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Gravity, Geometry and Physics

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Abstract

Geometry is present in physics at many levels, most prominently in the theory of gravity. In these introductory lectures geometrical concepts like manifolds, geodesics, curvature and topology are introduced as a tool to describe and interpret physical phenomena in space-time, including the gravitational field itself. The focus is not only on possible static structures, e.g. black holes, but also on dynamical effects like waves and traveling domain walls. Although quantum gravity itself is outside the scope of these lectures, it is briefly discussed how geometry can be used even in that context to characterise configurations dominating the path-integral in various circumstances, including typical quantum processes like vacuum tunneling. Many of these geometrical methods can also be used in other branches of physics by linking the dynamics of a system with the geometry of its configuration space; some examples are mentioned.

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Chapter 1

Gravity and Geometry

1.1 The gravitational force

Gravity is the most universal force in nature. As far as we can tell from observations and experiments every object, every particle in the universe attracts any other one by a force proportional to its mass. For slow moving bodies at large distances this is a central force, inversely proportional to the square of the distance. As the action is reciprocal, and since according to Newton action and reaction forces are equal in magnitude, the expression for the gravitational force between two objects of mass M_1 and M_2 at a distance R is then determined to have the unique form

$$F = G \frac{M_1 M_2}{R^2}. \tag{1.1}$$

The constant of proportionality, Newton's constant of gravity, has dimensions of acceleration per unit of mass times an area. Therefore its numerical value obviously depends on the choice of units. In the MKS system this is

$$G = 6.672\,59(85) \times 10^{-11} \, \mathrm{m}^3 \, \mathrm{kg}^{-1} \, \mathrm{s}^{-2}. \tag{1.2}$$

It is also possible, and sometimes convenient, to fix the unit of mass in such a way that Newtons constant has the numerical value $G = 1$. In the natural system of units, in which also the velocity of light and Planck's constant are unity: $c = \hbar = 1$, this unit of mass is the Planck mass m_P:

$$\begin{aligned} m_P = \sqrt{\hbar c / G} &= 2.176\,71 \times 10^{-8} \, \mathrm{kg} \\ &= 1.221\,047 \times 10^{19} \mathrm{GeV}/c^2. \end{aligned} \tag{1.3}$$

Newton's law of gravity (1.1) is valid for any two massive bodies, as long as they are far apart and do not move too fast with respect to one another. In particular, it describes the motions of celestial bodies like the moon circling the earth, or the planets orbiting the sun, as well as those of terrestrial objects like apples falling from a tree, or canon balls in free flight. Ever since Newton this unification of celestial and terrestial mechanics has continued to impress people and has had a tremendous impact on our view of the universe. It is the origin and basis for the belief in the general validity of physical laws independent of time and place.

1.2 Fields

Although as a force gravity is universal, Newton's law (1.1) itself has only limited validity. Like Coulomb's law for the electrostatic force between two fixed charges, Newton's law holds strictly speaking only for static forces between bodies at rest. Moreover, and unlike the electric forces, there are modifications at smaller, finite distances which can be observed experimentally.

For example, if the gravitational force would have a pure $1/R^2$ dependence, the orbits of particles around a very heavy central body would be conic sections: ellipses, parabola's or hyperbola's, depending on the energy and angular momentum, in accordance with Kepler's laws. The observation of an excess in the precession of the perihelion of the orbit of Mercury around the sun by LeVerrier in 1845, and improved by Newcomb in 1882 [1], was one of the first clear indications that this is actually not the case, and that the gravitational force is more complicated.

The exact form of the gravitational forces exerted by moving bodies is a problem with many similarities to the analogous problem in electrodynamics. The understanding of electrodynamical phenomena greatly improved with the introduction of the concept of local field of force. This concept refers to the following characteristics of electrodynamical forces:

- the influence of electric charges and currents, and of magnetic poles, extends throughout empty space;
- the force on a standard test charge or test magnet at any given time depends on its location with respect to the source of the field and the relative state of motion;
- changes in the sources of the fields give rise to changes in the force on test objects at a later time, depending on the distance; the speed of propagation of disturbances in empty space is finite.

In the case of electrodynamics this speed turned out to be a universal constant, the speed of light c. One of the most striking consequences of these properties, which follow directly from the mathematical description of the fields as expressed by Maxwell's equations, is the existence of electromagnetic waves, local variations in the fields which transport energy, momentum and angular momentum over large distances through empty space. Maxwell's predictions were magnificently verified in the experiments of Hertz, confirming the reality and characteristics of electromagnetic waves [2]. From these results it became clear, for example, that light is just a special variety of such waves. Therefore optical and electromagnetic phenomena have a common origin and can be described in a single theoretical framework.

The concept of field has found other applications in physics, for example in the phenomenological description of fluids and gases. Treating them as continua, one can describe the local pressure and temperature as fields, with local variations in these quantities propagating through the medium as sound waves or heat waves. As in electrodynamics, also in gravity the concept of field has taken a central place. What Maxwell achieved for electromagnetism, was accomplished by Einstein in the case of gravity [3]: to obtain a complete description of the forces in terms of space- and time-dependent fields. This field theory is known as the general theory of relativity. One of the most remarkable aspects of this theory is, that it provides an interpretation of gravitational phenomena in terms of the geometry of space-time. Many detailed presentations of the geometrical description of space-time can be found in the literature, including a number of books included in the references at the end of these lecture notes [6]-[14]. A general outline of the structure of the theory is presented in the following sections.

1.3 Geometrical interpretation of gravity

The first step in describing a local field, both in electrodynamics and gravity, is to specify the potentials. The force on a test particle at a certain location is then computed from the gradients of these potentials. In the case of gravity these quantities have not only a dynamical interpretation, but also a geometrical one. Whereas the electromagnetic field is described in terms of one scalar and one vector potential, together forming the components of a four vector[1] $A_\mu(x)$, the gravitational field is described by 10 potentials, which can be assembled in a symmetric four tensor $g_{\mu\nu}(x)$. This tensor has a geometrical interpretation as the metric of space-time, determining invariant space-time intervals ds in terms of local coordinates x^μ:

$$ds^2 \,=\, g_{\mu\nu}(x)\, dx^\mu dx^\nu. \tag{1.4}$$

In the absence of gravity, space-time obeys the rules of Minkowski geometry well-known from special relativity. In cartesian coordinates (ct, x, y, z) the metric of Minkowski space is

$$\eta_{\mu\nu} \,=\, \mathrm{diag}(-1, +1, +1, +1). \tag{1.5}$$

[1] We use greek letters $\mu, \nu, ...$, taking values (0,1,2,3), to denote the space-time components of four vectors and tensors.

Then the invariant ds^2 takes the form

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.6}$$

For a test particle, there is a simple physical interpretation of invariant space-time intervals measured along its world line, in terms of proper time measured by a clock in the local rest frame of the particle. As world lines are time-like, any interval along a world line satisifies $ds^2 < 0$, and the proper time interval $d\tau$ measured by a clock at rest with respect to the particle is given by

$$c^2 d\tau^2 = -ds^2. \tag{1.7}$$

The local matrix inverse of the metric is denoted by $g^{\mu\nu}(x)$:

$$g^{\mu\nu}(x)\, g_{\nu\lambda}(x) = \delta^\mu_\lambda, \tag{1.8}$$

where δ^μ_λ is the Kronecker delta, and the Einstein summation convention has been used. This convention implies that one should automatically perform a full summation over any repeated upper and lower indices, like ν on the left-hand side of this equation.

¿From the gradients of the potentials, i.e. of the metric tensor, one constructs a quantity known as the connection, with components $\Gamma_{\mu\nu}^{\ \ \lambda}$ given by the expression:

$$\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right). \tag{1.9}$$

At this point this expression may simply be considered a definition. It is a useful definition because this combination of gradients of the metric occurs very often in equations in general relativity. The deeper mathematical reason for this stems from the transformation properties of expressions for physical quantities under local coordinate transformations. However, it is not necessary to discuss these aspects at this point.

A physical argument for the importance of the connection is, that it gives directly the gravitational acceleration of a test particle in a given coordinate system by the following prescription, due to Einstein: the world lines of point-like test particles in general relativity are time-like geodesics of space-time, the latter being considered as a continuous pseudo-euclidean space[2] with metric $g_{\mu\nu}$. Time-like geodesics are lines along which the total proper time between two fixed events $\int d\tau$ is an extremum; such world lines are solutions of the equation of motion

[2]The term pseudo-euclidean here refers to the fact that the metric is not positive-definite, as it would be in a true euclidean space; space-time has a lorentzian signature $(-, +, +, +)$, which implies that a general metric $g_{\mu\nu}$ has one negative (time-like) and three positive (space-like) eigenvalues.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\lambda\nu}{}^{\mu}\,\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \,=\, 0. \tag{1.10}$$

A geometrical interpretation of the connection is therefore that it defines geodesics. As concerns its dynamical interpretation, there is some similarity of eq.(1.10) with the Lorentz force in electrodynamics in that the proper acceleration depends on the four velocity $\dot{x}^\mu$ (we use an overdot to denote a derivative w.r.t. proper time). However, whereas the Lorentz force is linear in the four velocity, the gravitational acceleration depends quadratically on the four velocity. The coefficients of this quadratic term are given by the connection, which therefore directly determines the force. Notice, that the gravitational acceleration does not involve the mass of the test particle. This is the mathematical content of the famous *equivalence principle*, which states that in a fixed gravitational field all bodies fall with the same acceleration, independent of their mass.

Eqs.(1.10) and (1.9) express the force on a test particle in terms of a field (the connection), and the field in terms of potentials (the metric). This provides the basis for the field theory of gravity formulated by Einstein in his theory of general relativity.

What is still lacking here of course is a set of equations determining the fields or potentials in terms of given sources, like Maxwell's equations allow one to compute the electromagnetic fields in terms of charges and currents. General relativity provides such equations; they are known as the Einstein equations, and they determine the gravitational fields arising from given distributions of mass-energy and momentum. These distributions, the sources of the gravitational fields, are described by a symmetric tensor $T_{\mu\nu}(x)$ with 10 independent components, equal to the number of independent components of the metric; $T_{\mu\nu}$ is called the energy-momentum tensor. In order to give a concise formulation of Einstein's equations, it is useful to first introduce one more geometrical concept, that of curvature.

1.4 Curvature

Curvature is a important concept, because it characterizes the space-time geometry in a way which is basically coordinate independent. In order to illustrate the idea, we first present a heuristic discussion. Then we turn to mathematical definitions, which allow us to write Einsteins equations in a compact way with a direct and elegant geometrical interpretation.

Curvature is defined by the properties of surfaces; it is a measure of their deviation from a flat plane. Curvature can be expressed conveniently in terms of the change of a vector when transported parallel to itself around a loop enclosing a two-dimensional surface element. In a plane, displacement of a vector parallel to itself will always result in the same vector when you return to the starting point: there is absolute parallelism, but no curvature. In a curved surface this is no longer true. Consider the example of a sphere, a surface of constant curvature. If a vector is moved southward from the north pole parallel to itself, along a great circle, it ends up on the equator cutting it at a right angle. Now transport it along the equator over an arc of angular extension θ, always keeping at a right angle. Finally, move it back to the north pole along a great circle, keeping it parallel to the great circle and to itself. It then ends up at the north pole rotated by an angle θ with respect to its original orientation. Thus parallel transport of a vector along a closed loop on a curved surface does not leave the vector invariant in general.

In the example of the sphere, the rotation of the vector is proportional to the distance traveled along the equator, as measured by the angle θ. This is also proportional to the area enclosed by the loop consisting of the two sections of great circles through the north pole and the arc of the equator. For a sphere with radius R, the change dV in a vector V when the loop encloses an area $dA = R^2 d\theta$ is

$$dV = V d\theta = \frac{V}{R^2}\, dA \equiv K V dA. \tag{1.11}$$

The constant of proportionality $K = 1/R^2$ between the relative change in the vector dV/V and the change in the area dA is a measure of the intrinsic curvature of the surface, known as the Gaussian curvature. Note, that the Gaussian curvature is large when the radius of the sphere is small, and that it vanishes in the limit $R \rightarrow \infty$, as one would expect intuitively.

This idea can now be generalized to situations in higher-dimensional spaces. Of course, in a space of three or more dimensions, there are many surfaces one can draw through a given point. Each of these surfaces may have its own curvature. Our first conclusion is therefore, that in general the curvature has many components.

Fortunately, it is only necessary to consider curvature components in surfaces which are linearly independent. To this end, let us choose a locally non-degenerate coordinate system x^μ and consider in particular the surfaces defined by keeping all coordinates constant except two, say x^μ and x^ν.

Let $d\Sigma^{\mu\nu} = -d\Sigma^{\nu\mu}$ be the area of an oriented surface element in the x^μ-x^ν-plane which contains the point with coordinates (x_0). Any vector with components V^λ transported parallel to itself around the loop defined by the circumference of this surface element now changes to first order by an amount

$$dV^\lambda = -\frac{1}{2}\, R_{\mu\nu\kappa}{}^{\lambda}(x_0)\, V^\kappa d\Sigma^{\mu\nu}. \tag{1.12}$$

The constants of proportionality $R_{\mu\nu\kappa}{}^{\lambda}(x_0)$ define a tensor known as the Riemann curvature tensor at the location (x_0). A comparison with eq.(1.11) shows, that the

Riemann tensor is a higher-dimensional generalization of the Gaussian curvature. From eq.(1.12) it follows immediately, that a space-time is completely flat if and only if all components of the Riemann curvature tensor vanish.

Defining parallel transport of a vector along a line element dx^λ in terms of the Riemann-Christoffel connection (1.9):

$$dV^\mu \;=\; -dx^\lambda \, \Gamma_{\lambda\nu}^{\;\;\;\mu} \, V^\nu, \tag{1.13}$$

a direct computation for an infinitesimal closed loop gives the result that the components of the Riemann tensor can be expressed as

$$R_{\mu\nu\kappa}^{\quad\;\lambda} \;=\; \partial_\mu \Gamma_{\nu\kappa}^{\;\;\;\lambda} \,-\, \partial_\nu \Gamma_{\mu\kappa}^{\;\;\;\lambda} \,-\, \left[\Gamma_\mu, \Gamma_\nu\right]_\kappa^{\;\;\lambda}, \tag{1.14}$$

where the square brackets denote an ordinary matrix commutator of two connection components, considered as 4×4 matrices $(\Gamma_\alpha)_\kappa^{\;\;\lambda}$, the first one with $\alpha = \mu$, the second one with $\alpha = \nu$.

Using eq.(1.9) for the connection, we can ultimately express the curvature in terms of the metric and its first and second derivatives. It may then be checked from this explicit expression that it identically satisfies the relation

$$R_{\mu\nu\lambda}^{\quad\;\kappa} \,+\, R_{\nu\lambda\mu}^{\quad\;\kappa} \,+\, R_{\lambda\mu\nu}^{\quad\;\kappa} \;=\; 0. \tag{1.15}$$

This relation is known as the Bianchi identity. It is analogous to the homogeneous Maxwell-equation, which implies that the electric and magnetic fields can be obtained from a four-vector potential A_μ. In a similar way the Bianchi identity for the curvature tensor implies that the curvature tensor can be obtained from a symmetric tensor $g_{\mu\nu}$.

We wish to emphasize here, that the curvature of space-time is *not* a measure for the strength of the gravitational fields or forces; rather, it is a measure for the *variation* of the gravitational fields in space and time, the gradients.

1.5 The Einstein equations

The Einstein equations are second order non-linear partial differential equations for the metric of space-time in terms of a given energy-momentum distribution. As a result, these equations describe variations of the fields in space and time, rather than the fields themselves. In view of the role of curvature as a measure for such variations and the geometrical interpetation of gravitational forces, it is quite natural that the Einstein equations should admit an interpretation as equations for the curvature of space-time. However, rather than the full Riemann tensor, it turns out that these equations involve only its contracted forms, the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R:

$$R_{\mu\nu} \;=\; R_{\mu\lambda\nu}^{\quad\;\lambda}, \qquad\qquad R \;=\; R_\mu^{\;\mu} \;=\; g^{\mu\nu}\, R_{\mu\nu}. \tag{1.16}$$

Note that we adhere to the general convention that contraction with an inverse metric can be used to raise an index of a vector or tensor, turning a covariant component into a contravariant one; similarly contraction of an upper index on a vector or tensor with the metric itself is used to lower it to make a covariant component out of a contravariant one.

The reason that only the contracted forms of the Riemann curvature appear in the Einstein equations is, that in four-dimensional space-time the Riemann curvature tensor has too many components, twice as many as the metric. Therefore fixing the metric components in terms of the full Riemann tensor would generally lead to an overdetermined system of equations. In contrast, the Ricci tensor is symmetric: $R_{\mu\nu} = R_{\nu\mu}$, and has 10 independent components, precisely equal to the number of metric components to be solved for. It is therefore to be expected that the Einstein equations take the form of an equation for the Ricci tensor. Indeed, Einstein's equations for the gravitational fields can be written in the simple form

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -\frac{8\pi G}{c^4}\, T_{\mu\nu}, \tag{1.17}$$

or equivalently:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T \right), \tag{1.18}$$

where $T = T_\mu^{\ \mu} = g^{\mu\nu} T_{\mu\nu}$. Notice the appearance of Newton's constant on the right-hand side of the equations. Of course, in empty space-time $T_{\mu\nu} = 0$; the second version of Einstein's equations (1.18) then implies that the Ricci tensor vanishes:

$$R_{\mu\nu} = 0. \tag{1.19}$$

Space-time geometries which satisfy these conditions are some-times called *Ricci-flat*. The most interesting aspect of these equations is, that an empty space-time can still support non-trivial gravitational fields: the Riemann curvature tensor can have non-zero components even if its contracted form, the Ricci tensor, is zero everywhere.

This observation is important, because it lies at the heart of the theoretical arguments for the existence of gravitational waves. Indeed, as we discuss in more detail later on, one can find non-trivial solutions of the Einstein equations representing lumps in the gravitational field which travel at the speed of light and transport a finite amount of energy and momentum per unit of volume from one flat part of empty space to another flat part of empty space.

There is a modified version of Einstein's equations in which the gravitational field itself creates a non-vanishing energy-momentum tensor, even in the absence of matter. This gravitational field behaves like a homogeneous continuum with constant density and pressure (one of them negative), satisfying the equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\, g_{\mu\nu}\Lambda, \tag{1.20}$$

with Λ a constant. Such an equation arises if the energy-momentum tensor is

$$T_{\mu\nu} = g_{\mu\nu}\Lambda. \tag{1.21}$$

The constant Λ is known as the *cosmological constant.* Contraction of eq.(1.20) with the inverse metric gives

$$R = \frac{32\pi G}{c^4}\,\Lambda. \tag{1.22}$$

This shows that in the presence of a cosmological constant empty space-time has a constant non-zero scalar curvature. Depending on the sign of the cosmological constant, eq.(1.20) then admits solutions of Einstein's equations corresponding to empty space-times in continuous expansion or contraction. This modified version of Einstein's equations is especially interesting with regard to cosmological applications.

Of course in the general case there is a contribution to the energy-momentum tensor from all matter and energy which happens to be present in the part of the universe we wish to describe. The precise form of the energy-momentum tensor depends on the kind of matter or radiation which constitutes the source of the gravitational field. However, it is very often justified and convenient to treat matter and radiation as an ideal fluid, the energy-momentum tensor of which depends only on the mass- and pressure-density and its flow, as decribed by the local four velocity of an element of fluid at location with coordinates (x). Denoting the mass density by $\rho(x)$, the pressure density by $p(x)$, and the local four velocity by $u^\mu(x)$, the energy-momentum tensor then takes the form

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)\, u_\mu u_\nu. \tag{1.23}$$

Comparison with eq.(1.21) shows, that the cosmological constant can be reobtained from an ideal fluid with $p = -\rho = \Lambda$.

1.6 The action principle

The Einstein equations (1.17) can be obtained from a variational principle; that is, they follow by requiring an action S to be stationary under variations $\delta g_{\mu\nu}$ of the metric: $\delta S = 0$. The action for the gravitational field equations, including a coupling to material energy-momentum, is

$$S = -\frac{1}{2c}\int d^4x\,\sqrt{-g}\,\left(\frac{c^4}{8\pi G}\,R + g^{\mu\nu}\Theta_{\mu\nu}\right). \tag{1.24}$$

Here $\Theta_{\mu\nu}(x)$ represents the specific source term, but generally it is not the energy-momentum tensor itself; rather, the energy-momentum tensor is given in terms of $\Theta_{\mu\nu}$ and its covariant trace $\Theta = \Theta_{\mu}^{\ \mu}$ by

$$T_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\Theta. \tag{1.25}$$

However, we consider $\Theta_{\mu\nu}$ to be the independent variable, hence by fiat there is no contribution from this tensor under variations with respect to the metric. Moreover, in eq.(1.24) $g(x) = \det g_{\mu\nu}(x)$, and the minus sign is necessary because of the lorentzian signature of the metric, which implies the product of its eigenvalues to be negative everywhere. The proof of the action principle can be stated in a few lines, but it requires some intermediate results which we discuss first.

To begin with, in stead of considering variations of the metric $g_{\mu\nu}$, one can equivalently use the variations of the inverse metric $g^{\mu\nu}$. Indeed, because $g_{\mu\lambda}g^{\lambda\nu} = \delta_{\mu}^{\lambda}$, it follows that

$$\delta g_{\mu\lambda} g^{\lambda\nu} \, + \, g_{\mu\lambda}\delta g^{\lambda\nu} \, = \, 0, \tag{1.26}$$

or

$$\delta g_{\mu\nu} \, = \, -g_{\mu\lambda}\delta g^{\lambda\kappa} g_{\kappa\nu}. \tag{1.27}$$

Thus the variation of the metric can be expressed in terms of the variation of the inverse metric and vice versa.

Next, we need a rule for computing the variation of the determinant g. This is obtained from the formula

$$\log\det M = \mathrm{Tr}\log M, \tag{1.28}$$

which holds for symmetric matrices M, as can easily be checked by diagonalizing M and using the rule that the logarithm of the product equals the sum of the logarithms of the eigenvalues. From this equation we then derive

$$\frac{1}{-g}\,\delta(-g) \, = \, g^{\mu\nu}\delta g_{\mu\nu} \, = \, -\delta g^{\mu\nu} g_{\mu\nu}. \tag{1.29}$$

Finally, we observe that the Ricci tensor $R_{\mu\nu}$ is an expression involving only the connection $\Gamma_{\mu\nu}^{\ \ \lambda}$. Hence

$$\delta R \, = \, \delta g^{\mu\nu}\, R_{\mu\nu} \, + \, g^{\mu\nu}\delta R_{\mu\nu}, \tag{1.30}$$

with the variation $\delta R_{\mu\nu}$ expressible purely terms of the variation of the connection $\delta\Gamma_{\mu\nu}^{\ \ \lambda}$. The total variation of the action now becomes

$$\delta S = -\frac{1}{2c}\int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\left[\left(\frac{c^4}{8\pi G}R_{\mu\nu}+\Theta_{\mu\nu}\right)-\frac{1}{2}g_{\mu\nu}\left(\frac{c^4}{8\pi G}R+\Theta\right)\right]$$
$$-\frac{1}{2}\int d^4x\,\sqrt{-g}\,\frac{\delta R}{\delta\Gamma_{\mu\nu}{}^{\lambda}}\,\delta\Gamma_{\mu\nu}{}^{\lambda}. \tag{1.31}$$

The last term is to be interpreted in the sense that we vary R with respect to the metric only via its dependence on the connection. However, it turns out that if we insert the expression (1.9) for the connection, this whole variation adds up to a total derivative. As the action principle applies to variations which vanish on the boundary, this term does not contribute to δS. If we then require the variation of the action to vanish, we indeed obtain the Einstein equations:

$$\begin{aligned}\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+\Theta_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\Theta &= \\ \frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+T_{\mu\nu} &= 0.\end{aligned} \tag{1.32}$$

The equation with the cosmological constant is obtained by making the replacement $g^{\mu\nu}\Theta_{\mu\nu}=\Theta\rightarrow 2\Lambda$ in the action.

Chapter 2

Geodesics

2.1 Curves and geodesics

In this chapter we discuss how basic information about the geometry of spaces or space-times —generically manifolds— can be obtained from studying their geodesics. The geodesics of euclidean differentiable manifolds are smooth curves of stationary proper length, meaning that to first order the proper length of an arc does not change under small variations of the curve. More generally and more precisely, given a metric on a manifold (euclidean or lorentzian) the geodesics are those smooth curves for which the proper interval between any two fixed points $\mathcal{P}$ and $\mathcal{P}'$ through which the curve passes is both well-defined and stationary. The interval among the fixed points can be either a maximum, a minimum or a saddle point.

A direct reason for geodesics to be important is that time-like geodesics represent the allowed trajectories of test particles in a fixed background space-time. But in general geodesics tell us interesting things about the symmetries and topology of a manifold, which often have wider implications for physics.

First we derive the fundamental equation characterizing geodesics. The proper interval $\Delta s\left[\mathcal{C}_{\mathcal{P}\mathcal{P}'}\right]$ of the arc $\mathcal{C}_{\mathcal{P}\mathcal{P}'}$ of an arbitrary smooth curve $\mathcal{C}$ between the two fixed points $(\mathcal{P}, \mathcal{P}')$ is given by the integral

$$\Delta s\left[\mathcal{C}_{\mathcal{P}\mathcal{P}'}\right] = \int_{\mathcal{C}_{\mathcal{P}\mathcal{P}'}} ds = \int_{\mathcal{C}_{\mathcal{P}\mathcal{P}'}} d\lambda \sqrt{\left| g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right|}. \tag{2.1}$$

This holds for all geodesics of euclidean manifolds, and time- or space-like geodesics of lorentzian manifolds; λ is a continuous parameter on some interval of the real line, parametrizing the curve between the points $\mathcal{P}$ and $\mathcal{P}'$ in any convenient way, provided the coordinates $x^{\mu}(\lambda)$ of points on the curve are monotonic differentiable functions of this parameter: the label of any point must be unique and the tangent vector $\dot{x}^{\mu} = dx^{\mu}/d\lambda$ must be well-defined. The parameter λ is then called an affine parameter. Note that the integral does not depend on the particular choice of affine parameter: it is invariant under differentiable changes of parametrization $\lambda \to \lambda'(\lambda)$. This freedom to choose for λ any smooth parametrization of the curve is often useful, but requires some care in interpreting results explicitly containing λ. As any invariant geometrical results with physical interpretation can not depend on the choice of affine parameter, the best way to proceed is to remove any λ-dependence from equations before drawing physical conclusions.

The proper interval $\Delta\, s[\mathcal{C}_{\mathcal{P}\mathcal{P}'}]$ is stationary if the integral does not change to first order under arbitrary smooth variations of the path while keeping the end points of the integral fixed:

$$\delta \int_{\mathcal{C}_{\mathcal{P}\mathcal{P}'}} d\lambda\, w[x(\lambda)] \,=\, 0, \tag{2.2}$$

where the integrand is $w[x(\lambda)] \,=\, \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}$. Now after a partial integration with vanishing boundary terms (the end points remain fixed) the variation of the integrand can be written

$$\delta w \,=\, \pm \delta x^\mu \left(-\frac{d}{d\lambda} \left[\frac{g_{\mu\nu}}{w} \frac{dx^\nu}{d\lambda} \right] \,+\, \frac{1}{2w} \frac{\partial g_{\lambda\nu}}{\partial x^\mu} \frac{dx^\lambda}{d\lambda} \frac{dx^\nu}{d\lambda} \right), \tag{2.3}$$

provided $w \neq 0$. The sign on the r.h.s. is positive in euclidean spaces or for space-like curves in lorentzian space-times, and negative for time-like curves in lorentzian space-times. Light-like curves in lorentzian space-time need special consideration, as $ds^2 = 0$ is equivalent to $wd\lambda = 0$.

The variational principle (2.2) now implies that for any smooth local variation δx^μ the induced variation δw must vanish if the curve $x^\mu(\lambda)$ is to describe a geodesic. For $w \neq 0$ we can divide the right-hand side of eq.(2.3) by w once more, and use the fact that $wd\lambda = ds$, the infinitesimal proper interval which is measured in the laboratory frame and independent of the choice of affine parameter λ. We can also replace the partial derivative of the metric in the second term by the connection; recall, that the connection was defined in eq.(1.9) as

$$\Gamma_{\lambda\nu}^{\;\;\;\mu} \,=\, \frac{1}{2}\, g^{\mu\kappa} \left(\frac{\partial g_{\lambda\kappa}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\kappa} \right). \tag{2.4}$$

This expression is actually the solution of the equation

$$\frac{Dg_{\mu\nu}}{Dx^\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma_{\lambda\mu}^{\;\;\;\kappa} g_{\kappa\nu} - \Gamma_{\lambda\nu}^{\;\;\;\kappa} g_{\mu\kappa} = 0, \tag{2.5}$$

stating that the metric is covariantly constant, a condition known as the metric postulate. From the equation above it follows that

$$\frac{\partial g_{\lambda\nu}}{\partial x^\mu} \,=\, \Gamma_{\mu\lambda}^{\;\;\;\kappa} g_{\kappa\nu} \,+\, \Gamma_{\mu\nu}^{\;\;\;\kappa} g_{\kappa\lambda}. \tag{2.6}$$

The vanishing of δw is now seen to imply a differential equation of the form (1.10) for the coordinates of a point moving on the curve as a function of the proper interval parameter s:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\kappa\nu}^{\ \ \mu} \frac{dx^\nu}{ds} \frac{dx^\kappa}{ds} = 0. \tag{2.7}$$

Note, that the above procedure eliminating the affine parameter λ is equivalent to a choice of rate of flow of the affine parameter $d\lambda$ equal to the proper interval ds measured in the coordinate frame $\{x^\mu\}$. If the signature of the manifold is lorentzian and the curve time-like this affine parameter is the proper time τ, and eq.(2.7) is the equation of motion of a test particle of arbitrary non-vanishing mass, in a fixed space-time with affine connection $\Gamma_{\kappa\nu}^{\ \ \mu}(x)$. Note also that the choice of $d\lambda = ds$ implies an identity

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \pm 1, \tag{2.8}$$

where again the plus sign holds for euclidean or space-like lorentzian curves, and the minus sign for time-like curves in lorentzian space-times. This is a constraint on the solutions of the geodesic equation (2.7).

