt{t}$ has time-independent moments. As the change $\bar{q} = x/\sqrt{\varepsilon}$ also indicates, one can thus assign to the position x a dimension $1/2$ in time unit (this also corresponds to assign a Hausdorff dimension two to a Brownian trajectory in higher dimensions).

Corrections to continuum limit

One can also study how perturbations to the limiting Gaussian distribution decrease with ε.

We can express the distribution of q in terms of $w(k) = \ln \tilde{\rho}(k)$. Correspondingly, we introduce $\bar{w}(k)$ the equivalent quantity for $\bar{q} = q - nw_1$. We then obtain the relation

$$\int \mathrm{d}k \ \mathrm{e}^{ik\bar{q}+iknw_1} \, \mathrm{e}^{nw(k)} = \int \mathrm{d}k \ \mathrm{e}^{ik\bar{q}+n\bar{w}(k)}$$

with

$$\bar{w}(k) = w(k) + ikw_1 \, .$$

With our assumptions, the expansion of the regular function $\bar{w}(k)$ in powers of k reads

$$\bar{w}(k) = -\frac{1}{2} w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r \, .$$

After the introduction of macroscopic variables, which for the Fourier variables corresponds to $k = \kappa\sqrt{\varepsilon}$, one finds

$$n\bar{w}(k) = t\omega(\kappa) \text{ with } \omega(\kappa) = -\frac{w_2}{2!}\kappa^2 + \sum_{r=3} \varepsilon^{r/2-1} \frac{(-i)^r}{r!} w_r \kappa^r \,.$$

One observes that, when $\varepsilon = t/n$ goes to zero, each additional power of κ goes with an additional power of $\sqrt{\varepsilon}$.

In the continuum limit, the distribution becomes

$$\Pi(t,x) = \frac{1}{2\pi} \int \mathrm{d}\kappa \, \mathrm{e}^{-i\kappa x} \, \mathrm{e}^{tw(\kappa)} \,.$$

Differentiating with respect to the time t, one obtains

$$\frac{\partial}{\partial t}\Pi(t,x) = \frac{1}{2\pi} \int \mathrm{d}\kappa \, w(\kappa) \, \mathrm{e}^{-i\kappa x} \, \mathrm{e}^{tw(\kappa)}$$

and in $w(\kappa)$, κ can then be replaced by the differential operator $i\partial/\partial x$.

One thus finds that $\Pi(t,x)$ satisfies the linear ‘partial differential equation’

$$\frac{\partial}{\partial t}\Pi(t,x) = \left[\frac{w_2}{2!}\left(\frac{\partial}{\partial x}\right)^2 + \sum_{r=3} \varepsilon^{r/2-1}\frac{1}{r!}w_r\left(\frac{\partial}{\partial x}\right)^r\right]\Pi(t,x).$$

In the expansion, each additional derivative implies an additional factor $\sqrt{\varepsilon}$ and, thus, the contributions that contain more derivatives decrease faster to zero.

Universality and fixed points of transformations

We now explain the universality property, that is, the existence of a limiting Gaussian distribution that is independent of the initial distribution, and its scaling properties by a quite different method, that is, without calculating the asymptotic distribution explicitly.

For simplicity, we assume that initially the number of time steps is of the form $n = 2^m$.

The idea then is to recursively combine the time steps two by two, decreasing the number of steps by a factor two at each iteration. This provides a very simple application of RG ideas to the derivation of universality properties.

At each iteration one thus replaces $\rho(q-q')$ by

$$[\mathcal{T}\rho](q-q') \equiv \int \mathrm{d}q'' \rho(q-q'')\rho(q''-q').$$

The transformation of the distribution $\rho(q)$ is non-linear but applied to the function $w(k)$, it becomes the linear transformation

$$[\mathcal{T}w](k) \equiv 2w(k).$$

This transformation has an important property: it is independent of m or n. In the language of dynamical systems, its repeated application generates a stationary, or invariant under time translation, Markovian dynamics.

