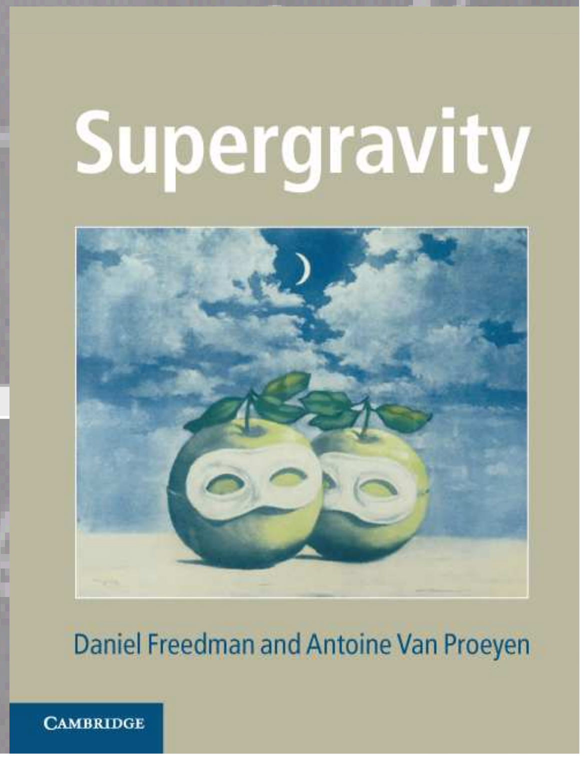
Tools for supersymmetry

Antoine Van Proeyen
KU Leuven

Wolfersdorf, September 2014

Based on some sections of the book
'Supergravity'



Plan for the lectures

1. Symmetries
2. Clifford algebras and spinors
3. Spinor properties
4. Duality and tools of gauge theories
5. Geometry and symmetries of supersymmetric theories and Kähler manifolds

Lecture 1: Symmetries

- What is a symmetry (1.2)
- Finite and infinitesimal transformations, generators, matrices, commutators, (1.2.2, 11.1.1)
- Spacetime symmetries (1.2.3)
- Noether currents and charges, energy-momentum tensors (1.3)
- Classical versus quantum transformations (1.4, 1.5)
- Gauge transformations (11.1.2)
- Counting degrees of freedom (intro ch.4)

What is a symmetry ?

- A rule : $\phi^{\,i}(x)$ satisfying eom $\rightarrow$ $\phi'^{\,i}(x)$ satisfying eom
- Symmetries that leave the action invariant: S[ϕ] = S[ϕ']. Not for dualities.
- Lagrangian invariant up to a total derivative
- Usually internal symmetries: leave Lagrangian invariant, spacetime symmetries not.

Finite and infinitesimal transformations

- ϕ' (x)= U ϕ (x)
- we will consider in this course only transformations connected to the identity
- typically defined by matrices t_A

$$\phi^i(x) \to \phi'^i(x) \equiv U(\Theta)^i{}_j \phi^j(x). \qquad U(\Theta) = \mathrm{e}^{-\Theta} = \mathrm{e}^{-\theta^A t_A}.$$

- Infinitesimal first order in the parameters:

$$\delta\phi = \phi' - \phi = -\Theta\phi$$

Generators in general

- In this chapter: symmetries that leave action invariant.
- Continuous, infinitesimal: Lie algebra.
- Extend: structure functions.
- General treatment:
  spacetime, internal symmetries and susy.

Global infinitesimal symmetries

$$\delta(\epsilon) = \epsilon^A T_A \,,$$

T_A operator,
can in Hamiltonian be defined by Poisson brackets.
First linear:

$$T_A\phi^i = -(t_A)^i{}_j\phi^j \,, \qquad [t_A, t_B] = f_{AB}{}^C t_C \,.$$

Transformations act on fields !!

$$\begin{aligned} \delta(\epsilon_1)\delta(\epsilon_2)\phi^i &= \epsilon_1^A T_A \epsilon_2^B \left[-(t_B)^i{}_j\phi^j\right] \\ &= \epsilon_1^A\epsilon_2^B(-t_B)^i{}_j T_A\phi^j \\ &= \epsilon_1^A\epsilon_2^B(-t_B)^i{}_j(-t_A)^j{}_k\phi^k \,. \end{aligned}$$

Leads to

$$[T_A, T_B] = f_{AB}{}^C T_C \,,$$

T_A is more general notation.
Ψ^α in a complex representation of a compact symmetry group, their conjugates $\bar\Psi_\alpha$, and fields ϕ^B in the adjoint:

$$\begin{aligned} T_A\Psi^\alpha &= -(t_A)^\alpha{}_\beta\Psi^\beta \,, \\ T_A\bar\Psi_\alpha &= \bar\Psi_\beta(t_A)^\beta{}_\alpha \,, \\ T_A\phi^B &= -f_{AC}{}^B\phi^C \,. \end{aligned}$$

Poincaré symmetries:

Space with $(x^\mu) = (t, \vec{x})$

Metric

$$\mathrm{d}s^2 = -\mathrm{d}t\mathrm{d}t + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = \mathrm{d}x^\mu \eta_{\mu\nu}\mathrm{d}x^\nu$$

Isometries (preserve metric)

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu$$

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

Expand

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \lambda^\mu{}_\nu + \mathcal{O}(\lambda^2) \\ &= \left(\mathrm{e}^{\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}\right)^\mu{}_\nu \end{aligned}$$

$$m_{[\rho\sigma]}{}^\mu{}_\nu \equiv \delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\rho\nu} = -m_{[\sigma\rho]}{}^\mu{}_\nu$$

Algebra SO(1, D-1)

$$\begin{aligned} [m_{[\mu\nu]}, m_{[\rho\sigma]}] &= \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} \\ &\quad -\eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]} \end{aligned}$$

Act on fields: ϕ(x)=ϕ´(x´)

$$\begin{aligned} \phi'(x) &= U(\Lambda)\phi(x) = \phi(\Lambda x) \\ U(\Lambda) &\equiv \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} L_{[\rho\sigma]}} \\ L_{[\rho\sigma]} &\equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho \end{aligned}$$

More general if not scalar fields

$$J_{[\rho\sigma]} = L_{[\rho\sigma]} \mathbb{1} + m_{[\rho\sigma]},$$

$$\begin{aligned} \psi'^i(x) &= U(\Lambda, a)^i{}_j \psi^j(x) \\ &= \left(\mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}\right)^i{}_j \psi^j(\Lambda x + a) \end{aligned}$$

for the Poincaré group

As operator

$$\delta = a^\mu P_\mu + \tfrac{1}{2}\lambda^{\mu\nu} M_{[\mu\nu]}$$

E.g; on fermions: $(m_{[\mu\nu]} = \frac{1}{2}\gamma_{\mu\nu})$

$$M_{[\mu\nu]}\Psi(x) \;=\; -(L_{[\mu\nu]} + \tfrac{1}{2}\gamma_{\mu\nu})\Psi(x)\,.$$

While

$$P_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}\,,$$

The nonlinear σ-model and Killing symmetries.

Not linear

$$T_A \phi^i = k^i_A(\phi)\,.$$

with $k_A = k^i_A(\phi)\frac{\partial}{\partial \phi^i}$:

$$[k_A, k_B] = f_{AB}{}^C k_C\,,$$

Noether currents

Generic infinitesimal

$$\delta\phi^i(x) \equiv \epsilon^A \Delta_A \phi^i(x)\,,$$

(constant parameters).

Transformation of Lagrangian:

$$\delta\mathcal{L} \equiv \epsilon^A \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Delta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Delta_A\phi^i\right] = \epsilon^A\partial_\mu K^\mu_A\,.$$

Leads to conserved currents

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i + K^\mu_A\,, \qquad \partial_\mu J^\mu{}_A \approx 0\,.$$

Noether currents for the spacetime symmetries

- translations : index A is another spacetime index. E.g. for scalars $T^{\mu}{}_{\nu} = \partial^{\mu}\phi\,\partial_{\nu}\phi + \delta^{\mu}_{\nu}\mathcal{L}.$
- for Lorentz transformations for scalars

$$M^{\mu}{}_{[\rho\sigma]} = -x_{\rho}T^{\mu}{}_{\sigma} + x_{\sigma}T^{\mu}{}_{\rho}.$$

- in general $M^{\mu}{}_{\rho\sigma} = -2x_{[\rho}T^{\mu}{}_{\sigma]} + m^{\mu}{}_{\rho\sigma}.$
- can be used to make energy-momentum tensor symmetric (and still prerserved)

$$\Theta^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{2}\partial_{\rho}\left(m^{\rho\mu\nu} - m^{\mu\rho\nu} - m^{\nu\rho\mu}\right)$$

To charges and quantum commutators

$Q_A = \int \mathrm{d}^{D-1}\vec{x}\, J^0{}_A(\vec{x},t).$

$\pi(\vec{x},0) = \delta S/\delta\partial_t\phi(\vec{x},0).$

when Poisson brackets can be defined (define momenta, …):