That this constraint is consistent with the geodesic equation follows by observing that

$$H(s) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{2.9}$$

is a constant of motion, or more generally a constant of geodesic flow:

$$\frac{dH}{ds} = g_{\mu\nu} \frac{dx^\mu}{ds} \left(\frac{d^2 x^\nu}{ds^2} + \Gamma_{\lambda\kappa}^{\ \ \nu} \frac{dx^\lambda}{ds} \frac{dx^\kappa}{ds} \right) = 0. \tag{2.10}$$

Finally light-like curves (also called null-curves) by definition obey

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \tag{2.11}$$

In this case the affine parameter λ can not be identified with proper time or distance. As eq.(2.11) is to hold on all points of the light-like curve, the quantity

$$H(\lambda) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{2.12}$$

must be a constant of light-like motion. Therefore eq.(2.10) still holds with s replaced by λ. From this it follows that it is correct to interpret the solutions of the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\kappa\nu}^{\ \ \mu} \frac{dx^\nu}{d\lambda} \frac{dx^\kappa}{d\lambda} = 0, \tag{2.13}$$

satisfying the constraint $H(\lambda) = 0$ as light-like geodesics.

Remark. Although we have shown that the geodesic equation implies the constancy of $H(\lambda)$ for all types of geodesics, the inverse statement is not true. In particular, for non-null geodesics (euclidean, and space-like or time-like lorentzian) the more general condition

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\kappa\nu}^{\;\;\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = B^{\mu}_{\;\;\nu}(x)\frac{dx^\nu}{d\lambda}, \tag{2.14}$$

with $B_{\mu\nu} = -B_{\nu\mu}$ an anti-symmetric tensor, is sufficient for H to be conserved. This happens for example for charged particles in an electro-magnetic field with field strength $F_{\mu\nu}$:

$$B_{\mu\nu}(x) = \frac{q}{m}F_{\mu\nu}(x), \tag{2.15}$$

where q/m is the charge-to-mass ratio of the particle. Similar extensions exists for particles with spin or non-abelian charges.

For light-like geodesics the equation can be further generalized to read

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\kappa\nu}^{\;\;\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = B^{\mu}_{\;\;\nu}(x)\frac{dx^\nu}{d\lambda} + \kappa\frac{dx^\mu}{d\lambda}, \tag{2.16}$$

with κ an arbitrary constant. It should be noted that if $\kappa \neq 0$, then H is only conserved by virtue of the fact that it vanishes.

Summary

The results of this discussion an be summarized as follows. All geodesic curves, in euclidean or lorentzian manifolds, satisfy the equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\kappa\nu}^{\;\;\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = 0, \tag{2.17}$$

where λ is an affine parameter. The quantity

$$H(\lambda) = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{2.18}$$

is conserved by the geodesic flow. In physical applications three cases are to be distinguished:

1. For euclidean manifolds and space-like geodesics of lorentzian manifolds one can take $d\lambda = ds$, the proper distance; these geodesics are then characterized by the condition $2H(s) = 1$.

2. Time-like geodesics of lorentzian manifolds can be parametrized by the proper time: $d\lambda = d\tau$; such geodesics satisfy $2H(\tau) = -1$.

3. Light-like (null) geodesics admit no such identification of λ with invariant measures of the arc length in an inertial frame ('laboratory frame'); null-geodesics are characterized by $H(\lambda) = 0$.

2.2 Canonical formulation

In D-dimensional space-time eq.(2.17) represents a set of D second order differential equations in the affine parameter λ providing a covariant description of geodesics. It is possible to replace this set by a set of $2D$ first order differential equations by going over to the canonical formulation of the dynamics of test particles, in which the number of independent variables is doubled. Introducing the momentum variables

$$p_\mu(\lambda) = g_{\mu\nu}(x)\frac{dx^\nu}{d\lambda}, \tag{2.19}$$

the geodesic equation (2.17) becomes

$$\frac{dp_\mu}{d\lambda} = \Gamma_\mu^{\ \lambda\nu} p_\lambda p_\nu. \tag{2.20}$$

The equations (2.19) and (2.20) constitute a pair of first-order differential equations equivalent to the second order geodesic equation (2.17). A powerful result is now obtained by observing that these two equations can be derived from the constant of motion $H(\lambda)$, eq.(2.18) as a pair of canonical hamiltonian equations with $H(\lambda)$ in the role of hamiltonian:

$$H(\lambda) = \frac{1}{2}\, g^{\mu\nu} p_\mu p_\nu. \tag{2.21}$$

With this definition

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}. \tag{2.22}$$

Then any function $F(x^\mu, p_\mu)$ on the $2D$-dimensional phase space spanned by the coordinates and momenta changes along a geodesic according to

$$\frac{dF}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial F}{\partial x^\mu} + \frac{dp_\mu}{d\lambda}\frac{\partial F}{\partial p_\mu} = \{F, H\}, \tag{2.23}$$

where the Poisson bracket of two functions (F, G) on the phase space is defined generally as

$$\{F, G\} = -\{G, F\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu}. \tag{2.24}$$

The anti-symmetry of the Poisson bracket automatically guarantees the conservation of the hamiltonian H by

$$\{H, H\} = 0. \tag{2.25}$$

A similar property holds for all quantities which are constant along geodesics:

$$\frac{dF}{d\lambda} = 0 \qquad \Leftrightarrow \qquad \{F, H\} = 0. \tag{2.26}$$

Formal properties of the Poisson bracket include its anti-symmetry and its linearity in each of the arguments:

$$\{F, \alpha_1 G_1 + \alpha_2 G_2\} = \alpha_1 \{F, G_1\} + \alpha_2 \{F, G_2\}. \tag{2.27}$$

Another important property is that it satisfies the Jacobi identity:

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0. \tag{2.28}$$

This property implies, that the Poisson bracket of two constants of geodesic flow is again a constant of geodesic flow: if $\{F, H\} = 0$ and $\{G, H\} = 0$, then

$$\{\{F, G\}, H\} = 0. \tag{2.29}$$

As H is itself constant on geodesics, the above results are sufficient to establish that the constants of geodesic flow form a Lie algebra, with the Poisson bracket as the Lie bracket.

The definition of the Poisson bracket (2.24) is not manifestly covariant, and therefore at first sight it seems to destroy the build-in covariance of the tensor calculus of differential geometry, and of general relativity in particular. However, note that in our definition the connection (2.4) is alway symmetric in its lower indices; therefore we can rewrite the Poisson bracket formula as

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \mathcal{D}_\mu G, \tag{2.30}$$

where for scalar functions F and G on the phase space we define the *covariant* derivative

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma_{\mu\nu}{}^{\lambda} p_\lambda \frac{\partial F}{\partial p_\nu}. \tag{2.31}$$

This equation preserves manifest covariance for scalar functions F and G; for example, if $J(x, p) = J^\mu(x) p_\mu$, then

$$\mathcal{D}_\mu J = \left(\partial_\mu J^\nu + \Gamma_{\mu\lambda}{}^{\nu} J^\lambda \right) p_\nu = \left(D_\mu J \right)^\nu p_\nu. \tag{2.32}$$

The manifestly covariant form of the Poisson bracket can also be extended to all completely symmetric tensors $T_{\mu_1 \ldots \mu_n}$ of rank n by contracting all n indices with the momentum p_λ to obtain the scalar

$$T(x,p) = \frac{1}{n!}\, T^{\mu_1 \cdots \mu_n}(x) p_{\mu_1} \ldots p_{\mu_n}. \tag{2.33}$$

Inserting this scalar into the covariant Poisson bracket, the bracket for a particular tensor component $T^{\mu_1 \cdots \mu_n}$ is obtained by taking, in the expression resulting on the right-hand side, the coefficient of the n-nomial in the momenta that one is interested in. For anti-symmetric index structures a similar result can be achieved using differential forms, or more generally Grassmann algebras [18, 19, 20].

2.3 Action principles

The canonical formulation of geodesic flow presented here can be cast into the form of a standard hamiltonian action principle. Namely, the canonical eqs.(2.22) define the critical points of the phase-space action

$$S = \int_1^2 d\lambda \left(\dot{x} \cdot p - H(x,p) \right). \tag{2.34}$$

By using the hamilton equation

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu},$$

the momentum variable can be eliminated in favour of the velocity of geodesic flow, to give the lagrangian form of the action

$$S = \int_1^2 d\lambda\, \frac{1}{2}\, g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu. \tag{2.35}$$

We observe, that this is not the action for the geodesics that we started from, eq.(2.1). In particular, it is not reparametrization invariant. A comparison shows, that the lagrangian in (2.35) is actually the square of $w(\lambda)$ used earlier, eq.(2.2). Clearly, the critical points are therefore the same, with the possible exception of light-like curves. Actually, the variational principle for the original action was not even well-defined in the case of light-like curves.

In this section we examine somewhat more closely the relation between the various action principles, and in particular the role of reparametrization invariance. It makes clear how one can derive the lagrangian and hamiltonian actions presented above within a unified framework.

The key to the general formulation is to incorporate reparametrization invariance in an action of the type (2.35) [21]. This is achieved by introduction of a new variable $e(\lambda)$, a function of the affine parameter, which acts as a gauge field in one dimension for local reparametrizations of the worldline. More precisely, the variable $e(\lambda)$ by definition transforms under reparametrizations $\lambda \rightarrow \lambda'$ as

$$e(\lambda) \to e'(\lambda') = e(\lambda)\,\frac{d\lambda}{d\lambda'}. \tag{2.36}$$

This is the transformation rule for the square root of a metric in one dimension (on a curve), for which reason the variable $e(\lambda)$ is called the einbein. Written in a slightly different way, eq.(2.36) states that $e(\lambda)d\lambda$ is defined to be invariant under reparametrizations. This makes it straightforward to write a reparametrization-invariant form of the quadratic action (2.35):

$$S[e;x] = \int_1^2 d\lambda\,\left(\frac{1}{2e}\,g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{c^2}{2}e\right). \tag{2.37}$$

The last term, proportional to the constant c^2, does not depend on the coordinates x^μ. However, it is reparametrization invariant and its inclusion is important for reasons given below.

Variation of the full action $S[e;x]$ w.r.t. the coordinates x^μ reproduces the geodesic equation (2.17) provided everywhere the differentials $d\lambda$ are replaced by $ed\lambda$. This replacement makes the geodesic equation reparametrization invariant, as $ed\lambda = e'd\lambda'$. On the other hand, the substitution $e(\lambda) = 1$, which breaks reparametrization invariance, makes the old and new equations fully identical. Such a substitution is allowed because it only serves to define the affine parameter λ in terms of quantities measurable by an observer in the locally euclidean or lorentzian laboratory frame, like proper length or proper time. Of course, such a choice necessarily breaks the reparametrization invariance, but it preserves the physical content of the equations. It amounts to a choice of gauge for reparametrizations in the true sense of the expression, with the result that modulo a constant the action $S[e;x]$ reduces to the simple quadratic action (2.35).

However, the action $S[e;x]$ contains more information, as one can also consider its variation w.r.t. the einbein e. Requiring stationarity under this variation gives a constraint on the solutions of the geodesic equation, which after multiplication by $2e^2$ takes the form

$$g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + e^2c^2 = 0. \tag{2.38}$$

With $e = 1$, comparison with the definition of H in eq.(2.9) shows that this constraint expresses the constancy of the hamiltonian along a geodesic:

$$2H(\lambda) = -c^2. \tag{2.39}$$

For euclidean manifolds one can always choose units such that $c^2 = -1$, but for lorentzian space-times $c^2 = (-1, 0, +1)$ corresponds to space-like, light-like or time-like geodesics respectively; in the latter case the constant c then represents the speed of light, which is unity when expressed in natural units.

Finally we derive the original reparametrization-invariant geodesic action (2.1) from $S[e;x]$. This is achieved by using eq.(2.38) to solve for e, instead of fixing a gauge:

$$e = \pm\sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu/c^2|}. \tag{2.40}$$

Substitution of this expression for e into the action $S[e;x]$ then leads back (modulo a possible sign) to the original action (2.1). Therefore we conclude that both this action and the quadratic action (2.35) follow from the same starting point provided by eq.(2.37).

2.4 Symmetries and Killing vectors

Often space-times of interest possess special symmetries. Such symmetries are useful in finding solutions of the geodesic equations, especially if they are continuous symmetries as these generate constants of geodesic flow. This is the content of Noether's theorem applied to the motion of test particles and its generalization to space-like curves; see for example refs.[11]-[15], [18, 19]. In this section the role of continuous symmetries is explored in some detail.

A manifold possesses a continuous symmetry, or isometry, if there are infinitesimal coordinate transformations leaving the metric invariant. Finite transformations can be obtained by reiteration of the infinitesimal transformations, and donot require separate discussion at this point. Consider an infinitesimal coordinate transformation characterised by local parameters $\xi^\mu(x)$:

$$x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x). \tag{2.41}$$

For the line element ds^2 to be invariant the metric must transform as a tensor; therefore by definition

$$g_{\mu\nu}(x) = g'_{\kappa\lambda}(x')\frac{\partial x'^\kappa}{\partial x^\mu}\frac{\partial x'^\lambda}{\partial x^\nu}. \tag{2.42}$$

On the other hand, invariance of the metric under a coordinate transformation implies

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x). \tag{2.43}$$

Combining the two equations (2.42) and (2.43) to eliminate the transformed metric $g'_{\mu\nu}$, and inserting the infinitesimal transformation (2.41), expansion to first order in ξ^μ leads to the covariant condition

$$D_\mu\xi_\nu + D_\nu\xi_\mu = 0. \tag{2.44}$$

This equation is known as the Killing equation, and the vector fields defined by the solutions $\xi^\mu(x)$ are called Killing vectors. The existence of a Killing vector is equivalent with the existence of an infinitesimal coordinate transformation (2.41) leaving the metric invariant.

Let us pause for a moment to reflect on the interpretation of this result. If the coordinate transformation is considered to be a passive transformation, the new coordinates represent a relabeling of the points of the manifold: x^{μ} and x'^{μ} in eq.(2.41) refer to the same point $\mathcal{P}$. Now consider the point $\mathcal{P}'$, which in the new coordinate system has the same value of the coordinates as the point $\mathcal{P}$ in the old coordinate system: $x'^{\mu}(\mathcal{P}') = x^{\mu}(\mathcal{P})$. Eq.(2.43) states that the new metric at the point $\mathcal{P}'$ has the same functional dependence on the new coordinates as the old metric on the old coordinates at the point $\mathcal{P}$. Thus in the new coordinate system the neighborhood of $\mathcal{P}'$ looks identical to the neighborhood of $\mathcal{P}$ in the old coordinates. In particular it is possible to map the points in the neighborhood of $\mathcal{P}$ one-to-one to points in the neighborhood of $\mathcal{P}'$ in such a way that all distances (intervals) are preserved. As this is in fact a coordinate-invariant statement, it implies a local indistinguishability of the two neighborhoods of the manifold (as long as it is not endowed with additional structures). This then is what is implied by a continuous symmetry of the manifold generated by the solutions of the Killing equation.

Now if two neighborhoods of the manifold are similar in the sense described above, then this similarity should also extend to the geodesics passing through the points mapped on each other by the symmetry operation. Indeed, as the new metric at the point $\mathcal{P}'$ has the same *functional dependence* on the new coordinates there as the old metric at $\mathcal{P}$ has on the old coordinates, and not just the same value, the connections and curvature components there also become identical after the coordinate transformation. The same holds for the solutions of the geodesic equations. Thus the symmetry transformation maps geodesics through the point $\mathcal{P}$ to geodesics through the point $\mathcal{P}'$, and the geodesic flow in the neighborhood of these points is completely similar.

The similarity of geodesic structure generated by symmetry operations implies the conservation of certain dynamical quantities under geodesic flow. This is most conveniently discussed in the canonical formalism of the geodesic flow presented above in sect.2.2. The quantity conserved is the generator of the symmetry transformations, obtained by contraction of the Killing vector with the canonical momentum:

$$J[\xi] = \xi^{\mu}(x) p_{\mu}. \tag{2.45}$$

This is a scalar quantity, hence its value is coordinate invariant. It generates the infinitesimal coordinate transformations through the Poisson bracket (2.24):

$$\delta x^{\mu} = -\xi^{\mu}(x) = \left\{ J[\xi], x^{\mu} \right\}. \tag{2.46}$$

One can similarly determine the variation of the momentum under the transformation generated by $J[\xi]$:

$$
\begin{aligned}
\delta p_\mu &= \{J[\xi], p_\mu\} \\
&= \partial_\mu J[\xi] \,=\, T_\mu{}^\nu(x) p_\nu
\end{aligned}
\tag{2.47}
$$

with $T_\mu{}^\nu = \partial_\mu \xi^\nu$. This represents a coordinate-dependent linear transformation of the momentum.

That the quantity $J[\xi]$ is conserved now follows from the observation that the hamiltonian is invariant under the phase-space transformations defined in eqs.(2.46), (2.47). Indeed, taking into account the canonical equations of motion (2.23) and (2.26), and using the Killing equation (2.44) we obtain

$$
\frac{dJ[\xi]}{d\lambda} \,=\, \{J[\xi], H\} \,=\, 0. \tag{2.48}
$$

This result was to be expected, as the hamiltonian depends on the coordinates only through the metric, which is invariant by definition of an isometry.

As the Killing equation (2.44) is linear it follows that the Killing vectors define a linear vector space: any linear combination of two Killing vectors is again a Killing vector. Let the dimension of this vector space be r; then any Killing vector can be expanded in terms of r linearly independent basis vectors $e_i(x)$, $i = 1, ..., r$:

$$
\xi(\alpha_i) = \alpha_1 e_1(x) + ... + \alpha_r e_r(x). \tag{2.49}
$$

Equivalently, any infinitesimal symmetry transformation (2.41) depends linearly on r parameters α_i, to which correspond an equal number of conserved generators

$$
J[\xi] = \alpha_1 J_1 + ... + \alpha_r J_r, \qquad \text{where} \quad J_i(x,p) = e_i^\mu(x) p_\mu. \tag{2.50}
$$

We now show that these generators define a Lie algebra through their Poisson brackets. The key observation is, that the Poisson bracket of any two constants of geodesic flow is itself a constant of geodesic flow. This is implied by the Jacobi identity for the Poisson bracket, which can be written as:

$$
\{\{F,G\},H\} \,=\, \{\{F,H\},G\} \,-\, \{\{G,H\},F\}. \tag{2.51}
$$

Thus, if the Poisson brackets of F and G with the hamiltonian H vanish, then the left-hand side vanishes as well, and $\{F,G\}$ is itself a constant of geodesic flow.

A second important observation is, that for two constants of motion which are linear in the momentum, like the J_i, the Poisson bracket is also linear in the momentum:

$$
\{J_i, J_j\} \,=\, \left(e_j^\nu \partial_\nu e_i^\mu - e_i^\nu \partial_\nu e_j^\mu \right) \, p_\mu. \tag{2.52}
$$

As the $\{J_i\}_{i=1}^r$ are supposed to form a complete set, the expression on the right-hand side can be expanded in terms of the basis elements, with the result that with a certain set of coefficients $f_{ij}{}^k = -f_{ji}{}^k$ we have

$$\{J_i, J_j\} = f_{ij}{}^k J_k. \tag{2.53}$$

An equivalent statement is that the basis vectors of the Killing space satisfy

$$e_j^\nu \partial_\nu e_i^\mu - e_i^\nu \partial_\nu e_j^\mu = f_{ij}{}^k e_k^\mu. \tag{2.54}$$

Eq.(2.53), combined with the observations that the J_i form a linear vector space and that the Poisson bracket is bilinear and anti-symmetric in its arguments complete the proof that the generators of the symmetry transformations define a Lie algebra with structure constants $f_{ij}{}^k$.

Of course the linear p.d.e.(2.54) encodes the same information. It can be interpreted in terms of the Lie-derivative, which is the operator comparing the value of a function at the point $\mathcal{P}$ with that at the point $\mathcal{P}'$ which has the same value of its coordinates after an arbitrary transformation of the form (2.41) as the point $\mathcal{P}$ had before the transformation: $x'^\mu(\mathcal{P}') = x^\mu(\mathcal{P})$. For a scalar $\phi(x)$, for which $\phi'(x') = \phi(x)$, this gives

$$\mathcal{L}_\xi [\phi] (x) \equiv \mathrm{lin}_\xi [\phi'(x) - \phi(x)] = \xi^\nu \partial_\nu \phi(x), \tag{2.55}$$

where the $\mathrm{lin}_\xi[Q]$ denotes the part of the expression Q linear in ξ. For a vector $v^\mu(x)$, for which

$$v'^\mu(x') = v^\nu(x) \frac{\partial x'^\mu}{\partial x^\nu}, \tag{2.56}$$

one finds similarly

$$(\mathcal{L}_\xi [v])^\mu (x) \equiv \mathrm{lin}_\xi [v'^\mu(x) - v^\mu(x)] = \xi^\nu \partial_\nu v^\mu(x) - \partial_\nu \xi^\mu v^\nu(x). \tag{2.57}$$

For higher-rank tensors the construction of the Lie-derivative gives the result

$$\begin{aligned} (\mathcal{L}_\xi [T])^{\mu_1 \dots \mu_k} (x) &\equiv \mathrm{lin}_\xi [T'^{\mu_1 \dots \mu_k}(x) - T^{\mu_1 \dots \mu_k}(x)] \\ &= \xi^\nu \partial_\nu T^{\mu_1 \dots \mu_k}(x) - \partial_\nu \xi^{\mu_1} T^{\nu \dots \mu_k}(x) - \dots - \partial_\nu \xi^{\mu_k} T^{\mu_1 \dots \nu}(x). \end{aligned} \tag{2.58}$$

Now comparing eqs.(2.54) and (2.57) we see that the Lie-algebra property of the Killing vectors may be expressed as

$$\mathcal{L}_{e_j}[e_i] = -\mathcal{L}_{e_i}[e_j] = f_{ij}{}^k e_k. \tag{2.59}$$

Finally, in the phase space where by definition $\partial_\nu p_\mu = 0$, we can also write the transformation (2.47) for the momentum as

$$\delta_i p_\mu \equiv \{J_i, p_\mu\} = \left(\mathcal{L}_{e_i}[p]\right)_\mu . \tag{2.60}$$

Thus we have established a close relationship between Poisson brackets and Lie derivatives, and between symmetries, Killing vectors and Lie algebras.

2.5 Phase-space symmetries and conservation laws

In the previous section it was established that continuous symmetries of manifolds are generated by Killing vectors and imply conservation laws for certain dynamical quantities (the generators J) along geodesics. However, not every constant of geodesic flow is necessarily connected with a Killing vector. This is most easily seen from the example of the hamiltonian itself, which is conserved but not of the form (2.45). Rather the hamiltonian is quadratic in the momenta. This leads us to consider the possibility of conservation laws for quantities which are higher order expressions in the momenta.

Let $J(x,p)$ be any function on the phase space of coordinates $x^\mu(\lambda)$ and momenta $p_\mu(\lambda)$, associated with a geodesic, which is non-singular in the momenta. In others words, we suppose it has an expansion of the type

$$J(x,p) = \sum_{k=0}^{\infty} \frac{1}{k!}\, J^{(k)\,\mu_1\ldots\mu_k}(x)\, p_{\mu_1}\ldots p_{\mu_k}. \tag{2.61}$$

Here the coefficients $J^{(k)}$ can be taken to be completely symmetric in all k indices. We ask under what conditions such a function can be a constant of geodesic flow:

$$\frac{dJ}{d\lambda} = \{J, H\} = 0. \tag{2.62}$$

To obtain the answer, insert the expansion (2.61) into this equation and compare terms at equal order of powers of momentum. At zeroth order one finds

$$\partial_\mu J^{(0)} = 0, \tag{2.63}$$

i.e. $J^{(0)}$ is a constant. Such a constant can always be absorbed in the definition of J and is of no consequence. From now on it will be dropped.

At first order one finds back the Killing equation (2.44):

$$D_\mu J^{(1)}_\nu + D_\nu J^{(1)}_\mu = 0.$$

These terms we have already discussed in detail.

Finally, for $k \geq 2$ we find a generalization of the Killing equation for the co-efficients $J^{(k)\,\mu_1\ldots\mu_k}(x)$:

$$D_{(\mu_1} J^{(k)}_{\mu_2 \ldots \mu_{k+1})} \,=\, 0. \tag{2.64}$$

The parentheses denote full symmetrization over all component indices enclosed. The solutions of these generalised Killing equations are called Killing tensors. It is now obvious that the hamiltonian is included, because the metric $g_{\mu\nu}$ is covariantly constant, and therefore automatically a solution of eq.(2.64) for $k = 2$. Moreover, this solution always exists, independent of the particular manifold considered. In addition, for specific manifolds (specific metrics $g_{\mu\nu}$) other solutions may exist as well. For example, in spherically symmetric spaces the square of the angular momentum:

$$L^2 = \frac{1}{2} \left(x^2 p^2 - (x \cdot p)^2 \right), \tag{2.65}$$

is conserved and quadratic in the momenta by construction. Other examples will be encountered later on.

Just as the Killing vectors, the Killing tensors generate symmetry transformations, not of the manifold but rather of the corresponding phase space spanned by the coordinates and the momenta (tangent vectors). In particular:

$$\delta x^\mu \,=\, \{J(x,p), x^\mu\} \,=\, -\,J^{(1)\,\mu}(x) \,-\, \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\, J^{(k)\,\mu\mu_2\ldots\mu_k}(x) p_{\mu_2} \ldots p_{\mu_k}. \tag{2.66}$$

We observe that the transformations generated by the higher-order Killing tensors are velocity dependent [18].

Finally we address the algebraic properties of the constants of geodesic flow. We have already remarked, that the Poisson brackets of two such constants yields another constant by the Jacobi identity (2.51). In general the rank of quantities on the left-hand side are additive minus one:

$$\left\{ J^{(k)}, J^{(l)} \right\} \,\sim\, J^{(k+l-1)}. \tag{2.67}$$

Now for $l = 1$ the rank of $J^{(k+l-1)}$ is the same as that of $J^{(k)}$. It follows, that
(i) for the constants of geodesic flow linear in the momentum, corresponding to Killing vectors ($k = l = 1$), the algebra is closed in itself: it is a Lie-algebra G;
(ii) for any fixed value of k, the brackets of a $J^{(1)}$ and a $J^{(k)}$ close on the $J^{(k)}$; however, the brackets of two elements $J^{(k)}$ generates an element of order $(2k-1)$. Hence in general the algebra (2.67) is not closed, unless one includes elements of arbitrary rank, i.e. all possible k. This implies that once higher-order invariants appear, one may expect an infinite series of them. Sometimes this happens indeed, as with the Virasoro algebra for the conformal transformation in two dimensions [22]. However, in many cases the series stops because the higher order invariants are Casimir-type of operators, or they become trivial in that they are

just products of commuting lower-order elements (commuting here is meant in the sense of vanishing Poisson brackets).

The Lie algebra G of Killing vectors is characterized by its structure constants $f_{ij}^{\ \ k}$. As the Poisson brackets of $J^{(k)}$ with all elements of the Lie algebra spanned by the $J^{(1)}$ are again of order k, one may choose a basis $\left\{ e_A^{(k)\mu_1 \ldots \mu_k}(x) \right\}$ for the Killing tensors of order k, labeled by the index A, and decompose the right-hand side of the Poisson bracket again in terms of the corresponding conserved charges $J_A^{(k)}$:

$$\left\{ J_i^{(1)}, J_A^{(k)} \right\} = g_{iA}^{\ \ \ B} \, J_B^{(k)}. \tag{2.68}$$

This defines a set of higher-order structure constants. The Jacobi identities then imply, that the matrices $(T_i)_A^{\ B} = -g_{iA}^{\ \ \ B}$ define a representation of the Lie algebra (2.53):

$$g_{jA}^{\ \ \ C} g_{iC}^{\ \ \ B} - g_{iA}^{\ \ \ C} g_{jC}^{\ \ \ B} = f_{ij}^{\ \ k} g_{kA}^{\ \ \ B} \qquad \Longleftrightarrow \qquad [T_i, T_j]_A^{\ B} = f_{ij}^{\ \ k} \, T_{kA}^{\ \ B}. \tag{2.69}$$

Therefore the constants of geodesic flow

$$J_A^{(k)}(x, p) = \frac{1}{k!} \, e_A^{(k)\mu_1 \ldots \mu_k}(x) p_{\mu_1} \ldots p_{\mu_k}, \tag{2.70}$$

span a representation space of the Lie algebra G for the representation $\{T_i\}$. The discussion of symmetries shows that group theory and Lie algebras can be important tools in the analysis of geodesic motion and the structure of manifolds.

2.6 Example: the rigid rotor

The above concepts and procedures can be illustrated by the simple example of a rigid rotor, which has a physical interpretation as a model for the low-energy behaviour of diatomic molecules. As such the example also serves to emphasize the usefulness of geometric methods in physics outside the context of general relativity.

Consider two particles of mass m_1 and m_2, interacting through a central potential $V(r)$ depending only on the relative distance $r = |\boldsymbol{r}_2 - \boldsymbol{r}_1|$. The lagrangian for this system is

$$L = \frac{m_1}{2} \, \dot{\boldsymbol{r}}_1^2 + \frac{m_2}{2} \, \dot{\boldsymbol{r}}_2^2 - V(r). \tag{2.71}$$

Defining as usual the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2 / M$, the lagrangian is separable in the center of mass coordinates defined by

$$M\boldsymbol{R} = m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2, \qquad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{2.72}$$

In terms of $(\boldsymbol{R}, \boldsymbol{r})$ the expression (2.71) becomes

$$L = \frac{M}{2}\dot{\boldsymbol{R}}^2 + \frac{\mu}{2}\dot{\boldsymbol{r}}^2 - V(r). \tag{2.73}$$

As the center of mass moves like a free particle of mass M, it is convenient to work in the rest frame of the center of mass: $\boldsymbol{R} = 0$. Then we are only left with the problem of describing the relative motion of the masses, described by the three coordinates represented by the vector $\boldsymbol{r}$.

Now suppose that the potential $V(r)$ has a minimum for some separation $r = r_0$, and rises steeply for all values of r near this minimum. Then the first excited vibration state of the molecule has an energy well above the ground state, and at low temperatures the only degrees of freedom that play a role in the dynamics are rotations of the molecule at fixed interatomic distance r_0. Thus the distance is frozen out as one of the dynamical degrees of freedom, and we are effectively left with only two coordinates: the angles (θ, φ) describing the relative orientation of the two masses; the potential $V(r) = V(r_0)$ is constant and may be ignored. This leaves as the effective action of the system

$$L_{eff} = \frac{\mu}{2} r_0^2 \left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right). \tag{2.74}$$

This is the lagrangian of a particle of mass μ moving on a spherical surface of radius r_0. In mathematical physicists' parlance the effective low energy degrees of freedom (θ, φ) are sometimes called the modular parameters or moduli of the system, and the sphere representing the effective low-energy configuration space is called the moduli space.