We now study the properties of the iterated transformation $\mathcal{T}^m$ for $m \to \infty$. A limiting distribution necessarily is a fixed point of the transformation. Thus, it corresponds to a function $w_*(k)$ (where the notation $_*$ is not related to complex conjugation) that satisfies

$$[\mathcal{T}w_*](k) \equiv 2w_*(k) = w_*(k).$$

Expanding in powers of k, one verifies that such a transformation has, with our assumptions, only the trivial fixed point $w_*(k) \equiv 0$.

But a larger class of fixed points becomes available if the transformation is combined with a renormalization of the distance scale, $q \mapsto \lambda q$, with $\lambda > 0$. We thus consider the transformation

$$[\mathcal{T}_\lambda w](k) \equiv 2w(k/\lambda)\,.$$

The transformation $\mathcal{T}_\lambda$ provides a simple example of a RG transformation, a concept that we describe in more detail in the framework of phase transitions.

Generic situation

The fixed point equation then becomes

$$[\mathcal{T}_\lambda w_*](k) \equiv 2w_*(k/\lambda) = w_*(k).$$

For the class of fast decreasing distributions, the functions $w(k)$ are regular at $k = 0$.

Thus, $w_*(k)$ has an expansion in powers of k of the form $(w(0)=0)$

$$w_*(k) = -iw_1 k - \tfrac{1}{2} w_2 k^2 + \sum_{\ell=3} \frac{(-i)^\ell}{\ell!} w_\ell k^\ell \,, \ w_2 > 0 \,.$$

In the generic situation $w_1 \neq 0$. Expanding the equation, at order k one finds

$$2w_1/\lambda = w_1 \ \Rightarrow \ \lambda = 2 \,.$$

Then, identifying the terms of higher degree, one obtains

$$2^{1-\ell} w_\ell = w_\ell \ \Rightarrow \ w_\ell = 0 \text{ for } \ell > 1 \,.$$

Therefore,

$$w_*(k) = -iw_1 k \,.$$

The fixed points form a one-parameter family, but the parameter w_1 can be absorbed into a normalization of the random variable q.

Since

$$\rho_*(q) = \frac{1}{2\pi}\int \mathrm{d}k\ \mathrm{e}^{ikq-iw_1k} = \delta(q-w_1),$$

fixed points correspond to the certain distribution $q = \langle q \rangle = w_1$. Since space and time are rescaled by the same factor 2, the fixed point corresponds to $q(t) = w_1 t$, the equation of the mean path.

Convergence and fixed point stability. For a non-linear transformation, a global stability analysis is often impossible. One can only linearize the transformation near the fixed point and perform a local study.

Here, this is not necessary since the transformation for $w(k)$ is linear. Setting

$$w(k) = w_*(k) + \delta w(k),$$

then,

$$[\mathcal{T}_2\delta w](k) = 2\delta w(k/2)\,.$$

The function δw is regular and, thus, can be expanded in a Taylor series of the form

$$\delta w(k) = \sum_{\ell=1} \frac{(-i)^\ell}{\ell!} \delta w_\ell k^\ell \,.$$

Then,

$$[\mathcal{T}_2 \delta w](k) = 2\delta w(k/2) = 2 \sum_{\ell=1} \frac{(-ik)^\ell}{\ell!} 2^{-\ell} \delta w_\ell \,.$$

The expression shows that the functions k^ℓ with $\ell > 0$, are the eigenvectors of the transformation $\mathcal{T}_2$ and the corresponding eigenvalues are

$$\tau_\ell = 2^{1-\ell}.$$

Since at each iteration the number of variables is divided by two, one can relate the eigenvalues to the behaviour as a function of the initial number n of variables. One defines an associated exponent

$$\alpha_\ell = \ln \tau_\ell / \ln 2 = 1 - \ell \,.$$

After m iterations, the component δw_ℓ is multiplied by n^{α_ℓ} since

$$\mathcal{T}_2^m k^\ell = 2^{m(1-\ell)} k^\ell = n^{\alpha_\ell} k^\ell .$$

The behaviour, for $n \to \infty$, of a component of δw on the eigenvectors thus depends on the sign of α_ℓ.

We now adopt the RG language to discuss eigenvalues and eigenvectors.

We examine the various values of ℓ:

(i) $\ell = 1 \Rightarrow \tau_1 = 1$, $\alpha_1 = 0$. If one adds a term δw proportional to the eigenvector k to $w_*(k)$, $\delta w(k) = -i\delta w_1 k$, then

$$w_1 \mapsto w_1 + \delta w_1 ,$$

which correspond to a new fixed point. This change has also the interpretation of a linear transformation on k or on the random variable q.