$\Delta_A\phi^i(x) = \{Q_A, \phi^i(x)\}$

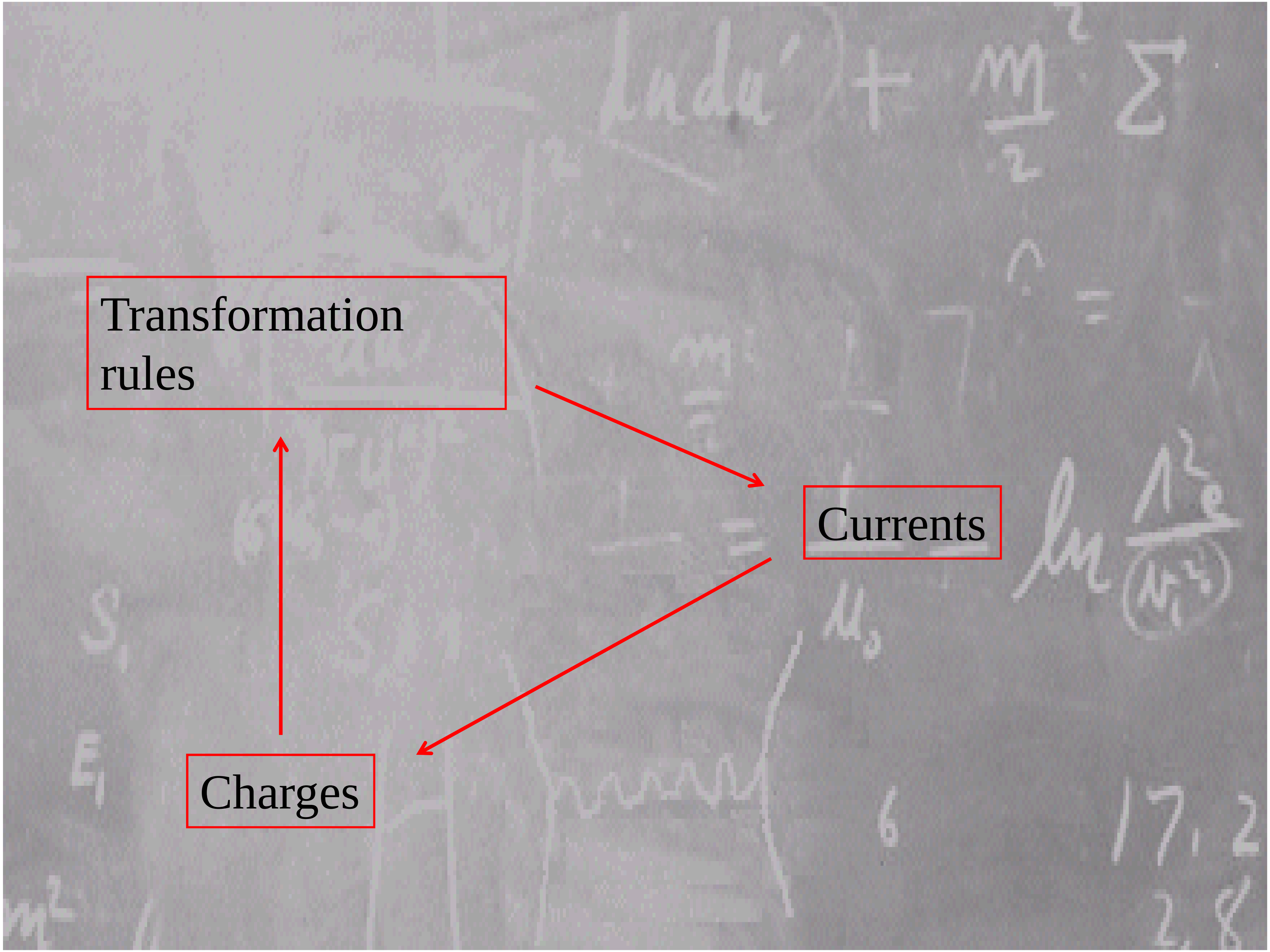
$\delta(\epsilon) = \epsilon^A\{Q_A,\phi\}$

$$\{Q_A\,,\,Q_B\} = f_{AB}{}^C Q_C.$$

From Poisson brackets to quantum operators

$\{A,B\} = C \;\rightarrow\; [\mathbf{A},\mathbf{B}]_{\mathsf{qu}} = \mathsf{i}\mathbf{C}\,, \qquad \hbar = 1.$

$$\begin{aligned} \Delta_A\mathbf{\Phi}^i &= -\mathsf{i}\left[\mathbf{Q}_A, \mathbf{\Phi}^i\right]_{\mathsf{qu}}\,, \\ [\mathbf{Q}_A\,,\,\mathbf{Q}_B]_{\mathsf{qu}} &= \mathsf{i}\, f_{AB}{}^C\mathbf{Q}_C. \end{aligned}$$


Transformation rules
Currents
Charges

Local symmetries and gauge fields

- Gauge theory $\delta(\epsilon) B_\mu{}^A \equiv \partial_\mu \epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A$

generic gauge symmetry T_A	parameter ϵ^A	gauge field B_μ^A
local translations P_a	ξ^a	e_μ^a
Lorentz transformations $M_{\{ab\}}$	λ^{ab}	$\omega_\mu{}^{ab}$
Supersymmetry Q_α	$\overline{\epsilon}^\alpha$	$\bar{\psi}_\mu^\alpha$
Internal symmetry T_A	θ^A	A_μ^A

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A$$

Exercises lecture 1

Start from matrices that satisfy $[t_A, t_B] = f_{AB}{}^C t_C$,

$$\delta(\theta)\phi = -\theta^A t_A \phi$$

$$[\delta(\theta_1), \delta(\theta_2)]\phi = \delta(\theta_3)\phi$$

what is θ_3 ?

$$\theta_3^C = \theta_2^B \theta_1^A f_{AB}{}^C$$

In general for ϵ bosonic or fermionic

$$\delta(\epsilon) = \epsilon^A T_A \, ,$$

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\phi^i = \epsilon_3^A T_A \phi^i$$

$$[T_A, T_B] = f_{AB}{}^C T_C \, ,$$

$$[T_A, T_B] = T_A T_B \mp T_B T_A$$

Exercises on chapter 1

- Ex 1.5: Show that the action

$$S = \int \mathrm{d}^D x\, \mathcal{L}(x) = -\tfrac{1}{2}\int \mathrm{d}^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + m^2\phi^i\phi^i\right]$$

is invariant under the transformation

$$\phi^i(x) \xrightarrow{\Lambda} \phi'^i(x) \equiv \phi^i(\Lambda x).$$

Important: fields transform, not the integration variables

- Ex.1.6: Compute the commutators $[L_{[\mu\nu]}, L_{[\rho\sigma]}]$ and show that they agree with that for matrix generators. $L_{[\rho\sigma]} \equiv 2x_{[\rho}\partial_{\sigma]}$ $\quad [L^{\mu\nu}, L_{\rho\sigma}] = 4\delta^{[\mu}{}_{[\rho}L_{\sigma]}{}^{\nu]}$

  Show that to first order in $\lambda^{\rho\sigma}$

$$\phi^i(x^\mu) - \tfrac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}\phi^i(x^\mu) = \phi^i(x^\mu + \lambda^{\mu\nu}x_\nu)$$

Exercise on Lorentz transformations

$$S[\bar{\Psi},\Psi] = \int \mathrm{d}^D x \bar{\Psi}(x)[\gamma^\mu \partial_\mu - m]\Psi(x)$$

$$\begin{aligned} \delta\Psi &= \tfrac{1}{2}\lambda^{\mu\nu}(-L_{\mu\nu}\Psi - \Sigma_{\mu\nu}\Psi)\,, \\ \delta\bar{\Psi} &= \tfrac{1}{2}\lambda^{\mu\nu}(-L_{\mu\nu}\bar{\Psi} + \bar{\Psi}\Sigma_{\mu\nu}) \end{aligned}$$

$\lambda^{\mu\nu}$ are the parameters of Lorentz transformations

$\Sigma_{\mu\nu}$ are some matrices in spinor space similar to γ^μ

Which fundamental relation of gamma matrices is required in order to have invariance of the action ?

exercise: improved energy-momentum tensor

- Suppose that we have calculated the energy-momentum tensor of a field, $T^{\mu\nu}$ and it is not symmetric. We know that there is also a Lorentz current that is preserved:

$$M^{\mu}{}_{\rho\sigma} = -2x_{[\rho}T^{\mu}{}_{\sigma]} + m^{\mu}{}_{\rho\sigma}\,, \qquad \partial_\mu M^{\mu}{}_{\rho\sigma} \approx 0$$

- Prove that

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\rho\left(m^{\rho\mu\nu} - m^{\mu\rho\nu} - m^{\nu\rho\mu}\right)$$

1. is preserved
2. is (weakly) symmetric

Exercise on gauge theories

Note rewriting of spinor quantity
with indices to be explained tomorrow

Starting from the SUSY-commutator relation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^a\epsilon_2 P_a = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1 P_a = -\tfrac{1}{2}\epsilon_2^\beta(\gamma^a)_{\beta\alpha}\epsilon_1^\alpha P_a$$

read of that

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}P_a\,, \qquad \rightarrow \qquad f_{\alpha\beta}{}^a = -\tfrac{1}{2}(\gamma^a)_{\alpha\beta} = f_{\beta\alpha}{}^a\,.$$

Obtain from this the supergravity transformation of the frame field e_μ^a, using that this is the gauge field of translations, and that ψ_μ^α is the gauge field of supersymmetry.

