An Introduction to Supersymmetry

Ulrich Theis

Institute for Theoretical Physics, Friedrich-Schiller-University Jena,
Max-Wien-Platz 1, D–07743 Jena, Germany
Ulrich.Theis@uni-jena.de

This is a write-up of a series of five introductory lectures on global supersymmetry in four dimensions given at the 13th "Saalburg" Summer School 2007 in Wolfersdorf, Germany.

Contents

1 Why supersymmetry?

When the Large Hadron Collider at CERN takes up operations soon, its main objective, besides confirming the existence of the Higgs boson, will be to discover new physics beyond the standard model of the strong and electroweak interactions. It is widely believed that what will be found is a (at energies accessible to the LHC softly broken) supersymmetric extension of the standard model. What makes supersymmetry such an attractive feature that the majority of the theoretical physics community is convinced of its existence?

First of all, under plausible assumptions on the properties of relativistic quantum field theories, supersymmetry is the unique extension of the algebra of Poincaré and internal symmtries of the S-matrix. If new physics is based on such an extension, it must be supersymmetric. Furthermore, the quantum properties of supersymmetric theories are much better under control than in non-supersymmetric ones, thanks to powerful non-renormalization theorems. The latter provide a solution to the problem that has plagued ordinary grand unified theories (GUTs), namely how to stabilize the large hierarchy of energy scales, ranging from the electroweak scale (10^2 GeV) to the GUT scale (10^{16} GeV) or the Planck scale (10^{19} GeV), against quantum corrections that drive the tree-level masses of bosonic particles to very high values, in contradiction to experimental bounds.[1]
Let us explain this problem and how it can be overcome by supersymmetry by studying a simple example. Consider the following Lagrangian for a complex scalar field $\varphi(x)$ and a Weyl spinor $\psi(x)$ (more about those in the next section):

$$\begin{aligned}\mathcal{L} = &-\partial_\mu \bar\varphi\, \partial^\mu \varphi - \mathrm{i}\bar\psi\bar\sigma^\mu\partial_\mu\psi - m_\varphi^2\bar\varphi\varphi - \tfrac{1}{2}m_\psi(\psi\psi + \bar\psi\bar\psi) \\ &- g(\varphi\psi\psi + \bar\varphi\bar\psi\bar\psi) - \mu(\varphi\bar\varphi^2 + \bar\varphi\varphi^2) - \lambda^2(\bar\varphi\varphi)^2 \ . \end{aligned} \tag{1.1}$$

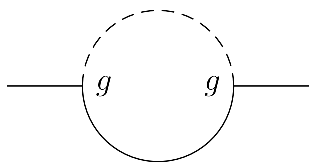
As we will show later, the corresponding action is supersymmetric (and known as the Wess–Zumino model in this case) if the masses and coupling constants satisfy the relations

$$m_\varphi = m_\psi = m \ , \quad \lambda = g \ , \quad \mu = mg \ , \tag{1.2}$$

but let us not assume these to hold for the time being.
If $m_\psi = \mu = 0$, i.e., when the tree-level fermionic mass terms and cubic scalar couplings are absent, the model is invariant under the chiral symmetry

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi \quad \bar\psi \to \mathrm{e}^{-\mathrm{i}\alpha}\bar\psi \ , \quad \varphi \to \mathrm{e}^{-2\mathrm{i}\alpha}\varphi \ , \quad \bar\varphi \to \mathrm{e}^{2\mathrm{i}\alpha}\bar\varphi \ . \tag{1.3}$$

This symmetry guarantees that no fermionic masses will be generated by perturbative quantum corrections. Indeed, for $m_\psi \neq 0$ the one-loop correction to m_ψ arises from the Feynman diagram



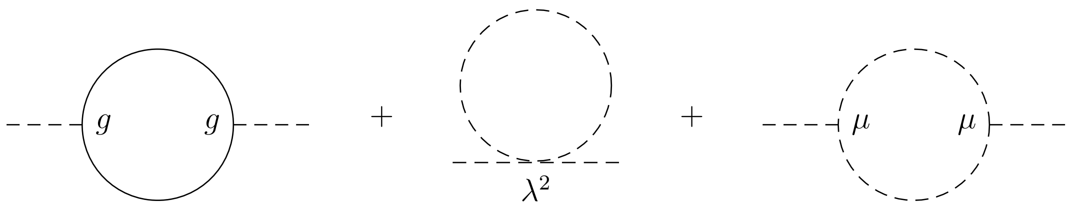
which gives

$$\delta m_\psi \ \propto \ g^2 \int_0^\Lambda d^4k \ \frac{m_\psi}{(k^2+m_\psi^2)(k^2+m_\varphi^2)} \ \propto \ g^2 m_\psi \log(\Lambda^2/m^2) \ , \tag{1.4}$$