An eigen-perturbation corresponding to the eigenvalue 1 and, thus to an exponent $\alpha_1 = 0$, is called marginal.

Quite generally, the existence of a one-parameter family of fixed points implies the existence of an eigenvalue $\tau = 1$ and, thus, an exponent $\alpha = 0$. Indeed, let us assume the existence of one-parameter family of fixed points $w_*(s)$,

$$\mathcal{T} w_*(s) = w_*(s)\,,$$

where $w_*(s)$ is a differentiable function of the parameter s. Then,

$$\mathcal{T}\,\frac{\partial w_*}{\partial s} = \frac{\partial w_*}{\partial s}.$$

(ii) $\ell > 1 \Rightarrow \tau_\ell = 2^{1-\ell} < 1$, $\alpha_\ell < 0$. The components of δw on such eigenvectors converge to zero for n or $m \to \infty$.

In the RG terminology, the eigen-pertubations that correspond to eigenvalues smaller in modulus than 1 and, thus, to negative exponents (more generally with a negative real part), are called irrelevant.

Universality, in the RG formulation, is a consequence of the property that all eigenvectors, but a finite number, are irrelevant.

Dimension of a random variable. To the random variable that has a limiting distribution, one can attach a dimension d_q defined by

$$d_q = \ln \lambda / \ln 2 \,. \tag{8}$$

This corresponds to dividing the sum by n^{d_q}. One here finds $d_q = 1$, which is consistent with $q(t) \propto t$.

Centred distribution

For a centred distribution, $w_1 = 0$, one has to expand to order k^2. One finds the equation

$$w_2 = 2w_2/\lambda^2.$$

Since the variance w_2 is strictly positive, except for a certain distribution, a case that we now exclude, the equation implies $\lambda = \sqrt{2}$.

Again, the coefficients w_ℓ vanish for $\ell > 2$ and the fixed points have the form

$$w_*(k) = -\tfrac{1}{2} w_2 k^2 .$$

Therefore, one finds the Gaussian distribution

$$\rho_*(q) = \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - w_2 k^2/2} = \frac{1}{\sqrt{2\pi w_2}} \, \mathrm{e}^{-q^2/2w_2} \, .$$

Since, in the transformation $\mathcal{T}$, the number n of time steps is divided by two, this value of the renormalization factor λ corresponds to dividing space by a factor $\sqrt{2}$. This is consistent with the scaling dimension of the space variable in time unit $x \propto \sqrt{t}$ of the Brownian motion:

$$d_q = \ln \lambda / \ln 2 = \tfrac{1}{2} \, .$$

The two essential asymptotic properties of the random walk, convergence toward a Gaussian distribution and scaling property are thus reproduced by the RG type analysis.

Fixed point stability. One can now study the stability of the fixed point corresponding to the transformation $\mathcal{T}_{\sqrt{2}}$. One sets

$$w(k) = w_*(k) + \delta w(k),$$

and looks for the eigenvectors and eigenvalues of the transformation

$$[\mathcal{T}_{\sqrt{2}}\,\delta w](k) \equiv 2\delta w(k/\sqrt{2}) = \tau \delta w(k).$$

Clearly, the eigenvectors have still the form

$$\delta w(k) = k^\ell \;\Rightarrow\; \tau_\ell = 2^{1-\ell/2}.$$

The correspondent exponent is

$$\alpha_\ell = \ln \tau_\ell / \ln 2 = 1 - \ell/2\,.$$

The values can be classified as:

(i) $\ell = 1 \Rightarrow \tau_1 = \sqrt{2}$, $\alpha_1 = \frac{1}{2}$. This corresponds to an unstable direction; a component on such a eigenvector diverges for $m \to \infty$.

In the RG terminology, a perturbation corresponding to a positive exponent α, and which thus moves away from the fixed point, is called relevant.

Here, a perturbation linear in k violates the condition $w_1 = 0$. One is then brought back to the study of fixed points with $w_1 \neq 0$.