(There are no other commutators of the form $[T_A, Q] = ...P^a$)

On-shell Degrees of freedom by initial conditions

- On-shell= nr. of helicity states
- count number of initial conditions, divide by 2. (coordinate + momenta describe one state)
- E.g. scalar: field equation $\partial_\mu \partial^\mu \phi = 0$. Initial conditions $\phi(t=0,x^i)$ and $\partial_0 \phi(t=0,x^i)$
- Dirac: first order: determines time derivatives: (for $D=4$) 4 initial conditions: 2 dof on shell
- PS: $\partial^i \partial^i \phi = 0$ has no normalizable solutions in R^{D-1} ; hence this gives $\phi=0$ (other way: we consider $\partial^i \partial^i = \vec{k}^2$).
- count only gauge inequivalent solutions. How: choose a gauge condition

On-shell degrees of freedom (dof)

- The Maxwell field A_μ (x), with field equation

$$\partial^\mu F_{\mu\nu} = 0 \,; \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

and symmetry transformation $\delta A_\mu = \partial_\mu \theta$

Split the spacetime indices in $\mu = (0,i)$; and consider the gauge $\partial^i A_i = 0$

1. Prove that this is a good gauge
2. Prove that the field has D-2 on-shell dof

3. Clifford algebras and spinors

- Determines the properties of
  - the spinors in the theory
  - the supersymmetry algebra
- We should know
  - how large are the smallest spinors in each dimension
  - what are the reality conditions
  - which bispinors are (anti)symmetric
    (can occur in superalgebra)

Lecture 2 and 3

- 2. **Clifford algebras and spinors**
  - Clifford algebra in a general dimension (3.1):
    Complete Clifford, Levi-Civita, practical basis (even dim), highest rank and chiral, odd-dim, symmetries of γ- matrices
  - Spinors in general dimension (3.2):
    including: spinor bilinears, spinor indices, Fierz relations, reality
- 3. **Spinor properties**
  - Majorana spinors (3.3) (and other reduced spinors):
    their dimension and properties
  - Majorana spinors in physical theories (3.4):
    field equations, Weyl versus Majorana, U(1) symmetries

3.1 The Clifford algebra in general dimension

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}$$

3.1.1 The generating γ matrices

Hermiticity $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$ (hermitian for spacelike)

representations related by conjugacy by unitary S

$$\gamma'^\mu = S\gamma^\mu S^{-1}$$

unique except one sign in odd D

explicit representation and dimension $2^{[\mathrm{Int}(D/2)]}$

3.1.2 The complete Clifford algebra

$$\gamma^{\mu_1\dots\mu_r} = \gamma^{[\mu_1}\dots\gamma^{\mu_r]}\,, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu$$

all traceless except 1, and product of all in odd D.

3.1.3 Levi-Civita symbol

$$\varepsilon_{012(D-1)} = 1\,, \qquad \varepsilon^{012(D-1)} = -1$$

Schouten identity $\delta_\mu{}^{[\nu}\varepsilon^{\rho\sigma\tau\lambda]}$

3.1.4 Practical γ – matrix manipulation

$$\gamma^\mu\gamma_\mu = D\,, \qquad \gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu$$

reversal symmetry of indices

products of matrices

$$\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \eta^{\mu\nu}$$

3.1.5 Basis of the algebra for even dimension $D = 2\,m$

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \ldots, \gamma^{\mu_1\cdots\mu_D}\}$$

with $\mu_1 < \mu_2 < \ldots < \mu_r$

reverse order list

$$\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \ldots, \gamma_{\mu_D\cdots\mu_1}\}\,.$$

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\,\delta^A_B$$

expansion for any matrix in spinor space M

$$M = \sum_A m_A \Gamma^A, \qquad m_A = \frac{1}{2^m}\mathrm{Tr}(M\Gamma_A)$$

3.1.6 The highest rank Clifford algebra element

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}\,,$$

which satisfies $\gamma_*^2 = \mathbb{1}$.

E.g. $D = 4$: $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$.

Projections

$$P_L = \tfrac{1}{2}(\mathbb{1} + \gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1} - \gamma_*)\,.$$

3.1.7 Odd spacetime dimension *D=2m+1*

γ matrices can be constructed in two ways from those in *D=2m*:

$$\gamma_\pm{}^\mu = (\gamma^0, \gamma^1, \ldots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*)$$

The set with all $\gamma^{\mu_1\cdots\mu_r}$ is overcomplete

$$\gamma_\pm^{\mu_1\cdots\mu_r} = \pm \mathrm{i}^{m+1} \frac{1}{(D-r)!} \varepsilon^{\mu_1\cdots\mu_D} \gamma_{\pm\,\mu_D\cdots\mu_{r+1}}$$

Ex. 3.16: prove this and the analogue for even dimensions, in particular D=4

Supersymmetry and symmetry of bi-spinors (intro)

- E.g. a supersymmetry on a scalar is a symmetry transformation depending on a spinor ε:

$$\delta(\epsilon)\phi(x) = \bar{\epsilon}\psi(x)$$

  - For the algebra we should obtain a GCT

$$[\delta(\epsilon_2), \delta(\epsilon_1)]\,\phi(x) = \underbrace{\bar{\epsilon}_1\gamma^\mu\epsilon_2}_{\xi^\mu}\partial_\mu\phi(x)$$

    - Then the GCT parameter

should be antisymmetric in the spinor parameters

$$\xi^\mu = \bar{\epsilon}_1\gamma^\mu\epsilon_2 = -\bar{\epsilon}_2\gamma^\mu\epsilon_1$$

Thus, to see what is possible, we have to know the symmetry properties of bi-spinors

3.1.8 Symmetries of γ - matrices

C is a matrix such that $C\gamma_{\mu_1 \ldots \mu_r}$ are all symmetric or antisymmetric, depending only on D and r.

See D=5 first

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}\,, \qquad t_r = \pm 1\,,$$

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0,3 **0,1**	2,1 **2,3**
1	0,1	2,3
2	0,1 **1,2**	2,3 **0,3**
3	1,2	0,3
4	**1,2** 2,3	**0,3** 0,1
5	**2,3**	**0,1**
6	2,3 **0,3**	0,1 **1,2**
7	0,3	1,2

the following suffices

$$C^T = -t_0 C\,, \qquad \gamma^{\mu T} = t_0 t_1\, C\gamma^\mu C^{-1}\,.$$

Explicitely in constructed rep:

$$C_+ = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots\,, \qquad t_0 t_1 = 1\,,$$
$$C_- = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots\,, \qquad t_0 t_1 = -1\,.$$

3.2 Spinors in general dimensions

3.2.1 Spinors and spinor bilinears

~~Dirac conjugate~~

$$\cancel{\bar{\lambda} = \lambda^{\dagger} \mathrm{i} \gamma^{0}}$$

Majorana conjugate

$$\bar{\lambda} = \lambda^{T} C$$

- with anticommuting spinors

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda$$

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0,3 **0,1**	2,1 **2,3**
1	0,1	2,3
2	0,1 **1,2**	2,3 **0,3**
3	1,2	0,3
4	**1,2** 2,3	**0,3** 0,1
5	**2,3**	**0,1**
6	2,3 **0,3**	0,1 **1,2**
7	0,3	1,2

Since symmetries of spinor bilinears are important for supersymmetry, we use the Majorana conjugate to define $\bar{\lambda}$.

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda$$

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0,3	2,1
	0,1	**2,3**
1	0,1	2,3
2	0,1	2,3
10	**1,2**	**0,3**
3 =11	1,2	0,3
4	**1,2**	**0,3**
	2,3	0,1
5	**2,3**	**0,1**
6	2,3	0,1
	0,3	**1,2**
7	0,3	1,2

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\ldots\Gamma^{(r_p)}\lambda \implies \bar{\chi} = t_0^p t_{r_1} t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\ldots\Gamma^{(r_2)}\Gamma^{(r_1)}$$

Important consequence in even dimensions:

$$\chi = P_L\lambda \;\to\; \bar{\chi} = \begin{cases} \bar{\lambda}P_L\,, & \text{for } D = 0,4,8,\ldots, \\ \bar{\lambda}P_R\,, & \text{for } D = 2,6,10,\ldots. \end{cases}$$

3.2.2 Spinor indices

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta\,, \qquad \lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}\,.$$

NW-SE convention

Note that $\mathcal{C}_{\alpha\beta}$ are components of C^{-1} and $\mathcal{C}^{\alpha\beta}$ of C^T.

Translations:

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta\,,$$

and also

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta}$$

Have symmetry $-t_1$: $(\gamma_\mu)_{\alpha\beta} = -t_1(\gamma_\mu)_{\beta\alpha}$.

3.2.3 Fierz rearrangement

based on completeness relation

$$M = 2^{-m} \sum_{k=0}^{[D]} \frac{1}{k!} \gamma_{\mu_1 \ldots \mu_k} \; \mathrm{Tr} \left(\gamma^{\mu_k \cdots \mu_1} M \right)$$

$$\begin{cases} [D] = D, & \text{for even } D\,, \\ [D] = (D-1)/2, & \text{for odd } D \end{cases}$$

$$\chi \bar{\lambda} = -2^{-m} \sum_{k=0}^{[D]} \frac{1}{k!} \gamma_{\mu_1 \ldots \mu_k} \left(\bar{\lambda} \gamma^{\mu_k \cdots \mu_1} \chi \right)$$

usually simplifies, e.g. *D=4*

$$\begin{aligned} P_L \chi \bar{\lambda} P_L &= -\tfrac{1}{2} P_L \left(\bar{\lambda} P_L \chi \right) + \tfrac{1}{8} P_L \gamma^{\mu\nu} \left(\bar{\lambda} \gamma_{\mu\nu} P_L \chi \right) \,, \\ P_L \chi \bar{\lambda} P_R &= -\tfrac{1}{2} P_L \gamma^{\mu} \left(\bar{\lambda} \gamma_{\mu} P_L \chi \right) \end{aligned}$$

3.2.4 Reality

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

In fact complex conjugation uses

$$\gamma^{\mu *} = -t_0 t_1\, B\gamma^\mu B^{-1}\,, \qquad B \equiv \mathrm{i}t_0 C\gamma^0$$

We use

$$\lambda^C \equiv B^{-1}\lambda^*\,, \qquad (\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0 t_1)\gamma_\mu\,.$$

It works like this:

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0 t_1)\overline{\chi^C} M^C \lambda^C$$