[1]Additional benefits such as gauge coupling unification and natural cold dark matter candidates will not be covered in these lectures.

where Λ is an ultraviolet momentum cut-off that regularizes the divergent integral. We learn that the mass correction is proportional to the tree-level mass and diverges only logarithmically. It therefore stays small for small m_ψ and vanishes if the fermions are massless at tree-level, i.e., when the chiral symmetry is present. In this sense fermion masses are natural, as no further fine-tuning is needed to keep them small if this property has been implemented by hand in the classical theory. Small fermion masses are protected by an approximate chiral symmetry.

The situation is completely different for the scalar masses. Their one-loop corrections arise from the diagrams



which correspond to

$$\delta m_\varphi^2 \propto \int_0^\Lambda d^4k \left[-g^2 \frac{k^2 - m_\psi^2}{(k^2 + m_\psi)^2} + \lambda^2 \frac{1}{k^2 + m_\varphi^2} - 2\mu^2 \frac{1}{(k^2 + m_\varphi^2)^2} \right]$$
$$\propto (\lambda^2 - g^2)\Lambda^2 + \text{log. divergent} \ . \tag{1.5}$$

The squared scalar masses are quadratically divergent and receive large corrections even if $m_\varphi = 0$ at tree-level. This is a problem in GUTs, where initially small Higgs masses are driven up to the GUT scale unless one carefully fine-tunes the bare masses and coupling constants to avoid this scenario. For this reason scalar masses are called unnatural.

But the expression for δm_φ^2 suggests a way out: If the Yukawa and quartic scalar coupling constants are equal, $\lambda = g$, the quadratic divergence is absent! Indeed, if all of the relations (1.2) are satisfied, the logarithmically divergent corrections vanish as well, and we have $\delta m_\varphi^2 = 0$ at one-loop order! This is no fine-tuning, as these relations are naturally enforced by supersymmetry. The latter requires an equal number of fermionic and bosonic degrees of freedom, such that the fermion loop, coming with a minus sign due to the anticommuting nature of the spinor fields, precisely cancels the boson loops. In this way supersymmetry can stabilize the small scale of electroweak symmetry breaking against the much larger scale where grand unification occurs, thereby solving the so-called hierarchy problem.[2]

Our argument relied on a one-loop analysis, but the conclusions derived above remain true to all orders in perturbation theory thanks to the aforementioned non-renormalization theorems in $N = 1$ supersymmetry, which will be the topic of the last lecture.

[2]This argument does *not* explain where the hierarchy of scales is coming from in the first place. One should therefore better speak of the naturalness problem.

2 Weyl spinors in D=4

The superspace approach to supersymmetric theories in four spacetime dimensions is most conveniently formulated in terms of Weyl spinors. As these are usually not covered in introductory courses on quantum field theory, we shall briefly list their properties.
In our conventions, the Dirac γ-matrices γ^μ with $\mu = 0, \ldots, 3$ satisfy the Clifford algebra

$$\{\gamma_\mu \, , \, \gamma_\nu\} = 2\, \eta_{\mu\nu} \mathbb{1} \ , \tag{2.1}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the Minowski metric. A particular representation of this algebra is the Weyl representation

$$\gamma^\mu = \mathrm{i} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ . \tag{2.2}$$

Here, the σ-matrices are given by

$$\sigma^\mu = (-\mathbb{1}, \vec{\tau}\,) \ , \qquad \bar{\sigma}^\mu = (-\mathbb{1}, -\vec{\tau}\,) \ , \tag{2.3}$$

where $\vec{\tau}$ are the three Pauli matrices. In the Weyl representation, we have

$$\gamma_5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \ , \tag{2.4}$$

such that Dirac spinors Ψ_D decompose into left- and right-handed two-component spinors with respect to the projectors $P_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma_5)$:

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \ . \tag{2.5}$$

Dotted and undotted Greek indices from the beginning of the alphabet run from 1 to 2. The Weyl spinors χ and $\bar{\lambda}$ form irreducible (and inequivalent) representations of the universal covering $\mathrm{SL}(2, \mathbb{C})$ of the Lorentz group; infinitesimally, we have

$$M_{\mu\nu}\Psi_D = \tfrac{1}{4}[\,\gamma_\mu \, , \, \gamma_\nu\,]\,\Psi_D = -\begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix} \begin{pmatrix} \chi \\ \bar{\lambda} \end{pmatrix} \ . \tag{2.6}$$

Here, we have introduced matrices

$$\sigma^{\mu\nu} = \tfrac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \ , \qquad \bar{\sigma}^{\mu\nu} = \tfrac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \ , \tag{2.7}$$

satisfying the same commutation relations as the Lorentz generators $M_{\mu\nu}$, namely

$$[\,\sigma_{\mu\nu} \, , \, \sigma_{\varrho\sigma}\,] = \eta_{\mu\varrho}\sigma_{\nu\sigma} - \eta_{\nu\varrho}\sigma_{\mu\sigma} + \eta_{\nu\sigma}\sigma_{\mu\varrho} - \eta_{\mu\sigma}\sigma_{\nu\varrho} \tag{2.8}$$

and analogously for $\bar{\sigma}_{\mu\nu}$. Using the SL$(2,\mathbb{C})$-invariant[3] ε-tensors

$$(\varepsilon^{\alpha\beta}) = -(\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{\dot\alpha\dot\beta}) = -(\varepsilon_{\dot\alpha\dot\beta}) \tag{2.9}$$

satisfying $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$, we can pull up and down spinor indices,

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta \ , \qquad \bar\lambda_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\bar\lambda^{\dot\beta} \ , \tag{2.10}$$

and form Lorentz invariants from two Weyl spinors of the same chirality:

$$\begin{aligned} \chi\psi &\equiv \chi^\alpha\psi_\alpha = \varepsilon^{\alpha\beta}\chi_\beta\psi_\alpha = -\varepsilon^{\alpha\beta}\chi_\alpha\psi_\beta = -\chi_\alpha\psi^\alpha = \psi^\alpha\chi_\alpha = \psi\chi \\ \bar\lambda\bar\psi &\equiv \bar\lambda_{\dot\alpha}\bar\psi^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\bar\lambda^{\dot\beta}\bar\psi^{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\,\bar\lambda^{\dot\alpha}\bar\psi^{\dot\beta} = -\bar\lambda^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\psi_{\dot\alpha}\bar\lambda^{\dot\alpha} = \bar\psi\bar\lambda \ . \end{aligned} \tag{2.11}$$

Here we have assumed that the spinors anticommute.
Complex conjugation reverses the order of fields and turns left-handed spinors into right-handed ones and vice versa. For example

$$(\chi\psi)^* = (\chi^\alpha\psi_\alpha)^* = (\psi_\alpha)^*(\chi^\alpha)^* = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\psi\bar\chi = \bar\chi\bar\psi \ . \tag{2.12}$$

In the Weyl representation, Majorana spinors are of the form

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \quad \text{with} \quad \bar\chi^{\dot\alpha} = (\chi^\alpha)^* \ . \tag{2.13}$$

The free action for a massive Majorana spinor then reads

$$\mathcal{L}_0 = -\tfrac{1}{2}\bar\Psi_M(\gamma^\mu\partial_\mu + m)\Psi_M = -\tfrac{\mathrm{i}}{2}\chi\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\chi - \tfrac{1}{2}m(\chi\chi + \bar\chi\bar\chi) \ , \tag{2.14}$$

where $\bar\Psi_M = \mathrm{i}\Psi_M^\dagger\gamma^0 = (\chi^\alpha, \bar\chi_{\dot\alpha})$ and $A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - \partial_\mu A B$.
Numerous identities satisfied by the σ-matrices can be found in appendix A.
Exercise: *Use the Schwinger–Dyson equation* $\langle \delta S/\delta\phi^i(x)\,\phi^j(x)\rangle = \mathrm{i}\delta^j_i\,\delta(x-y)$ *in showing that for the Lagrangian* (1.1) *subject to the conditions* (1.2) *the fermion propagators are given by*

$$\big\langle \psi_\alpha(x)\,\bar\psi_{\dot\alpha}(y)\big\rangle_0 = \frac{1}{\Box - m^2}\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\delta(x-y) \ , \quad \big\langle \psi_\alpha(x)\,\psi^\beta(y)\big\rangle_0 = \frac{\mathrm{i}m}{\Box - m^2}\,\delta^\beta_\alpha\,\delta(x-y) \ .$$