Up to an overall scale factor μr_0^2, the lagrangian L_{eff} represents an action of the form (2.35) in a 2-dimensional space with euclidean signature, described by the coordinates $x^1 = \theta$, $x^2 = \varphi$ and metric

$$g_{ij} = \left(\begin{array}{cc} 1 & 0 \\ 0 & \sin^2\theta \end{array} \right). \tag{2.75}$$

As the geodesics on a sphere are great circles, the great circles provide the general solution to the equations of motion. The spherical symmetry implies that the system is invariant under 3-dimensional rotations (rotations about each of three independent body axes of the sphere) and hence there are three Killing vectors generating these rotations. The corresponding constants of motion are the 3 components of angular momentum $\boldsymbol{L} = (L_1, L_2, L_3)$, given explicitly by

$$\begin{aligned} L_1 &= -\sin\varphi\, p_\theta - \cot\theta\cos\varphi\, p_\varphi, \\ L_2 &= \cos\varphi\, p_\theta - \cot\theta\sin\varphi\, p_\varphi, \\ L_3 &= p_\varphi. \end{aligned} \tag{2.76}$$

They span the Lie-algebra so(3) of the 3-dimensional rotation group:

$$\{L_i, L_j\} \,=\, \varepsilon_{ijk}\, L_k. \tag{2.77}$$

As the Poisson brackets of $\boldsymbol{L}$ with the hamiltonian vanish, the hamiltonian can depend only on the Casimir constant of the rotation group, i.e. the total angular momentum squared $\boldsymbol{L}^2$. A direct computation shows, that up to the constant potential the effective hamiltonian is in fact proportional to the Casimir invariant:

$$H_{eff} \,=\, \frac{\boldsymbol{L}^2}{2\mu r_0^2}. \tag{2.78}$$

Chapter 3

Dynamics of space-time

3.1 Classical solutions of the gravitational field equations

The Einstein equations (1.17) describe the gravitational fields generated by a specific distribution of material sources, as specified by the energy-momentum tensor $T_{\mu\nu}$. In two or three dimensions, these fields carry at best a finite number of physical degrees of freedom, related to the sources and/or the topology [23]. In four or higher-dimensional space-time the gravitational field is a fully dynamical system with infinitely many degrees of freedom. In particular, dynamical gravitational fields can exist in the absence of material sources, in empty space-time. In the geometrical interpretation of general relativity such fields represent the intrinsic dynamics of space-time itself.

In eq.(1.12) the curvature tensor was defined in terms of the parallel transport of a vector around a closed curve. A manifold is called flat if after parallel transport around an arbitrary closed curve in the manifold the image of any vector coincides with the original. Therefore a necessary and sufficient condition for a manifold to be flat near a given point is that the components of the Riemann tensor $R_{\mu\nu\kappa}{}^{\lambda}$ vanish there. This condition is more general than to require that the metric takes the form

$$ds^2 = \pm dx_0^2 + \sum_{i=1}^{n} dx_i^2, \tag{3.1}$$

with the sign depending on the euclidean or lorentzian signature of the manifold. Indeed, in any differentiable manifold one can always find a local coordinate system in the neighborhood of a given point, in which the line element can be diagonalized to this form, but this is not generally the case globally. Even in cases in which it can be done globally, eq.(3.1) is not the most general solution of the Einstein equations for flat empty space-time, because it is still allowed to perform an arbitrary local change of co-ordinates in this metric. The flatnesscriterion introduced above, amounting to the vanishing of the Riemann-curvature, is more useful as it holds in any co-ordinate system and in any topology.

The existence of non-trivial solutions of the Einstein equations with vanishing cosmological constant in matter-free regions of space-time is possible because vanishing of the Ricci-tensor does not imply a vanishing of all components of the Riemann-tensor. A formal mathematical criterion for distinguishing solutions of the Einstein equations in empty space from those with material sources is through the Weyl curvature tensor, defined as the completely traceless part of the Riemann tensor:

$$\begin{aligned} W_{\mu\nu\kappa\lambda} &= R_{\mu\nu\kappa\lambda} - \frac{1}{(d-2)} \left(g_{\mu\kappa} R_{\nu\lambda} - g_{\nu\kappa} R_{\mu\lambda} - g_{\mu\lambda} R_{\nu\kappa} + g_{\nu\lambda} R_{\mu\kappa} \right) \\ &\quad + \frac{1}{(d-1)(d-2)} \left(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa} \right) R. \end{aligned} \tag{3.2}$$

Here d is the dimensionality of the manifold. If we similarly define $W_{\mu\nu}$ as the traceless part of the Ricci tensor:

$$W_{\mu\nu} = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R, \qquad W_{\mu\nu} = W_{\nu\mu}, \qquad W_{\mu}^{\ \mu} = 0, \tag{3.3}$$

then we obtain a complete decomposition of the Riemann tensor in terms of traceless components and the Riemann scalar:

$$\begin{aligned} R_{\mu\nu\kappa\lambda} &= W_{\mu\nu\kappa\lambda} + \frac{1}{(d-2)} \left(g_{\mu\kappa} W_{\nu\lambda} - g_{\nu\kappa} W_{\mu\lambda} - g_{\mu\lambda} W_{\nu\kappa} + g_{\nu\lambda} W_{\mu\kappa} \right) \\ &\quad + \frac{1}{d(d-1)} \left(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa} \right) R. \end{aligned} \tag{3.4}$$

This decomposition allows a classification of the solutions of the Einstein equations according to which of the independent components $(W_{\mu\nu\kappa\lambda}, W_{\mu\nu}, R)$ vanish and which don't. In particular, metrics for which only the Weyl tensor is non-vanishing are solutions of the source-free Einstein equations $R_{\mu\nu} = 0$. On the other hand, there also exist manifolds for which the only non-zero component is the Riemann scalar R, whilst the Weyl tensor and traceless Ricci tensor vanish: the n-spheres $x_0^2 + x_1^2 + \ldots + x_n^2 = \rho^2$, and all two-dimensional surfaces. For the n-spheres this follows essentially from the spherical symmetry, which implies the absence of any prefered direction, in combination with the permutation symmetries between the components of the Riemann curvature tensor.

For two-dimensional manifolds, it is always possible to write the Riemann tensor as

$$R_{\mu\nu\kappa\lambda} = \frac{1}{2}\, g\, \varepsilon_{\mu\nu}\, \varepsilon_{\kappa\lambda}\, R = \frac{1}{2}\, (g_{\mu\kappa} g_{\nu\lambda} - g_{\nu\kappa} g_{\mu\lambda})\, R. \qquad (3.5)$$

For the (torsion-free) riemannian and pseudo-riemannian manifolds we consider in higher dimensions ($d \geq 3$), the number of independent components of the Riemann tensor $R_{\mu\nu\kappa\lambda}$, taking account of all its symmetries, is given by:

$$N[R_{\mu\nu\kappa\lambda}] = \frac{d^2(d^2-1)}{12}. \qquad (3.6)$$

This is counted as follows: the Riemann tensor is anti-symmetric in each of the index pairs $[\mu\nu]$ and $[\kappa\lambda]$. Hence $R_{[\mu\nu][\kappa\lambda]}$ may be considered as a square matrix of dimension $n = d(d-1)/1$. Also as an $(n \times n)$-matrix it is symmetric under interchange of these index pairs. As such it has $n(n+1)/2$ independent elements. Finally the cyclic symmetry in the first three (lower) indices of the fully covariant components of the Riemann tensor expressed by the Bianchi identity (1.15) adds $d(d-1)(d-2)(d-3)/4!$ algebraic constraints, eliminating an equal number of independent components. Combining these results one finds precisely the number $N[R_{\mu\nu\kappa\lambda}]$ in eq.(3.6) above.

For $d \geq 3$ the number of components of the traceless Ricci tensor and Riemann scalar is

$$N[W_{\mu\nu}] = \frac{d(d+1)}{2} - 1. \qquad N[R] = 1. \qquad (3.7)$$

(for $d = 2$ the Riemann curvature tensor has only one independent component, corresponding to the Riemann scalar). Therefore for $d \geq 3$ the number of independent components of the Weyl tensor is

$$N[W_{\mu\nu\kappa\lambda}] = \frac{1}{12}\, d(d+1)(d+2)(d-3). \qquad (3.8)$$

Clearly this vanishes for $d = 3$, and is positive non-zero only for $d \geq 4$. Thus we find indeed that non-trivial dynamical solutions to the source-free Einstein equations can exist only in space-times of dimensions $d \geq 4$. As the minimal case $d = 4$ happens to be the number of dimensions of our (macroscopic) world, we observe that for this case the Weyl tensor and the Ricci tensor each have 10 algebraically independent components. Therefore half of the components of the curvature are determined by the material sources, whilst the other half describes the intrinsic dynamics of space-time.

In the geometrical interpretation of the gravitational field and its dynamics, Einstein's famous principle of the equivalence of accelerations and local gravitational fields can be formulated as the hypothesis that space-time is a manifold, in each point of which one can establish a locally flat (cartesian) coordinate system of the form (3.1). Of course a fully geometric interpretation of all of physics, and also the equivalence principle, breaks down in those points or regions where material sources are located[1]. But also solutions of the source-free Einstein equations can violate the equivalence principle locally if there are singularities — for example points where curvature invariants become infinite and/or the number of dimensions changes. Physics at these points is no longer described, at least not completely, by the Einstein equations, and space-time is not necessarily a manifold in the strict sense. In some cases it can still be a manifold with boundaries, where specific boundary conditions hold; but these boundary conditions then represent external data and the gravitational field is not determined by general relativity alone.

Nevertheless singular solutions are important because they are relevant for physics as we observe it in the universe: cosmological solutions of the Friedmann-Robinson-Walker type with a 'Big Bang'-singularity, and black-hole solutions describing stellar collapse. The singularities in these cases are not necessarily physical, as the short-distance physics may be regulated by Planck-scale modifications of the theory, which are presently unobservable. However that may be, here we simply accept the possibility of singularities and discuss examples of both non-singular and singular geometries arising as solutions of the Einstein equations. They illustrate many interesting aspects of gravitational physics.

3.2 Plane fronted waves

To show that in four dimensions non-trivial gravitational fields can indeed propagate through empty space-time, we now consider a class of solutions known in the literature as plane-fronted gravitational waves [4, 5]. This name derives from the property that there is a co-ordinate system in which these field configurations (i) move with the speed of light along a straight line; (ii) have a flat planar wave front, and (iii) are of finite duration. When discussing the physical properties of these solutions it will become clear that these wave-like solutions actually behave more like dynamical domain walls.

We construct a planar wave solution with the direction of motion along the x-axis, and the transverse directions defined by the (y, z)-plane. Thus we look for solutions of the Einstein equations of the form

$$g_{\mu\nu}(\vec{x}, t) = \left(\begin{array}{cccc} -1 & & & \mathbf{0} \\ & 1 & & \\ & & f^2(x,t) & \\ \mathbf{0} & & & g^2(x,t) \end{array} \right). \tag{3.9}$$

The corresponding space-time line-element is

[1] Attempts to explain matter as a manifestation of physics in higher-dimensional space-time, as in Kaluza-Klein models or the more sophisticated string theories, aim to redress this situation.

$$ds^2 = -c^2dt^2 + dx^2 + f^2(x,t)dy^2 + g^2(x,t)dz^2. \tag{3.10}$$

We observe the following properties of these metrics:
– in the (x,t)-plane space-time is flat (Minkowskian);
– in the y- and z-directions space-time is generally not flat, but the value of the metric depends only on the distance along the x-axis and on time, not on the values of y or z themselves. Hence in the co-ordinate system chosen the components of these gravitational potentials are the same everywhere in the (y,z)-plane. In particular they do not fall off to unity at spatial infinity, showing that the space-time is not asymptotically Minkowskian at large space-like distances. Of course, they share this property with plane electromagnetic waves, which at a fixed time also have the same amplitude everywhere in the transverse directions;
– the metric has no inverse if $f = 0$ or $g = 0$; then the connections are singular. We must take special care when this happens.

With the *Ansatz* (3.9) for the metric one can compute the components of the connection using eq.(1.9). Using the notation $x^\mu = (ct, x, y, z)$ for $\mu = (0,1,2,3)$, the non-vanishing components have been assembled in the following list:

$$\boxed{\begin{array}{ll} {\Gamma_{02}}^2 = {\Gamma_{20}}^2 = \dfrac{f_t}{cf}, & {\Gamma_{22}}^0 = \dfrac{1}{c}\, f f_t, \\[2ex] {\Gamma_{03}}^3 = {\Gamma_{30}}^3 = \dfrac{g_t}{cg}, & {\Gamma_{33}}^0 = \dfrac{1}{c}\, g g_t, \\[2ex] {\Gamma_{12}}^2 = {\Gamma_{21}}^2 = \dfrac{f_x}{f}, & {\Gamma_{22}}^1 = -f f_x, \\[2ex] {\Gamma_{13}}^3 = {\Gamma_{31}}^3 = \dfrac{g_x}{g}, & {\Gamma_{33}}^1 = -g g_x. \end{array}} \tag{3.11}$$

The next step in solving the Einstein equations is to compute the components of the Riemann curvature tensor using eq.(1.14). The only independent covariant components $R_{\mu\nu\lambda\kappa} = {R_{\mu\nu\lambda}}^\rho g_{\rho\kappa}$ not identically zero are the following ones:

$$\boxed{\begin{array}{ll} R_{0202} = \dfrac{1}{c^2}\, f f_{tt}, & R_{0303} = \dfrac{1}{c^2}\, g g_{tt}, \\[2ex] R_{1212} = f f_{xx}, & R_{1313} = g g_{xx}, \\[2ex] R_{0212} = \dfrac{1}{c}\, f f_{tx}, & R_{0313} = \dfrac{1}{c}\, g g_{tx}, \\[2ex] R_{2323} = fg(f_x g_x - \dfrac{1}{c^2}\, f_t g_t). & \end{array}} \tag{3.12}$$

Before discussing the general solution of the Einstein equations for this class of metrics, we first use the above results to answer the question which of the metrics singled out by the *Ansatz* (3.9) describe flat Minkowski space. If we require all components of the curvature tensor (3.12) to vanish independently, the most general solution for $f(x,t)$ and $g(x,t)$ is

$$\begin{aligned} f(x,t) &= a + px + rct, \\ g(x,t) &= b + qx + lct, \end{aligned} \tag{3.13}$$

with the additional constraint

$$pq = rl. \tag{3.14}$$

Thus metrics of the form (3.9), and corresponding line-elements (3.10), describe a flat space-time if f and g are linear functions of x and t, with the additional constraint (3.14) between their co-efficients.

The components of the Ricci tensor $R_{\mu\nu}$ now follow by taking a covariant trace as in eqs.(1.16). The non-vanishing ones are

$$\boxed{\begin{aligned} R_{00} &= \frac{1}{c^2}\left(\frac{f_{tt}}{f} + \frac{g_{tt}}{g}\right), \\ R_{01} &= R_{10} = \frac{1}{c}\left(\frac{f_{tx}}{f} + \frac{g_{tx}}{g}\right), \\ R_{11} &= \frac{f_{xx}}{f} + \frac{g_{xx}}{g}, \\ R_{22} &= f\left(f_{xx} - \frac{1}{c^2} f_{tt}\right) + \frac{f}{g}\left(f_x g_x - \frac{1}{c^2} f_t g_t\right), \\ R_{33} &= g\left(g_{xx} - \frac{1}{c^2} g_{tt}\right) + \frac{g}{f}\left(f_x g_x - \frac{1}{c^2} f_t g_t\right), \end{aligned}} \tag{3.15}$$

In the absence of material sources Einstein's equations require the Ricci tensor to vanish. Assuming f and g themselves not to vanish identically, some manipulation of these equations then leads to the following five necessary and sufficient conditions:

$$\begin{aligned} f_{xx} - \frac{1}{c^2} f_{tt} &= 0, \\ g_{xx} - \frac{1}{c^2} g_{tt} &= 0, \\ f_x g_x - \frac{1}{c^2} f_t g_t &= 0, \\ g f_{xx} + f g_{xx} &= 0, \\ g f_{tx} + f g_{tx} &= 0. \end{aligned} \tag{3.16}$$

The first two equations are free wave equations in two-dimensional (x, t) space. They imply that f and g can be decomposed into left- and right-moving waves:

$$\begin{aligned} f(x,t) &= f_+(x+ct) + f_-(x-ct), \\ g(x,t) &= g_+(x+ct) + g_-(x-ct), \end{aligned} \tag{3.17}$$

where $(f_-(u), g_-(u))$ and $(f_+(v), g_+(v)$ are functions only of the single lightcone variable $u = x - ct$, and $v = x + ct$, respectively. Each of these components then satisfies the wave equation by itself, and at this stage they can be taken to be completely independent. The only ambiguity here is that a possible constant term can be divided in an arbitrary way between the left- and right-moving components of the solution.

If we now substitute this decomposition of f and g in the third equation (3.16), and assume that the first derivatives of the components with respect to u or v (i.e. $f'_\pm$ and $g'_\pm$) are all non-zero, then the only solutions of the Einstein equations correspond to flat Minkowski space. This can be seen by dividing the equation by $g'_+ g'_-$ to obtain

$$\frac{f'_+}{g'_+} = -\frac{f'_-}{g'_-} = \lambda, \tag{3.18}$$

where λ on the right-hand side of the last equality is a finite, non-zero constant. This last equality holds because the two previous expressions are functions of different variables, hence they can be identically equal only if they are constant. It then follows, that

$$f'_+ = \lambda g'_+, \qquad f'_- = -\lambda g'_-. \tag{3.19}$$

Inserting these results into the last two equations (3.16), addition and subtraction gives the equivalent conditions

$$(\lambda g + f)\, g''_+ = 0, \qquad (\lambda g - f)\, g''_- = 0. \tag{3.20}$$

The solution $g''_+ = g''_- = 0$ implies $f''_+ = f''_- = 0$. This leads to linear dependence of f and g on x and t: they are of the form (3.13), (3.14) and correspond to flat Minkowski space.

On the other hand, in case one of the second derivatives $g''_\pm$ is non-zero, one of the co-efficients $(\lambda g \pm f)$ must vanish; note that they cannot vanish simultaneously for non-zero f and g. Hence eq.(3.20) gives either

$$f = \lambda g \qquad \text{and} \qquad g''_+ = f''_+ = 0, \tag{3.21}$$

or

$$f = -\lambda g \qquad \text{and} \qquad g''_- = f''_- = 0. \tag{3.22}$$

But combining this with the results (3.19), we find again that all second derivatives must vanish and that the geometry is that of flat space-time.

We conclude that in order to get non-trivial solutions, at least one of the first derivatives $f'_\pm$ or $g'_\pm$ must vanish. The last three equations (3.16) then imply that, after absorbing any constants terms in the remaining components, the only solutions are

$$f(x,t) = f_+(x+ct), \qquad g(x,t) = g_+(x+ct), \tag{3.23}$$

or

$$f(x,t) = f_-(x-ct), \qquad g(x,t) = g_-(x-ct), \tag{3.24}$$

with the additional condition

$$\frac{f''}{f} + \frac{g''}{g} = 0. \tag{3.25}$$

Although f and g represent waves traveling either to the right or to the left at the velocity of light, the non-linear character of the Einstein equations exemplified by the additional condition (3.25) do not allow us to make arbitrary superpositions of these solutions. Interestingly, viewed as a two-dimensional theory in (x,t)-space f and g represent chiral bosons. Wherever necessary in the following we chose to work with right-moving fields $f(x,t) = f_-(u)$ and $g(x,t) = g_-(u)$ for definiteness.

An example of a simple solution of eq.(3.25) is [9]:

$$\begin{aligned} f(x,t) &= \cos k(x-ct) = \cos ku, \\ g(x,t) &= \cosh k(x-ct) = \cosh ku. \end{aligned} \tag{3.26}$$

Note that at $u = (2n+1)\pi/2k$ the component f has a zero, whilst the solution for $g(x,t)$ grows indefinitely with time or distance. However, this does neither imply that the geometry is singular, nor that the gravitational fields become arbitrarily strong: the apparent problems turn out to be coordinate artifacts. This point is discussed in detail in the next section.

Let us consider a wave which reaches $x = 0$ at time $t = 0$, and extends to $x = -L$, i.e. the solution (3.26) holds for $-L \leq u \leq 0$. If $|kL| < \pi/2$ the metric components f^2 and g^2 are strictly positive. For $u \geq 0$ space-time is flat; we can choose a co-ordinate system in which $f = g = 1$. Note that the wave solution (3.26) assumes these same values at $u = 0$, whilst the first derivatives match as well. As a result the solutions are smoothly connected and satisfy the Einstein equations even at the boundary $u = 0$.

For $u \leq -L$ space-time is again flat, but now we have to use the general solution (3.13):

$$\begin{array}{rcl} f(x,t) & = & a + p(x-ct), \\ g(x,t) & = & b + q(x-ct), \end{array} \qquad (x-ct) \leq -L. \tag{3.27}$$

The co-efficients $(a, b; p, q)$ in this domain of (x, t) values are to be determined again by requiring the metric to be continuous and differentiable at the boundary $u = -L$ between the different domains of the solution. This condition is sufficient to guarantee that the Einstein equations are well-defined and satisfied everywhere. Imposing these requirements we find

$$\begin{array}{rclcrcl} \cos kL & = & a - pL, & & k\sin kL & = & p \\ \cosh kL & = & b - qL, & & -k\sinh kL & = & q. \end{array} \tag{3.28}$$

For f and g in the domain $(x - ct) \leq -L$ we obtain the result

$$\begin{array}{rcl} f(x,t) & = & \cos kL + k\,(L+x-ct)\,\sin kL, \\ g(x,t) & = & \cosh kL - k\,(L+x-ct)\,\sinh kL. \end{array} \tag{3.29}$$

Conclusion: we have constructed a solution of the Einstein equations in empty space which interpolates between two Minkowski half-spaces, connected by a finite region of non-zero curvature, where the gravitational fields have non-zero gradients and test particles are accelerated with respect to the initial inertial frame. After passage of the wave the test particles have acquired a finite, but constant velocity. This shows that they have been accelerated, but the acceleration has ceased after $t > (x + L)/c$.

The plane-fronted wave constructed here is a well-defined solution of the equation $R_{\mu\nu} = 0$ everywhere, including the wave fronts $x = ct$ and $x = ct - L$, as the contributions of the metric components $g_{yy} = f^2$ and $g_{zz} = g^2$ always compensate each other in eq.(3.25). However the Riemann curvature itself, which coincides with the Weyl curvature in this case, does not vanish everywhere. Indeed, we find from eq.(3.12)

$$R_{0202} = R_{1212} = \pm R_{0212} = ff'',$$

$$R_{0303} = R_{1313} = \pm R_{0313} = gg'', \tag{3.30}$$

$$R_{2323} = 0.$$

Inserting the explicit solution for f and g for each of the three regions $x > ct$, $-L < x - ct < 0$ and $x < ct - L$ one finds that the curvature is zero in the two Minkowski half-spaces before and after the wave, as expected, but finite and non-zero in the interior of the wave. Indeed, the second derivatives f'' and g'' have a finite discontinuity on the wave front of magnitude $\mp k^2$ at $x = ct$, and $-k^2 \cos kL$, resp. $k^2 \cosh kL$, at $x = ct - L$. This finite discontinuity then also appears in the curvature components, where it must be attributed to a finite discontinuity in the Weyl curvature, and thus can not be attributed to a (singular) distribution of sources. But the Riemann-Christoffel connections, which depend only on the first derivatives of the metric, are continuous. Then the geodesics and the motions of test particles in the space-time described by the wave are continuous, and it follows that the jump in the Weyl curvature presents no physical inconsistencies.

3.3 Nature of the space-time

The planar wave constructed above can be characterised as a region of non-vanishing curvature sandwiched between two flat space-times and propagating with the speed of light —either in the positive or the negative x-direction— such that one of the two flat space-time regions grows at the expense of the other. In this section I establish the relation between the two flat space-times, before and after the wave. The analysis shows that they are not related by a simple Lorentz transformation: a set of equidistant test particles carrying synchronised ideal clocks and thus defining an inertial co-ordinate system before the wave are neither equidistant nor synchronised in *any* inertial co-ordinate system after the wave. Therefore the two Minkowski space-times on either side of the region of non-zero curvature are inequivalent, in the sense that the Lorentz group of rigid transformations leaving the inproduct of four-vectors invariant in these flat regions are embedded differently in the group of local co-ordinate transformations which constitutes the symmetry of the full theory.

This situation is also encountered in scalar field theories with broken symmetry, when there is a boundary separating two domains with different vacuum expectation values of the field. One can even show that in some models these two forms of spontaneous symmetry breaking —of a scalar field and of the gravitational field— are connected [24]. Therefore we find a new interpretation for the planar gravitational waves: they act as domain boundaries separating different, inequivalent flat-space solutions ('vacua') of the Einstein equations. These domain boundaries are dynamical: they are not static but move at the speed of light; for this reason they are sometimes refered to as 'shock waves'. However, the planar wave solutions discussed here have finite extent, whereas usually the term shock-wave is reserved for waves with a delta-function profile.

Regions of space-time enclosed by such dynamical boundaries clearly grow or shrink at the speed of light. If we consider our planar wave solution as an approximation to a more general domain-wall geometry with a radius of curvature large compared to the characteristic scale of the region of space-time we are interested in, then we can use it to describe the dynamics of gravitational domain structure, which may have interesting implications for example for the physics of the early universe.

Having sketched this general picture, I now turn to describe the sandwich structure of the planar wave geometry more precisely. A convenient procedure is to introduce new co-ordinates $X^\mu = (cT, X, Y, Z)$ related to the original $x^\mu = (ct, x, y, z)$ by

$$\begin{aligned} cT &= ct - \frac{\Lambda}{2}, \\ X &= x - \frac{\Lambda}{2}, \\ Y &= fy, \\ Z &= gz, \end{aligned} \tag{3.31}$$

where in terms of the original coordinates x^μ

$$\Lambda(x^\mu) = y^2 ff' + z^2 gg'. \tag{3.32}$$

This coordinate transformation takes an even simpler form when expressed in light-come variables

$$\begin{aligned} U &= X - cT = x - ct = u, \\ V &= X + cT = x + ct - \Lambda = v - \Lambda. \end{aligned} \tag{3.33}$$

Indeed the line-element (3.10) now becomes

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 + f^2dy^2 + g^2dz^2 \\ &= dUdV + K(U,Y,Z)dU^2 + dY^2 + dZ^2, \end{aligned} \tag{3.34}$$

with

$$K(U,Y,Z) = \frac{f''}{f}Y^2 + \frac{g''}{g}Z^2 = \frac{g''}{g}\left(Z^2 - Y^2\right). \tag{3.35}$$

Both for $t < x/c$ and $t > (x+L)/c$ we have $f'' = g'' = 0$, and therefore the line element (3.34) is manifestly flat Minkowskian on both sides of the planar wave. However, inside our particular wave solution one has

$$\frac{f''}{f} = -\frac{g''}{g} = -k^2. \tag{3.36}$$

Then $K = k^2(Z^2 - Y^2)$, independent of U. The coordinate system X^μ with the line-element (3.34) is the closest one to an inertial frame at all times for an observer at rest in the origin one can construct. Indeed, the point $x = y = z = 0$ corresponds to $X = Y = Z = 0$ at all times, and both before and after the wave the X^μ define a global inertial frame. Furthermore, inside the wave the new coordinate system deviates from an inertial frame only for large values of $(Z^2 - Y^2)$. We also observe that in this coordinate system the line element (3.34) never becomes singular, showing the previous zero's of $\det g$, corresponding to zero's of $f(u)$, to be coordinate artifacts indeed.

Computing the various geometrical quantities: metric, connections and curvature components, in the new coordinates produces the following summary of results:

– The only non-vanishing components of the inverse metric are:

$$\begin{aligned} & g^{UV} = g^{VU} = 2, \quad g^{VV} = 4k^2(Y^2 - Z^2), \\ & g^{YY} = g^{ZZ} = 1. \end{aligned} \tag{3.37}$$

– The only non-vanishing components of the connection are:

$$\begin{aligned} & \Gamma_{UU}^{\;\;\;Y} = k^2 Y, \qquad \Gamma_{YU}^{\;\;\;V} = \Gamma_{UY}^{\;\;\;V} = -2k^2 Y, \\ & \Gamma_{UU}^{\;\;\;Z} = -k^2 Z, \quad \Gamma_{ZU}^{\;\;\;V} = \Gamma_{UZ}^{\;\;\;V} = 2k^2 Z. \end{aligned} \tag{3.38}$$

– The only non-vanishing covariant components of the curvature are

$$R_{UYUY} = -R_{UZUZ} = -k^2. \tag{3.39}$$

Thus in these coordinates the curvature components are constant inside the wave, and we conclude that our particular planar wave solution is actually like a block-wave: at the boundaries $U = 0$, $U = -L$ the curvature components jump from zero to the values (3.39) and back to zero, whilst in between they are constant. This makes it easy to visualize how our particular solution (3.26) can be used to construct general planar wave solutions with polarization along the Y- and Z-axes: essentially it amounts to glueing together sufficiently narrow block waves of fixed curvature to obtain solutions in which the curvature components vary according to some arbitrary profile.

The result (3.39) also allows one to verify once more by a one-line computation that the Ricci tensor vanishes identically for these planar-wave geometries.

With these results in hand, our next aim is to establish the relation between the flat half-spaces before and after the wave. We have already observed that the origin $x = y = z = 0$ becomes the origin $X = Y = Z = 0$ for all times t or T, respectively. Next observe, that the coordinate transformation (3.31) is not a Lorentz transformation: the coordinates (t, x, y, z), defining an inertial frame before the passage of the wave, represent a non-inertial frame after passage. As a check, note that the jacobian of the transformation is

$$\left| \frac{\partial X^\mu}{\partial x^\nu} \right| = fg \neq 1, \qquad \text{for } u = x - ct \neq 0. \tag{3.40}$$

Indeed, a system of synchronized, equidistant clocks at rest in the first coordinate system, which operationally defines an inertial frame before the wave arrives, is neither synchronized nor equidistant in the final inertial frame. Moreover, in general it cannot be made so by a simple Lorentz transformation: the clocks have been accelerated with respect to each other by the wave, and ultimately define a non-inertial frame at late times. This is verified by the explicit calculation of geodesics in the next section.