(ii) $\ell = 2 \Rightarrow \tau_2 = 1, \quad \alpha_2 = 0$. A vanishing eigenvalue characterizes a marginal perturbation. Here, the perturbation only modifies the value of w_2 and, again, has an interpretation as a linear transformation on the random variable.

(iii) $\ell > 2 \Rightarrow \tau_\ell = 2^{1-\ell/2} < 1\,, \quad \alpha_\ell = 1 - \ell/2 < 0$. Finally, all perturbations $\ell > 2$ correspond to stable directions in the sense that their amplitudes converge to zero for $m \to \infty$ and are irrelevant.

Redundant perturbations. In the examples examined here, the marginal perturbations correspond to simple changes in the normalization of the random variables. In many problems, this normalization plays no role. One can then consider that fixed points corresponding to different normalizations should not be distinguished. From this viewpoint, in both cases one has found really only one fixed point. The perturbation corresponding to the vanishing eigenvalue is then no longer called marginal but redundant, in the sense that it changes only an arbitrary normalization.

Other fixed points. Other values of $\lambda = 2^{1/\mu}$, correspond formally to new fixed points of the form $|k|^{\mu}$, $0 < \mu < 2$ ($\mu > 2$ is excluded because the coefficient of k^2 is strictly positive). However, these fixed points are no longer regular functions of k. They correspond to distributions that have no second moment $\langle q^2 \rangle$ and thus no variance: they decay only algebraically for large values of q. In the RG terminology, they correspond to different universality classes, distributions with other decay properties.

Random walk on the lattice of points with integer coordinates. The analysis can also be generalized to a random walk on the points of integer coordinate. The main difference is that $w(k)$ is a periodic function of period 2π. However, at each iteration the period is multiplied by a factor $\lambda > 1$. Thus, asymptotically, the period diverges and, at least for continuous observables, the discrete character of the initial lattice disappears.

In the d-dimensional lattice $\mathbb{Z}^d$, if the random walk has hypercubic symmetry, the leading term in the expansion of $w(\mathbf{k})$ for $\mathbf{k}$ small is again $\frac{1}{2}w_2\mathbf{k}^2$ because it is the only quadratic hypercubic invariant. Therefore, asymptotically the random walk is Brownian with rotation symmetry. The lattice structure is only apparent in the first irrelevant perturbation because there exists two independent cubic invariant monomials of degree four:

$$\sum_{\mu=1}^{d} k_\mu^4, \quad \left(\mathbf{k}^2\right)^2 .$$

Brownian motion and path integral

If one is interested only in the asymptotic properties of the distribution, which have been shown to be independent of the initial transition probability, one can obtain them, in the continuum limit, starting directly from Gaussian transition probabilities of the form (assuming rotation symmetry)

$$\rho(q) = \frac{1}{(2\pi w_2)^{1/2}}\,\mathrm{e}^{-q^2/2w_2}\,.$$

In the case of a certain initial position $q = q_0 = 0$, an iteration of the evolution equation then leads to

$$P_n(q) = \frac{1}{(2\pi w_2)^{n/2}} \int \mathrm{d}q_1\,\mathrm{d}q_2\,\ldots\mathrm{d}q_{n-1}\,\mathrm{e}^{-\mathcal{S}(q_0,q_2,\ldots,q_n)} \tag{9}$$

with $q_n = q$ and

$$\mathcal{S}(q_0, q_2, \ldots, q_n) = \sum_{\ell=1}^{n} \frac{(q_\ell - q_{\ell-1})^2}{2w_2}.$$

We then introduce macroscopic time variables,

$$\tau_\ell = \ell\varepsilon\,, \quad \tau_n = n\varepsilon = t\,,$$

and a continuous, piecewise linear path $x(\tau)$ (Fig. 1)

$$x(\tau) = \sqrt{\varepsilon}\left[q_{\ell-1} + \frac{\tau - \tau_{\ell-1}}{\tau_\ell - \tau_{\ell-1}}\left(q_\ell - q_{\ell-1}\right)\right] \quad \text{for } \tau_{\ell-1} \le \tau \le \tau_\ell\,.$$