Verify the tree-level Ward identities

$$\big\langle \delta_\xi\big(\psi_\alpha(x)\,\bar\varphi(y)\big)\big\rangle_0 = \big\langle \delta_\xi\big(\varphi(x)\,\psi^\beta(y)\big)\big\rangle_0 = 0$$

for the infinitesimal supersymmetry transformations

$$\delta_\xi\varphi = -\xi\psi \ , \qquad \delta_\xi\psi_\alpha = \xi_\alpha(m\bar\varphi + g\bar\varphi^2) - \mathrm{i}(\sigma^\mu\bar\xi)_\alpha\partial_\mu\varphi \ .$$

[3] For $M \in \mathrm{SL}(2,\mathbb{C})$ it is $M\varepsilon M^t = \det M\,\varepsilon = \varepsilon$.

3 The supersymmetry algebra

Drawing on the Coleman–Mandula theorem, which states that the most general algebra of bosonic symmetry generators of the S-matrix is given by the direct sum of the Poincaré algebra and a compact Lie algebra of internal symmetries, Haag, Lopuszanski and Sohnius proved that the unique[4] extension including fermionic generators with anticommutation relations consists of the supersymmetry (or super-Poincaré) algebra:

$$\{Q^i_\alpha\,,\,\bar{Q}_{\dot\alpha j}\} = 2\delta^i_j\sigma^\mu_{\alpha\dot\alpha}P_\mu\ ,\quad \{Q^i_\alpha\,,\,Q^j_\beta\} = \varepsilon_{\alpha\beta}Z^{ij}\ ,\quad \{\bar{Q}_{\dot\alpha i}\,,\,\bar{Q}_{\dot\beta j}\} = \varepsilon_{\dot\alpha\dot\beta}\bar{Z}_{ij}\ , \tag{3.1}$$

with indices i, j ranging from 1 to some number N. For $N > 1$ one speaks of N-extended supersymmtry. The $4N$ supercharges $Q^i_\alpha = (\bar{Q}_{\dot\alpha i})^\dagger$ are Weyl spinors,

$$[\,M_{\mu\nu}\,,\,Q^i_\alpha\,] = \mathrm{i}\sigma_{\mu\nu\alpha}{}^\beta Q^i_\beta\ ,\qquad [\,M_{\mu\nu}\,,\,\bar{Q}_{\dot\alpha i}\,] = -\mathrm{i}\bar{Q}_{\dot\beta i}\,\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\alpha}\ , \tag{3.2}$$

cf. (2.6), and translation-invariant:

$$[\,P_\mu\,,\,Q^i_\alpha\,] = [\,P_\mu\,,\,\bar{Q}_{\dot\alpha i}\,] = 0\ . \tag{3.3}$$

The so-called central charges $Z^{ij} = -Z^{ji} = (\bar{Z}_{ij})^\dagger$ can occur only for $N > 1$ and commute among themselves and with all other generators:

$$[\,Z^{ij}\,,\,\text{everything}\,] = [\,\bar{Z}_{ij}\,,\,\text{everything}\,] = 0\ . \tag{3.4}$$

The supersymmetry algebra (3.1) has an outer automorphism group $\mathrm{U}(N)$, under which the supercharges and central charges transform according to their index structure:

$$Q^i_\alpha \to U^i{}_j\,Q^j_\alpha\ ,\quad \bar{Q}_{\dot\alpha i} \to \bar{Q}_{\dot\alpha j}\,(U^{-1})^j{}_i\ ,\quad Z^{ij} \to U^i{}_k\,U^j{}_\ell\,Z^{k\ell}\ . \tag{3.5}$$

This automorphism group is being referred to as R-symmetry.

Exercise: *The graded commutator* $[\,A\,,B\}=AB-(-)^{|A||B|}BA$*, where* $|A|=0$ *if* A *is bosonic and* $|A|=1$ *if* A *is fermionic, satisfies the Bianchi identities*

$$(-)^{|A||C|}\,[\,A\,,[\,B\,,C\}\}+(-)^{|B||A|}\,[\,B\,,[\,C\,,A\}\}+(-)^{|C||B|}\,[\,C\,,[\,A\,,B\}\}=0\ .$$

Use this to determine from the supersymmetry algebra the commutator $[\,M_{\mu\nu}\,,\,P_\varrho\,]$.