¿From this analysis it will be clear, that in principle it is possible to determine the passage, and measure the characteristics, of a planar gravitational wave by measuring the acceleration it imparts to test masses in the laboratory. Indeed, such measurements can determine the functions (Λ, f, g) from the relation between (t, x, y, z) and (T, X, Y, Z). Although in this form the statement holds only for waves of the type discussed, the principles for other types of waves are the same, even if the functions characterizing the relation between the initial and final Minkowski spaces are generally different.

3.4 Scattering of test particles

In order to study the response of test masses to passing gravitational waves, it is necessary to solve for the geodesics of space-time in the gravitational field generated by the wave. We are particularly interested in the worldlines of particles initially at rest in a specific inertial frame (the laboratory). Observing frame gives the scattering data, from which the properties of the gravitational wave can then be inferred relatively easily.

In principle it is very easy to find geodesics: they correspond to the world-lines of particles in free fall. Consider therefore a particle at rest in the initial Minkowski frame; its four co-ordinates are

$$x^\mu(\tau) = (c\tau, \xi_1, \xi_2, \xi_3), \tag{3.41}$$

where the three vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ fixing the position of the particle is constant; note also that by definition for a particle at rest in an inertial frame the co-ordinate time equals the proper time τ.

In the co-ordinate system we use, a particle with co-ordinates (3.41) is actually always in free fall, its co-ordinates in the x^μ-frame do not change if it enters a non-trivial gravitational field. Of course, in an arbitrary gravitational field the x^μ-frame is not an inertial frame, and even if at late times space-time is flat again the particle will have acquired a finite, generally non-vanishing four velocity. However, its position in the final Minkowski frame can be found straightforwardly by applying the co-ordinate transformation (3.31).

Let us first confirm the above statements by a simple mathematical argument. Then we turn to apply the result to find the actual motion of the test particle in the final Minkowski frame. We begin by noting that in the x^μ-frame the metric for the gravitational wave is always of the form (3.9), and therefore the only non-vanishing components of the connection are those given in eq.(3.11). In particular, in the x^μ-frame the components ${\Gamma_{00}}^\mu$ always vanish.

On the other hand, for a particle with the worldline (3.41), the four velocity is purely time-like and constant:

$$u^\mu = \frac{dx^\mu}{d\tau} = (c, 0, 0, 0). \tag{3.42}$$

Then the four acceleration in this frame vanishes identically, as does the gravitational force:

$${\Gamma_{\nu\lambda}}^\mu \, u^\nu u^\lambda = 0. \tag{3.43}$$

As a result the geodesic equation (1.10) is satisfied, proving that (3.41) represents the worldline of a particle at *all* times.

What we are particularly interested in, is the position and velocity of the particle at the moment it comes out of the gravitational wave. It has already been remarked that the origin $\boldsymbol{\xi} = \mathbf{0}$ remains at rest and coincides with the origin of the final Minkowski frame. Therefore at late times the origin has laboratory coordinates

$$X_0^\mu(\tau) = (c\tau, 0, 0, 0)\,. \tag{3.44}$$

Let us compare this with the four co-ordinates of a particle away from the origin in the (y, z)-plane: $\boldsymbol{\xi} = (0, \xi_2, \xi_3) \neq \mathbf{0}$. For such a particle the values of the metric components at late times are

$$\begin{array}{rcl} f(0,\tau) & = & a - pc\tau, \\ g(0,\tau) & = & b - qc\tau, \end{array} \qquad c\tau \geq L. \tag{3.45}$$

Here $(a, b; p, q)$ have the values quoted before, eq.(3.28). For Λ one finds in the same domain $c\tau \geq L$:

$$\Lambda = ap\xi_2^2 + bq\xi_3^2 - \left(p^2\xi_2^2 + q^2\xi_3^2\right) c\tau. \tag{3.46}$$

As a result, the Minkowski co-odinates of this particle at late proper times are

$$X_1^\mu(\tau) = (cT_1, X_1, Y_1, Z_1), \tag{3.47}$$

with

$$\begin{aligned}
cT_1 &= -\frac{1}{2}\left(ap\xi_2^2 + bq\xi_3^2\right) + \frac{c\tau}{2}\left(2 + p^2\xi_2^2 + q^2\xi_3^2\right), \\
X_1 &= -\frac{1}{2}\left(ap\xi_2^2 + bq\xi_3^2\right) + \frac{c\tau}{2}\left(p^2\xi_2^2 + q^2\xi_3^2\right), \\
Y_1 &= a\xi_2 - p\xi_2 c\tau, \\
Z_1 &= b\xi_3 - q\xi_3 c\tau.
\end{aligned} \tag{3.48}$$

Note, that clocks which were originally synchronized —same value of $t = \tau$ at early proper time— are no longer so after passage of the gravitational wave: T_1 is not equal to $T_0 = \tau$ at late proper time. Before we can calculate the position of the test particle at synchronized co-ordinate time $T = \tau$, in the final Minkowski frame, we first have to correct for this gravitationally induced position-dependent time shift.

To determine the instantaneous proper distance between the particle with worldine $X_1^\mu(\tau)$ and the origin, we have to ask at what proper time $\bar{\tau}$ the time co-ordinate T_1 equals the time T_0 measured by a clock at rest in the origin:

$$T_1(\bar{\tau}) = \tau, \tag{3.49}$$

with the result

$$c\bar{\tau} = \frac{2c\tau + ap\xi_2^2 + bq\xi_3^2}{2 + p^2\xi_2^2 + q^2\xi_3^2}. \tag{3.50}$$

The position at this proper time, $\bar{X}_1^\mu = X_1^\mu(\bar{\tau})$, can now be evaluated:

$$\begin{aligned}
\bar{X}_1 &= \frac{-\left(ap\xi_2^2 + bq\xi_3^2\right) + c\tau\left(p^2\xi_2^2 + q^2\xi_3^2\right)}{2 + p^2\xi_2^2 + q^2\xi_3^2}, \\
\bar{Y}_1 &= \xi_2\left(\frac{(aq - bp)\, q\xi_3^2 + 2\,(a - pc\tau)}{2 + p^2\xi_2^2 + q^2\xi_3^2}\right), \\
\bar{Z}_1 &= \xi_3\left(\frac{-\,(aq - bp)\, p\xi_2^2 + 2\,(b - qc\tau)}{2 + p^2\xi_2^2 + q^2\xi_3^2}\right),
\end{aligned} \tag{3.51}$$

Finally we can also determine the velocity of the test particle:

$$\frac{\boldsymbol{v}}{c} = \frac{1}{2 + p^2\xi_2^2 + q^2\xi_3^2}\left(p^2\xi_2^2 + q^2\xi_3^2, -2p\xi_2, -2q\xi_3\right), \tag{3.52}$$

and as a result the kinetic energy is

$$\frac{1}{2}\,M\boldsymbol{v}^2 \;=\; Mc^2\,\frac{\left(p^2\xi_2^2+q^2\xi_3^2\right)\,\left(4+p^2\xi_2^2+q^2\xi_3^2\right)}{\left(2+p^2\xi_2^2+q^2\xi_3^2\right)^2}. \tag{3.53}$$

From this analysis we draw the conclusion, that not only in general test particles are accelerated by the gravitational wave, but also that the acceleration and hence the final velocity depend on the position of the particle. Therefore no Lorentz transformation with any fixed boost velocity v can bring all test particles initially at rest w.r.t. one another back to rest simultaneously after passage of the wave. This result explicitly demonstrates the inequivalence of the initial and final Minkowski space-times.

3.5 Symmetry breaking as a source of gravitational waves

The planar gravitational wave solutions described in this chapter arise as transition regions between flat domains of space-time, which are related by a coordinate transformation that cannot be reduced to a (linear) Lorentz transformation. We have therefore interpreted them as domain boundaries between regions of space-time trapped in inequivalent classical vacua of the gravitational field (in the absence of a cosmological constant). This is possible because flat space (zero curvature) is associated with a value of the gravitational field, the metric $g_{\mu\nu}$, which in an appropriate coordinate system (gauge) is constant, non-vanishing and Lorentz-invariant, but not invariant under general coordinate transformations. In this sense the general coordinate invariance of the Einstein equations is broken spontaneously by the vacuum solutions corresponding to Minkowski space-time.

A domain structure arising from the presence of degenerate but inequivalent classical vacua is a well-known consequence of spontaneous symmetry breaking in scalar field theories. In this section this point is elaborated on and it is shown, that dynamical domain boundaries in scalar field theories (scalar-type shock waves, if one prefers) can actually be sources of planar gravitational waves of the kind analysed in detail above [24].

The simplest scalar field theory with a continuously degenerate set of classical vacua is that of a complex scalar field with a 'mexican hat' type of potential, as described by the lagrangian

$$\frac{1}{\sqrt{-g}}\,\mathcal{L} \;=\; -g^{\mu\nu}\,\partial_\mu\phi^*\partial_\nu\phi \;-\; \frac{\lambda}{2}\left(\phi^*\phi-\frac{\mu^2}{\lambda}\right)^2, \tag{3.54}$$

In this lagrangian $\hbar c\lambda$ is a dimensionless coupling constant and μ is an inverse length ($\mu\hbar/c$ is a mass). In the natural system of units in which $\hbar = c = 1$ we can take λ to be dimensionless and μ to represent a mass. In the following equations one then can substitute $c = 1$ everywhere.

This lagrangian (3.54) is invariant under rigid $U(1)$ phase shifts $\phi \to \phi e^{i\eta}$, with η constant. As a result for $\mu^2 > 0$ there is an infinitely degenerate set of solutions of the equations of motion minimizing the potential (refered to as classical vacua), of the form

$$\phi \, = \, \frac{\mu}{\sqrt{\lambda}}\, e^{i\theta}, \tag{3.55}$$

with θ an arbitrary real constant. Clearly the solutions themselves are not invariant under the $U(1)$ transformations: the classical vacuum solutions break the $U(1)$ symmetry spontaneously. However, because of the $U(1)$ symmetry of the dynamics these solutions are physically indistinguishable, and the absolute value of θ is unobservable. On the other hand, variations in the vacuum angle between different regions of space-time can have observable effects [26, 27].

Parametrizing the complex scalar field as $\phi(x) = \rho(x)e^{i\theta(x)}/\sqrt{2}$, the lagrangian becomes

$$\frac{1}{\sqrt{-g}}\,\mathcal{L} \, = \, -\frac{1}{2}\, g^{\mu\nu}\partial_\mu\rho\partial_\nu\rho \, - \, \frac{\lambda}{8}\left(\rho^2 - \frac{2\mu^2}{\lambda}\right)^2 \, - \, \frac{1}{2}\,\rho^2\, g^{\mu\nu}\partial_\mu\theta\partial_\nu\theta. \tag{3.56}$$

This shows that in the spontaneously broken mode of the $U(1)$ symmetry the theory desribes actually two real scalar fields: the field $\chi = \rho - \mu\sqrt{2/\lambda}$, obtained by shifting ρ by its classical vacuum value $\mu\sqrt{2/\lambda}$, with a mass (or inverse coherence length) equal to μ, which can become arbitrarily large; and the strictly massless field $\sigma = \mu\theta\sqrt{2/\lambda}$, representing the Goldstone excitations arising from spontaneous symmetry breaking [26, 27].

Clearly at energies small compared to the scale μ set by the mass of the field χ the excitations of this massive field play no role and one only has to consider the Goldstone field σ, described by the effective lagrangian

$$\frac{-1}{\sqrt{-g}}\,\mathcal{L} \, = \, \frac{c^4}{16\pi G}\, R \, + \, \frac{1}{2}\, g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma, \tag{3.57}$$

where we have added the Einstein-Hilbert action for the gravitational field as this also describes a low-energy degree of freedom. Note that with the potential defined in eqs.(3.54), (3.56) there is no cosmological constant in the presence of the vacuum value of the scalar field, $\rho = \mu\sqrt{2/\lambda}$ or equivalently $\chi = 0$.

The field equations derived from this lagrangian are the Klein-Gordon equation for the Goldstone field σ:

$$\Box^{cov}\sigma \, = \, \frac{1}{\sqrt{-g}}\,\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\sigma \, = \, 0. \tag{3.58}$$

and the Einstein-equations with a source term from the energy-momentum of the Goldstone field:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} \partial_\mu \sigma \partial_\nu \sigma. \tag{3.59}$$

Remarkably this system of equations admits exact planar wave solutions similar to the free (sourceless) Einstein equations. Using the same Ansatz (3.10) for the metric we find running planar gravitational waves $(f_\pm(x \pm ct), g_\pm(x \pm ct))$ in combination with a scalar wave $\sigma_\pm(x \pm ct)$ running in the same direction. The continuity conditions require that this scalar wave interpolates continuously between a domain in which $\sigma = \sigma_0$ before the wave and a domain where $\sigma = \sigma_1$ after the wave, where both (σ_0, σ_1) are constants.

The simplest solution of this kind is a profile of σ interpolating linearly between these values:

$$\begin{aligned} \sigma_-(u) &= \sigma_0, && \text{for } u \geq 0; \\ \sigma_-(u) &= \sigma_0 - ku\mu\sqrt{2/\lambda}, && \text{for } -L \leq u \leq 0; \\ \sigma_-(u) &= \sigma_1 = \sigma_0 + kL\mu\sqrt{2/\lambda}, && \text{for } u \leq -L, \end{aligned} \tag{3.60}$$

with similar expressions for the left-moving solutions $\sigma_+(v)$. It follows that the slope parameter k in (3.60) is given by

$$k = \frac{\sigma_1 - \sigma_0}{\mu L} \sqrt{\frac{\lambda}{2}}. \tag{3.61}$$

At this point it is useful to recall that σ actually represents the phase of the complex scalar field ϕ. Hence with a fixed value of the modulus ρ a monotonic linear increase of σ implies a pure monochromatic oscillation of the real and imaginary components of the scalar field:

$$\phi(u) = \phi_0 \, e^{iku} = \frac{\mu}{\sqrt{\lambda}} \, e^{ik(x-ct)}, \tag{3.62}$$

where the last equality assumes ρ to equal its classical vacuum value and the gauge choice $\sigma_0 = 0$. It follows that the slope parameter k actually represents the wavenumber of these 'large' oscillations of ϕ interpolating between different classical vacuum values.

It remains to solve the Einstein equations (3.59). For the right-moving wave solution $(\sigma_-(u), f_-(u), g_-(u))$ these equations reduce to a slightly modified version of eqs.(3.16), the only modification being a constant source term in eq.(3.25):

$$\frac{f''}{f} + \frac{g''}{g} = -\frac{16\pi G\mu^2}{c^4 \lambda} k^2, \qquad \text{for } -L \leq u \leq 0. \tag{3.63}$$

Outside the region $-L \leq u \leq 0$ the right hand side vanishes, and we have the trivial flat-space solutions $f'' = g'' = 0$.

It is not very difficult to establish the existence of solutions to equation (3.63). In fact there are more solutions than the purely gravitational case, because first there are the planar gravitational block waves of the type

$$f(u) = \cos \kappa_1 u, \qquad g(u) = \cosh \kappa_2 u, \tag{3.64}$$

with the constraint between the parameters (κ_1, κ_2):

$$\kappa_1^2 - \kappa_2^2 = \frac{16\pi G\mu^2}{c^4\lambda}\, k^2. \tag{3.65}$$

These waves have a purely gravitational contribution from a non-vanishing Weyl tensor. But in addition there are also purely oscillatory solutions

$$f(u) = \cos \kappa_1 u, \qquad g(u) = \cos \kappa_2 u, \tag{3.66}$$

with a relation between the wavenumbers (κ_1, κ_2) of the type

$$\kappa_1^2 + \kappa_2^2 = \frac{16\pi G\mu^2}{c^4\lambda}\, k^2. \tag{3.67}$$

It is quite clear that these purely oscillatory waves require a non-vanishing of the Ricci tensor: they can exist only in the presence of the scalar field $\sigma_-(u)$, because in the limit $k \to 0$ one finds simultaneously that both $\kappa_{1,2} \to 0$. This is in contrast to the hyperbolic type of solution (3.65), where $k = 0$ only implies $\kappa_1 = \kappa_2$. Therefore one might think of the first type of solution as *radiative* waves, which have kept the quadrupole character known from the propagation of small perturbations in the linearized version of gravitational field theory, whereas the second type are *matter-induced* waves with a dipole-like behaviour.

By performing the coordinate transformation (3.31) we can again construct a reference frame which is manifestly Minkowskian before and after the wave, and has constant curvature components inside the wave. This is based on the choice of the simple solution (3.60) for the scalar field, with constant slope parameter k, i.e. a monochromatic ϕ-wave. As in the case of the purely gravitational waves, more general solutions with smooth behaviour at the wavefront can be contructed.

The main lesson to be learnt from the results in this section is that the interpretation of the gravitational waves as moving domain boundaries between inequivalent classical vacua has its parallel in the case of scalar fields with broken symmetry, and that these scalar waves are actually accompanied by planar gravitational waves of the type constructed above.

3.6 Coupling to the electro-magnetic field

The coupling of planar gravitational waves to scalar fields of the Goldstone type is the result of the existence of field configurations with constant slope (in domain boundaries), for which the energy-momentum tensor has constant non-zero components in half of the directions of space-time. A similar type of energy-momentum density can be generated by constant electric and magnetic fields and again one finds that these are accompanied by gravitational waves [25].

We consider the simplest case of constant electric and magnetic field perpendicular to each other, for example in the (y, z)-plane:

$$\vec{E} = (0, E, 0), \qquad \vec{B} = (0, 0, B), \tag{3.68}$$

and a metric which is non-flat only in the (zz) direction:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + g^2 dz^2, \tag{3.69}$$

These fields provide a solution of the combined Einstein and Maxwell equations:

$$R_{mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} = -\frac{8\pi\varepsilon_0 G}{c^2} \left(F_{\mu\lambda} F_{\nu}^{\ \lambda} - \frac{1}{4} g_{\mu\nu} F^2 \right), \tag{3.70}$$

and

$$D_\mu F^{\mu\nu} = 0, \tag{3.71}$$

provided the electric and magnetic field are constant and equal modulo c in magnitude:

$$E = \pm cB = \text{constant}, \tag{3.72}$$

whilst the metric component g_{zz} represents a planar wave

$$g(x \pm ct) = \cos \frac{1}{2} k(x \pm ct), \qquad k^2 = \frac{32\pi\varepsilon_0 G}{c^4} E^2. \tag{3.73}$$

Here the sign in the argument (i.e. left or right moving wave) depends on the sign in (3.72).

It is straightforward to check that this is a solution of the Einstein equation by substituting the g as above and $f = 1$ in eq.(3.15) and writing out the energy-momentum tensor of the electro-magnetic field:

$$T_{\mu\nu} = \varepsilon_0 c^2 \begin{pmatrix} \frac{1}{2}(E^2+c^2B^2) & cBE & 0 & 0 \\ cBE & \frac{1}{2}(E^2+c^2B^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(E^2-c^2B^2) & 0 \\ 0 & 0 & 0 & \frac{g^2}{2}(E^2-c^2B^2) \end{pmatrix}. \tag{3.74}$$

Observe, that the metric component g_{zz} itself becomes

$$g_{zz} = g^2 = \frac{1}{2}\left(1 + \cos k(x \pm ct)\right), \tag{3.75}$$

ans oscillates with frequency

$$\omega = kc = \frac{3E}{c}\sqrt{8\pi\varepsilon_0 G}. \tag{3.76}$$

It may seem surprising and worrying that constant electric and magnetic field generate gravitational waves of such large amplitudes; however, note that in laboratory units the frequency is

$$\nu = 1.3 \times 10^{-19}\,\frac{E}{[1\text{V/m}]}\ \text{Hz}. \tag{3.77}$$

For ordinary electric fields in the laboratory this means that it takes a full lifetime of the universe or longer to go through one complete cycle. Even at the highest fields imaginable in theory, when the vacuum breaks down due to electron-positron pair creation:

$$E_{crit} = \frac{\pi m_e c^2}{e\lambda_e} = 4.1 \times 10^{18}\ \text{V/m}, \tag{3.78}$$

where e is the electron charge, m_e its mass and λ_e its Compton wave length, the frequency can not get higher than about 1 Hz. Therefore the effect will not be measurable under ordinary circumstances.

Nevertheless it does seem remarkable that constant electro-magnetic fields generate time-dependent (oscillating) gravitational fields. This effect, as well as the scalar-generated waves discussed in the previous section, has quite close similarities to the Josephson effect in superconductors, where a constant potential generates an oscillating current. In that case the current is related to the change in the phase of the electron condensate, which is mathematically analogous to the phase of the complex scalar field represented by the Goldstone mode σ.

Chapter 4

Black holes

4.1 Horizons

The example of gravitational shock waves/domain walls has shown that space-time in four dimensions is a truely dynamical arena for physics even in the absence of matter, although adding matter —for example in the form of scalar or electromagnetic fields— makes the possible structures even richer. In this chapter we turn to another class of very interesting structures in four-dimensional space-time: static or quasi-static solutions of the Einstein equations which represent gravitating extended bodies, whose static fields become strong enough to capture permanently anything that gets sufficiently close to the central core. As these quasi-static objects even capture light they are called black holes, but it has turned out that their properties are much more interesting and peculiar than this ominous but dull sounding name suggests.

Black holes possess two characteristic features. The first, which has given them their name, is the existence of a horizon: a closed surface through which objects can fall without ever being able to return. The second is the presence of a singularity, a locus of points inside the horizon where the curvature becomes infinite. It has even been conjectured by some authors that these two properties are so closely linked that any space-time singularity should always be hidden behind a horizon. This conjecture bears the name of the Cosmic Censorship hypothesis. Although as an unqualified mathematical theorem the hypothesis is certainly not correct, it could be true for singularities that arise in realistic processes from the gravitational collapse of macroscopic bodies, like imploding massive stars.

In this chapter we restrict our attention to the special black-hole solutions that might be called classical 'eternal' black holes, in the sense that at least outside the horizon they describe stationary gravitational fields, and do not require for the description of their behaviour the inclusion of their formation from collapsing matter. In fact they are solutions of the source-free Einstein or coupled Einstein-

Maxwell equations and are characterised for an outside observer completely in terms of their mass, their angular momentum and their electro-magnetic charge. One might alternatively think of them as black holes that have been formed in the infinite past and settled in a stationary state in which all trace of their previous history has been lost, e.g. by emitting gravitational radiation.

Because of this restriction these lectures do not include a detailed account of the process of gravitational collapse as studied by astrophysicists, but they do allow one to isolate and study in detail the special aspects of the physics of black-holes. It may even be that these very special simple solutions are relevant to elementary particle physics near the Planck scale, but that leads one to consider the problems of quantum gravity, a subject we will not deal with at length in these lectures. The interested reader can find more information on many aspects of black-hole structure and formation in the literature [28, 29].

4.2 The Schwarzschild solution

The simplest of all stationary black-hole solutions of the source-free Einstein equations is that for the static spherically symmetric space-time, asymptotically flat at spatial infinity, first described in the literature by Schwarzschild[1] [30]. We present a derivation of this solution making full use of the symmetries and their connection to Killing vectors explained in chapter 2.

In view of the spherical symmetry we introduce polar coordinates (r, θ, φ) in three-dimensional space and add a time coordinate t, measuring asymptotic Minkowski time at $r \rightarrow \infty$. The Schwarzschild solution is obtained by requiring that there exists at least one coordinate system parametrized like this in which the following two conditions hold:
(i) the metric components are t-independent;
(ii) the line-element is invariant under the three-dimensional rotation group $SO(3)$ acting on three-vectors $\boldsymbol{r}$ in the standard (linear) way.
The first requirement is of course equivalent to invariance of the metric under time-shifts $t \rightarrow t + \delta t$. Thus formulated both conditions take the form of a symmetry requirement.

In chapter (2) we discussed how symmetries of the metric are related to Killing vectors. Rotations of three-vectors are generated by the differential operators $\boldsymbol{L}$ of orbital angular momentum:

$$
\begin{aligned}
L_1 &= -\sin\varphi \frac{\partial}{\partial\theta} - \cot\theta\cos\varphi \frac{\partial}{\partial\varphi}, \quad L_2 = \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta\sin\varphi \frac{\partial}{\partial\varphi}, \\
L_3 &= \frac{\partial}{\partial\varphi}.
\end{aligned}
\tag{4.1}
$$

[1] The solution was found independently at the same time in [31].

Similarly, time-shifts are generated by the operator

$$L_0 = \frac{\partial}{\partial t}. \tag{4.2}$$

According to the discussion in sect.(2.4) the coefficients of the symmetry operators $L_A = L_A^\mu \partial_\mu$ are to be components of a set of Killing vectors L_A, $A = 0, 1, 2, 3$:

$$D_\mu L_{A\nu} + D_\nu L_{A\mu} = 0. \tag{4.3}$$

These can equivalently be characterised as a set of constants of geodesic flow as in eq.(2.45):

$$J_A = L_A^\mu p_\mu. \tag{4.4}$$

The relation with the metric is, that the Poisson brackets of these constants of geodesic flow with the hamiltonian are to vanish:

$$\{J_A, H\} = 0, \qquad H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{4.5}$$

The general solution of these conditions for the hamiltonian H is t-independent, whilst it can depend on the angular coordinates only through the Casimir invariant

$$\boldsymbol{L}^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}. \tag{4.6}$$

It follows that the metric must be of the form[2]

$$ds^2 = -h^2(r)dt^2 + g^2(r)dr^2 + k^2(r)d\Omega^2, \tag{4.7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the angular distance on the unit sphere, and the coefficients $h(r)$, $g(r)$ and $k(r)$) are functions of the radial coordinate r only. Indeed, the hamiltonian derived from this metric is

$$\begin{aligned} H &= -\frac{1}{2h^2} p_t^2 + \frac{1}{2g^2} p_r^2 + \frac{1}{2k^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2\right) \\ &= -\frac{1}{2h^2} p_t^2 + \frac{1}{2g^2} p_r^2 + \frac{\boldsymbol{L}^2}{2k^2}. \end{aligned} \tag{4.8}$$

Note that the set of functions (h, g, k) can not be unique, as one can always perform a coordinate transformation $r \rightarrow \bar{r}(r)$ to another metric of the same general form (4.7) with coefficients $(\bar{h}, \bar{g}, \bar{k})$. There are several standard options to remove this freedom; to begin with, we choose one in which the spherical

[2]In this chapter we employ natural units with $c = 1$.

symmetry is manifest and the spatial geometry is as close to that of flat space as possible. That is, we take r as the solution of

$$k^2(r) = r^2 g^2(r). \tag{4.9}$$

Then the line element becomes

$$ds^2 = -h^2(r)dt^2 + g^2(r)\left(dr^2 + r^2 d\Omega^2\right) = -h^2(r)dt^2 + g^2(r)d\boldsymbol{r}^2. \tag{4.10}$$

Clearly it is rotation invariant, and the three-dimensional space-like part of the line element is conformal to that of flat space in polar coordinates, with the conformal factor $g(r)$ depending only on the radius, and not on the orientation or the time. In the literature the coordinates chosen here to parametrize the Schwarzschild metric are called isotropic coordinates.

Up to the freedom of radial reparametrizations, the form of the line element (4.10) is completely determined by the symmetry requirements. To fix the radial dependence of the remaining coefficients $h(r)$ and $g(r)$ we finally substitute the metric into the Einstein equations. For the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2\, g_{\mu\nu} R$ we find that it is diagonal with components

$$\begin{aligned}
G_{tt} &= \frac{h}{g^3}\left(2h(g'' + \frac{2}{r}g') - \frac{hg'^2}{g}\right), \\
G_{rr} &= -\frac{1}{gh}\left(2h'g' + \frac{hg'^2}{g} + \frac{2}{r}(hg)'\right), \\
G_{\theta\theta} &= \frac{G_{\varphi\varphi}}{\sin^2\theta} = -\frac{r^2}{gh}\left(h''g + hg'' + \frac{1}{r}(hg)' - \frac{hg'^2}{g}\right),
\end{aligned} \tag{4.11}$$

In the absence of sources or a cosmological constant these expressions have to vanish. As boundary conditions at infinity, where space-time must become asymptotically flat, we require that $r \to \infty$ implies $h \to 1$ and $g \to 1$. Then we have three differential equations for two unknown functions, and therefore the equations must be degenerate. One can check, that this is the case if the following two relations are satisfied:

$$hg = g + rg', \qquad \left[r^3(hg)'\right]' = 0. \tag{4.12}$$

Indeed, in this case the various components of $G_{\mu\nu}$ become proportional:

$$G_{rr} = -\frac{g^2}{h^2}G_{tt} = -\frac{1}{r^2}G_{\theta\theta} = -\frac{1}{r^2\sin^2\theta}G_{\varphi\varphi}. \tag{4.13}$$

The conditions (4.12) are met for any $g(r)$ of the form

$$g(r) = a + \frac{b}{r} + \frac{c}{r^2}, \qquad h(r) = \frac{g + rg'}{g}. \tag{4.14}$$

It turns out that eqs.(4.14) indeed provide solutions of the Einstein equations satisfying the correct boundary conditions. They are most easily obtained from the first Einstein equation (4.11): for $hg \neq 0$ it is possible to factor out all h-dependence and obtain an equation for g only:

$$gg'' + \frac{2}{r}gg' - \frac{1}{2}g'^2 = 0. \tag{4.15}$$

There is a unique solution with the required normalization for $r \to \infty$:

$$g(r) = \left(1 + \frac{m}{2r}\right)^2, \tag{4.16}$$

where m is an undetermined constant of integration. Eq.(4.12) then immediately provides us with the solution for h:

$$h(r) = \frac{2r - m}{2r + m}. \tag{4.17}$$

The complete solution for the static, spherically symmetric space-time in isotropic coordinates therefore is

$$ds^2 = -\left(\frac{2r - m}{2r + m}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right). \tag{4.18}$$

Clearly this solution matches Minkowski space at $r \to \infty$ by construction, and h and g change monotonically (decreasing and increasing, respectively) as a function of r untill $r = m/2$. At that value $h = 0$, and the metric has a zero mode. Then the derivation of the solution we have presented breaks down, and its continuation to values $r < m/2$ requires a careful interpretation, discussed below. In any case, it turns out that this singularity of the metric is not physical, as invariants of the curvature remain finite and well-defined there. Indeed, in the following we construct coordinate systems in which the metric is perfectly regular at these points and can be continued to regions inside the surface $r = m/2$. In the isotropic coordinate system the apparent singularity of the metric for $r = 0$ is another coordinate artifact. It disappears if one brings spatial infinity to a finite distance by an appropriate coordinate transformation. Details are given in the next section.