Note: $(\lambda^C)^C = -t_1\lambda$, $(\gamma_*)^C = (-)^{D/2+1}\gamma_*\,.$

The Dirac conjugate of a spinor is

$$\overline{\lambda^C}$$

3.3 Majorana spinors

- A priori a spinor ψ has $2^{\mathrm{Int}[D/2]}$ (complex) components
- Using e.g. 'left' projection $P_L = (1+\gamma_*)/2$
  'Weyl spinors' $P_L\ \psi = \psi$ **if *D* is even** (otherwise trivial)
- In some dimensions (and signature) there are reality conditions
  $\psi = \psi^C = B^{-1}\ \psi^*$
  consistent with Lorentz algebra: 'Majorana spinors'
- consistency requires $t_1 = -1$.

often described as:
Dirac conjugate = Majorana conjugate

$$\overline{\lambda^C} = \bar{\lambda}$$

Other types of spinors

- If $t_1=1$: Majorana condition not consistent
- Define other reality condition (for an even number of spinors):

$$\chi^i = \varepsilon^{ij}(\chi^j)^C$$

- 'Symplectic Majorana spinors'
- In some dimensions Weyl and Majorana can be combined, e.g. reality condition for Weyl spinors: 'Majorana-Weyl spinors'

$D = 2$ mod 8:

Majorana: $\psi^C = \psi\,,$ Weyl: $P_{L,R}\psi = \psi$

$D = 4$ mod 4

$$(P_L\psi)^C = P_R\psi\,, \qquad (P_R\psi)^C = P_L\psi$$

Possibilities for susy depend on the properties of irreducible spinors in each dimension

- Dependent on signature. Here: Minkowski
- M: Majorana
  MW: Majorana-Weyl
  S: Symplectic
  SW: Symplectic-Weyl

Dim	Spinor	min.# comp
2	MW	1
3	M	2
4	M	4
5	S	8
6	SW	8
7	S	16
8	M	16
9	M	16
10	MW	16
11	M	32

3.4 Majorana OR Weyl fields in D=4

- Any field theory of a Majorana spinor field Ψ can be rewritten in terms of a Weyl field $P_L\Psi$ and its complex conjugate.
- Conversely, any theory involving the chiral field $\chi=P_L\chi$ and its conjugate $\chi^C=P_R\chi^C$ can be rephrased as a Majorana equation if one defines the Majorana field $\Psi=P_L\chi+P_R\chi^C$.
- Supersymmetry theories in D=4 are formulated in both descriptions in the physics literature.

U(1) symmetries

- Note that for Majorana fields we cannot have U(1) transformations $\cancel{\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\theta}\Psi}$
- but we can have $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\gamma_* \theta}\Psi$
- but this is not a symmetry of the massive action

$$\int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

- only for the massless action.
- Note that in terms of chiral fermions:

$$P_{L,R}\Psi \to P_{L,R}\Psi' = \mathrm{e}^{\pm\mathrm{i}\theta}\Psi$$

Dirac and Majorana mass terms

- Decompose a Dirac field as $\psi = \frac{1}{\sqrt{2}}(\lambda_1 + \mathrm{i}\lambda_2)$
- Obtain that the Dirac kinetic term can be writtten as

$$S_{\mathrm{kin}} = -\frac{1}{2}\int \mathrm{d}^4x \left(\bar{\lambda}_1 \partial\!\!\!/ \lambda_1 + \bar{\lambda}_2 \partial\!\!\!/ \lambda_2\right) = -\int \mathrm{d}^4x\, \overline{\psi^C} \partial\!\!\!/ \psi$$

- Rewrite phase (em) transformations as

$$\delta\psi = \mathrm{i}\theta\psi\,, \qquad \delta\lambda_1 = -\theta\lambda_2\,, \quad \delta\lambda_2 = -\theta\lambda_1$$

- Rewrite mass terms

$$S_m = \frac{1}{2}\int \mathrm{d}^4x \left(m_{11}\bar{\lambda}_1\lambda_1 + 2m_{12}\bar{\lambda}_1\lambda_2 + m_{22}\bar{\lambda}_2\lambda_2\right)$$

$$S_m = \int \mathrm{d}^4x \left(m\overline{\psi^C}\psi + \frac{1}{2}\mu\bar{\psi}\psi + \frac{1}{2}\mu^*\overline{\psi^C}\psi^C\right)$$

which of these respect the em symmetry ?

Dirac and Majorana mass terms and sterile neutrinos

$$\psi = \frac{1}{\sqrt{2}}(\lambda_1 + \mathrm{i}\lambda_2)$$

$$S_{\mathrm{kin}} = -\tfrac{1}{2}\int \mathrm{d}^4x\, \left(\bar\lambda_1 \partial\!\!\!/ \lambda_1 + \bar\lambda_2 \partial\!\!\!/ \lambda_2\right) = -\int \mathrm{d}^4x\, \overline{\psi^C}\partial\!\!\!/ \psi$$

$$S_m = \int \mathrm{d}^4x\, \left(m\overline{\psi^C}\psi + \tfrac{1}{2}\mu\bar\psi\psi + \tfrac{1}{2}\mu^*\overline{\psi^C}\psi^C\right)$$

- Derive the field equation $\partial\!\!\!/ \psi - m\psi - \mu^*\psi^C = 0$
- Terminology: Dirac mass: m; Majorana mass μ
- Chiral fermions: $\psi = P_L\psi$. Rewrite in Majorana notation. Which mass terms can survive ?
- Conclude: one massive chiral fields can have only Majorana mass terms. These neutrinos should then be gauge invariant ('sterile neutrinos')

Exercises on gamma matrix products

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} =$$

$$\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho\nu_1\nu_2} =$$

$$\gamma_\nu\gamma^\mu\gamma^\nu =$$

$$\gamma_\rho\gamma^{\mu\nu}\gamma^\rho =$$

$$\gamma_\rho\gamma^{\mu_1\mu_2\cdots\mu_r}\gamma^\rho =$$

Exercise Lorentz generators

- Prove that $\Sigma_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu}$

are good Lorentz generators.

1. They satisfy

$$[\Sigma^{\mu\nu}, \gamma^{\rho}] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^{\mu}\eta^{\nu\rho} - \gamma^{\nu}\eta^{\mu\rho}$$

2. They respect the Lorentz algebra (normalization correct)

Exercises with the Levi-Civita tensor

Exercise: map of susy and sugra

- Take the following ingredients from field theory:
  - supersymmetry theories can have at most 16 real supercharges (spinor parameters)
  - supergravity at most 32.
- Make now a map of possible supersymmetric (and supergravity) theories that are possible in dimensions $D \geq 4$: dimension vertically, and number of generators horizontally.

The map: dimensions and # of supersymmetries

D	spinor	32	24	20	16	12	8	4
11	M							
10	MW							
9	M							
8	M							
7	S							
6	SW							
5	S							
4	M							

The map: dimensions and # of supersymmetries

D	susy	32		24	20	16		12	8	4
11	M	M								
10	MW	IIA	IIB			I				
9	M	N=2				N=1				
8	M	N=2				N=1				
7	S	N=4				N=2				
6	SW	(2,2)		(2,1)		(1,1)	(2,0)		(1,0)	
5	S	N=8		N=6		N=4			N=2	
4	M	N=8		N=6	N=5	N=4		N=3	N=2	N=1
		SUGRA				SUGRA/SUSY		SUGRA	SUGRA/SUSY	

Exercise: ‘cyclic identities’

(related to Ex. 3.27)

for consistency of SUSY YM, string actions, brane actions

$$(\gamma_\mu)_{\alpha(\beta}(\gamma^\mu)_{\gamma\delta)} = 0$$

more convenient: $$\gamma_\mu\lambda_{[1}\bar{\lambda}_2\gamma^\mu\lambda_{3]} = 0$$

valid for

- D=2 with Majorana-Weyl spinors
- D=3 Majorana spinors
- D=4 Majorana spinors
- D=6: symplectic Majorana-Weyl (a bit tricky with indices)
- D=10: Majorana-Weyl spinors

Prove for D=4 !

Part of Ex. 3.29

- Prove the Fierz identity

$$P_L\chi\bar\lambda P_R = -\frac{1}{2}P_L\gamma^\mu\left(\bar\lambda\gamma_\mu P_L\chi\right).$$

you will need the identity

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_*\,.$$

and

$$\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots\nu_p}\varepsilon^{\mu_1\ldots\mu_n\rho_1\ldots\rho_p} = -\,p!\,n!\,\delta_{\nu_1\ldots\nu_p}^{\rho_1\ldots\rho_p}$$

Exercise on chapter 3

- Ex. 3.40: Rewrite

$$S[\Psi] = -\tfrac{1}{2}\int \mathrm{d}^D x\,\bar{\Psi}[\gamma^\mu\partial_\mu - m]\Psi(x)$$

as

$$\begin{aligned} S[\psi] &= -\tfrac{1}{2}\int \mathrm{d}^4 x\;[\bar{\Psi}\gamma^\mu\partial_\mu - m]\,(P_L + P_R)\Psi \\ &= -\int \mathrm{d}^4 x\,\left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_R\Psi\right] . \end{aligned}$$

and prove that the Euler-Lagrange equations are

$$\not{\partial} P_L\Psi = mP_R\Psi\,, \qquad \not{\partial} P_R\Psi = mP_L\Psi\,.$$

Derive $\Box\, P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above

SUSY algebra

when you know that $\left\{Q_\alpha, Q_\beta\right\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta}P_\mu$

and Q is Majorana, prove that

$$\left\{Q_\alpha, Q^{\dagger\beta}\right\}_{\mathsf{qu}} = \frac{1}{2}\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu$$

- prove that the quantum algebra implies $\mathrm{Tr}(QQ^\dagger) = P^0$

The chiral multiplet

- Ex. 6.11 : Consider the theory of the chiral multiplet after elimination of F. Show that the action

$$S = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi - \overline{W}' W' - \tfrac{1}{2} \bar{\chi} (P_L W'' + P_R \overline{W}'') \chi \right]$$

is invariant under the transformation rules

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi\,, & \delta \bar{Z} &= \frac{1}{\sqrt{2}} \bar{\epsilon} P_R \chi \\ \delta P_L \chi &= \frac{1}{\sqrt{2}} P_L (\not{\partial} Z + F)\epsilon, & \delta P_R \chi &= \frac{1}{\sqrt{2}} P_R (\not{\partial} \bar{Z} + \bar{F})\epsilon \\ F &\equiv -\overline{W}'(\bar{Z}), & \bar{F} &= -W'(Z) \end{aligned}$$

Show that the commutator on the scalar is

$$[\delta_1, \delta_2] Z = -\tfrac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z$$

but is modified on the fermion as follows:

$$[\delta_1, \delta_2] P_L \chi = \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_L \left[-\tfrac{1}{2} \partial_\mu \chi + \tfrac{1}{4} \gamma_\mu (\not{\partial} + \overline{W}'') \chi \right]$$

We find the spacetime translation plus an extra term that vanishes for any solution of the equations of motion.