One verifies that $\mathcal{S}$ can be written as (with the notation $\dot{x}(\tau) \equiv \mathrm{d}x/\mathrm{d}\tau$)

$$\mathcal{S}\big(x(\tau)\big) = \frac{1}{2w_2}\int_0^t \big(\dot{x}(\tau)\big)^2 \mathrm{d}\tau$$

with the boundary conditions

$$x(0) = 0\,, \quad x(t) = \sqrt{\varepsilon}q = \mathbf{x}\,.$$

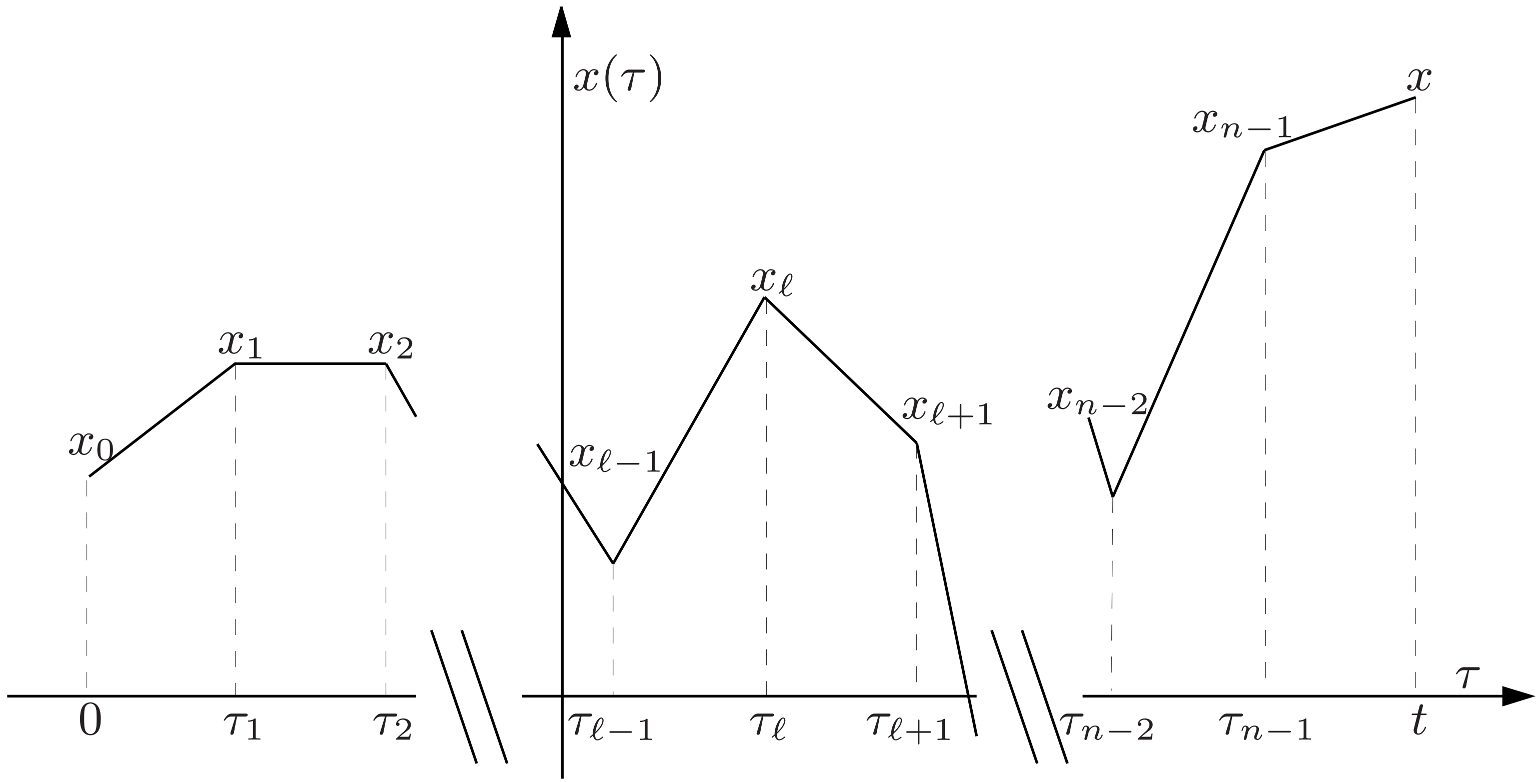

Fig. 1 Path contributing to the integral (9) $(d = 1)$ with $x_\ell \equiv x(\tau_\ell)$.