4.3 Discussion

Modulo radial reparametrizations $r \to \bar{r}(r)$ the line-element (4.18) is the most general static and spherically symmetric solution of the source-free Einstein equations matching flat Minkowski space-time at spatial infinity ($r \to \infty$). Therefore in the non-relativistic weak-field range ($v \ll 1, r \gg m$) it should reproduce Newton's law for the gravitational field of a spherically symmetric point mass of magnitude M, according to which the acceleration of a test particle at distance r is

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{GM}{r^3}\boldsymbol{r} \tag{4.19}$$

As the spherically symmetric solutions of the Einstein equations (4.18) are distinguished only by a parameter m, in the Newtonian limit this parameter must be related to the mass M, thereby representing the equivalent gravitational mass of the object described by the spherically symmetric metric (4.18).

We first show that our solutions reproduce Newtons law in the limit $r \gg m$, $v \ll 1$ $(= c)$. The equation of motion for a test particle is the geodesic equation (2.7). For the spatial coordinates x^i this becomes

$$\begin{aligned} \frac{d^2x^i}{d\tau^2} &= \left(\frac{dx^0}{d\tau}\right)^2 \Gamma_{00}{}^i + (\text{terms} \sim O(v/c)) \\ &\approx -\frac{h}{g^2}\,\partial_i h = -\frac{m}{r^3}\,x^i\,\frac{h}{g^3}. \end{aligned} \tag{4.20}$$

Note that in the limit $v \ll 1$ one has $dx^0 = dt \approx d\tau$, whilst for $r \to \infty$ the factor $h/g^3 \to 1$. Upon the identification $m = GM$ we thus indeed reproduce Newton's equation (4.19).

Taking into account the dimensions of Newton's constant, as follows from (4.19) and is expressed in eq.(1.2), and our natural units in which $c = 1$, we find that m has the dimension of a length, consistent with the observation that m/r should be dimensionless. We have already noted that in isotropic coordinates the metric component h vanishes at half this distance: $r = r_0 = m/2$. Presently this only signals that the isotropic system cannot be extended beyond this range, but the surface $r = r_0$ does have a special physical significance: it defines the location of the horizon of a spherically symmetric black hole. This characteristic value $r_0 = m/2$ of the radial coordinate is known as the Schwarzschild radius.

For the range $m/2 < r < \infty$ the Schwarzschild solution in the form (4.18) is well-defined, and it matches the asymptotic Minkowski metric for large values of r. We noted that on the Schwarzschild sphere $r = m/2$ the metric is singular, but as a solution of the Einstein equations it can be continued to smaller values of r and then it is again well-defined in the domain $0 < r < m/2$. However, it turns out that this domain of r-values does not in any sense represent the physical interior of the horizon, as one might expect naively. Rather, it easy to establish that the solution (4.18) is invariant under the transformation

$$r \to \frac{m^2}{4r}, \tag{4.21}$$

which maps the domain $(0, m/2)$ to $(m/2, \infty)$. Therefore the isotropic coordinate system actually represents a double cover of the region between the Schwarzschild radius and the asymptotic region $r \to \infty$. Note that under this transformation $h(r) \to -h(r)$, which can be compensated by an accompanying time-reversal $t \to -t$. From these observations it follows that the apparent singularity for $r = 0$ disappears as it simply describes the another asymptotic region like that for $r \to \infty$ in the original coordinate system.

The existence of such a coordinate system presenting a double cover of the space outside the black hole might lead one to suspect, that a test particle thrown in radially towards the spherical Schwarzschild surface after a long enough time would return to an asymptotic Minkowski region. However, this does not happen; in the first place, as a simple calculation to be presented later shows, in terms of the time measured by an observer at asymptotic infinity it takes an infinite period of time for a test particle starting at any finite radial distance $r > m/2$ with any finite velocity even to reach the Schwarzschild surface; certainly such an observer will never see it start up and set off towards infinity again. Apart from this, leaving the Schwarzschild surface would also take an infinite amount of asymptotic Minkowski-time. Clearly, these effects are due to the vanishing of $h(r)$ on the horizon, which implies that an infinite amount of asymptotic Minkowski time t has to pass during any finite period of proper time τ. But this is only the minor part of the argument; more important is the second reason, to wit that time-like geodesics do not flow from values $r > m/2$ to values $r < m/2$. Instead, as will become clear they flow into a new region of space-time described by complex values of r.

Some calculational details are given later, but the above arguments at least indicate that an observer at spatial infinity is essentially disconnected from the domain of space-time beyond the Schwarzschild sphere. Therefore the Schwarzschild surface is called a *horizon*: from spatial infinity one cannot look beyond it. In contrast, things are radically different for an observer moving with the test particle towards the horizon. For such an observer in free fall, co-moving with the test particle on a radial geodesic and starting from an arbitrary point at a finite distance from the Schwarzschild surface, only a finite amount of proper time passes until the radius $r = m/2$ is reached, although for large radial distances the required time grows linearly with the distance.

Moreover, having reached the horizon the particle (and the observer) can continue falling inwards without noticing anything particular, at least initially. The only remarkable effect they could discover once they have passed beyond the Schwarzschild surface is the impossibility to turn back to spatial infinity, no matter how powerful the engines they have at their disposal to accelerate and propel them. Therefore their passage to the inside of the Schwarzschild sphere represents an irreversible event: they are permanently trapped inside the Schwarzschild sphere. For this reason the Schwarzschild space-time is called a black hole: nothing, not even radiation, can get out once it has fallen through the horizon, at least not in the domain of classical relativity.

To describe this motion of test particles (i.e. the flow of geodesics) through the Schwarzschild sphere, one must pass to a different coordinate system, one better suited to particles in free fall rather than to the description of the space-time from the point of view of an asymptotic Minkowskian observer at rest at spatial infinity. We make this passage in two steps. First we show that in other coordinate systems there is a different continuation of space-time beyond the Schwarzschild surface, one in which the interior geometry inside the horizon is distinct from the outside geometry and can not be mapped back isometrically to the exterior. After that we show by explicit calculation of the geodesics, that this alternative continuation can be taken to decribe the true physical situation for a test particle falling through the Schwarzschild sphere.

4.4 The interior of the Schwarzschild sphere

To go beyond the region of space-time covered by the isotropic coordinates, we perform a radial coordinate transformation which instead of making the spatial part of the metric conformal to flat space, as in (4.9), (4.10), makes the angular part directly isomorphic to the two-sphere of radius $\bar{r}$:

$$\bar{r} = rg(r), \qquad \bar{k}^2(\bar{r}) = \bar{r}^2. \tag{4.22}$$

Then the metric takes the form

$$ds^2 = -\bar{h}^2(\bar{r})\, dt^2 + \bar{g}^2(\bar{r})\, d\bar{r}^2 + \bar{r}^2 d\Omega^2. \tag{4.23}$$

The explicit form of the radial transformation (4.22) gives for the coefficients

$$\bar{h}^2(\bar{r}) = \frac{1}{\bar{g}^2(\bar{r})} = 1 - \frac{2m}{\bar{r}}. \tag{4.24}$$

This leads to the standard form of the Schwarzschild solution as presented in the original papers [30, 31]:

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right) dt^2 + \frac{1}{\left(1 - \frac{2m}{\bar{r}}\right)} d\bar{r}^2 + \bar{r}^2 d\Omega^2. \tag{4.25}$$

As to lowest order the transformation (4.22) does not include a rescaling, we have for large r and $\bar{r}$:

$$\frac{1}{\bar{r}} = \frac{1}{r} + O\,(\frac{1}{r^2}). \tag{4.26}$$

Therefore in the asymptotic region the Newtonian equation of motion for a test-particle in the field of a point mass M still holds in the same form after replacing r by $\bar{r}$, and in the non-relativistic long-distance limit the two coordinate systems are indistinguishable.

Note that the Schwarzschild radius $r = m/2$ is now located at $\bar{r} = 2m$. Again the metric components are singular there, but now there is a zero mode in the (tt)-component and an infinity in the (rr)-component, in such a way that the determinant of the metric remains finite on the horizon. On the other hand, the determinant g does have a real zero for $\bar{r} = 0$. This metric singularity was not present in the isotropic coordinate system, as it did not cover this region of space-time, but it turns out to represent a real physical singularity: the curvature becomes infinite there in a coordinate-independent way.

A peculiarity of the metric (4.23) is that the components $\bar{h}^2$ and $\bar{g}^2$ are not positive definite, but change sign at the Schwarzschild radius. This implies that the role of time and radial distance are interchanged for $\bar{r} < 2m$: $\bar{r}$ is the time coordinate and t the radial coordinate. At the same time the metric components have become explicitly time dependent (they are functions of $\bar{r}$), and therefore the metric is no longer static inside the horizon. In return, the metric is now invariant under radial shifts $t \rightarrow t + \delta t$. Hence it is the radial momentum p_t which now commutes with the Hamiltonian (4.8), the generator of proper-time translations, and is conserved.

This strange result is the price one has to pay for continuing the Schwarzschild geometry to the interior of the Schwarzschild sphere. It is quite clear that this could never heve been achieved in the isotropic coordinate system (4.18), at least not for real values of r. Together with the appearance of the curvature singularity at $\bar{r} = 0$ this shows beyond doubt that the region inside the Schwarzschild sphere $\bar{r} < 2m$ has no counter part in the standard isotropic coordinate system.

To gain a better understanding of the difference between the continuation to $r < m/2$ in isotropic coordinates and $\bar{r} < 2m$ in Schwarzschild coordinates it is useful to perform yet another radial coordinate transformation to a dimensionless coordinate ρ defined by [32]

$$\tanh \frac{\rho}{2} = h(r), \tag{4.27}$$

with the result that

$$\bar{r} = r g(r) = 2m \cosh^2 \frac{\rho}{2}. \tag{4.28}$$

Then the Schwarzschild metric becomes

$$ds^2 = -\tanh^2 \frac{\rho}{2}\, dt^2 + 4m^2 \cosh^4 \frac{\rho}{2} \left(d\rho^2 + d\Omega^2 \right). \tag{4.29}$$

As $\cosh \rho/2 \geq 1$ this expression for the metric, like the isotropic coordinate system, is valid only outside the horizon $\bar{r} \geq 2m$. Moreover, continuing to negative values of ρ we reobtain the double cover of the exterior of the Schwarzschild black hole:

$$\tanh\left(\frac{-\rho}{2}\right) \,=\, -h(r) \,=\, h(\frac{m^2}{4r}). \tag{4.30}$$

Now the interior of the Schwarzschild sphere can also be reparametrized in a similar way, taking account of the fact that h^2 has become negative:

$$\bar{r} \,=\, m\,(1+\cos\sigma) \,=\, 2m\cos^2\frac{\sigma}{2}, \qquad \tan^2\frac{\sigma}{2} \,=\, -\bar{h}^2(\bar{r}). \tag{4.31}$$

This parametrization clearly exists only for $0 \leq \bar{r} \leq 2m$. In these coordinates the metric takes the form

$$ds^2 \,=\, \tan^2\frac{\sigma}{2}\,dt^2 \,+\, 4m^2\cos^4\frac{\sigma}{2}\,\left(-d\sigma^2 + d\Omega^2\right). \tag{4.32}$$

The time-like nature of σ now is obvious. It is also clear, that this coordinate system presents an infinitely repeated *periodic* covering of the interior of the Schwarzschild sphere: the fundamental domain may be chosen to be $0 \leq \sigma \leq \pi$, corresponding to $2m \geq \bar{r} \geq 0$; but continuation of σ beyond this region leads to repeated covering of the same set of $\bar{r}$-values.

Comparing these two parametrizations of the exterior and the interior of the Schwarzschild sphere, it is now obvious that they are related by analytic continuation [32]: the interior of the sphere is not described by negative values of ρ, which correspond to $r < m/2$ in isotropic coordinates, but by imaginary values of ρ:

$$\sigma \,=\, \pm i\rho, \tag{4.33}$$

which is like a Wick rotation changing the space-like radial coordinate ρ to the time-like coordinate σ. In terms of the original isotropic coordinates it can be described as the analytic continuation of r to complex values

$$r \,=\, \frac{m}{2}\,e^{i\sigma}. \tag{4.34}$$

Thus in the interior of the Schwarzschild surface the modulus of r is constant, but its phase changes, whilst outside its phase is constant and the modulus changes. Note that whereas the period of the fundamental domain of σ in describing the interior region of the Schwarzschild sphere is π, the values of the isotropic radial coordinate r are periodic in σ with period 2π. Again it seems that the isotropic coordinates cover the Schwarzschild solution twice. We have more to say about that in the following. Finally we observe, that as indicated in eq.(4.33) there are two ways to perform the analytic continuation from ρ to σ, i.e. from the exterior to the interior of the Schwarzschild space-time. Also this fact has non-trivial significance for the geometry of the Schwarzschild space-time.

4.5 Geodesics

The analysis presented so far makes it clear that the geometry of the Schwarzschild space-time is quite intricate. Indeed, at the Schwarzschild sphere $r = m/2$ (or $\bar{r} = 2m$ and $\sigma = \rho = 0$, respectively) several branches of the space-time meet: two branches of the exterior connected to an asymptotic Minkowski region, and two branches of the interior connecting the horizon with the space-time singularity. It is therefore of importance to understand the flow of geodesics in the neighborhood of the horizon, and see which branches test particles moving in the Schwarzschild space-time follow. We begin with time-like geodesics, representing the motion of test particles of unit mass. For such a particle one has in asymptotic Minkowski space:

$$2H \,=\, g^{\mu\nu}\, p_\mu p_\nu \,=\, -1. \tag{4.35}$$

This relation was already argued from the definition of proper time in sect.(2.1), eq.(2.8) and below. As H is conserved, the relation holds everywhere on the particle's worldline.

The solution of the geodesic equations for Schwarzschild space-time is much simplified by the large number of its symmetries. They provide us with sufficiently many additional constants of motion to allow a complete solution of the equations of motion of test particles in terms of their energy and angular momentum. The first of these constants of motion is the momentum component p_t, conserved because the metric is t-independent:

$$p_t \,=\, -h^2(r)\,\frac{dt}{d\tau} \,=\, -\epsilon \quad \Leftrightarrow \quad \frac{dt}{d\tau} \,=\, \frac{\epsilon}{h^2(r)}. \tag{4.36}$$

At space-like infinity $h^2 \rightarrow 1$; also, with τ defined as proper time, $dt = d\tau$ for a particle at rest there. Therefore a particle at rest at $r \rightarrow \infty$ has $\epsilon = 1$, equal to the (unit) rest mass of the test particle; moreover, as is intuitively obvious and argued more precisely below, if it is not at rest it must have $\epsilon > 1$. Indeed, particles with $\epsilon < 1$ cannot reach spatial infinity and in this sense they are in bound states. The factor ϵ gives the usual special relativistic (kinematical) time-dilation for a moving particle in asymptotic Minkowski space.

In contrast, for finite radial distance $r_0 < r < \infty$ the factor $h^2(r)$ describes the gravitational redshift: the time dilation due only to the presence of a gravitational field. It represents a universal effect, not influenced by the state of motion of the particle.

The conservation of angular momentum gives us three conservation laws, which are however not independent:

$$\begin{aligned} J_1 &= -\sin\varphi\, p_\theta - \cot\theta\, \cos\varphi\, p_\varphi, \\ J_2 &= \cos\varphi\, p_\theta - \cot\theta\, \sin\varphi\, p_\varphi, \\ J_3 &= p_\varphi, \end{aligned} \tag{4.37}$$

with $J_1^2 + J_2^2 + J_3^2 = \boldsymbol{L}^2 \equiv \ell^2$, a constant. Here the separate momentum components are

$$p_\theta = k^2(r)\, \frac{d\theta}{d\tau}, \qquad p_\varphi = k^2(r) \sin^2\theta\, \frac{d\varphi}{d\tau}. \tag{4.38}$$

If one orients the coordinate system such that the angular momentum is in the x^3-direction, then $\theta = \pi/2 =$ constant, and therefore $p_\theta = 0$ and $p_\varphi = J_3 = \ell$. Then the angular coordinates as a function of proper time are given by

$$\theta = \pi/2, \qquad \frac{d\varphi}{d\tau} = \frac{\ell}{k^2(r)}. \tag{4.39}$$

It remains to solve for the radial coordinate[3] r. For this we use the conservation of the hamiltonian, setting $2H = -1$:

$$p_r^2 = \left(g^2\, \frac{dr}{d\tau} \right)^2 = g^2 \left(\frac{\epsilon^2}{h^2} - \frac{\ell^2}{k^2} - 1 \right). \tag{4.40}$$

Note, that consistency of this equation in the region outside the horizon requires

$$\frac{\epsilon^2}{h^2} \geq \frac{\ell^2}{k^2} + 1. \tag{4.41}$$

It shows, why in the region $r \rightarrow \infty$ the time dilation factor has to be greater or equal to unity: $\epsilon \geq 1$. At the same time, for non-zero values of angular momentum ℓ the radial distance r can in general not become arbitrary low: like in the ordinary Kepler problem there is a centrifugal barrier to be overcome. For details we refer to the literature [11, 28, 29].

Combining eq.(4.40) with that for the angle φ, by eliminating τ we directly obtain the orbital equation

$$\frac{dr}{d\varphi} = \pm \frac{k^2}{\ell g} \sqrt{\frac{\epsilon^2}{h^2} - \frac{\ell^2}{k^2} - 1}. \tag{4.42}$$

It is clear that an orbit can have a point of closest approach, or furthest distance, only if $dr/d\varphi = dr/d\tau = 0$. The only physical solutions are points where relation (4.41) becomes an equality. When the equality holds identically, for all points of the orbit, the orbit is circular. One can show that the smallest radius allowed for a

[3] Note that each of these equations is still true for arbitrary radial parametrization.

stable circular orbit corresponds in Schwarzschild coordinates to $\bar{r} = 2m\ell_0^2 = 6m$.

To the set of time-like trajectories we add the light-like geodesics, representing the orbits of massless particles. In that case the proper-time interval vanishes identically on any geodesic, that is to say $H = 0$. We parametrize the trajectories with an affine parameter λ and obtain

$$p_t = -h^2 \frac{dt}{d\lambda} \equiv -\gamma, \tag{4.43}$$

allowing one to replace the affine parameter by t through

$$d\lambda = \frac{h^2}{\gamma} dt. \tag{4.44}$$

One can again choose the plane of motion to be $\theta = \pi/2$, and the conservation of angular momentum holds in the form

$$\frac{\gamma r^2}{h^2} \frac{d\varphi}{dt} = r^2 \frac{d\varphi}{d\lambda} = \omega, \tag{4.45}$$

where ω is a constant. The radial equation now becomes

$$h^2 = g^2 \left(\frac{dr}{dt} \right)^2 + k^2 \left(\frac{d\theta}{dt} \right)^2 + k^2 \sin^2\theta \left(\frac{d\varphi}{dt} \right)^2, \tag{4.46}$$

or using the choice $\theta = \pi/2$ above:

$$g^2 \left(\frac{dr}{dt} \right)^2 = h^2 \left(1 - \frac{\omega^2 h^2 k^2}{\gamma^2 r^4} \right). \tag{4.47}$$

Like in the massive case, if $\omega \neq 0$ one can eliminate the affine parameter λ also in favour of φ instead of t.

An interesting case, as concerns the exploration of the horizon and the connection between the interior and the exterior of the Schwarzschild sphere, is that of radial motion: $\ell = 0$ or $\omega = 0$, respectively. Let us first consider the time-like geodesic for a particle starting from rest near $r = \infty$, which implies $\epsilon = 1$ and $dr/dt \leq 0$. Then

$$\frac{dt}{d\tau} = \frac{1}{h^2}, \qquad \frac{dr}{d\tau} = -\frac{1}{gh}\sqrt{1-h^2}. \tag{4.48}$$

It is easy to solve these equations in the Schwarzschild coordinates; indeed, with $(1 - \bar{h}^2) = 2m/\bar{r}$ one finds a solution valid in the whole range $0 < \bar{r} < \infty$, both outside and inside the horizon:

$$\frac{\bar{r}(\tau)}{2m} = \left[\frac{3}{4m}(\tau_0 - \tau)\right]^{\frac{2}{3}}, \tag{4.49}$$

where τ_0 is the proper time at which the particle reaches the singularity at $\bar{r} = 0$. Taking this time as fixed, τ denotes the proper time, prior to τ_0, at which the particle is at radial distance $\bar{r}$. Observe, that this time is finite for any finite radial distance $\bar{r}$; only when starting from infinity it takes an infinite proper time to reach the singularity. Nothing special happens at the horizon $\bar{r} = 2m$: it is crossed at proper time

$$\tau_H = \tau_0 - \frac{4m}{3}, \tag{4.50}$$

a proper period $4m/3$ before the singularity is reached (of course, in terms of the time-coordinate $\bar{r}$ inside the horizon, it takes a finite period $2m$).

Solving for t as a function of proper time, we have to distinguish two cases: $\bar{r} > 2m$ and $\bar{r} < 2m$. Namely, introducing the new variable

$$y = \left[\frac{3}{4m}(\tau_0 - \tau)\right]^{\frac{1}{3}}, \tag{4.51}$$

we obtain the differential equation

$$-\frac{1}{4m}\frac{dt}{d\tau} = \frac{y^4}{y^2 - 1}, \tag{4.52}$$

the solution of which depends on whether the r.h.s. is positive or negative. We find for the two cases the solutions

$$\frac{t_0 - t}{4m} = \frac{\tau_0 - \tau}{4m} + \left[\frac{3}{4m}(\tau_0 - \tau)\right]^{\frac{1}{3}} + \begin{cases} \operatorname{arccoth}\left[\frac{3}{4m}(\tau_0 - \tau)\right]^{\frac{1}{3}}, & r > 2m; \\ \operatorname{arctanh}\left[\frac{3}{4m}(\tau_0 - \tau)\right]^{\frac{1}{3}}, & r < 2m. \end{cases} \tag{4.53}$$

In both cases the left-hand side becomes infinite at the horizon, i.e. at proper time τ_H. This confirms that an external observer at infinity will never actually see the particle reach the horizon. Similarly, the distance measured from the horizon to the singularity at $r = 0$ is infinite in terms of the coordinate t.

Now consider what happens in isotropic coordinates. In terms of a new variable

$$x = \sqrt{\frac{2r}{m}}, \tag{4.54}$$

where we choose the positive branch of the square root, integration of the equation for $r(\tau)$ is elementary and gives

$$\frac{2}{m}\,(\tau_0 - \tau) \,=\, \frac{1}{3}\left(x + \frac{1}{x}\right)^3. \qquad (4.55)$$

Clearly this is symmetric under interchange $x \rightarrow 1/x$, corresponding to inversion w.r.t. the horizon $r = m/2$. Indeed, both at $r \rightarrow \infty$ and $r = 0$ we see that $\tau \rightarrow -\infty$. This confirms our earlier remark that a particle released from space-like infinity does not return there. Instead, it will cross the horizon $x = 1$ at $\tau = \tau_H$, as in the previous calculation. Then continuing r to complex values (4.34), we find

$$x \,=\, e^{\frac{i}{2}\sigma}, \qquad \frac{\tau_0 - \tau}{4m} = \frac{1}{3}\cos^3\frac{\sigma}{2}. \qquad (4.56)$$

The singularity $\sigma = \pi$ is then reached for $\tau = \tau_0$, as expected. In terms of the radial coordinate ρ, related by the pseudo-Wick rotation (4.33) the equation for the radial distance in terms of proper time outside the horizon becomes

$$\frac{\tau_0 - \tau}{4m} \,=\, \frac{1}{3}\cosh^3\frac{\rho}{2}. \qquad (4.57)$$

The analytic continuation of coordinates in the complex plane on crossing the horizon is manifest.

Next we consider incoming light-like radial geodesics. The equation is

$$\frac{dr}{dt} \,=\, -\frac{h}{g}. \qquad (4.58)$$

Let t_0 refer to the time at which the massless particle (e.g., a photon) passes a fixed point on the radius, e.g. $\bar{r}_0$ in Schwarzschild coordinates. Then the solution of the geodesic equation for t is

$$\frac{t - t_0}{2m} \,=\, \frac{\bar{r}_0 - \bar{r}}{2m} \,-\, \ln\left|\frac{\bar{r} - 2m}{\bar{r}_0 - 2m}\right|. \qquad (4.59)$$

With the absolute value as argument of the logarithm this equation is valid both outside and inside the horizon. The time for the photon to reach the horizon from any point outside diverges, as does the distance it has to move from horizon to the curvature singularity, as measured by the coordinate t.

4.6 Extended Schwarzschild geometry

As we have already noticed that for a massive particle falling across the horizon nothing very special happens, it is unsatisfactory that we do not have a non-singular description of the same process for massless particles. To improve the description of light-like geodesics at the end of the last section, it is not sufficient to perform a radial coordinate transformation: one has to eliminate the

t-coordinate and replace it by one which does not become infinite on the horizon. Moreover, to describe light-like geodesics it would be convenient to have a metric where the light-cone structure is manifest by having the radial part of the metric conformal to radial Minkowski space: a coordinate system in which (r,t) are replaced by coordinates (u,v) such that the radial part of the metric becomes

$$-h^2(r)dt^2 \,+\, g(r)^2dr^2 \,\rightarrow f(u,v)\left(-du^2+dv^2\right). \tag{4.60}$$

Such a coordinate system was first proposed by Kruskal and Szekeres [33]; defining the new coordinates such as to be dimensionless, the line element is written in the form

$$\frac{ds^2}{4m^2} \,=\, f(u,v)\,\left(-dv^2+du^2\right) \,+\, g(u,v)\,d\Omega^2, \tag{4.61}$$

with the standard choice for (u,v) defined in terms of the Schwarzschild coordinates $(\bar{r},t)$ by:

$$u^2\,-\,v^2 \,=\, \left(\frac{\bar{r}}{2m}-1\right)\,e^{\frac{\bar{r}}{2m}}, \qquad \frac{u}{v} \,=\, \begin{cases} \tanh\dfrac{t}{4m}, & \text{if}\,|v|<|u|; \\[2ex] \coth\dfrac{t}{4m}, & \text{if}\,|v|>|u|. \end{cases} \tag{4.62}$$

In these coordinates the horizon $\bar{r}=2m$, $t=\pm\infty$ is located at $u=\pm v$, whilst the curvature singularity $\bar{r}=0$ corresponds to the *two* branches of the hyperbola $u^2-v^2=-1$, one in the past and one in the future of an observer at space-like infinity. More generally, from the first result one concludes that the surfaces $\bar{r}=$ *constant* are mapped to hyperbola's $u^2-v^2=constant$, where the last constant is positive outside and negative inside the horizon. These hyperbola's are worldlines of particles in circular orbit, i.e. being subject to a constant acceleration.

Similarly hypersurfaces $t=constant$ correspond to $u=constant\times v$, represented by straight lines in a (u,v)-diagram. Note also that the singularity $\bar{r}=0$ consists of a set of two *space-like* surfaces in (u,v)-coordinates.

More explicitly, we can write (u,v) in terms of (r,t) as

$$\begin{aligned} u \,&=\, \pm\sqrt{\frac{\bar{r}}{2m}-1}\,e^{\frac{\bar{r}}{4m}}\,\cosh\frac{t}{4m}, \\ v \,&=\, \pm\sqrt{\frac{\bar{r}}{2m}-1}\,e^{\frac{\bar{r}}{4m}}\,\sinh\frac{t}{4m}, \end{aligned} \tag{4.63}$$

outside the horizon $(\bar{r}\geq 2m)$; and

$$u = \pm\sqrt{1-\frac{\bar{r}}{2m}}\, e^{\frac{\bar{r}}{4m}}\, \sinh\frac{t}{4m},$$
$$v = \pm\sqrt{1-\frac{\bar{r}}{2m}}\, e^{\frac{\bar{r}}{4m}}\, \cosh\frac{t}{4m}, \tag{4.64}$$

within the horizon ($\bar{r} \leq 2m$). We note once again, that these coordinates produce two solutions for each region of the space-time geometry. In the above conventions the dimensionless coefficient functions are given implicitly by

$$f(u,v) = \frac{8m}{\bar{r}}\, e^{-\frac{\bar{r}}{2m}}, \qquad g(u,v) = \frac{\bar{r}^2}{4m^2}. \tag{4.65}$$

We can also compare with the coordinate systems (4.29) and (4.32), describing explicitly the double cover of the Schwarzschild space-time:

$$u = e^{\frac{1}{2}\cosh^2\frac{\rho}{2}}\, \sinh\frac{\rho}{2}\, \cosh\frac{t}{4m}, \qquad v = e^{\frac{1}{2}\cosh^2\frac{\rho}{2}}\, \sinh\frac{\rho}{2}\, \sinh\frac{t}{4m}, \tag{4.66}$$

for the exterior region $|v| < |u|$; and

$$u = e^{\frac{1}{2}\cos^2\frac{\sigma}{2}}\, \sin\frac{\sigma}{2}\, \sinh\frac{t}{4m}, \qquad v = e^{\frac{1}{2}\cos^2\frac{\sigma}{2}}\, \sin\frac{\sigma}{2}\, \cosh\frac{t}{4m}, \tag{4.67}$$

for the interior region $|v| > |u|$. From these expressions we infer, that the exterior regions $\rho > 0$ and $\rho < 0$, corresponding to $r > m/2$ and $r < m/2$ in isotropic coordinates, and the two interior regions $\sigma > 0$ and $\sigma < 0$ as well, coincide with the regions $(u,v) > 0$ and $(u,v) < 0$ in Kruskal-Szekeres coordinates, respectively. Hence the double cover of Schwarzschild space-time given by the two solutions for the Kruskal-Szekeres coordinates above correspond exactly to the double cover we found in isotropic and (ρ,σ)-coordinates.