Lecture 4: Duality and tools of gauge theories

1. Electromagnetic duality and the symplectic group (4.2.4)
2. Soft algebras and covariant translations:
   -first example in SUSY gauge theory (6.3.1)
   -general formulation (first part of 11.1.3)
3. zilch symmetries and open algebras:
   - first example in Wess-Zumino multiplet (6.2.2)
   - general formulation (continuation of 11.1.3),
4. Covariant derivatives, curvatures and their transformations (11.2)
5. (if time allows): modification for spacetime symmetries (11.3): general coordinate transformations, covariant derivatives and curvatures in gravity theories.

Electromagnetic duality and the symplectic group

$$\mathcal{L} = -\tfrac{1}{4}(\mathrm{Re}\, f_{AB}) F^A_{\mu\nu} F^{\mu\nu\, B} + \tfrac{1}{8}(\mathrm{Im}\, f_{AB}) \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^B_{\rho\sigma}$$

coupling constants or functions of scalars

$$F^{\pm A}_{\mu\nu} \equiv \tfrac{1}{2}\left(F^A_{\mu\nu} \mp \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\, A}\right)$$

$$\mathcal{L}(F^+, F^-) = -\tfrac{1}{4}\left(f_{AB} F^{-\,A}_{\mu\nu} F^{\mu\nu\,-\,B} + f^*_{AB} F^{+\,A}_{\mu\nu} F^{\mu\nu\,+\,B}\right)$$

$$G^{\mu\nu\,-}_{A} \equiv -2\mathrm{i}\frac{\delta S(F^+, F^-)}{\delta F^{-A}_{\mu\nu}} = \mathrm{i} f_{AB} F^{\mu\nu\,-\,B}$$

$$\begin{array}{rcll} \partial^\mu\, \mathrm{Im}\, F^{A-}_{\mu\nu} & = & 0 & \text{Bianchi identities} \\ \partial_\mu\, \mathrm{Im}\, G^{\mu\nu-}_{A} & = & 0 & \text{Equations of motion.} \end{array}$$

Invariance under Gl(*2m*,ℝ)

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

For consistency:

$$G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1} F'^-$$

$$\rightarrow \boxed{\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}}$$

should be symmetric

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})$$

Vector field strengths are in *2m* – symplectic vectors

More on duality transformations

- Not symmetries of the action if $B \neq 0$.
- Two applications:

  \- symmetries : those induced by transformations of the scalars. In extended sugra: all symmetries of the scalars are of this form, embedded in Sp

  - constants (spurionic quantities) change: like in M-theory: dualities between theories
- charges are in symplectic vectors.

  If quantized: charges: Sp(.., Z).

6.3. SUSY gauge theories

6.3.1 SUSY Yang-Mills vector multiplet

$$\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{d}^4x \left[-\tfrac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \tfrac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \tfrac{1}{2}D^A D^A\right], \\
\delta A^A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \\
\delta\lambda^A &= \left[-\tfrac{1}{4}\gamma^{\rho\sigma}F^A_{\rho\sigma} + \tfrac{1}{2}\mathrm{i}\gamma_* D^A\right]\epsilon, \\
\delta D^A &= \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\gamma^\mu D_\mu\lambda^A, \qquad D_\mu\lambda^A \equiv \partial_\mu\lambda^A + \lambda^C A_\mu{}^B f_{BC}{}^A
\end{aligned}$$

$$\begin{aligned}
\delta(\theta)A^A_\mu &= \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A, \\
\delta(\theta)\lambda^A &= \theta^C\lambda^B f_{BC}{}^A, \\
\delta(\theta)D^A &= \theta^C D^B f_{BC}{}^A
\end{aligned}$$

$$\begin{aligned}
[\delta_1, \delta_2]\, A^A_\mu &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 F^A_{\nu\mu}, \\
[\delta_1, \delta_2]\, \lambda^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu\lambda^A, \\
[\delta_1, \delta_2]\, D^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu D^A.
\end{aligned}$$

11.1.3. Modified symmetry algebras: soft algebra

Not mathematical Lie algebra

- When extra gauge symmetries, gauged by the vector multiplets, the derivatives become covariant

$$\left[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\right]\phi = \xi^\mu D_\mu\phi = \xi^\mu\partial_\mu\phi - \xi^\mu A_\mu^A T_A\phi\,, \qquad \xi^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1$$

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^\mu)_{\alpha\beta}(P_\mu - A_\mu^A T_A)$$

$$f_{\alpha\beta}{}^A = \tfrac{1}{2}A_\mu^A(\gamma^\mu)_{\alpha\beta}\,,$$

The algebra is ‘soft’:

structure constants become structure functions.

Modified Jacobi identities

For a solution: become again constants.

Leads to e.g. AdS or central charges.

The chiral multiplet

- Ex. 6.11 : Consider the theory of the chiral multiplet after elimination of F. Show that the action

$$S = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \partial\!\!\!/ P_L \chi - \overline{W}' W' - \tfrac{1}{2} \bar{\chi} (P_L W'' + P_R \overline{W}'') \chi \right]$$

is invariant under the transformation rules

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi \,, & \delta \bar{Z} &= \frac{1}{\sqrt{2}} \bar{\epsilon} P_R \chi \\ \delta P_L \chi &= \frac{1}{\sqrt{2}} P_L (\partial\!\!\!/ Z + F) \epsilon, & \delta P_R \chi &= \frac{1}{\sqrt{2}} P_R (\partial\!\!\!/ \bar{Z} + \bar{F}) \epsilon \\ F &\equiv -\overline{W}'(\bar{Z}), & \bar{F} &= -W'(Z) \end{aligned}$$

Show that the commutator on the scalar is

$$[\delta_1, \delta_2] Z = -\tfrac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z$$

but is modified on the fermion as follows:

$$[\delta_1, \delta_2] P_L \chi = \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_L \left[-\tfrac{1}{2} \partial_\mu \chi + \tfrac{1}{4} \gamma_\mu (\partial\!\!\!/ + \overline{W}'') \chi \right]$$

We find the spacetime translation plus an extra term that vanishes for any solution of the equations of motion.

Calculating the algebra

- Very simple on *Z*
- On fermions: more difficult; needs Fierz rearrangement
- With auxiliary field: algebra satisfied for all field configurations
  Without auxiliary field: satisfied modulo field equations.
- auxiliary fields lead to
  - transformations independent of e.g. the superpotential
  - algebra universal : ‘closed off-shell’
  - useful in determining more general actions
  - in local SUSY: simplify couplings of ghosts

The commutator of two symmetries of the action is a symmetry

A symmetry: $$S_{,i}\delta(\epsilon)\phi^i = 0\,, \qquad\qquad S_{,i} \equiv \frac{\delta S}{\delta \phi^i}$$

$$\cancel{S_{,ij}\delta(\epsilon_1)\phi^i\delta(\epsilon_2)\phi^j} + S_{,i}\delta(\epsilon_2)\delta(\epsilon_1)\phi^i = 0$$

$$-\ 1 \leftrightarrow 2$$

$$S_{,i}\,\underbrace{\left[\delta(\epsilon_2)\delta(\epsilon_1)\right]\phi^i}\ = 0$$

is a symmetry !

Zilch symmetries and open algebras

$$\delta_{\text{triv}}\phi^i = \epsilon\,\eta^{ij}\frac{\delta S}{\delta\phi^j}$$

$$\delta_{\text{triv}}S = \frac{\delta S}{\delta\phi^i}\epsilon\,\eta^{ij}\frac{\delta S}{\delta\phi^j} = 0 \quad \text{if} \quad \eta^{ij} = -\eta^{ji}$$

Therefore: transformations not uniquely determined.

But may include Zilch symmetries:

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,\phi^i = \text{susy algebra} + \eta^{ij}(\epsilon_1, \epsilon_2)\,\frac{\delta S}{\delta\phi^j}$$

‘Closed on-shell’ or ‘open algebra’

If basis without second term:

‘closed off-shell’, or ‘closed algebra'.

11.2 Covariant quantities

- Terminology: gauge fields ↔matter fields.
- For the latter $\delta(\epsilon)\phi^i(x) = \epsilon^A (T_A \phi^i)(x)$

  do not involve derivatives of the gauge parameters.

A **covariant quantity** is a local function that transforms under all local symmetries with no derivatives of a transformation parameter.

Note for below: special care needed for local translations.

Will be discussed afterwards.