Moreover,

$$P_n(q) = \frac{1}{(2\pi w_2)^{1/2}} \int \left(\prod_{\ell=1}^{n-1} \frac{\mathrm{d}x(\tau_\ell)}{(2\pi w_2 \varepsilon)^{1/2}} \right) \mathrm{e}^{-\mathcal{S}(x)} \,.$$

In the continuum limit $\varepsilon \to 0$, $n \to \infty$ with t fixed, the expression becomes a representation of the distribution of the continuum limit,

$$\Pi(t,x) \sim \varepsilon^{-1/2} P_n(q),$$

in the form of a path integral, which we denote symbolically

$$\Pi(t,x) = \int [\mathrm{d}x(\tau)]\, \mathrm{e}^{-\mathcal{S}(x(\tau))}\,,$$

where $\int [\mathrm{d}x(\tau)]$ means sum over all continuous paths that start from the origin at time $\tau = 0$ and reach x at time t. The trajectories that contribute to the path integral correspond to a Brownian motion, a random walk in continuum time and space. The representation of the Brownian motion by path integrals, initially introduced by Wiener, is also called Wiener integral.

Detailed balance: introduction

The condition of detailed balance is needed for one of the exercises, but it also characterizes an important class of random dynamical systems. It relates a Markov process and its time-reversed one. Expressed in terms of the transition probability $\rho(\mathbf{q},\mathbf{q}')$, it reads

$$\rho(\mathbf{q},\mathbf{q}')P_\infty(\mathbf{q}') = \rho(\mathbf{q}',\mathbf{q})P_\infty(\mathbf{q}), \tag{10}$$

where P_∞ is a probability distribution:

$$P_\infty(\mathbf{q}) \geq 0\,, \quad \int \mathrm{d}^d q\, P_\infty(\mathbf{q}) = 1\,.$$

We now integrate equation (10) over $\mathbf{q}'$ and use the condition of conservation of probabilities (1)

$$\int \mathrm{d}^d q' \rho(\mathbf{q}',\mathbf{q}) = 1\,.$$

We obtain

$$P_\infty(\mathbf{q}) = \int \mathrm{d}^d q' \, \rho(\mathbf{q}, \mathbf{q}') P_\infty(\mathbf{q}').$$

Thus, the distribution $P_\infty(\mathbf{q})$ is the asymptotic distribution of the process.

If $P_\infty(\mathbf{q})$ is strictly positive, which, for example, is implied by the condition $\rho(\mathbf{q}, \mathbf{q}') > 0$, one can define the kernel

$$\mathcal{T}(\mathbf{q}, \mathbf{q}') = P_\infty^{-1/2}(\mathbf{q}) \rho(\mathbf{q}, \mathbf{q}') P_\infty^{1/2}(\mathbf{q}') = \mathcal{T}(\mathbf{q}', \mathbf{q}), \tag{11}$$

which corresponds to a real symmetric operator. This operator plays the role of the transfer matrix in statistical lattice models, the time of the stochastic process becoming the space of the lattice. With a few additional weak technical conditions, one shows that this operator has a discrete real spectrum.

In translation-invariant examples, detailed balance is formally satisfied if $\rho(\mathbf{q}) = \rho(-\mathbf{q}) \, \mathrm{e}^{\mathbf{c} \cdot \mathbf{q}}$, but then the function P_∞ is not normalizable and not a distribution. Nevertheless, the spectrum of ρ is real.

Exercises

Exercise 1

Study the local stability of the Gaussian fixed point

$$\rho_{\mathrm{G}}(q) = \mathrm{e}^{-q^2/2}/\sqrt{2\pi}\,,$$

by starting directly from the equation

$$[\mathcal{T}_\lambda \rho](q) = \lambda \int \mathrm{d}q' \rho(q')\rho(\lambda q - q'). \tag{12}$$

Determine the value of the renormalization factor λ for which the Gaussian probability distribution ρ_{G} is a fixed point of $\mathcal{T}_\lambda$.