The Kruskal-Szekeres coordinates were introduced, and are especially useful, because they represent the light-like geodesics in a very simple way. In particular, the radial light-like geodesic are given by

$$du^2 = dv^2 \quad \Rightarrow \quad \frac{du}{dv} = \pm 1. \tag{4.68}$$

These correspond to straight lines in the (u,v)-diagram parallel to the diagonals. The special light rays $u = \pm v$ are the asymptotes of the singularity and describe the horizon itself: they can be interpreted as photons permanently hovering on the horizon. Any other light-like radial geodesic eventually hits the singularity, either in the past or in the future, at finite (u,v), although this is in the region before $t = -\infty$ or after $t = \infty$ in terms of the time of an observer at spatial infinity.

As the light cones have a particularly simple representation in the (u, v)-diagram, for all regions inside and outside the horizon, it follows that time-like and space-like geodesics are distinguished by having their derivatives $|du/dv| > 1$ for the time-like case, and $|du/dv| < 1$ for the space-like ones. One can check by explicit computation, that a particle starting from rest at any distance r and falling into the black hole has a time-like world line everywhere, also inside the horizon: du/dv can change on the world line of a particle, but never from time-like to space-like or vice versa. Therefore all time-like geodesics must cross the horizon (light-like) and the singularity (space-like) sooner or later. If they don't reach or start from spatial infinity, they must come from the past singularity across the past horizon, and then cross the future horizon to travel towards the final singularity, all within a finite amount of proper time.

The Kruskal-Szekeres coordinate system also teaches us more about the topology of the Schwarzschild geometry: how the various regions of the space-time are connected. First of all it shows that there is a past and a future singularity inside a past and a future horizon ($|u| < |v|$), depending on v being positive or negative. In this respect it is helpful to observe, that according to eq.(4.64) for $\bar{r} < 2m$ the sign of v is fixed for all t. In the literature these regions are sometimes refered to as the *black hole*, the interior of the horizon containting the future singularity, and the *white hole* containing the past singularity. In a standard convention, these regions of space-time are classified as regions II (future Schwarzschild sphere) and IV (past Schwarzschild sphere), respectively.

There are also two exterior regions connected to asymptotic Minkowski space. They are distinguished similarly by $u > 0$ and $u < 0$, as for $\bar{r} > 2m$ the sign of u is fixed for all t: one cannot reach negative u starting from positive u or vice versa by any time-like or light-like geodesic. In the same standard convention these regions are refered to as region I and region III, respectively. Being connected by space-like geodesics only, the two sheets of the exterior of the Schwarzschild space-time are not in causal contact and cannot physically communicate. However, the surfaces $t = constant$ do extend from $u = \infty$ to $u = -\infty$; indeed, the regions are space-like connected at the locus $u = v = 0$, where the two branches of the horizon intersect. To understand the topology of Schwarzschild space-time, we must analyze this connection in some more detail; this can be done using the technique of embedding surfaces, as explained e.g. in ref.[11].

Consider a surface $t = constant$ and fix a plane $\theta = \pi/2$. We wish to understand how the regions $u > 0$ and $u < 0$ are connected for this plane. The Schwarzschild line element restricted to this plane is

$$ds^2 = \frac{d\bar{r}^2}{1 - \dfrac{2m}{\bar{r}}} + \bar{r}^2 d\varphi^2. \qquad (4.69)$$

Now we embed this two-dimensional surface in a three-dimensional flat, euclidean space:

$$ds^2 \,=\, dx^2 \,+\, dy^2 \,+\, dz^2, \tag{4.70}$$

by taking $x = \bar{r}\cos\varphi$, $y = \bar{r}\sin\varphi$, and defining the surface $z = F(\bar{r})$ such that

$$ds^2 \,=\, \left(1 + F'^2(\bar{r})\right)\, d\bar{r}^2 \,+\, \bar{r}^2 d\varphi^2. \tag{4.71}$$

Comparison of eqs. (4.71) and (4.69) then implies

$$\frac{z^2}{16m^2} \,=\, \left(\frac{F(\bar{r})}{4m}\right)^2 \,=\, \frac{\bar{r}}{2m} \,-\, 1. \tag{4.72}$$

This defines a surface of revolution, obtained by taking the parabola $z^2 = 8mx - 16m^2$ in the (x,z)-plane and rotating it around the z-axis. It is a single surface, with looks like a throat, of which the upper half $z > 0$ corresponds to

$$\sqrt{\frac{\bar{r}}{2m} - 1} \geq 0, \qquad \text{or} \quad u \geq 0, \tag{4.73}$$

whilst the lower half $z < 0$ describes the branch

$$\sqrt{\frac{\bar{r}}{2m} - 1} \leq 0, \qquad \text{or} \quad u \leq 0, \tag{4.74}$$

The two regions are connected along the circle $\bar{r} = 2m$ at $z = 0$. Of course, the spherical symmetry implies that the same topology would be observed for any choice of angle θ; any plane with $u > 0$ is connected to one with $u < 0$ in this way. Although no time-like travel is possible classically between the regions on each side of the throat, the possibility of particles tunneling via quantum processes between the two regions poses a curious and interesting question to theories of quantum gravity.

4.7 Charged black holes

The spherically symmetric solution of the source-free Einstein equations can be extended to a solution of the coupled Einstein-Maxwell equations including a static spherically symmetric electric field (in the rest frame of the black hole)[34]. As the only solution of this type in flat Minkowski space is the Coulomb field, we expect a solution of the coupled equations to approach the Coulomb potential at space-like infinity:

$$A \,=\, A_\mu dx^\mu \,=\, \frac{q}{4\pi r p(r)}\, dt, \tag{4.75}$$

with $p(r) \rightarrow 1$ for $r \rightarrow \infty$. This is a solution of the Maxwell equation

$$D_\mu F^{\mu\nu} \,=\, 0, \tag{4.76}$$

in a spherically symmetric static space-time geometry. The Maxwell field strength appearing in this equations is

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = \frac{q}{4\pi} \frac{p + rp'}{r^2 p^2}\, dt \wedge dr. \qquad (4.77)$$

In isotropic coordinates (4.10) the energy-momentum tensor of this electro-magnetic field reads

$$T_{\mu\nu}[F] = \left(F_{\mu\lambda} F_\nu^{\;\lambda} - \frac{1}{4} g_{\mu\nu} F^2 \right) = \frac{q^2}{32\pi^2} \frac{(p + rp')^2}{r^4 h^2 p^4} \operatorname{diag}\left(\frac{h^2}{g^2}, -1, r^2, r^2 \sin^2\theta \right). \qquad (4.78)$$

This has the same general structure as the Einstein tensor (4.13):

$$G_{\mu\nu} = \frac{1}{gh} \left(2h'g' + \frac{hg'^2}{g} + \frac{2}{r} (hg)' \right) \operatorname{diag}\left(\frac{h^2}{g^2}, -1, r^2, r^2 \sin^2\theta \right), \qquad (4.79)$$

which was derived under the conditions (4.12), with the general solution (4.14). Now requiring the coupled Einstein-Maxwell equations

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu}[F], \qquad (4.80)$$

this reduces to a single condition on the radial functions (h, g, p), with the solution

$$\begin{aligned} p(r) &= g(r) = 1 + \frac{m}{r} + \frac{\nu^2}{4r^2}, \\ h(r) &= \frac{g + rg'}{g} = \frac{4r^2 - \nu^2}{4r^2 + 4mr + \nu^2}. \end{aligned} \qquad (4.81)$$

Here the constant ν^2 is defined by

$$\nu^2 = m^2 - e^2, \qquad e^2 = \frac{q^2 G}{4\pi}. \qquad (4.82)$$

It may be checked that this solution satisfies the Maxwell equations in this gravitational fields as well. As $g(r)$ can also be written in the form

$$g(r) = \left(1 + \frac{m}{2r} \right)^2 - \frac{e^2}{4r^2}, \qquad (4.83)$$

we find that the Schwarzschild solution is reobtained in the limit $e^2 \to 0$. For $\nu^2 > 0$ the zero of the metric coefficient h^2 is now shifted to

$$r = \frac{\nu}{2}. \qquad (4.84)$$

As in the Schwarzschild case this is a point where two asymptotically flat regions of space-time meet: the metric is again invariant under the transformation

$$r \to \frac{\nu^2}{4r}, \qquad t \to -t, \tag{4.85}$$

under which $h(r)dt \to h(r)dt$. Moreover, also like in the Schwarzschild geometry, the sphere $r = \nu/2$ actually defines a horizon and time-like geodesics are continued into the interior of this horizon, which is not covered by the (real) isotropic coordinate system.

A Schwarzschild-like coordinate system which allows continuation into the interior of the horizon exists. It was found by Reissner and Nordstrom [34] and is related to the isotropic coordinates by a radial coordinate transformation

$$\bar{r} = rg(r) = r + m + \frac{\nu^2}{4r}. \tag{4.86}$$

Then the metric of the charged spherically symmetric black hole takes the form

$$ds^2 = -\bar{h}^2 dt^2 + \bar{g}^2 d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{4.87}$$

where the coefficients $(\bar{h}, \bar{g})$ are again inversely related:

$$\bar{h}^2(\bar{r}) = \frac{1}{\bar{g}^2(\bar{r})} = 1 - \frac{2m}{\bar{r}} + \frac{e^2}{\bar{r}^2} = \left(1 - \frac{m}{\bar{r}}\right)^2 - \frac{\nu^2}{\bar{r}^2}. \tag{4.88}$$

The horizon is now seen to be located at $\bar{r} = m + \nu$; in fact, there is a second horizon at $\bar{r} = m - \nu$, provided $m > e$. As in general $\bar{h}^2$ and $\bar{g}^2$ are not positive definite, at each of these horizons the role of radial and time coordinate is interchanged. Only for $\nu = 0$, i.e. for $m = e$, the situation is different, as the two horizons coincide and no interchange of time- and space-like coordinates occurs. This special case of Reissner-Nordstrom solution is refered to in the literature as an *extremal* charged black hole.

The result of the second horizon for non-extremal charged black holes is, that the real curvature singularity that is found to exist at $r = 0$ is a time-like singularity, for $m > e$. This is expected from the Coulomb solution in flat Minkowski space, which contains a time-like singularity of the electric field at $r = 0$. Therefore such a black hole looks more like a point particle than the Schwarzschild type of solution of the Einstein equations.

The interior region between the horizons can be obtained from the isotropic coordinates by complex analytic continuation of r from the outer horizon by

$$r = \frac{\nu}{2} e^{i\sigma}. \tag{4.89}$$

The domain of the argument σ is chosen as $[0, \pi]$. Indeed, for these values of r the radial Reissner-Nordstrom coordinate is

$$\bar{r} = m + \nu \cos\sigma, \tag{4.90}$$

and takes all values between the two horizons $\bar{r} = m \pm \nu$ precisely once. In this parametrization the isotropic Reissner-Nordstrom metric becomes

$$ds^2 = \frac{\nu^2 \sin^2\sigma}{(m + \nu\cos\sigma)^2}\, dt^2 + (m + \nu\cos\sigma)^2 \left(-d\sigma^2 + d\Omega^2\right). \tag{4.91}$$

Again we observe the role of σ as the time-like variable, with t space-like between the horizons. For $\sigma = \pi$ the isotropic coordinate r becomes negative:

$$\sigma = \pi \to r = -\frac{\nu}{2}. \tag{4.92}$$

Thus we discover that the Reissner-Nordstrom space-time can be extended consistently to negative values of the radial coordinate. In particular the curvature singularity at $\bar{r} = 0$ is seen to be mapped to

$$r = -\frac{m \pm e}{2}. \tag{4.93}$$

This shows first of all, that there are two (time-like) branches of the singularity, and secondly that r can become negative for non-extremal black holes with $m > e$. The two branches are found also upon analytic continuation of σ in (4.91) to imaginary values:

$$\sigma \to \pm i\rho. \tag{4.94}$$

This leads to a parametrization of the Reissner-Nordstrom geometry for the exterior regions of the form

$$ds^2 = -\frac{\nu^2 \sinh^2\rho}{(m \pm \nu\cosh\rho)^2}\, dt^2 + (m \pm \nu\cosh\rho)^2 \left(d\rho^2 + d\Omega^2\right), \tag{4.95}$$

where the plus sign holds for the exterior region connected to asymptotic flat Minkowski space, and the minus sign to the region connected to the curvature singularity. Because of the double-valuedness of the analytic continuation (4.94) there are two copies of each of these exterior regions involved.

4.8 Spinning black holes

The Schwarzschild and Reissner-Nordstrom black holes are static and spherically symmetric w.r.t. an asymptotic Minkowski frame at spatial infinity. There also exist extensions of these solutions of Einstein and coupled Einstein-Maxwell equations which are neither static nor spherically symmetric, but represent rotating black holes with or without charge [35, 36]. These solutions have an axial symmetry around the axis of rotation, and are stationary: in the exterior region the metric is time-independent, and the rate of rotation and all other observable properties of the black hole are constant in time.

The standard choice of coordinates for these metrics, with the axis of rotation being taken as the z-axis and reducing to those of Schwarzschild, or Reissner and Nordstrom for the spherically symmetric case of no rotation, is that of Boyer and Lindquist [37]. Denoting these coordinates by $(t, \bar{r}, \theta, \varphi)$ to emphasize their relation with the spherical coordinate systems for the non-rotating metrics, the line element for reads

$$\begin{aligned} ds^2 &= \frac{-\bar{\Delta}^2}{\bar{r}^2 + a^2\cos^2\theta}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\bar{r}^2 + a^2\cos^2\theta}{\bar{\Delta}^2}\left(d\bar{r}^2 + \bar{\Delta}^2 d\theta^2\right) \\ &\quad + \frac{(\bar{r}^2 + a^2)^2\sin^2\theta}{\bar{r}^2 + a^2\cos^2\theta}\left(d\varphi - \frac{a}{\bar{r}^2 + a^2}dt\right)^2. \end{aligned} \tag{4.96}$$

Here a is a constant parametrizing the deviation of this line element from the usual diagonal form of the Schwarzschild-Reissner-Nordstrom metric; its physical interpretation is that of the total angular momentum per unit of mass:

$$J = ma. \tag{4.97}$$

As one can always choose coordinates such that the angular momentum points along the positive z-axis, the angular momentum per unit of mass may be taken positive: $a > 0$. For convenience of discussion this will be assumed in the following.

The quantity $\bar{\Delta}^2$ in the line element (4.96) is short hand for

$$\bar{\Delta}^2 = (\bar{r} - m)^2 + a^2 + e^2 - m^2. \tag{4.98}$$

Note that, like the metric coefficients $\bar{h}^2$ and $\bar{g}^2$ we have encountered earlier, this quantity is not positive definite everywhere. Also, the zero's of this function determine the location of the horizons, of which there are two (an outer and an inner one) for $m^2 > a^2 + e^2$:

$$\bar{r}_{\pm H} = m \pm \sqrt{m^2 - a^2 - e^2}. \tag{4.99}$$

The gravitational field represented by the metric (4.96) is accompanied by an electro-magnetic field, the vector potential of which can be chosen as the one-form

$$A = \frac{q\bar{r}}{4\pi\bar{\rho}^2}\left(dt - a\sin^2\theta d\varphi\right). \tag{4.100}$$

For convenience of notation we have here introduced the function $\bar{\rho}$ defined as

$$\bar{\rho}^2 = \bar{r}^2 + a^2 \cos^2 \theta. \tag{4.101}$$

These fields are solutions of the coupled Einstein-Maxwell equations (4.76) and (4.80), with the Maxwell field strength given by the two form

$$\begin{aligned} F &= -\frac{q}{4\pi \bar{\rho}^4} \left(\bar{r}^2 - a^2 \cos^2 \theta \right) dr \wedge \left(dt - a \sin^2 \theta d\varphi \right) \\ & \quad + \frac{qa}{2\pi} \frac{\bar{r} \cos\theta \sin\theta}{\bar{\rho}^4} d\theta \wedge \left(a dt - (\bar{r}^2 + a^2) d\varphi \right), \end{aligned} \tag{4.102}$$

and with the identification of the quantity e^2 as in eq.(4.82):

$$e^2 = \frac{q^2 G}{4\pi}.$$

To determine the properties of the space-time described by the Kerr-Newman line element we need to analyse the geodesic flow. Because the spherical symmetry of the Schwarzschild and Reissner-Nordstrom solutions is absent, and only axial symmetry remains, there is one less independent constant of motion from angular momentum: only the component along the axis of rotation (the z-axis, say) is preserved. Fortunately this is compensated by the appearance of a new constant of motion, quadratic in momenta, which replaces the Casimir invariant of total angular momentum [39, 40]. As a result one can still completely solve the geodesic equations in terms of first integrals of motion. We now give some details.

First we write down the geodesic hamiltonian, effectively representing the inverse metric $g^{\mu\nu}$:

$$H = \frac{1}{2\bar{\rho}^2} \left[\bar{\Delta}^2 p_r^2 + p_\theta^2 + \left(a \sin\theta \, p_t + \frac{1}{\sin\theta} \, p_\varphi \right)^2 - \frac{1}{\bar{\Delta}^2} \left((\bar{r}^2 + a^2) \, p_t + a \, p_\varphi \right)^2 \right]. \tag{4.103}$$

Clearly, to be well-defined we must assume $\theta \neq (0, \pi)$, as the z-axis represents a coordinate singularity in the Boyer-Lindquist system where the angle φ is not well-defined. The above hamiltonian is a constant of geodesic flow; recall, that for time-like geodesics it takes the value $2H = -1$, whilst for light-like ones $H = 0$.

Next there are two Killing vectors, respresenting invariance of the geometry under time-shifts and rotations around the z-axis:

$$\begin{aligned} \varepsilon &= -p_t = -g_{tt} \frac{dt}{d\lambda} - g_{t\varphi} \frac{d\varphi}{d\lambda}, \\ \ell &= p_\varphi = g_{\varphi t} \frac{dt}{d\lambda} + g_{\varphi\varphi} \frac{d\varphi}{d\lambda}. \end{aligned} \tag{4.104}$$

Of course, for time-like geodesics the affine parameter λ maybe taken to be proper time τ. These equations can be viewed as establishing a linear relation between between the constants of motion (ε, ℓ) and the velocities $(dt/d\lambda, d\varphi/d\lambda)$:

$$\begin{pmatrix} \varepsilon \\ \ell \end{pmatrix} = \tilde{G} \begin{pmatrix} \dot{t} \\ \dot{\varphi} \end{pmatrix}, \tag{4.105}$$

with the dot denoting a derivative w.r.t. the affine parameter λ. The determinant of this linear transformation is

$$\det \tilde{G} = -g_{tt} g_{\varphi\varphi} + g_{t\varphi}^2 = \bar{\Delta}^2 \sin^2 \theta, \tag{4.106}$$

and changes sign whenever $\bar{\Delta}^2$ does. Therefore the zero's of $\bar{\Delta}^2$ define the locus of points where one of the eigenvalues of $\tilde{G}$ changes sign, from a time-like to a space-like variable or vice-versa; at the same time, the coefficient of p_r^2 in the hamiltonian (4.103) changes sign, confirming that these values represent the horizons of the Kerr-Newman space-time. It follows from this analysis, that outside the outer horizon, in the region connected to space-like infinity, $\det \tilde{G} > 0$.

Now if from eqs.(4.104) the angular proper velocity $d\varphi/d\lambda$ is eliminated, one finds the relation

$$\varepsilon - \Omega \ell = \frac{\det \tilde{G}}{g_{\varphi\varphi}} \frac{dt}{d\lambda}, \tag{4.107}$$

where the quantity Ω with the dimensions of an angular velocity is defined by

$$\begin{aligned} \Omega &= -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = a \, \frac{\bar{r}^2 + a^2 - \bar{\Delta}^2}{(\bar{r}^2 + a^2)^2 - \bar{\Delta}^2 a^2 \sin^2 \theta} \\ &= \frac{a}{\bar{r}^2 + a^2} \cdot \frac{1}{1 + \bar{\Delta}^2 \bar{\rho}^2 \, (\bar{r}^2 + a^2)^{-1} \, (\bar{r}^2 + a^2 - \bar{\Delta}^2)^{-1}}. \end{aligned} \tag{4.108}$$

Observe, that it is proportional to a with a constant of proportionality which for large but finite $\bar{r}$ is positive. Hence Ω represents an angular velocity with the same orientation as the angular momentum J. With a canonical choice of coordinate system this quantity is positive, and therefore $\Omega > 0$ as well.

It also follows from eq.(4.107) that for time-like geodesics outside the horizon, with $\lambda = \tau$ and therefore $dt/d\lambda > 0$, an inequality holds of the form

$$\varepsilon - \Omega \ell > 0. \tag{4.109}$$

For large $\bar{r}$ near infinity one finds in fact the stronger inequality $\varepsilon - \Omega \ell > 1$.

Returning to the solution of the geodesic equations, we take into account the existence of a constant of motion K such that $2K = K^{\mu\nu} p_\mu p_\nu$, with $K^{\mu\nu}$ the contravariant components of a Killing tensor:

$$D_{(\mu} K_{\nu\lambda)} = 0. \tag{4.110}$$

The explicit expression for K is

$$
\begin{aligned}
K \; = \; & \frac{1}{2\bar{\rho}^2} \left(-\bar{\Delta}^2 a^2 \cos^2\theta \, p_r^2 \, + \, \bar{r}^2 p_\theta^2 \right) \, + \, \frac{\bar{r}^2 \sin^2\theta}{2\bar{\rho}^2} \left(a p_t + \frac{p_\varphi}{\sin^2\theta} \right)^2 \\
& + \frac{a^2 \cos^2\theta}{2\bar{\rho}^2 \bar{\Delta}^2} \left((\bar{r}^2 + a^2) p_t + a p_\varphi \right)^2 .
\end{aligned}
\tag{4.111}
$$

That K defines a constant of geodesic flow is most easily established by checking that its Poisson bracket with the hamiltonian H (4.103) vanishes:

$$
\{K, H\} \, = \, 0. \tag{4.112}
$$

Using these constants of motion, it is straightforward to establish the solutions for time-like geodesics in the form

$$
\begin{aligned}
\bar{\rho}^2 \frac{dt}{d\tau} \quad & = \; \ell a \left(1 - \frac{\bar{r}^2 + a^2}{\bar{\Delta}^2} \right) \, + \, \varepsilon \frac{(\bar{r}^2 + a^2)^2 \, - \, \bar{\Delta}^2 a^2 \sin^2\theta}{\bar{\Delta}^2}, \\
\bar{\rho}^2 \frac{d\varphi}{d\tau} \quad & = \; \ell \, \frac{\bar{\Delta}^2 - a^2 \sin^2\theta}{\bar{\Delta}^2 \sin^2\theta} \, - \, \varepsilon a \left(1 - \frac{\bar{r}^2 + a^2}{\bar{\Delta}^2} \right), \\
\bar{\rho}^4 \left(\frac{dr}{d\tau} \right)^2 \quad & = \; -\bar{\Delta}^2 \left(2K + \bar{r}^2 \right) \, + \, \left(\ell a - \varepsilon (\bar{r}^2 + a^2) \right)^2, \\
\bar{\rho}^4 \left(\frac{d\theta}{d\tau} \right)^2 \quad & = \; 2K - a^2 \cos^2\theta - \, a^2 \sin^2\theta \left(\frac{\ell}{a \sin^2\theta} - \varepsilon \right)^2 .
\end{aligned}
\tag{4.113}
$$

Note that for a particle starting from $r = \infty$ one has $\varepsilon \geq 1$. These solutions are well-defined everywhere except on the horizons and at points where $\bar{\rho}^2 = 0$. At the latter locus the curvature invariant $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ becomes infinite and there is a real physical singularity. At the horizons both t and φ change infinitely fast, signalling infinite growth of these coordinates w.r.t. an observer in asymptotic flat Minkowski space.

As with the infinite amount of coordinate time t to reach the horizon in the Schwarzschild and Reissner-Nordstrom solutions, this merely signifies a coordinate singularity. However, here it is not only t but also φ which becomes infinite. To gain a better understanding of the reason for this rotation of geodesics, first notice that according to the second equation (4.113) even for $\ell = 0$ the proper angular velocity $d\varphi/d\tau$ does not vanish in general, because $\varepsilon \neq 0$. Hence there are no purely radial geodesics: any incoming test particle picks up a rotation. Now computing the angular velocity as measured by an observer at rest at spatial infinity from the first two equations (4.113) one finds:

$$\frac{d\varphi}{dt} = -\frac{\varepsilon g_{t\varphi} + \ell g_{tt}}{\varepsilon g_{\varphi\varphi} + \ell g_{t\varphi}}. \tag{4.114}$$

It follows immediately, that for a geodesic without orbital angular momentum: $\ell = 0$, the observed angular velocity equals the special reference value (4.108):

$$\left. \frac{d\varphi}{dt} \right|_{\ell=0} = \Omega. \tag{4.115}$$

At the horizon $\bar{r} = \bar{r}_{+H}$, where $\bar{\Delta}^2 = 0$, this becomes

$$\left. \frac{d\varphi}{dt} \right|_{\left[\begin{array}{c} \ell = 0 \\ \bar{r} = \bar{r}_{+H} \end{array} \right]} = \Omega_H = \frac{a}{\bar{r}_{+H}^2 + a^2}. \tag{4.116}$$

Comparison with the expression for Ω, eq.(4.108), shows that outside the horizon the inequality

$$\Omega \leq \Omega_H, \tag{4.117}$$

holds, with equality only on the horizon. As these quantities are finite, the time t can increase near the horizon at an infinite rate only if φ increases at a proportional rate. For this reason both t and φ have to grow without bound as $\bar{r}$ approaches $\bar{r}_{+H}$.

At space-like infinity the Killing vectors (4.104) define time translations and plane rotations. However, their nature changes if one approaches the horizon. Consider the time translation generated by $p_t = -\varepsilon$. The corresponding Killing vector has contravariant and covariant components, given by

$$\xi_t^\mu = -\delta_t^\mu \quad \Leftrightarrow \quad \xi_{t\mu} = -g_{t\mu}. \tag{4.118}$$

Here δ_ν^μ is the usual Kronecker delta symbol. Similarly the components of the Killing vector for plane rotations, generated by p_φ, are

$$\xi_\varphi^\mu = \delta_\varphi^\mu \quad \Leftrightarrow \quad \xi_{\varphi\mu} = g_{\varphi\mu}. \tag{4.119}$$

It follows, that these Killing vectors have norms given by

$$(\xi_t)^2 = g_{tt} = \frac{-\bar{\Delta}^2 + a^2 \sin^2\theta}{\bar{\rho}^2}, \tag{4.120}$$

and

$$(\xi_\varphi)^2 = g_{\varphi\varphi} = \sin^2\theta \, \frac{(\bar{r}^2 + a^2)^2 - \bar{\Delta}^2 a^2 \sin^2\theta}{\bar{\rho}^2}, \tag{4.121}$$

whilst their inner product is

$$\xi_t \cdot \xi_\varphi \,=\, -g_{t\varphi} \,=\, a \sin^2\theta \, \frac{\bar{r}^2 + a^2 - \bar{\Delta}^2}{\bar{\rho}^2}. \tag{4.122}$$

From these equations several observations follow. First, as the norms and inner products of Killing vectors are coordinate independent quantities, we can write the expressions for the reference angular velocity Ω, eq.(4.108), and for the observed angular velocity, eq.(4.114), in a coordinate independent form:

$$\Omega \,=\, \frac{\xi_t \cdot \xi_\varphi}{\xi_\varphi^2}, \qquad \frac{d\varphi}{dt} \,=\, \frac{\xi_t \cdot \xi_\varphi \, \varepsilon \,-\, \xi_t^2 \, \ell}{\xi_\varphi^2 \, \varepsilon \,-\, \xi_t \cdot \xi_\varphi \, \ell}. \tag{4.123}$$

The last equality is equivalent to

$$(\varepsilon - \Omega\ell) \, \frac{d\varphi}{dt} \,=\, \Omega\varepsilon \,-\, \ell \, \frac{\xi_t^2}{\xi_\varphi^2} \,=\, \Omega \left(\varepsilon - \Omega\ell \, \frac{\xi_t^2}{\Omega^2 \xi_\varphi^2} \right). \tag{4.124}$$

Second, with $a > 0$ and excluding the z-axis $\theta = (0, \pi)$ —where the Boyer-Lindquist coordinates are not well-defined and Ω vanishes— Ω and the invariant inner product $\xi_t \cdot \xi_\varphi$ are positive outside the horizon. Also the positivity of $\det \tilde{G}$ in the exterior region, eq.(4.106), can be expressed in the invariant form

$$\Omega^2 \geq \frac{\xi_t^2}{\xi_\varphi^2} \qquad \Leftrightarrow \qquad \frac{\xi_t^2}{\Omega^2 \xi_\varphi^2} \,\leq\, 1, \tag{4.125}$$

with the equality holding only on the horizon. Then for $\ell \geq 0$ we find from eq.(4.124) that $d\varphi/dt \geq \Omega$, whilst for $\ell < 0$ the angular velocity is less than this reference value: $d\varphi/dt < \Omega$. As long as ξ_t is time-like this is all that can be established; in fact, with $\ell < 0$ and $\xi_t^2 < 0$, the absolute value of the angular velocity $d\varphi/dt$ can become arbitrarily large.

However, it is easily established that the norm of ξ_t changes sign on the surface

$$\bar{r}(\theta) \,=\, \bar{r}_0(\theta) \,=\, m \,+\, \sqrt{m^2 - e^2 - a^2 \cos^2\theta}, \tag{4.126}$$

which, with the exception of the points where the surface cuts the z-axis: $\theta = (0, \pi)$, lies *outside* the horizon. The region between this surface and the horizon is called the ergosphere, and ξ_t is space-like there: $\xi_t^2 > 0$. It then follows directly from eqs.(4.124) and (4.125) that in this region the angular velocity is always positive:

$$\frac{d\varphi}{dt} \,>\, 0, \qquad \text{for } \xi_t^2 \,>\, 0. \tag{4.127}$$

Thus we have established, that inside the ergosphere any test particle is dragged along with the rotation of the black hole and no free-falling observer can be non-rotating there. Related to this, both ξ_t and ξ_φ are space-like Killing vectors there. Nevertheless, outside the horizon there still exists time-like combinations of these vectors even inside the ergosphere:

$$\chi = \xi_t - \Omega \xi_\varphi, \tag{4.128}$$

which has non-positive norm: $\chi^2 \leq 0$ as a result of the inequality (4.125), with equality holding only on the horizon $\bar{r} = \bar{r}_H$.