Covariant derivatives and curvatures

$$\begin{aligned}\mathcal{D}_\mu\phi^i &\equiv (\partial_\mu - \delta(B_\mu))\,\phi^i \\ &= \left(\partial_\mu - B_\mu^A T_A\right)\phi^i\,.\end{aligned}$$

is a covariant quantity.

Stronger: Gauge transformations and covariant derivatives commute on fields on which the algebra is off-shell closed.

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= -\delta(R_{\mu\nu})\,, \\ R_{\mu\nu}{}^A &= 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A\,.\end{aligned}$$

is a covariant quantity.

Remark as intro on (super)-Poincaré gauge theory

(anti)commutators	structure constants	third parameter
$\left[M_{\{ab\}}, M_{\{cd\}}\right] = 4\eta_{[a[c}M_{\{d]b]\}}$	$f_{\{ab\}\{cd\}}{}^{\{ef\}} = 8\eta_{[c[b}\delta^{[e}_{a]}\delta^{f]}_{d]}$	$\lambda_3^{ab} = 2\lambda_1{}^a{}_c\lambda_2^{cb}$
$\left[P_a, M_{\{bc\}}\right] = 2\eta_{a[b}P_{c]}$	$f_{a,\{bc\}}{}^d = 2\eta_{a[b}\delta^d_{c]}$	$\xi_3^a = -\lambda_2^{ab}\xi_{1b} + \lambda_1^{ab}\xi_{2b}$
$[P_a, P_b] = 0$		
$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}(\gamma^a)_{\alpha\beta}P_a$	$f_{\alpha\beta}{}^a = -\frac{1}{2}(\gamma^a)_{\alpha\beta}$	$\xi_3^a = \frac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1$
$\left[M_{\{ab\}}, Q\right] = -\frac{1}{2}\gamma_{ab}Q$	$f_{\{ab\},\alpha}{}^\beta = -\frac{1}{2}(\gamma_{ab})_\alpha{}^\beta$	$\epsilon_3 = \frac{1}{4}\lambda_1^{ab}\gamma_{ab}\epsilon_2 - \frac{1}{4}\lambda_2^{ab}\gamma_{ab}\epsilon_1$
$[P_a, Q] = 0$		

$$R_{\mu\nu}{}^A = 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A$$

translations : $R_{\mu\nu}{}^a = 2\partial_{[\mu}e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}e_{\nu]b} + \frac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu$

We will further define the spin connection such that R(P)=0 !

11.3.1 Gauge transformations for the Poincaré group

Poincaré on scalars

$$\delta(a,\lambda)\phi(x) = \left[a^\mu\partial_\mu - \tfrac{1}{2}\lambda^{\mu\nu}L_{[\mu\nu]}\right]\phi(x) = \left[a^\mu + \lambda^{\mu\nu}x_\nu\right]\partial_\mu\phi(x) = \xi^\mu(x)\partial_\mu\phi(x)$$

Orbital part can be included in $\xi^\mu(x)$.
Is change of basis from $a^\mu(x)$ and $\lambda^{ab}(x)$ to
$\xi^\mu(x) = a^\mu(x) + \lambda^{\mu\nu}(x)x_\nu$ and $\lambda^{ab}(x)$.

spinors: global

$$\delta(a,\lambda)\Psi(x) = \left[a^\mu + \lambda^{\mu\nu}x_\nu\right]\partial_\mu\Psi(x) - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\Psi(x)$$

Local

$$\delta(\xi,\lambda)\Psi(x) = \xi^\mu(x)\partial_\mu\Psi(x) - \tfrac{1}{4}\lambda^{ab}(x)\gamma_{ab}\Psi(x)$$

Vectors

$$\begin{aligned}
\delta(\xi,\lambda)V_\mu(x) &= \xi^\nu(x)\partial_\nu V_\mu(x) + V_\nu(x)\partial_\mu\xi^\nu(x) \\
\delta(\xi,\lambda)V_a &= \xi^\mu(x)\partial_\mu V_a(x) + V_b(x)\lambda^b{}_a(x)\,.
\end{aligned}$$

Lesson:
Local Poincaré transformations

- Local translations are replaced by general coordinate transformations.
- Local Lorentz transformations: Only fields carrying local frame indices transform under local Lorentz transformations. The transformation rule involves the appropriate matrix generator.

at the end : use $\xi^a = e^a{}_\mu \xi^\mu$ (and λ^{ab}) as parameters

11.3.2. Covariant derivatives and general coordinate transformations

There is a problem: $\mathcal{D}_\mu \phi = \partial_\mu \phi - e^a_\mu(x)\partial_a \phi(x) = 0$

1. Remove gct from the sum over all symmetries:
all the others are called 'standard gauge transformations'.

$$\begin{aligned} \mathcal{D}_\mu \phi^i &\equiv (\partial_\mu - \delta(B_\mu))\,\phi^i \\ &= \left(\partial_\mu - B^A_\mu T_A\right)\phi^i\,. \end{aligned}$$

2. We will always impose the constraint $R_{\mu\nu}(P^a)=0$

3. We replace translations with
'covariant coordinate transformations'

$$\delta_{\mathsf{cgct}}(\xi) = \delta_{\mathsf{gct}}(\xi) - \delta(\xi^\mu B_\mu)$$

Covariant general coordinate transformations

$$\delta_{\mathsf{cgct}}(\xi) \equiv \delta_{\mathsf{gct}}(\xi) - \delta(\xi^\mu B_\mu)$$

Action on various fields

- Scalars: $\delta_{\mathsf{cgct}}(\xi)\phi = \xi^\mu \mathcal{D}_\mu \phi = \xi^a \mathcal{D}_a \phi\,.$
- Gauge fields: $\delta_{\mathsf{cgct}}(\xi) B_\mu^A = \xi^\nu R_{\nu\mu}{}^A\,.$
- Frame field:

$$\delta_{\mathsf{cgct}}(\xi) e_\mu^a = \underbrace{\partial_\mu \xi^a + \xi^c B_\mu{}^B f_{Bc}{}^a}_{\text{gauge rule for translations}} - \cancel{\xi^\nu R_{\mu\nu}{}^a}\,.$$

11.3.3 Covariant derivatives and curvatures in a gravity theory

- Some gauge fields have extra (non-gauge) terms

$$\delta(\epsilon)B_\mu^A = \partial_\mu\epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A + \epsilon^B \mathcal{M}_{\mu B}{}^A$$

- E.g. $$\delta A_\mu^A = \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A - \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A$$
- Covariant curvature

$$\widehat{R}_{\mu\nu}^A = 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A - 2B_{[\mu}{}^B \mathcal{M}_{\nu]B}{}^A$$

- The covariant quantities in gravity have flat indices:

$$\mathcal{D}_a\phi = e_a{}^\mu \mathcal{D}_\mu\phi\,, \qquad \widehat{R}_{ab}^A = e_a{}^\mu e_b{}^\nu \widehat{R}_{\mu\nu}^A$$

- Gauge fields do not appear naked in covariant quantities: either in covariant derivative or in curvature.

Lesson:
Transformations of covariant quantities

1. The covariant derivative $\mathcal{D}_a$ of a covariant quantity is a covariant quantity, and so is the curvature $\widehat{R}_{ab}$
2. The gauge transformation of a covariant quantity does not involve a derivative of a parameter.
3. If the algebra closes on the fields, then the transformation of a covariant quantity is a covariant quantity,
   i.e. gauge fields only appear either included in covariant derivatives or in curvatures.

Lecture 5: Geometry and symmetries of supersymmetric theories and Kähler manifolds

1. The nonlinear σ-model (7.11)
2. Symmetries and Killing vectors (7.12)
3. Scalars and geometry (12.5)
4. Local description of complex and Kähler manifolds (13.1)
5. Mathematical structure of Kähler manifolds (13.2)
6. (if time allows): Symmetries of Kähler metrics (13.4)

Intro: 7. Differential geometry
7.2 Scalars, vector, tensors

$$
\begin{aligned}
\phi'(x') &= \phi(x) \\
U'^{\mu}(x') &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} U^{\nu}(x) \\
\omega'_{\mu}(x') &= \frac{\partial x^{\nu}}{\partial x'^{\mu}} \omega_{\nu}(x) \\
T'^{\mu}_{\nu}(x') &= \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} T^{\sigma}_{\rho}(x) \,.
\end{aligned}
$$

General coordinate transformations : infinitesimal $x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$

$$
\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = \mathcal{L}_{\xi}\phi = \xi^{\mu}\partial_{\mu}\phi \,, \\
\delta U^{\mu}(x) &\equiv U'^{\mu}(x) - U^{\mu}(x) = \mathcal{L}_{\xi}U^{\mu} = \xi^{\rho}\partial_{\rho}U^{\mu} - (\partial_{\rho}\xi^{\mu})U^{\rho} \,, \\
\delta\omega_{\mu}(x) &\equiv \omega'_{\mu}(x) - \omega_{\mu}(x) = \mathcal{L}_{\xi}\omega_{\mu} = \xi^{\rho}\partial_{\rho}\omega_{\mu} + (\partial_{\mu}\xi^{\rho})\omega_{\rho} \,, \\
\delta T^{\mu}_{\nu}(x) &\equiv T'^{\mu}_{\nu}(x) - T^{\mu}_{\nu}(x) = \mathcal{L}_{\xi}T^{\mu}_{\nu} = \xi^{\rho}\partial_{\rho}T^{\mu}_{\nu} - (\partial_{\rho}\xi^{\mu})T^{\rho}_{\nu} + (\partial_{\nu}\xi^{\rho})T^{\mu}_{\rho}
\end{aligned}
$$