Setting $\rho = \rho_{\mathrm{G}} + \delta\rho$, expand equation (12) to first order in $\delta\rho$. Show that the eigenvectors of the linear operator acting on $\delta\rho$ have the form

$$\delta\rho_p(q) = (\mathrm{d}/\mathrm{d}q)^p\,\rho_{\mathrm{G}}(q), \quad p > 0\,.$$

Infer the corresponding eigenvalues.

Exercise 2

Random walk on a circle. To exhibit the somewhat different asymptotic properties of a random walk on compact manifolds, it is proposed to study random walk on a circle. One still assumes translation invariance. The random walk is then specified by a transition function $\rho(q-q')$, where q and q' are two angles corresponding to positions on the circle. Moreover, the function $\rho(q)$ is assumed to be periodic and continuous. Determine the asymptotic distribution of the walker position. At initial time $n = 0$, the walker is at the point $q = 0$.

Exercise 3

Another universality class

One considers now the transition probability $\rho(q - q')$ with

$$\rho(q) = \frac{2}{3\pi} \frac{2 + q^2}{(1 + q^2)^2}.$$

The initial distribution is again

$$P_0(q) = \delta(q).$$

Evaluate the asymptotic distribution $P_n(q)$ for $n \to \infty$.

Exercise 4

Random walk and detailed balance. One considers a Markovian process in continuum space with, as transition probability, the Gaussian function

$$\rho(q, q') = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(q - \lambda q')^2\right],$$

where λ is a real parameter with $|\lambda| < 1$.

Show that $\rho(q, q')$ satisfies the condition of detailed balance (10). Infer the asymptotic distribution at time n when $n \to \infty$. Associate to $\rho(q, q')$ a real symmetric operator $\mathcal{T}$ as in equation (11),

$$\mathcal{T}(q, q') = P_\infty^{-1/2}(q)\rho(q, q')P_\infty^{1/2}(q') = \mathcal{T}(q', q).$$

Continuous phase transitions. Universality

We now discuss continuous (second order) phase transitions in statistical systems with short range interactions. We work in the framework of ferromagnetic systems but, as a consequence of the universality property of critical phenomena, the results that are derived apply to many other transitions that are not magnetic, like the liquid–vapour, binary mixtures, superfluid Helium *etc*.... Surprisingly enough, they also apply to the statistical properties of polymers, or self-avoiding random walk on a lattice.

For these statistical systems, the correlation length, which characterizes the decay at large distance of connected correlation functions, diverges at the transition temperature: a distance scale, large with respect to the microscopic scales (range of forces, lattice spacing), is generated dynamically. A non-trivial large distance physics then appears that has properties largely independent of the details of the microscopic interactions. This is the phenomenon that we want to investigate.

As long as the correlation length remains finite, macroscopic quantities, like the mean spin, have the behaviour predicted by the central limit theorem; in the infinite volume limit, they tend toward certain values with decreasing Gaussian fluctuations. This result can be understood in the following way: the initial microscopic degrees of freedom can be replaced by independent mean spins, attached to volumes having the correlation length as linear size. Therefore, it is natural to first study the properties of Gaussian models.

At the transition temperature T_c, and in the several phase region, the arguments are no longer valid. Nevertheless, one may wonder whether the asymptotic Gaussian measure can then be simply replaced by a perturbed Gaussian measure, that is, whether the residual correlations between mean spins can be treated perturbatively. Such an approximation can be called classical or quasi-Gaussian.

The appearance of a large scale at the transition generates a non-trivial large distance physics. The quasi-Gaussian approximation predicts that it has remarkably universal large distance properties at T_c, independent to a large extent of symmetries, of dimension of space... Moreover, within the quasi-Gaussian approximation, universality extends to the critical domain: $|T - T_c| \ll T_c$ and small magnetic field.

A systematic study of corrections to the quasi-Gaussian approximation then allows verifying its consistency and its domain of validity. The special role of space dimension 4 emerges, which separates the higher dimensions where the approximation is justified, to lower dimensions where it cannot be valid.

For simplicity, the discussion will first be restricted to models with a discrete Ising-like $\mathbb{Z}_2$ reflection symmetry. Indeed, below T_c or in a magnetic field, models with continuous symmetries have special properties due to the presence of Goldstone modes, which require a specific analysis.