4.9 The Kerr singularity

We continue the analysis of black-hole geometry with a brief discussion of the nature of the singularity of spinning black holes. To begin, we have already noticed that, like the spherical charged black hole, the spinning Kerr-Newman black holes have two horizons, an innner and an outer one. As the singularity $\bar{\rho}^2 = 0$ is located inside the inner horizon, we expect the singularity to have a time-like character.

In order to elucidate the topological structure of the space-time near the singularity, we introduce new coordinates in two steps. First we define the Kerr coordinates $(\tilde{V}, \tilde{\varphi})$ as the solution of the differential equations

$$d\tilde{V} = dt + \frac{(\bar{r}^2 + a^2)}{\bar{\Delta}^2}\, d\bar{r}, \qquad d\tilde{\varphi} = d\varphi + \frac{a}{\bar{\Delta}^2}\, d\bar{r}. \tag{4.129}$$

With these definitions the Kerr-Newman line element becomes

$$\begin{aligned} ds^2 &= -\frac{\bar{\Delta}^2}{\bar{\rho}^2} \left(d\tilde{V} - a \sin^2\theta d\tilde{\varphi} \right)^2 + \frac{\sin^2\theta}{\bar{\rho}^2} \left[(\bar{r}^2 + a^2) d\tilde{\varphi} - a d\tilde{V} \right]^2 \\ &\quad + 2 \left(d\tilde{V} - a \sin^2\theta\, d\tilde{\varphi} \right) d\bar{r} + \bar{\rho}^2\, d\theta^2. \end{aligned} \tag{4.130}$$

Now we turn from quasi-spherical coordinates to quasi-Cartesian ones, known as Kerr-Schild coordinates, by defining

$$\begin{aligned} x &= (\bar{r} \cos\tilde{\varphi} - a \sin\tilde{\varphi})\, \sin\theta, \\ y &= (\bar{r} \sin\tilde{\varphi} + a \cos\tilde{\varphi})\, \sin\theta \\ z &= r \cos\theta, \end{aligned} \tag{4.131}$$

and we reparametrize the time coordinate $t \rightarrow \tilde{t}$ using

$$d\tilde{t} = d\tilde{V} - d\bar{r}. \tag{4.132}$$

We then find that at fixed $(\theta, \tilde{\varphi})$ the transformation

$$x \;\rightarrow\; x' \,=\, x \,-\, 2\bar{r}\cos\tilde{\varphi},$$
$$y \;\rightarrow\; y' \,=\, y \,-\, 2\bar{r}\sin\tilde{\varphi}, \tag{4.133}$$
$$z \;\rightarrow\; z' \,=\, -z,$$

is equivalent to $\bar{r} \rightarrow \bar{r}' = -\bar{r}$. This shows, that like the Reissner-Nordstrom case, the negative values of $\bar{r}$ are to be taken as part of the physical domain of values.

Next one introduces the coefficients k^i $(i = 1, 2, 3)$ by the relation

$$\boldsymbol{k}\cdot d\boldsymbol{r} \,\equiv\, \frac{-\bar{r}\,(xdx + ydy) \,+\, a\,(xdy - ydx)}{\bar{r}^2 + a^2} \,-\, \frac{zdz}{\bar{r}} \,-\, d\tilde{t}, \tag{4.134}$$

and scalar function $u(\boldsymbol{r})$:

$$u(\boldsymbol{r}) \,=\, \frac{-\bar{\Delta}^2 + \bar{r}^2 + a^2}{\bar{\rho}^2} \,=\, \frac{2m\bar{r} - e^2}{\bar{r}^2 + \dfrac{a^2 z^2}{\bar{r}^2}}. \tag{4.135}$$

With these definitions the metric can be written in the hybrid form

$$ds^2 \,=\, dx_\mu^2 \,+\, u(\boldsymbol{r})\,\left(\boldsymbol{k}\cdot d\boldsymbol{r}\right)^2, \tag{4.136}$$

where the first term on the right-hand side is to be interpreted as in flat space-time, with $x^\mu = (x, y, z, \tilde{t})$. Therefore u and $\boldsymbol{k}$ parametrize the deviation from flat space-time.

Having introduced this new parametrization, the physical singularity $\bar{\rho}^2 = 0$, equivalent to $\bar{r} = 0$ and $\theta = \pi/2$, is now seen to be mapped to the ring

$$x^2 \,+\, y^2 \,=\, a^2, \qquad z \,=\, 0, \tag{4.137}$$

with the solution

$$x \,=\, -a\sin\tilde{\varphi}, \qquad y \,=\, a\cos\tilde{\varphi}. \tag{4.138}$$

A consequence of this structure is, that in contrast to the Schwarzschild geometry, geodesic crossing the horizon do not necessarily encounter the singularity, but can pass it by.

4.10 Black-holes and thermodynamics

When a test particle falls into the ergosphere of a black hole, the constant of motion $\xi_t \cdot p \,=\, \varepsilon$ becomes a momentum component, rather than an energy. This provides a means of extracting energy from a spinning black hole, as first proposed by Penrose [41]: if a particle enters the ergosphere with energy ε, this becomes a momentum; during its stay in the ergosphere the momentum can be changed, for example by a decay of the particle into two or more fragments, in such a way that the final momentum of one of the fragments is larger than the original momentum. If this fragment would again escape, which is allowed as the ergosphere is located outside the horizon, it leaves with a different energy ε' corresponding to its changed momentum. This energy is now larger than the energy ε with which the original particle entered. By this process it is possible to extract energy from a black hole, even in classical physics.

It would seem that the Penrose process violates the conservation of energy, if there would be no limit to the total amount of energy that can be extracted this way. However, it can be established that such a violation is not possible and that the process is self-limiting: emission of energy is always accompanied by a decrease in the angular momentum (and/or electric charge) of the black hole in such a way, that after extraction of a finite amount of energy the black hole becomes spherical. After that there is no longer an ergosphere and no energy extraction by the Penrose process is possible.

We will now show how this comes about for the simplest case of an electrically neutral black hole of the Kerr type. In this case the location of the horizon is at

$$\bar{r}_H = m + \sqrt{m^2 - a^2} = m + \frac{1}{m} \sqrt{m^4 - J^2}. \tag{4.139}$$

Then the angular momentum of test particles moving tangentially to the horizon is

$$\Omega_H = \frac{a}{\bar{r}_H^2 + a^2} = \frac{J}{2m} \cdot \frac{1}{m^2 + \sqrt{m^4 - J^2}}. \tag{4.140}$$

This quantity can be interpreted in terms of the area of the spherical surface formed by horizon of the black hole:

$$A_H = \int\int_{\bar{r}=\bar{r}_H} d\theta d\varphi \sqrt{g_{\theta\theta} g_{\varphi\varphi}} = 4\pi \left(\bar{r}_H^2 + a^2 \right) = \frac{4\pi a}{\Omega_H}. \tag{4.141}$$

Eq.(4.140) thus provides an expression for the area of the horizon in terms of the mass and the angular momentum of the black hole:

$$\frac{A_H}{8\pi} = m^2 + \sqrt{m^4 - J^2}. \tag{4.142}$$

Now we observe that
(i) the area A_H increases with increasing mass;
(ii) at fixed area A_H the mass reaches a minimum for $J = 0$;
(iii) by absorbing infalling test particles (including Penrose-type processes) the black-hole area A_H never decreases.

The first two statements are obvious from eq.(4.142). The last one follows if we consider the effect of changes in the mass and angular momentum of the black hole. We assume that the absorption of a test particle with energy ε and angular momentum ℓ changes a Kerr black hole of total mass m and angular momentum J to another Kerr black hole of mass $m' = m+\varepsilon$ and angular momentum $J' = J+\ell$. This assumption is justified by the uniqueness of the Kerr solution for stationary rotating black holes.

Now under arbitrary changes $(\delta m, \delta J)$ the corresponding change in the area is

$$\begin{aligned} \frac{\delta A_H}{8\pi} &= \frac{2m\delta m}{\sqrt{m^4 - J^2}} \left(m^2 + \sqrt{m^4 - J^2} \right) - \frac{J\delta J}{\sqrt{m^4 - J^2}} \\ &= \frac{J}{\Omega_H \sqrt{m^4 - J^2}} \left(\delta m - \Omega_H \delta J \right). \end{aligned} \tag{4.143}$$

For an infalling test particle only subject to the gravitational force of the black hole we have derived the inequality (4.109). Moreover, on the horizon such a particle has been shown to have the angular velocity Ω_H w.r.t. a stationary observer at infinity, independent of the value of ℓ. Hence with $\delta m = \varepsilon$ and $\delta J = \ell$ we obtain

$$\frac{\delta A_H}{8\pi} = \frac{J}{\Omega_H \sqrt{m^4 - J^2}} \left(\varepsilon - \Omega_H \ell \right) > 0. \tag{4.144}$$

Thus we can indeed conclude that by such adiabatic processes, adding infinitesimal amounts of energy and angular momentum at a time, the area of the black hole never decreases.

Casting the relation (4.143) in the form

$$\delta m = \frac{\kappa}{8\pi} \delta A_H + \Omega_H \delta J, \tag{4.145}$$

where κ is given by:

$$\kappa = \frac{4\pi}{m A_H} \sqrt{m^4 - J^2}, \tag{4.146}$$

it bears a remarkable resemblance to the laws of thermodynamics which states that changes of state described by equilibrium thermodynamics are such that for isolated systems variations in energy U, entropy S and angular momentum J are related by

$$\delta U = T \delta S + \Omega \delta J, \tag{4.147}$$

where T is the temperature and Ω the total angular momentum, and moreover that in such processes one always has $\delta S > 0$. The important step needed in a full identification of these relations is to associate the area A_H with the entropy and the quantity κ, called the surface gravity, with the temperature:

$$\delta S = c \delta A_H, \qquad T = \frac{\kappa}{8\pi c}, \tag{4.148}$$

where c is an unknown constant of proportionality. Now if black holes have a finite temperature, they should emit thermal radiation. Using a semi-classical approximation, it was shown by Hawking [38], that quantum fluctuations near the horizon of a black hole indeed lead to the emission of particles with a thermal spectrum. This is true irrespective of the existence of an ergosphere, therefore it covers the case of vanishing angular momentum as well. The temperature of the radiation Hawking found was

$$T_H = \frac{\kappa}{2\pi}. \tag{4.149}$$

This fixes the unknown constant above to the value

$$c = \frac{1}{4}. \tag{4.150}$$

It seems therefore, that the thermodynamical description of black holes is more than a mere analogy. The issue of interpreting the entropy

$$S = \frac{A_H}{4} + S_0, \tag{4.151}$$

where S_0 is an unknown additive constant, in statistical terms by counting microstates is hotly debated and has recently received a lot of impetus from developments in superstring theory.

The extension of these results to the case of charged black holes is straightforward. The relation between horizon area A_H and angular velocity Ω_H remains the as in (4.141), but in terms of the macroscopic observables (m, J, e) associated with the black hole the area is now

$$\frac{A_H}{8\pi} = m^2 - \frac{e^2}{2} + \sqrt{m^4 - J^2 - m^2 e^2}. \tag{4.152}$$

The changes in these observable quantities by absorbing (possibly charged) test particles are then related by

$$\delta m = \frac{\kappa}{8\pi}\,\delta A_H + \Omega_H \delta J + \Phi_H \delta e, \tag{4.153}$$

where the electric potential at the horizon is

$$\Phi_H = \frac{4\pi e \bar{r}_H}{A_H}, \tag{4.154}$$

and the constant κ is now defined as

$$\kappa = \frac{4\pi}{m A_H}\sqrt{m^4 - J^2 - m^2 e^2}. \tag{4.155}$$

Again, by sending test particles into the black hole the area can never decrease: $\delta A_H > 0$. In this way the thermodynamical treatment is generalized to include processes in which the electric charge changes.

We conclude this discussion with establishing the gravitational interpretation of the constant κ, associated with the temperature in the thermodynamical description. This quantity is known as the surface gravity, because in the spherically symmetric case it represents the (outward) radial acceleration necessary to keep a test particle hovering on the horizon without actually falling into the black hole.

For the case of the Schwarzschild black hole this is easily established. Define the covariant acceleration as

$$\eta^{\mu} = \frac{Du^{\mu}}{D\tau}. \tag{4.156}$$

As the four velocity of a test particle has negative unit norm: $u^2 = -1$, it follows that in the four-dimensional sense the acceleration is orthogonal to the velocity:

$$\eta \cdot u = 0. \tag{4.157}$$

Also, a particle at rest in the frame of an asymptotic inertial observer at infinity, has a velocity four-vector of the form

$$u^{\mu} = \left(\frac{1}{\sqrt{-g_{tt}}}, 0, 0, 0 \right). \tag{4.158}$$

Combining these two equations and taking into account the spherical symmetry, a particle at rest must have a purely radial acceleration given by

$$\eta^{\bar{r}} = -\frac{1}{2}\, \partial_{\bar{r}} g_{tt} = \frac{m}{\bar{r}^2}. \tag{4.159}$$

Remarkably, we recover Newtons law! It follws in particular that on the horizon

$$\eta^{\bar{r}}_{H} = \frac{1}{4m} = \kappa. \tag{4.160}$$

This quantity is actually a proper scalar invariant, as can be seen from writing it as

$$\kappa = \left[\eta \sqrt{-\xi_t^2} \right]_{\bar{r}_H}, \tag{4.161}$$

with η the invariant magnitude of the four acceleration:

$$\eta = \sqrt{g_{\mu\nu} \eta^{\mu} \eta^{\nu}}. \tag{4.162}$$

For the case of the spinning black holes the proper generalization of this result gives the surface gravity as in eq.(4.155).

Chapter 5

Topological invariants and self-duality

5.1 Topology and topological invariants

Gravity is a gauge theory, and as such it has many similarities to Yang-Mills theories. In particular the nature of the gravitational vacuum is complicated by the possibilities of non-trivial topology, quantum tunneling and spontaneous symmetry breaking. In this section we briefly introduce some concepts and notions of topology.

In the geometrical context the global topology of a manifold describes properties characterising its connectedness, like the presence of non-contractable loops, spheres, etc. Clearly the way that various parts of a manifold are connected determines whether it can be covered by a single globally defined coordinate system, or requires a specific minimal number of coordinate charts glued together in a smooth way, i.e. such that in the overlap region of two charts the transformation from one coordinate system to the other is differentiable. Generically more than one coordinate chart is necessary, as in the case of a sphere in which conventionally the northern and southern hemisphere are mapped separately on the plane, with the maps overlapping in the region of the equator. Thus general manifolds are to be considered as glued together from a number of smoothly deformed regions of flat space, often in a non-trivial way.

This has important consequences. For example, if one computes the integral of a certain function over the manifold, this integral must necessarily be decomposed into a sum of integrals over each coordinate patch. Even in the case in which the manifold as a whole has no boundary, the various coordinate patches do have boundaries. Thus, in the application of the generalized Gauss-Stokes theorem for the integral of a total divergence of a vector field $K_\mu(x)$ over a manifold $\mathcal{M} = \sum_i \mathcal{M}_i$, with $\mathcal{M}_i$ the pieces of the manifold which can be covered with a single coordinate chart, the integral becomes a sum of integrals over the boundaries of the coordinate patches:

$$\int_{\mathcal{M}} \nabla \cdot K = \sum_i \int_{\mathcal{M}_i} \nabla \cdot K^{(i)} = \sum_i \int_{\partial \mathcal{M}_i} K_n^{(i)}, \tag{5.1}$$

where $K_\mu^{(i)}(x^{(i)})$ are the components of the vector field in the i-th coordinate patch, and de subscript n denotes the component normal to the boundary $\partial\mathcal{M}_i$.

Now if the complete manifold $\mathcal{M}$ has no boundary, all the boundaries $\partial\mathcal{M}_i$ are internal boundaries and these are integrated over twice with opposite orientation. Therefore, if the integrand is the same in each case, the contributions of the integrals over these internal boundaries cancel exactly. However, in the change of coordinates from one region $\mathcal{M}_i$ to the next the value of the integrand $K_n^{(i)}$ can sometimes change and the cancellation may be spoiled. Then the non-trivial topology manifests itself in an apparent violation of the generalized Gauss-Stokes theorem: integrals over a total divergence do not have to vanish in the absence of an external boundary. Of course, if K_μ is a smooth vector field, like the physically observable electric field, it can not jump on the boundary $\partial\mathcal{M}_i$: a violation of the Gauss-Stokes theorem requires that the vector field cannot be smooth. Only particular kinds of physically relevant vector fields can satisfy this requirement without leading to physical or mathematical inconsistencies: vector fields which change in a controllable way, with the change between two coordinate patches not leading to observable or measurable consequences. Basically such vector fields must be gauge fields, as these can change by unobservable gauge transformations on the boundary of coordinate regions.

Example

We illustrate these general and rather abstract arguments by a simple and well-known example of vector fields on the two-sphere. The standard coordinate system for a two-sphere of radius r is the polar coordinate system $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$. In these coordinates the line element on the sphere reads

$$ds^2 = r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right). \tag{5.2}$$

From this expression it can be seen that the metric is singular —it has a zero mode in the φ-direction— at the north and south poles, $\theta = (0, \pi)$. This coordinate singularity is avoided by covering the upper and lower hemispheres $H_\pm$, corresponding to $0 \leq \theta \leq \pi/2$ and $\pi/2 \leq \theta \leq \pi$, separately with a cartesian coordinate system $(x_\pm, y_\pm)$, obtained for example by projection on the disc $x^2 + y^2 \leq r^2$:

$$\begin{aligned} x_+ &= r \tan\frac{\theta}{2}\cos\varphi, \\ y_+ &= r \tan\frac{\theta}{2}\sin\varphi, \end{aligned} \qquad \text{for } H_+, \tag{5.3}$$

and

$$
\left.\begin{aligned}
x_- &= r \cot \frac{\theta}{2} \cos \varphi, \\
y_- &= -r \cot \frac{\theta}{2} \sin \varphi,
\end{aligned}\right. \qquad \text{for } H_-. \tag{5.4}
$$

The minus sign in the expression for y_- is necessary to have (x_-, y_-) form a right-handed coordinate system on the exterior of the southern hemisphere; it could be absorbed in a redefinition $\varphi \rightarrow -\varphi$. Thus the orientation of the cartesian coordinates is preserved in region of overlap of the two coordinate systems, which includes the equator $\theta = \pi/2$:

$$
\begin{aligned}
x_+ = x_- &= r \cos \varphi, \\
y_+ = -y_- &= r \sin \varphi,
\end{aligned} \tag{5.5}
$$

but which actually can be extended to the entire sphere with the exclusion of the north and south poles by continuing $(x_\pm, y_\pm)$ outside the disc $x^2 + y^2 \leq r^2$.

In the cartesian coordinates $(x_\pm, y_\pm)$ the line element becomes

$$
ds^2 = \frac{4r^2}{(r^2 + x_\pm^2 + y_\pm^2)^2} \left(dx_\pm^2 + dy_\pm^2 \right). \tag{5.6}
$$

This expression is manifestly non-singular, except in the limit $(x_\pm, y_\pm) \rightarrow \infty$ corresponding to one of the poles in each case. Observe, that the north pole is now a regular point of the coordinate system (x_+, y_+) and the south pole of the system (x_-, y_-).

Now consider a vector field on the sphere defined in components by

$$
A_{\pm\mu} = \frac{N}{r^2 + x_\pm^2 + y_\pm^2} \left(-y_\pm, x_\pm \right), \tag{5.7}
$$

where N is a normalization constant. This vector field can be represented as a one-form

$$
\begin{aligned}
A_\pm &= \frac{N}{r^2 + x_\pm^2 + y_\pm^2} \left(-y_\pm dx_\pm + x_\pm dy_\pm \right) \\
&= \frac{N}{2} \left(\pm 1 - \cos \theta \right) d\varphi.
\end{aligned} \tag{5.8}
$$

The last expression in terms of the polar coordinates shows, that in the region of overlap the vector fields $A_\pm$ differ by an exact one-form:

$$
A_+ - A_- = N d \arctan \left(\frac{y_+}{x_+} \right) = N d \arctan \left(\frac{y_-}{x_-} \right) = N d\varphi. \tag{5.9}
$$

It follows, that the two-form $F = dA_+ = dA_-$ is unique and well-defined on the entire sphere:

$$\begin{aligned} F &= \tfrac{1}{2}\, F_{\mu\nu}\, dx^\mu \wedge dx^\nu \\ &= \frac{2Nr^2}{(r^2 + x_\pm^2 + y_\pm^2)^2}\, dx \wedge dy \;=\; \frac{N}{2}\, \sin\theta\, d\theta \wedge d\varphi. \end{aligned} \tag{5.10}$$

The dual of the two-form F is defined to be the zero-form

$$\tilde{F} = \frac{1}{2\sqrt{g}}\, \varepsilon^{\mu\nu} F_{\mu\nu} = \frac{N}{2r^2}, \tag{5.11}$$

with the tensor density $\varepsilon^{\mu\nu}$ representing the two-dimensional permutation symbol. As this is a constant, it is closed: $d\tilde{F} = 0$, or in components:

$$\partial^\mu F_{\mu\nu} = 0. \tag{5.12}$$

This shows that we have constructed a non-trivial solution of the free Maxwell equations on the sphere, representing a constant magnetic field of strength $B = N/2r^2$. This solution can be extended trivially to a static spherically symmetric solution of Maxwell's equations in three dimensions, by taking the radial component of the vector field to vanish: $A_r = 0$. In three dimensions this solution represents a magnetic monopole of strength $N/2$ located in the origin.

Note that we can write the gauge transformation connecting the vector fields $A_\pm$ in the region of overlap in terms of the elements $g(\varphi) = \exp iN\varphi$ of a $U(1)$ field $g(x, y)$ in the standard form

$$A_+ = g^{-1} A_- g + \frac{1}{i}\, g^{-1} dg. \tag{5.13}$$

For this $U(1)$ field to be well-defined in the whole region of overlap of the domains of $A_\pm$ —the whole sphere minus the poles— it must return to the same value after a rotation $\varphi \to \varphi + 2\pi$. This is the case only if N is an integer.

The number N has a direct topological interpretation: it represents the winding number of the map from the equator[1] of the sphere to the $U(1)$ group manifold, which is the unit circle in the complex plane. Therefore we see that the interpretation of the vector field $A_\pm$ as a connection for a $U(1)$ bundle (a magnetic vector potential) on the sphere requires the winding number of the gauge transformation (the magnetic charge) to be quantized, a well-known result first obtained by Dirac.

---

[1]Note that the equator cannot be contracted to a point on the sphere whilst staying completely inside the region of overlap where the gauge transformation is defined, as in this contraction it must always pass one of the poles.

The example of the vector field $A_\pm$ on the two-sphere illustrates how the naive Stokes' theorem can be violated: even though F is a locally exact two-form ($F = dA_\pm$ in $H_\pm$) well-defined and smooth over the entire sphere, and even though the sphere has no boundary, the integral of F over the sphere is non-zero. This is because F is not an exact two-form globally, and the vector field makes a jump on the boundary between the regions $H_\pm$:

$$\begin{aligned}\int_{S^2} F &= \int_{S^2} d^2x\,\sqrt{g}\,\tilde{F} &&= \int_{S^2} d^2x\,\frac{1}{2}\,\varepsilon^{\mu\nu}\,F_{\mu\nu} \\ &= \int_{H_+} F + \int_{H_-} F &&= \oint_C (A_+ - A_-) = 2\pi N.\end{aligned} \tag{5.14}$$

In the line integral the loop $C = \partial H_+ = -\partial H_-$ denotes the equator, with the minus sign in front of ∂H_- deriving from the reversal of the orientation of φ. This induces the minus sign in front of A_- in the loop integral.

Finally we observe that a smooth deformation of the sphere will not change the value of the integral (5.14): if $\mathcal{M}$ denotes any closed surface which encloses the magnetic charge, and Ω is the volume between this surface and the two-sphere S^2, then

$$\begin{aligned}\int_{\mathcal{M}} F - \int_{S^2} F &= \int_{\partial\Omega} F \\ &= \int_{\Omega} dF = 0,\end{aligned} \tag{5.15}$$

by the second Maxwell equation in the absence of magnetic charge (the Bianchi identity) inside Ω. Geometrically we can understand this result, because the first line of eq.(5.14) shows that in components the integral is manifestly independent of the metric. Hence a local change of the metric, which is equivalent to a local deformation, does not affect the value of the integral. Such a quantity is called a topological invariant.

5.2 Topological invariants in gravity

In gravity topology can play a role at various levels. At the macroscopic classical level one can consider multiply connected universes and wormholes, whilst at the microscopic Planck scale space-time topology may be subject to quantum fluctuations ('space-time foam'); in analogy with other field theories like sigma models and Yang-Mills theories, it is expected that the quantum tunneling processes between different topologies are dominated by finite-action solutions of Euclidean gravity, the gravitational instantons.

One way to characterise topologically non-trivial solutions of the gravitational field equations is by the value of topologically invariant integrals over certain polynomials of the curvature tensor. In four dimensions there are essentially two independent topological invariants of this type: the first Pontrjagin number and the Euler number. To define these we first introduce the dual and double dual curvatures with components:

$$\tilde{R}^{\mu\nu}{}_{\kappa}{}^{\lambda} = \frac{1}{2\sqrt{g}}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\rho\sigma\kappa}{}^{\lambda}, \qquad \bar{R}^{\mu\nu}{}_{\kappa\lambda} = \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\kappa\lambda\alpha\beta}\,R_{\rho\sigma}{}^{\alpha\beta}. \tag{5.16}$$

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the permutation symbol, a tensor density requiring a factor $\sqrt{g}$ to guarantee the transformation character of the dual curvature as a tensor under local coordinate transformations. The definition (5.16) is adapted to manifolds with euclidean signature, but it can easily be generalized to include the lorentzian case.

The first Pontrjagin number of a compact manifold $\mathcal{M}$ without boundary involves the dual curvature and is defined by the expression

$$p_1(\mathcal{M}) = \frac{1}{64\pi^2}\int_{\mathcal{M}} d^4x\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu\kappa}{}^{\lambda}R_{\rho\sigma\lambda}{}^{\kappa}, \tag{5.17}$$

whilst the Euler number is defined in a similar way using the double dual of the Riemann tensor:

$$\chi(\mathcal{M}) = \frac{1}{128\pi^2}\int_{\mathcal{M}} d^4x\,\sqrt{g}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{\rho\sigma\tau\upsilon}\,R_{\mu\nu}{}^{\rho\sigma}\,R_{\kappa\lambda}{}^{\tau\upsilon}. \tag{5.18}$$

For a general proof that both expressions are topological invariants it is more convenient to use the formulation of differential geometry in terms of the vierbein and spin-connection, rather than the metric and the Riemann-Christoffel connection:

$$R^{ab}_{\mu\nu} = e^{\kappa a}e_{\lambda}{}^{b}\,R_{\mu\nu\kappa}{}^{\lambda} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - [\omega_\mu,\omega_\nu]^{ab}. \tag{5.19}$$

In this formulation, the traces in the definition of the Pontrjagin and Euler number are to be taken over the tangent space (i.e. local SO(4)) indices (ab), and we can write both topological invariants in the form

$$c(\mathcal{M}) = \frac{1}{32\pi^2}\int_{\mathcal{M}} d^4x\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(R_{\mu\nu}\Lambda R_{\rho\sigma}) \tag{5.20}$$

where depending on the topological invariant condidered the linear operator Λ acting on anti-symmetric SO(4) tensors is either the identity or the SO(4) duality operator:

$$\Lambda R^{\mu\nu ab} = R^{ab}_{\mu\nu}, \qquad \text{or} \qquad \Lambda R^{\mu\nu ab} = \frac{1}{2}\,\varepsilon^{abcd}\,R^{cd}_{\mu\nu}, \tag{5.21}$$

respectively. From the expression (5.20) it is clear, that the square root of the metric has canceled inside the integral: the integral depends on the $\omega_\mu{}^{ab}$. the topological invariance of $c(\mathcal{M})$ is now straightforward: one proves that the value of the integral is the same for any two spin connections which can be smoothly deformed into each other. Let ω denote the original spin connection, and ω_1 the deformed one. The difference $\eta = \omega_1 - \omega$ is a covariant SO(4) tensor. We define a path in the space of connections linking ω and ω_1 parametrized by a real number $0 \leq \lambda \leq 1$:

$$\omega(\lambda) = \omega + \lambda\eta. \tag{5.22}$$

We show that the integral expression (5.20) is independent of λ:

$$\frac{dc(\mathcal{M})}{d\lambda} = \frac{1}{16\pi^2} \int_{\mathcal{M}} d^4x\, \varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr} \left(\frac{dR_{\mu\nu}(\lambda)}{d\lambda} \Lambda R_{\rho\sigma}(\lambda) \right) = 0. \tag{5.23}$$

The last result follows, because the integrand is a total derivative which is smooth and well-defined everywhere if η is; to check this, consider the explicit form of the deformed curvature tensor $R(\lambda)$;

$$R_{\mu\nu}^{ab}(\lambda) = R_{\mu\nu}^{ab} + \lambda\,(D_\mu \eta_\nu - D_\mu \eta_\nu) - \lambda^2\,[\eta_\mu, \eta_\mu]^{ab}\,. \tag{5.24}$$

Here the covariant derivativer of η is constructed in terms of the original undeformed spin connection. Its derivative with respect to λ is then

$$\frac{d}{d\lambda} R_{\mu\nu}^{ab}(\lambda) = D_\mu \eta_\nu - D_\nu \eta_\mu - 2\lambda\,[\eta_\mu, \eta_\nu]^{ab}\,. \tag{5.25}$$

The last term involving the commutator of two η's is precisely the one necessary to complete the covariant derivatives of η to covariant derivatives $D^{(\lambda)}$ w.r.t. the full deformed connection $\omega(\lambda)$:

$$\frac{d}{d\lambda} R_{\mu\nu}^{ab}(\lambda) = D_\mu^{(\lambda)} \eta_\nu - D_\nu^{(\lambda)} \eta_\mu. \tag{5.26}$$

Inserting this in the integral in eq.(5.23), one obtains

$$\begin{aligned} \frac{dc(\mathcal{M})}{d\lambda} &= \frac{1}{16\pi^2} \int_{\mathcal{M}} d^4x\, \mathrm{Tr} \left(\frac{dR_{\mu\nu}(\lambda)}{d\lambda} \Lambda R_{\rho\sigma}(\lambda) \right) \\ &= -\frac{1}{8\pi^2} \int_{\mathcal{M}} d^4x\, \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \left[D_\mu^{(\lambda)} \eta_\nu \Lambda R_{\rho\sigma}(\lambda) \right] \\ &= \frac{1}{8\pi^2} \int_{\mathcal{M}} d^4x\, \partial_\mu \left(\varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\,[\eta_\nu \Lambda R_{\rho\sigma}(\lambda)] \right). \end{aligned} \tag{5.27}$$

To obtain the last line we have used the Bianchi identity for the SO(4)-curvature:

$$\varepsilon^{\mu\nu\rho\sigma}\, D_\nu^{(\lambda)} R_{\rho\sigma}^{ab}(\lambda) = 0, \tag{5.28}$$

which guarantees that the SO(4)-covariant derivative can be taken outside the trace in the integrand. As both η and the curvature are covariant tensors, the invariant SO(4)-traces do not have a discontinuity across the boundaries between the different coordinate patches, and the integrand in eq.(5.27) is a globally defined total derivative on $\mathcal{M}$. If the manifold has no boundary, the integral then vanishes, thus proving that the original quantity $c(\mathcal{M})$ is independent of the connection indeed.