7.3 The algebra and calculus of differential forms

definition and exterior derivative

$$\omega^{(p)} = \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\wedge\ldots \mathrm{d}x^{\mu_p}$$

$$\mathrm{d}\omega^{(p)} = \frac{1}{p!}\partial_\mu\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu}\wedge \mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\wedge\ldots \mathrm{d}x^{\mu_p}$$

insertion p → p-1 form

$$i_V\omega^{(p)} = \frac{1}{(p-1)!}V^\mu\omega_{\mu\mu_1\ldots\mu_{p-1}}\mathrm{d}x^{\mu_1}\ldots \mathrm{d}x^{\mu_{p-1}}$$

Lie derivative on forms

$$\mathcal{L}_V = \mathrm{d}i_V + i_V\mathrm{d}$$

Intro: 7.9 Connections and covariant derivatives

$$\begin{aligned} \nabla_\mu V^\rho &= \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu \,, \\ \nabla_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho \,, \end{aligned}$$

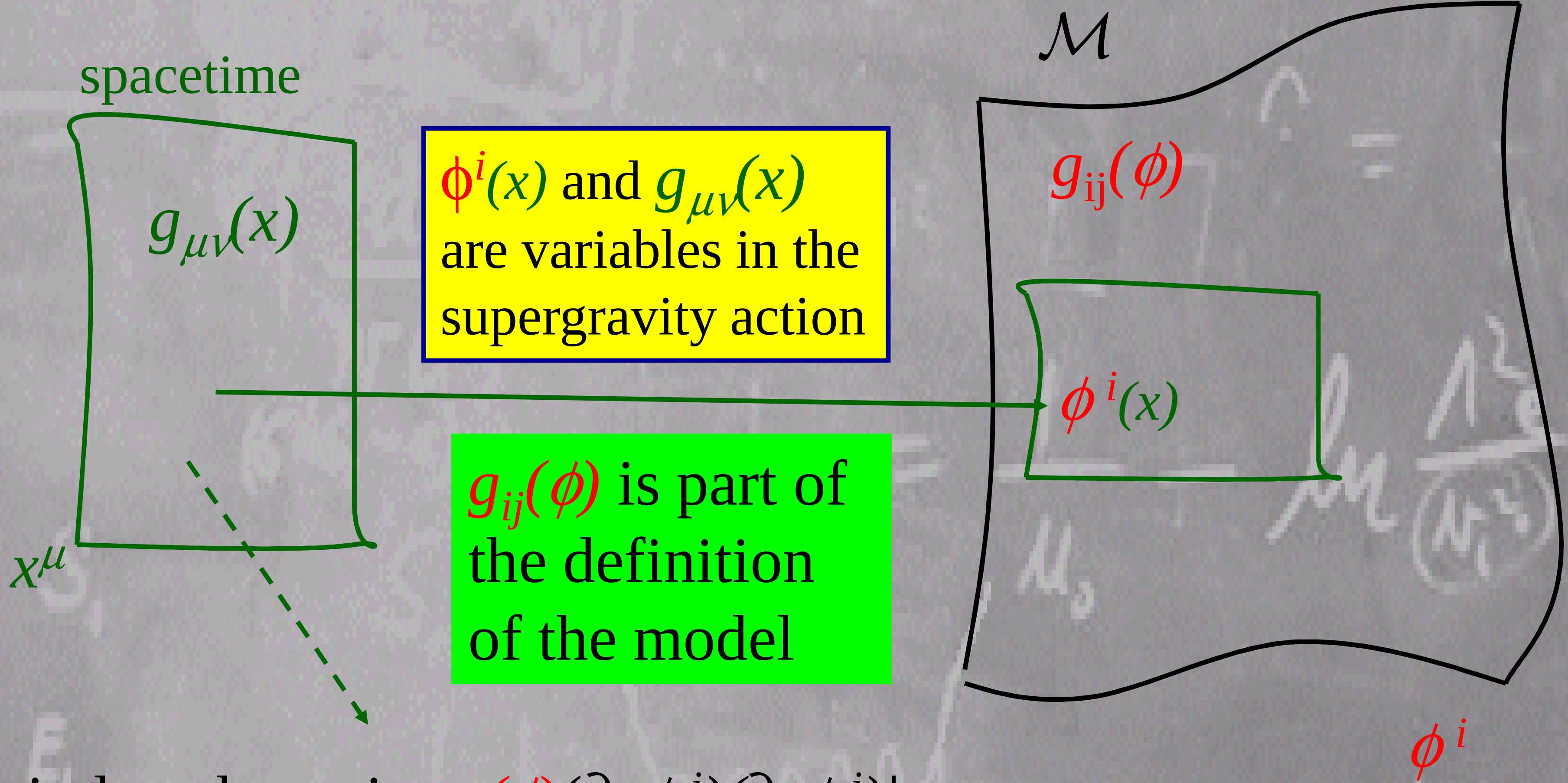
metric postulate

$$\nabla_\mu g_{\nu\rho} \equiv \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0$$

if there is no 'torsion' $\quad \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g) = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

7.11 The nonlinear σ-model



$\phi^i(x)$ and $g_{\mu\nu}(x)$ are variables in the supergravity action

$g_{ij}(\phi)$ is part of the definition of the model

induced metric $g_{ij}(\phi)(\partial_\mu\phi^i)(\partial_\nu\phi^j)|_{\phi=\phi(x)}$

appears in action $S[\phi] = -\frac{1}{2}\int \mathrm{d}^D x\, g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j$

7.12 Symmetries and Killing vectors
7.12.1 σ– model symmetries

Symmetries of action $S[\phi] = -\frac{1}{2}\int \mathrm{d}^D x\, g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j$

can be parametrized as a general form $\delta(\theta)\phi^i = \theta^A k_A{}^i(\phi)$

Each $k_A{}^i$ (for every value of A) should satisfy

$$\nabla_i k_{jA} + \nabla_j k_{iA} = 0\,, \qquad k_{iA} = g_{ij}k^j_A\,, \qquad \nabla_i k_{jA} = \partial_i k_{jA} - \Gamma^k_{ij}(g)k_{kA}$$

Solutions are called ‘Killing vectors’ $k_A \equiv k^j_A \frac{\partial}{\partial\phi^j}$

and satisfy an algebra $[k_A\,,\,k_B] = f_{AB}{}^C k_C$

7.12.2 Symmetries of the Poincaré plane

Poincaré plane (X, Y>0)

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2}$$

SL$(2,\mathbb{R})$ transformations act as nonlinear maps

$$Z \to Z' = \frac{aZ+b}{cZ+d} = X' + \mathrm{i}Y'$$

12.5 Scalars and geometry

- Scalar manifold can have isometries (symmetry of kinetic energy $ds^2 = g_{ij}\, d\phi^i\, d\phi^j$)
- usually extended to symmetry of all equations of motions

  ('U-duality group')
- The connection between scalars and vectors in the matrix $\mathcal{N}_{AB}(\phi)$ (or $f_{AB}(\phi)$)

  $\Rightarrow$ isometries act also as duality transformations
  $\Rightarrow$ restriction of possible U-duality groups:
  in $D=4$, $\mathcal{N} \geq 2$: U-duality group $\subset$ Sp($2m$)

  for theories with m vectors (from vector multiplets or supergravity mult.)
- A subgroup of the isometry group (at most of dimension m) can be gauged.

Homogeneous / Symmetric manifolds

- If isometry group G connect all points of a manifold $\rightarrow$ homogeneous manifold.
  Such a manifold can be identified with the coset G/H, where H is the isotropy group: group of transformations that leave a point invariant
- If the algebras $\mathfrak{g}$ of G and $\mathfrak{h}$ of H have the structure

$$\forall g \in \mathfrak{g} \; : \; g = h + k\,, \qquad h \in \mathfrak{h}\,, \qquad k \in \mathfrak{k}\,,$$

$$\forall h_1, h_2 \in \mathfrak{h},\; k_1, k_2 \in \mathfrak{k} \; : \; [h_1, h_2] \in \mathfrak{h}\,, \qquad [h_1, k_1] \in \mathfrak{k}\,, \qquad [k_1, k_2] \in \mathfrak{h}$$

then the manifold is symmetric.
The curvature tensor is covariantly constant

Geometries in supergravity

$$\mathcal{L} = \sqrt{g} g^{\mu\nu} (\partial_\mu \varphi^u)(\partial_\nu \varphi^v) g_{uv}(\varphi)$$

- Scalar manifolds for theories with more than 8 susys are symmetric spaces
- Scalar manifolds for theories with 4 susys ($\mathcal{N}=1$, $D=4$, or lower D) are Kähler
- Scalar manifolds for theories with 8 susys are called ‘special manifolds’.
  Include real, special Kähler, quaternionic manifolds
  They can be symmetric, homogeneous, or not even that

The map of geometries

- With > 8 susys: symmetric spaces

d	32	24	20	16		12
9	$\frac{S\ell(2)}{SO(2)} \otimes O(1,1)$			$\frac{O(1,n)}{O(n)} \otimes O(1,1)$		
8	$\frac{S\ell(3)}{SU(2)} \otimes \frac{S\ell(2)}{U(1)}$			$\frac{O(2,n)}{U(1)\times O(n)} \otimes O(1,1)$		
7	$\frac{S\ell(5)}{USp(4)}$			$\frac{O(3,n)}{USp(2)\times O(n)} \otimes O(1,1)$		
6	$\frac{O(5,5)}{USp(4)\times USp(4)}$	$\frac{SO(5,1)}{SO(5)}$		$\frac{O(4,n)}{O(n)\times SO(4)} \otimes O(1,1)$	$\frac{O(5,n)}{O(n)\times USp(4)}$	
5	$\frac{E_6}{USp(8)}$	$\frac{SU^*(6)}{USp(6)}$		$\frac{O(5,n)}{USp(4)\times O(n)} \otimes O(1,1)$		
4	$\frac{E_7}{SU(8)}$	$\frac{SO^*(12)}{U(6)}$	$\frac{SU(1,5)}{U(5)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SU(4)\times SO(n)}$		$\frac{SU(3,n)}{U(3)\times SU(n)}$