Effective statistical field theory

We have shown that the large distance properties of the simple random walk can be described by a path integral.

Heuristic arguments of the kind we have given for the random walk, lead us then to expect that, even if the initial statistical system is defined in terms of random variables associated to the sites of a space lattice, and which take only a finite set of values (like, *e.g.*, the classical spins of the Ising model), when the correlation length is large, the large distance properties of the system can be inferred from an effective statistical field theory in continuum space.

The effective statistical field theory is defined in terms of a random real field $\sigma(x)$ in continuum space, $x \in \mathbb{R}^d$, and a functional measure on fields of the form $\mathrm{e}^{-\mathcal{H}(\sigma)}/\mathcal{Z}$, where $\mathcal{H}(\sigma)$ is called the Hamiltonian in statistical physics (a denomination borrowed from the statistical theory of classical gases) and the normalization $\mathcal{Z}$ is the partition function.

Moreover, the partition function is given by the field integral

$$\mathcal{Z} = \int [\mathrm{d}\sigma(x)] \, \mathrm{e}^{-\mathcal{H}(\sigma)},$$

where the dependence in the temperature T is included in $\mathcal{H}(\sigma)$.

Field integrals are the generalization to d space dimensions of path integrals, and the symbol $[\mathrm{d}\sigma(x)]$ stands for summation over all fields $\sigma(x)$.

The essential condition of short range interactions in the initial statistical system translates into the property of locality of the field theory: $\mathcal{H}(\sigma)$ can be chosen as a space-integral over a linear combination of monomials in the field $\sigma(x)$ and its derivatives.

We assume also space translation and rotation invariance and, to discuss a specific case, $\mathbb{Z}_2$ reflection symmetry (like in the Ising model):

$$\mathcal{H}(\sigma) = \mathcal{H}(-\sigma).$$

In d space dimensions, a typical form then is

$$\mathcal{H}(\sigma) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \sigma(x)\right)^2 + \tfrac{1}{2} r \sigma^2(x) + \frac{g}{4!}\sigma^4(x) + \cdots\right].$$

(∇_x is the gradient vector with components $\partial/\partial x_\mu$, $\mu = 1, \ldots, d$.)

As a systematic expansion of corrections to the mean field approximation indicates, the coefficients of $\mathcal{H}(\sigma)$, like above r, g..., are regular functions of the temperature T near the critical temperature T_c.

Correlation functions

Physical observables involve field correlation functions (generalized moments of the field distribution),

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle \equiv \frac{1}{\mathcal{Z}}\int [\mathrm{d}\sigma(x)]\sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\,\mathrm{e}^{-\mathcal{H}(\sigma)}.$$

They can be derived by functional differentiation from the generating functional (generalized partition function) in an external field $H(x)$,

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)]\,\exp\left[-\mathcal{H}(\sigma) + \int \mathrm{d}^d x\, H(x)\sigma(x)\right],$$

as

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle = \frac{1}{\mathcal{Z}(0)}\,\frac{\delta}{\delta H(x_1)}\frac{\delta}{\delta H(x_2)}\cdots\frac{\delta}{\delta H(x_n)}\mathcal{Z}(H)\Big|_{H=0}\,.$$

Connected correlation functions

More relevant physical observables are the connected correlation functions (generalized cumulants). The n-point function $W^{(n)}(x_1, x_2, \ldots, x_n)$ can be derived by functional differentiation from the free energy $\mathcal{W}(H) = \ln \mathcal{Z}(H)$:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\delta}{\delta H(x_1)} \frac{\delta}{\delta H(x_2)} \cdots \frac{\delta}{\delta H(x_n)} \mathcal{W}(H) \Big|_{H=0} .$$

Translation invariance then implies

$$W^{(n)}(x_1, x_2, \ldots, x_n) = W^{(n)}(x_1 + a, x_2 + a, \ldots, x_n + a) \quad \forall a .$$

Connected correlation functions have the so-called cluster property: if in a connected n-point function one separates the points $x_1, \ldots, x_n$ into two non-empty sets, the function vanish when the distance between the two sets goes to infinity. It is the large distance behaviour of connected correlation functions in the critical domain near T_c that may exhibit universal properties.