It is of interest to note, that the Pontrjagin density can actually be written locally as a total covariant divergence:

$$\frac{1}{32\pi^2}\,\mathrm{Tr}\left(R\tilde{R}\right) \,=\, D_\mu J_P^\mu \,=\, \frac{1}{\sqrt{g}}\,\partial_\mu(\sqrt{g}J_P^\mu), \tag{5.29}$$

where the current $J_P^\mu(x)$ is defined by the Chern-Simons form

$$\begin{aligned} \sqrt{g}\,J_P^\mu \;&=\; \frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\Gamma_\nu\partial_\rho\Gamma_\sigma - \frac{2}{3}\,\Gamma_\nu\Gamma_\rho\Gamma_\sigma\right) \\ &=\; \frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\omega_\nu\partial_\rho\omega_\sigma - \frac{2}{3}\,\omega_\nu\omega_\rho\omega_\sigma\right). \end{aligned} \tag{5.30}$$

To interpret the middle expression correctly, one should regard the Christoffel symbols as a set of four (4×4) matrices $(\Gamma_\mu)_\nu^{\;\lambda}$, and the Riemann tensor (or its dual) as a set of six (4×4) matrices $(R_{\mu\nu})_\kappa^{\;\lambda}$. The trace is then taken over a contravariant upper and a covariant lower space-time index. In the spin connection the trace is over the local SO(4) tangent-space indices. This result implies directly, that the only non-trivial contributions to the Pontrjagin number come from the boundaries of the coordinate patches:

$$p_1(\mathcal{M}) \,=\, \frac{1}{16\pi^2}\sum_{i=1}^{N}\int_{K_i} d^4x^{(i)}\,\partial_\mu\left[\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\Gamma_\nu^{(i)}\partial_\rho\Gamma_\sigma^{(i)} - \frac{2}{3}\,\Gamma_\nu^{(i)}\Gamma_\rho^{(i)}\Gamma_\sigma^{(i)}\right)\right]. \tag{5.31}$$

where the $x^{(i)}$ and $\Gamma^{(i)}$ denote the coordinates and connections in the i-th coordinate chart K_i. That they can come from these boundaries indeed, follows because the current is not covariant — only the variation of its divergence is. Therefore we can construct another simple proof of the topological invariance of the Pontrjagin number: under a deformation of the metric (or the vierbein), the variation of the current can be written as

$$\delta(\sqrt{g}J_P^\mu) \,=\, \frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\left[\mathrm{Tr}\left(R_{\rho\sigma}\delta\Gamma_\nu\right) \,-\, \partial_\nu\mathrm{Tr}\left(\Gamma_\rho\delta\Gamma_\sigma\right)\right], \tag{5.32}$$

or a similar form in terms of the spin connection. Because of the contraction with the ε-tensor the last term does not contribute to the divergence of the variation $\delta(\sqrt{g}J_P^\mu)$. But as the first term is covariant it does not produce a non-vanishing contribution to the integral by its boundary terms: its boundary terms cancel pairwise between adjoining patches. Therefore the Pontrjagin number is invariant under smooth deformations of the manifold:

$$\delta p_1(\mathcal{M}) \,=\, 0. \tag{5.33}$$

As there is no similar Chern-Simons type current for the Euler number, this proof cannot be extended to that case.

5.3 Self-dual solutions of the Einstein equations

There is an interesting class of space-time geometries for which the two topological invariants — the Pontrjagin and the Euler number — are identical, or identical up to a sign: the (anti) self-dual geometries satisfying

$$\tilde{R}^{\mu\nu}{}_{\kappa}{}^{\lambda} \,=\, \frac{1}{2\sqrt{g}}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\rho\sigma\kappa}{}^{\lambda} \,=\, \pm\,R^{\mu\nu}{}_{\kappa}{}^{\lambda}. \tag{5.34}$$

Because of the symmetry of the fully covariant Riemann tensor $R_{\mu\nu\rho\sigma}$ under interchange of the index pairs $(\mu\nu)$ and $(\rho\sigma)$, the double dual $\bar{R}_{\mu\nu\rho\sigma}$ then also equals the dual (up to a possible sign), and the two topological invariants are equivalent indeed.

The interesting aspect of these (anti) self-dual geometries is, that they are automatically solutions of the Einstein equations in empty space or space-time —depending on the signature— with zero cosmological constant. To see this, construct the Ricci tensor and note that it vanishes due to the Bianchi identity:

$$R_{\mu}^{\ \nu} \,=\, R_{\mu\lambda}{}^{\nu\lambda} \,=\, \pm\frac{1}{2\sqrt{g}}\,\varepsilon^{\nu\lambda\rho\sigma}\,R_{\mu\lambda\rho\sigma} \,=\, 0. \tag{5.35}$$

Then automatically also the Riemann curvature scalar vanishes, and therefore these solutions have vanishing action, at least if the space-time manifold has no boundary. Euclidean solutions of this type are identified as gravitational instantons.

A considerable number of such (anti) self-dual four-geometries are known, but only two represent regular, compact euclidean four-manifolds without boundary: the four-torus T^4, which has a flat metric, but is non-simply connected, and the famous K_3 manifold, which is non-flat but simply connected. Although no explicit expression for the K_3 metric in terms of coordinates is known, its topological invariants can be calculated using cohomology arguments and index theory, which gives $\chi(K_3) = p_1(K_3) = 24$.

All other self-dual solutions of the Einstein equations violate at least one of the other assumptions: the manifold is not compact or it has a non-zero boundary. The earlier proof of the topological invariance of the Pontrjagin and Euler number then fails: in general there can appear additional contributions from infinity or from the boundaries.[2] In this class of solutions one finds the Eguchi-Hanson metric [43], and the multi-center solutions of Gibbons and Hawking [44].

Taub-NUT

A relatively simple and interesting example of these solutions is the (anti) self-dual Taub-NUT metric. It is the simplest example of the class of Gibbons-Hawking solutions. As it also has some interesting applications outside general relativity, I have chosen to discuss it here in some detail.

The Taub-NUT manifold is a four-dimensional Riemannian manifold, involving on a single scale parameter m, with Euclidean signature and (anti) self-dual Riemann curvature, depending on the sign of m. A convenient set of coordinates to describe the manifold are the four-dimensional polar coordinates $(r, \theta, \psi, \varphi)$ with ranges $0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$, $0 \leq \varphi \leq 2\pi$, in terms of which the metric is given by the line element

$$ds^2 = \left(1 + \frac{2m}{r} \right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) + \frac{4m^2}{\left(1 + \frac{2m}{r} \right)} \left(d\psi + \cos\theta d\varphi \right)^2 . \tag{5.36}$$

The parameter m, with dimension of length, can take both positive or negative values. Upon factoring out m^2 and rescaling the radial coordinate $|r/m| \to r$, the expression (5.36) becomes

$$\frac{1}{m^2} ds^2 = \left(1 \pm \frac{2}{r} \right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) + \frac{4}{\left(1 \pm \frac{2}{r} \right)} \left(d\psi + \cos\theta d\varphi \right)^2 , \tag{5.37}$$

where the sign represents the sign of m. As m^2 has been reduced to a multiplicative constant, one can choose units of length such that $m^2 = 1$. The right hand side of eq.(5.37) then directly defines the metric. In the following discussion of the Taub-NUT geometry this choice of metric is made, but it is straightforward to reinstate the general m-dependence when desired.

In the appendix of this chapter we have collected the explicit expressions for the metric, the components of the Riemann-Christoffel connections and the covariant components of the Riemann curvature tensor. From these expressions one may check, that the curvature is self-dual for positive, and anti self-dual for negative $m = -1$.

The inverse metric is encoded in the hamiltonian

[2] Extensions of the construction of topological invariants including boundary contributions can be found in the literature, see for example the review [42].

$$2H = \frac{p_r^2}{1 \pm \dfrac{2}{r}} + \frac{p_\theta^2}{r(r \pm 2)} + \frac{1}{4}\left(1 \pm \frac{2}{r}\right) p_\psi^2 + \frac{(p_\varphi - \cos\theta\, p_\psi)^2}{r(r \pm 2)\sin^2\theta}, \tag{5.38}$$

which describes the geodesic flow. The canonical momenta appearing in this expression are derived from (5.37):

$$\begin{aligned} p_r &= \left(1 \pm \frac{2}{r}\right) \dot{r}, & p_\theta &= r^2 \left(1 \pm \frac{2}{r}\right) \dot{\theta}, \\ p_\psi &= \frac{4}{\left(1 \pm \dfrac{2}{r}\right)} \left(\dot{\psi} + \cos\theta\, \dot{\varphi}\right), & p_\varphi &= \cos\theta\, p_\psi + r^2 \sin^2\theta \left(1 \pm \frac{2}{r}\right) \dot{\varphi}, \end{aligned} \tag{5.39}$$

where as before an upper dot denotes a derivative w.r.t. to the affine parameter λ.

First integrals of motion can be constructed with the help of the constants of geodesic flow, as explained in chapter 2. Indeed, the Taub-NUT manifold admits a set of three independent Killing vectors which, together with the conservation of the hamiltonian (5.38), allow complete solution of the geodesic equations. It is actually convenient to express the Killing vectors as a set of four dependent ones, instead of three independent ones, as this leads to simplifications both in computations and physical interpretations. The contravariant components $\xi^{(s)\mu}$, $s = 1, 2, 3, 4$, of these Killing vectors can be read off from the constants of geodesic flow $J^{(s)} = \xi^{(s)\mu} p_\mu$, with

$$\begin{aligned} J^{(1)} &= -\sin\varphi\, p_\theta + \frac{\cos\varphi}{\sin\theta}\, p_\psi - \cot\theta\cos\varphi\, p_\varphi, & J^{(3)} &= p_\varphi, \\ J^{(2)} &= \cos\varphi\, p_\theta + \frac{\sin\varphi}{\sin\theta}\, p_\psi - \cot\theta\sin\varphi\, p_\varphi, & J^{(4)} &= p_\psi. \end{aligned} \tag{5.40}$$

Solving these equations we get for the angular components of the four-momentum:

$$\begin{aligned} p_\theta &= -\sin\varphi\, J^{(1)} + \cos\varphi\, J^{(2)}, \\ p_\varphi &= J^{(3)}, \qquad p_\psi = J^{(4)}, \end{aligned} \tag{5.41}$$

whilst between the constants of geodesic flow there is the additional relation:

$$J^{(4)} = \cos\theta\, J^{(3)} + \sin\theta \left(\cos\varphi\, J^{(1)} + \sin\varphi\, J^{(2)}\right). \tag{5.42}$$

The radial momentum can be computed from the hamiltonian:

$$p_r^2 = \left(1 \pm \frac{2}{r}\right)\left[2H - \frac{1}{4}\left(1 \pm \frac{2}{r}\right) J_4^2 \right.$$
$$\left. - \frac{1}{r(r \pm 2)}\left(\left(\sin\varphi\, J^{(1)} - \cos\varphi\, J^{(2)}\right)^2 + \frac{1}{\sin^2\theta}\left(J^{(3)} - \cos\theta\, J^{(4)}\right)^2\right)\right]. \tag{5.43}$$

Before discussing some features of the solutions for the geodesics, we digress briefly to study the infinitesimal transformations genereated by the constants of motion; this gives useful general insight into the geometry of the Taub-NUT manifold [45]. From the last two equations (5.41) it is clear, that $J^{(3)}$ and $J^{(4)}$ generate shifts in the angles φ and ψ, i.e. mutually commuting rotations around two orthogonal axes. As $J^{(1,2)}$ are independent of ψ, they commute with the rotations in the ψ directions generated by $J^{(4)}$. However, they do not commute with $J^{(3)}$ or with each other. A direct computation shows that their Poisson brackets realize the commutations relations of ordinary angular momentum:

$$\left\{J^{(i)}, J^{(j)}\right\} = \varepsilon^{ijk}\, J^{(k)}, \qquad i,j,k = (1,2,3). \tag{5.44}$$

As the generators $J^{(i)}$ change the angle ψ in a non-trivial way, it follows that they do not just realize three-dimensional rotations on the submanifold spanned by the coordinates (θ, φ). Rather, three-dimensional rotations on the spherical surface defined the (θ, φ) coordinate directions are generated by a set of related dynamical quantities, the restricted angular momentum three-vector $\boldsymbol{I}$ with components

$$\begin{aligned} I^{(1)} &= J^{(1)} - \frac{\cos\varphi}{\sin\theta}\, J^{(4)}, \quad I^{(3)} = J^{(3)}. \\ I^{(2)} &= J^{(2)} - \frac{\sin\varphi}{\sin\theta}\, J^{(4)}, \end{aligned} \tag{5.45}$$

The components of the restricted angular momentum three vector satisfy the same Poisson bracket relations (5.44) as the components of the total angular momentum $\boldsymbol{J}$:

$$\left\{I^{(i)}, I^{(j)}\right\} = \varepsilon^{ijk}\, I^{(k)}, \tag{5.46}$$

but with two crucial differences: (i) they induce rotations only in the (θ, φ)-directions, whilst commuting with the angle ψ; (ii) unlike the components of $\boldsymbol{J}$ they are not conserved on geodesics. This discussion of symmetries shows, that it is convenient to think of the Taub-NUT manifold locally as a (3+1) dimensional Euclidean geometry, with a three-dimensional space spanned by the spherical coordinates (r, θ, φ) and an internal compact one-dimensional space characterized by the coordinate ψ, but with non-trivial interplay between the flow of geodesics in these directions.

The identity (5.42) shows, that in this three-dimensional sense $J^{(4)}$ is the projection of $\boldsymbol{J}$ in the radial direction, hence orthogonal to the spherical surface spanned by (θ, φ):

$$J^{(4)} = \hat{\boldsymbol{r}} \cdot \boldsymbol{J} = \frac{\boldsymbol{r} \cdot \boldsymbol{J}}{r}. \tag{5.47}$$

The contribution to the total angular momentum orthogonal to $\hat{\boldsymbol{r}}$ can be identitfied as the orbital angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + J^{(4)} \hat{\boldsymbol{r}}, \tag{5.48}$$

in the sense that

$$\boldsymbol{L} = (1 \pm \frac{2}{r})\, \boldsymbol{r} \times \dot{\boldsymbol{r}}. \tag{5.49}$$

Thus $\boldsymbol{r} \cdot \boldsymbol{L} = 0$ indeed. However, in components

$$\begin{aligned}
L^{(1)} &= J^{(1)} - \sin\theta \cos\varphi\, p_\psi = -\sin\varphi\, p_\theta - \cot\theta \cos\varphi\, (p_\varphi - \cos\theta\, p_\psi), \\
L^{(2)} &= J^{(2)} - \sin\theta \sin\varphi\, p_\psi = \cos\varphi\, p_\theta - \cot\theta \sin\varphi\, (p_\varphi - \cos\theta\, p_\psi), \\
L^{(3)} &= J^{(3)} - \cos\theta\, p_\psi = p_\varphi - \cos\theta\, p_\psi,
\end{aligned} \tag{5.50}$$

and therefore the orbital angular momentum is not identical with the restricted angular momentum $\boldsymbol{I}$. In fact the Poisson brackets of $\boldsymbol{L}$ do not obey the angular momentum algebra. Also $\boldsymbol{L}$ is not conserved, although its magnitude $|\boldsymbol{L}|$ is, as guaranteed by the conservation of $\boldsymbol{J}$ and $J^{(4)}$, and the orthogonality to $\boldsymbol{J}$ of the contributions from $J^{(4)}$ and $\boldsymbol{L}$ in eq.(5.48). This clarifies the non-trivial dynamical relation between motion in the (θ, φ) and ψ-directions.

Finally, as $\boldsymbol{J}$ and $J^{(4)}$ are conserved separately along geodesics, it follows from (5.47) that the three-dimensional position vector $\boldsymbol{r}$ moves at a fixed angle w.r.t. the axis of total angular momentum, and the geodesic is restricted to lie on the cone traced out by rays from the origin around $\boldsymbol{J}$ at a fixed opening angle

$$\beta = \arccos \frac{J^{(4)}}{|\boldsymbol{L}|}. \tag{5.51}$$

Obviously, this is a generalization of the motion being confined to the equatorial plane, corresponding to $\beta = \pi/2$ ($J^{(4)} = 0$), in three-dimensional central force fields, like in the Kepler problem. Again this testifies as to the non-trivial role of the ψ-momentum $p_\psi = J^{(4)}$ in the determination of the orbits.

5.4 Runge-Lenz vector and Yano tensors

It is possible to obtain more information about the nature of geodesics by observing that there exists a triplet of constants of motion $K^{(i)}$, $i = 1, 2, 3$, which are quadratic in the momenta, and which transform among themselves as a vector $\boldsymbol{K}$ under three-dimensional rotations [45]. This conserved vector is very similar to the Runge-Lenz vector in the Kepler problem. It is conserved along geodesics, and for $J^{(4)} \neq 0$ it satisfies the identity

$$\boldsymbol{r} \cdot \left(\boldsymbol{K} \pm \frac{H\boldsymbol{J}}{J^{(4)}} \right) = \frac{1}{2} \left(\boldsymbol{J}^2 - J^{(4)2} \right). \tag{5.52}$$

As $(\boldsymbol{K}, H, \boldsymbol{J}, J^{(4)})$ are all constant, it follows that the three-dimensional position vector $\boldsymbol{r}$ is constrained to lie in a plane. We have already seen, that for $J^{(4)} \neq 0$ this vector takes values on a cone with opening angle β (5.51) around the axis of angular momentum. Combining these two results it follows, that the geodesics are actually conic sections obtained by intersecting the cone (5.51) with the plane (5.52).

It remains to construct the vector $\boldsymbol{K}$ and prove the relation (5.52). First it should be pointed out, that as they are quadratic functions of the momenta:

$$K^{(i)} = \frac{1}{2} K^{(i)\,\mu\nu}\, p_\mu p_\nu, \tag{5.53}$$

their existence is implied by a triplet of symmetric second rank Killing tensors $K^{(i)}_{\mu\nu}$ as in eq.(2.64):

$$D_{(\mu} K^{(i)}_{\nu\lambda)} = 0. \tag{5.54}$$

Such a triplet of Killing tensors can indeed be found. Instead of directly giving the explicit expressions, we construct them in terms of a set of simpler objects, known as Yano tensors [47]. A Yano tensor is an anti-symmetric tensor $f_{\mu\nu} = -f_{\nu\mu}$ satisfying the Killing-like equation

$$D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} = 0. \tag{5.55}$$

As a result, the covariant derivative of a Yano tensor is completely anti-symmetric:

$$D_\mu f_{\nu\lambda} = \frac{1}{3} \left(D_\mu f_{\nu\lambda} + D_\nu f_{\lambda\mu} + D_\lambda f_{\mu\nu} \right) = D_{[\mu} f_{\nu\lambda]}. \tag{5.56}$$

Given one or more Yano tensors $f^{(a)}_{\mu\nu}$ one can construct symmetric Killing tensors simply by symmetrized multiplication:

$$K^{(ab)}_{\mu\nu} = \frac{1}{2} \left(f^{(a)\,\lambda}_{\mu} f^{(b)}_{\lambda\nu} + f^{(b)\,\lambda}_{\mu} f^{(a)}_{\lambda\nu} \right). \tag{5.57}$$

These symmetric tensors satisfy eq.(5.54) by virtue of the Yano condition (5.55). It turns out that the Taub-NUT manifold admits four such Yano tensors [46], transforming as a singlet Y and as a triplet f_i under three-dimensional rotations. Their explicit expressions are given by the two-forms

$$\begin{aligned} Y &= 4(d\psi + \cos\theta\, d\varphi) \wedge dr \,+\, 2r(r \pm 1)(r \pm 2)\, \sin\theta\, d\theta \wedge d\varphi, \\ f_i &= -\,\varepsilon_{ijk}\,(1 \pm \frac{2}{r})\, dx_j \wedge dx_k \,\pm\, 4\,(d\psi + \cos\theta\, d\varphi)\, \wedge dx_i. \end{aligned} \tag{5.58}$$

A straightforward calculation shows, that the covariant exterior derivative of these two forms gives:

$$\begin{aligned} \nabla Y &= D_\mu Y_{\nu\lambda}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \,=\, r(r \pm 2)\, \sin\theta\, dr \wedge d\theta \wedge d\varphi, \\ \nabla f_i &= 0. \end{aligned} \tag{5.59}$$

By studying the individual terms, one establishes the stronger results

$$D_\mu Y_{\nu\lambda} = D_\lambda Y_{\mu\nu} = D_\nu Y_{\lambda\mu}, \tag{5.60}$$

and

$$D_\mu f_{i\nu\lambda} = 0. \tag{5.61}$$

Thus both two-forms indeed obey the Yano condition (5.55), with the triplet of tensors f_i actually being covariantly constant. It follows immediately that the triplet of symmetric tensors, defined by

$$K_{i\,\mu\nu} \,=\, \frac{1}{2}\, \left(Y_\mu^{\ \lambda} f_{i\,\lambda\nu} + f_{i\,\mu}^{\ \ \lambda} Y_{\lambda\nu} \right), \tag{5.62}$$

form a vector of Killing tensors. For this triplet of Killing tensors eq.(5.52) can be proven. First note, that as a two-form

$$\boldsymbol{r} \cdot \boldsymbol{f} \,=\, -\,r^2(r \pm 2)\, \sin\theta\, d\theta \wedge d\varphi \,\pm\, 4r(d\psi + \cos\theta\, d\varphi)\, \wedge dr. \tag{5.63}$$

Performing matrix multiplication and contracting with the momenta to get an expression of the form (5.53), it follows that

$$\begin{aligned}
\boldsymbol{r}\cdot\boldsymbol{K} &= \boldsymbol{r}\cdot(\boldsymbol{f}\,Y) = \boldsymbol{r}\cdot(Y\,\boldsymbol{f}) = \frac{1}{2}\,(\boldsymbol{r}\cdot\boldsymbol{K})^{\mu\nu}\;p_\mu p_\nu \\
&= \mp\,\frac{r^2}{2(r\pm 2)}\,p_r^2 \mp \frac{1}{8}\,(r\pm 2)\,p_\psi^2 + \frac{1}{2}\left(\frac{r\pm 1}{r\pm 2}\right)\left[p_\theta^2 + \frac{1}{\sin^2\theta}\,(p_\varphi - \cos\theta\,p_\psi)^2\right] \\
&= \mp\,rH + \frac{1}{2}\left(\boldsymbol{J}^2 - J^{(4)\,2}\right).
\end{aligned} \tag{5.64}$$

Using eq.(5.47) this gives the desired result (5.52). Note, that this equation remains true also for $J^{(4)} = 0$, in which case the geodesic flow in the three-dimensional (r,θ,φ) space is a along a hyperbola in the plane perpendicular to $\boldsymbol{J}$. If $\boldsymbol{J}$is along the z-axis ($\theta = \pi/2$), then the vanishing of p_ψ implies that at the same time ψ is constant.

The constants of geodesic flow constructed above may be interpreted as an analogue of the Runge-Lenz vector for the Taub-NUT geometry, because they can be written in the manifest three-vector form

$$\boldsymbol{K} = \frac{1}{2}\left(1 \pm \frac{2}{r}\right)\,\dot{\boldsymbol{r}}\times\boldsymbol{J} \mp \hat{\boldsymbol{r}}\left(H - \frac{1}{4}\,J^{(4)\,2}\right) \mp \frac{1}{4}\,J^{(4)}\boldsymbol{J}. \tag{5.65}$$

Finally we comment upon the existence of the covariantly constant two-forms f_i. They have the property that as anti-symmetric two-tensors each of them squares to minus the identity. In fact, they realize the quaternion algebra

$$f_i f_j = -\delta_{ij} + \varepsilon_{ijk} f_k. \tag{5.66}$$

Thus the two-forms f_i define a triplet of almost complex structures, and the four-dimensional real geometry of the Taub-NUT manifold can be recast in terms of analytic complex or quaternionic geometry. Indeed, the Taub-NUT manifold belongs to the class of complex manifolds known as hyper-Kähler manifolds. An important consequence of being Kähler or hyper-Kähler is, that these manifolds have $N = 2$, resp. $N = 4$ supersymmetric extensions [48, 49, 50]. Such supersymmetries give much additional insight into the properties of manifolds like Taub-NUT [51].

Appendix

In this appendix I have collected the components of the connection and curvature tensor of the Taub-NUT manifold in the coordinate system in which the metric takes the form (5.37), with $m^2 = 1$. The non-zero components of the Riemann-Christoffel connection $\Gamma_{\mu\nu}{}^{\lambda}$ are:

$$\begin{aligned}
&\Gamma_{rr}{}^{r} = \frac{\mp 1}{r(r \pm 2)}, && \Gamma_{\theta\theta}{}^{r} = -\frac{r(r \pm 1)}{(r \pm 2)}, \\
&\Gamma_{r\theta}{}^{\theta} = \Gamma_{\theta r}{}^{\theta} = \frac{r \pm 1}{r(r \pm 2)}, && \Gamma_{\theta\varphi}{}^{\varphi} = \Gamma_{\varphi\theta}{}^{\varphi} = \cot\theta \left(1 - \frac{2}{(r \pm 2)^2}\right), \\
&\Gamma_{r\varphi}{}^{\varphi} = \Gamma_{\varphi r}{}^{\varphi} = \frac{r \pm 1}{r(r \pm 2)}, && \Gamma_{\theta\psi}{}^{\varphi} = \Gamma_{\psi\theta}{}^{\varphi} = \frac{-2}{\sin\theta (r \pm 2)^2}, \\
&\Gamma_{r\varphi}{}^{\psi} = \Gamma_{\varphi r}{}^{\psi} = -\frac{\cos\theta}{(r \pm 2)}, && \Gamma_{\theta\varphi}{}^{\psi} = \Gamma_{\varphi\theta}{}^{\psi} = -\frac{1}{2}\sin\theta - \frac{\cos^2\theta}{\sin\theta}\left(1 - \frac{2}{(r \pm 2)^2}\right), \\
&\Gamma_{r\psi}{}^{\psi} = \Gamma_{\psi r}{}^{\psi} = \frac{\pm 1}{r(r \pm 2)}, && \Gamma_{\theta\psi}{}^{\psi} = \Gamma_{\psi\theta}{}^{\psi} = \frac{2\cot\theta}{(r \pm 2)^2},
\end{aligned} \tag{5.67}$$

and

$$\begin{aligned}
&\Gamma_{\varphi\varphi}{}^{r} = \mp\frac{4r}{(r \mp 2)^3}\cos^2\theta - \frac{r(r \pm 1)}{(r \pm 2)}\sin^2\theta, && \Gamma_{\varphi\varphi}{}^{\theta} = -\sin\theta\cos\theta\left(1 - \frac{4}{(r \pm 2)^2}\right), \\
&\Gamma_{\varphi\psi}{}^{r} = \Gamma_{\psi\varphi}{}^{r} = \mp\frac{4r\cos\theta}{(r \pm 2)^3}, && \Gamma_{\varphi\psi}{}^{\theta} = \Gamma_{\psi\varphi}{}^{\theta} = \frac{2\sin\theta}{(r \pm 2)^2}, \\
&\Gamma_{\psi\psi}{}^{r} = \mp\frac{4r}{(r \pm 2)^3}.
\end{aligned} \tag{5.68}$$

From these expressions one can compute the components of the Riemann curvature tensor

$$R_{\mu\nu\kappa}{}^{\lambda} = \partial_\mu \Gamma_{\nu\kappa}{}^{\lambda} - \partial_\nu \Gamma_{\mu\kappa}{}^{\lambda} - [\Gamma_\mu, \Gamma_\nu]_\kappa{}^{\lambda}.$$

To allow easy check of the (anti) self-duality (5.34), we give here the completely covariant components $R_{\mu\nu\kappa\lambda}$; the non-vanishing ones are

$$
\begin{aligned}
R_{r\theta r\theta} &= \frac{\pm 1}{(r \pm 2)}, & R_{r\varphi r\varphi} &= \pm\frac{\sin^2\theta}{(r\pm 2)}\left(1 - \frac{8\cot^2\theta}{(r\pm 2)^2}\right),\\
R_{r\varphi r\psi} &= \mp\frac{8\cos\theta}{(r\pm 2)^3}, & R_{r\psi r\psi} &= \mp\frac{8}{(r\pm 2)^3},\\
R_{r\varphi\theta\varphi} &= \frac{6r}{(r\pm 2)^2}\,\sin\theta\cos\theta, & R_{r\psi\theta\varphi} &= \frac{4r\sin\theta}{(r\pm 2)^2},\\
R_{r\varphi\theta\psi} &= \frac{2r\sin\theta}{(r\pm 2)^2}, & R_{r\theta\varphi\psi} &= -\frac{2r\sin\theta}{(r\pm 2)^2},\\
R_{\theta\varphi\theta\varphi} &= \mp\frac{2r^2\sin^2\theta}{(r\pm 2)}\left(1 - \frac{2\cot^2\theta}{(r\pm 2)^2}\right), & R_{\theta\psi\theta\psi} &= \pm\frac{4r^2}{(r\pm 2)^3},\\
R_{\theta\varphi\theta\psi} &= \pm\frac{4r^2\cos\theta}{(r\pm 2)^3}, & R_{\varphi\psi\varphi\psi} &= \pm\frac{4r^2\sin^2\theta}{(r\pm 2)^3}.
\end{aligned}
\tag{5.69}
$$

All other components not related by permutation symmetry of the indices are zero.

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