- 8 susys: very special, special Kähler and quaternionic spaces

$d=6$	$d=5$	$d=4$
$\frac{O(1,n)}{O(n)} \times \mathcal{QM}$	$\mathcal{VSR} \times \mathcal{QM}$	$\mathcal{SK} \times \mathcal{QM}$

U(1) part in holonomy group

SU(2)=USp(2) part in holonomy group

- 4 susys: Kähler: **U(1)** part in holonomy group

13. Complex manifolds

13.1 The local description of complex and Kähler manifolds

- Use complex coordinates

$$\{z^a\} = \{z^\alpha, \bar{z}^{\bar{\alpha}}\} \qquad a = 1, \ldots, 2n;\ \alpha, \bar{\alpha} = 1, \ldots, n$$

$$ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}$$

Hermitian manifold

define fundamental 2-form $\Omega = -2\mathrm{i} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$

Kähler manifold: closed fundamental 2-form

$$d\Omega = -\mathrm{i}(\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}}) dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + \text{c.c.} = 0$$

Properties of metric, connection, curvature for Kähler manifolds

- metric derivable from a 'Kähler potential'

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z})$$

- connections have only unmixed components

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\delta}} \partial_\beta g_{\gamma\bar{\delta}}\,, \qquad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = g^{\delta\bar{\alpha}} \partial_{\bar{\beta}} g_{\delta\bar{\gamma}}.$$

- curvature components related to

$R_{\bar{\delta}\gamma}{}^{\alpha}{}_{\beta} = \partial_{\bar{\delta}} \Gamma^{\alpha}_{\beta\gamma}$ (two holomorphic indiees up and down, and symmetric in these pairs)

- Ricci tensor $R_{ab} = g^{cd} R_{acbd} = R_{ba}$

$$R_{\alpha\bar{\beta}} = g^{\bar{\gamma}\gamma} R_{\alpha\bar{\gamma}\bar{\beta}\gamma} = -R_{\alpha\bar{\beta}\gamma}{}^{\gamma} = -\partial_\alpha \partial_{\bar{\beta}} (\log \det g_{\gamma\bar{\delta}})$$

13.2 Mathematical structure of Kähler manifolds

- starts from a complex structure
  - almost complex: tensor on tangent space $J_i{}^k J_k{}^j = -\delta_i{}^j$
  - Nijenhuis tensor vanishes. In presence of a torsion-free connection, this is implied by covariant constancy of complex structure $$\nabla_k J_i{}^j = \partial_k J_i{}^j - \Gamma^\ell_{ki} J_\ell{}^j + \Gamma^j_{k\ell} J_i{}^\ell = 0$$
  - metric hermitian : $JgJ^T = g$
    and Levi-Civita connection of this metric is used above
- Then the Kähler form is $\Omega = -J_{ij}\mathrm{d}\phi^i \wedge \mathrm{d}\phi^j\,, \qquad J_{ij} = J_i{}^k g_{kj}$
- In complex coordinates $z = (1 - \mathrm{i}J)\phi\,, \qquad \bar{z} = (1 + \mathrm{i}J)\phi$

$$J = \begin{pmatrix} \mathrm{i}\delta_\alpha{}^\beta & 0 \\ 0 & -\mathrm{i}\delta_{\bar{\alpha}}{}^{\bar{\beta}} \end{pmatrix}.$$

13.4 Symmetries of Kähler metrics

13.4.1 Holomorphic Killing vectors and moment maps

$$\delta\phi^i = \theta k^i(\phi) \qquad \text{or} \qquad \delta z^\alpha = \theta k^\alpha(z,\bar z)$$

- require vanishing Lie derivatives of metric *and* of complex structure.
- Implies that in complex coordinates
  - the Killing vector is holomorphic
  - Lie derivative of Killing form vanishes

    $\rightarrow$ Killing vectors determined by real moment map $\mathcal{P}$

PS: a Kähler manifold is a symplectic manifold due to the existence of the Kähler 2-form. Moment map is generating function of a canonical transformation

$$\begin{aligned}
0 &= \mathcal{L}_k\Omega = (i_k\mathrm{d} + \mathrm{d}i_k)\,\Omega = \mathrm{d}i_k\Omega \\
i_k\Omega &= -2\mathrm{d}\mathcal{P} \\
k_\alpha &= g_{\alpha\bar\beta}\,k^{\bar\beta}(\bar z) = \mathrm{i}\partial_\alpha\,\mathcal{P}(z,\bar z)\,, \\
k_{\bar\alpha} &= g_{\beta\bar\alpha}\,k^{\beta}(z) = -\mathrm{i}\partial_{\bar\alpha}\,\mathcal{P}(z,\bar z)\,.
\end{aligned}$$

Kähler transformations and the moment map

- Kähler potential is not unique: $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z,\bar{z})$
- Kähler transformations

$$K(z,\bar{z}) \to K'(z,\bar{z}) = K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$$

- Also for symmetries

$$\delta K = \theta \left(k^\alpha \partial_\alpha + k^{\bar{\alpha}} \partial_{\bar{\alpha}} \right) K(z,\bar{z}) = \theta \left[r(z) + \bar{r}(\bar{z}) \right]$$

$$\mathcal{P}(z,\bar{z}) = \mathrm{i} \left[k^\alpha \partial_\alpha K(z,\bar{z}) - r(z) \right] = -\mathrm{i} \left[k^{\bar{\alpha}} \partial_{\bar{\alpha}} K(z,\bar{z}) - \bar{r}(\bar{z}) \right] .$$

Exercises on duality

- dual of dual is identity
- Electromagnetic $E_i = F_{i0}\,, \qquad B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$
  ex. 4.8: recognise the transformations
  $$F^{\mu\nu} \to F'^{\mu\nu} = \frac{\mathrm{i}}{g^2}\tilde{F}^{\mu\nu}\,.$$
  Take f_{AB} = -i Z = 1/g^2 and find the duality transformation that gives this transformation. How does Z change under a general duality transformation ?
- How does g change under the specific one of electromagnetic duality ? (related to ex. 4.15)

Exercise on covariant derivatives

- Symmetries of the nonlinear σ-model are generated by Killing vectors $k_A{}^i(\phi)$.
  Suppose that the symmetry is gauged.
  Show that the covariant derivative

$$\mathcal{D}_\mu \phi^i = \partial_\mu \phi^i - A_\mu^A k_A^i$$

transforms as

$$\delta \mathcal{D}_\mu \phi^i = \theta^A \mathcal{D}_\mu k_A^i = \theta^A \left(\partial_j k_A^i\right) \mathcal{D}_\mu \phi^j .$$

1. using the theorems
2. doing the full calculation

Exercise on chapter 7

- **Ex. 7.48**: Consider for the Poincaré plane Z and $\bar{Z}$ as the independent fields, rather than X and Y, and use the line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2}$$

The metric components are

$$g_{ZZ} = g_{\bar{Z}\bar{Z}} = 0\,, \qquad g_{Z\bar{Z}} = g_{\bar{Z}Z} = -\frac{2}{(Z-\bar{Z})^2}$$

Show that the only non-vanishing components of the Christoffel connection are $\Gamma_{ZZ}{}^{Z}$ and its complex conjugate. Calculate them and then show that there are three Killing vectors,

$$k_1^Z = 1\,, \qquad k_2^Z = Z\,, \qquad k_3^Z = Z^2$$

each with conjugate. Show that their Lie brackets give a Lie algebra whose non-vanishing structure constants are

$$f_{12}{}^{1} = 1\,, \qquad f_{13}{}^{2} = 2\,, \qquad f_{23}{}^{3} = 1$$

This is a standard presentation of the Lie algebra of

$$\mathfrak{su}(1,1) = \mathfrak{so}(2,1) = \mathfrak{sl}(2)$$

Exercises on chapter 13

- Ex. 13.14: Show that the metric of the Poincaré plane of complex dimension 1 is a Kähler metric.
  What is the Kähler potential?
- Ex. 13.18: Consider CP^{1}with Kähler potential $K = \ln(1 + z\bar{z})$
  - Check that there are 3 Killing vectors
  - that satisfy the su(2) algebra

$$[k_A\,,\,k_B] = \varepsilon_{ABC}k_C$$

$$\begin{aligned} k_1 &= -\mathrm{i}\frac{1}{2}(1-z^2)\frac{\partial}{\partial z} + c.c.\,, \\ k_2 &= \frac{1}{2}(1+z^2)\frac{\partial}{\partial z} + c.c.\,, \\ k_3 &= -\mathrm{i}z\frac{\partial}{\partial z} + c.c.\,. \end{aligned}$$

- Ex. 13.20: Apply $\delta K = \theta^A\left(k_A^z\partial_z + k_A^{\bar{z}}\partial_{\bar{z}}\right)K(z,\bar{z}) = \theta^A\left[r_A(z) + \bar{r}_A(\bar{z})\right]$
  to obtain $r_1 = \frac{1}{2}\mathrm{i}z\,, \qquad r_2 = \frac{1}{2}z\,, \qquad r_3 = -\frac{1}{2}\mathrm{i}$

  Note that the Kähler potential is invariant under k_3, but still $r_3 \neq 0$.
  Its value is fixed by the 'equivariance relation'

$$k_A{}^\alpha g_{\alpha\bar{\beta}}k_B{}^{\bar{\beta}} - k_B{}^\alpha g_{\alpha\bar{\beta}}k_A{}^{\bar{\beta}} = \mathrm{i}f_{AB}{}^C\mathcal{P}_C$$