4 Supersymmetry multiplets

In this section we construct irreducible representations of the super-Poincaré algebra using Wigner's method of induced representations. This consists of first constructing representations of the little group leaving a given momentum vector invariant and then applying Lorentz boosts to obtain representations of the full super-Poincaré group. Note

[4] In the massless case it can be further extended to the superconformal algebra.

that as a consequence of $[\,Q\,,\,P^2\,]=0$ states in multiplets of unbroken supersymmetry are degenerate in mass. We shall treat only massless states in some detail; we briefly comment on massive states and central charges at the end of this section.
Let us consider one-particle states with fixed light-like momentum $p^\mu=(E,0,0,E)$. The little group of p^μ is generated by the operators

$$N_1=M_{10}+M_{13}\ ,\qquad N_2=M_{20}+M_{23}\ ,\qquad J_3=M_{12}\ . \tag{4.1}$$

They span the algebra E_2. N_1 and N_2 being non-compact generators, they must be trivially realized in any finite-dimensional unitary representation. Thus, we can classify states by their J_3 eigenvalues λ, the helicities, which are half-integers:

$$J_3|\lambda\rangle=\lambda|\lambda\rangle\ ,\quad \lambda\in\tfrac{1}{2}\mathbb{Z}\ . \tag{4.2}$$

From the commutator

$$[\,J_3\,,\,\bar{Q}_{\dot\alpha i}\,]=-\mathrm{i}\bar{Q}_{\dot\beta i}\,\bar\sigma_{12}{}^{\dot\beta}{}_{\dot\alpha}=-\tfrac{1}{2}\bar{Q}_{\dot\beta i}\,\tau^{3\dot\beta}{}_{\dot\alpha} \tag{4.3}$$

we infer that $\bar{Q}_{\dot 1 i}$ lowers the helicity by $\frac{1}{2}$, while $\bar{Q}_{\dot 2 i}$ raises it by the same amount.
On states with momentum p^μ, we have

$$\{Q^i_\alpha\,,\,\bar{Q}_{\dot\alpha j}\}=2E\,\delta^i_j(\sigma^3-\sigma^0)_{\alpha\dot\alpha}=4E\,\delta^i_j\begin{pmatrix}1&0\\0&0\end{pmatrix}\ , \tag{4.4}$$

implying that Q^i_2 and its hermitean conjugate $\bar{Q}_{\dot 2 i}$ must be trivially realized. From (3.1) it then follows that the central charges Z^{ij} vanish:

$$\{Q^i_\alpha\,,\,Q^j_\beta\}=0\ ,\quad \{\bar{Q}_{\dot\alpha i}\,,\,\bar{Q}_{\dot\beta j}\}=0\ . \tag{4.5}$$

Thus, $\bar{Q}_{\dot 1 i}$ and Q^i_1 satisfy the same anticommutation relations as N copies of fermionic creation and annihilation operators, respectively, familiar from quantum mechanics. We can build multiplets by successive application of creation operators on a highest weight state $|\lambda\rangle$ with

$$Q^i_1|\lambda\rangle=Q^i_2|\lambda\rangle=\bar{Q}_{\dot 2 i}|\lambda\rangle=0\ . \tag{4.6}$$

For simplicity, we take $|\lambda\rangle$ to be a singlet of the R-symmetry group. We can then apply up to N different creation operators $\bar{Q}_{\dot 1 i}$ to $|\lambda\rangle$; monomials of more than N such operators vanish identically due to their nilpotency. In this way, we obtain the following states:

states	helicity	# components
$\|\lambda\rangle$	λ	1
$\bar{Q}_{\dot 1 i}\|\lambda\rangle$	$\lambda-1/2$	N
$\vdots$	$\vdots$	$\vdots$
$\bar{Q}_{\dot 1 i_1}\ldots\bar{Q}_{\dot 1 i_k}\|\lambda\rangle$	$\lambda-k/2$	$\binom{N}{k}$
$\vdots$	$\vdots$	$\vdots$
$\bar{Q}_{\dot 1 1}\ldots\bar{Q}_{\dot 1 N}\|\lambda\rangle$	$\lambda-N/2$	1

The total number of states is $\sum_{k=0}^{N} \binom{N}{k} = 2^N$. However, if we want to realize such a multiplet in a relativistic quantum field theory, we in general have to add the CPT conjugated states with opposite chiralities in order to achieve CPT invariance. Only when $\lambda = N/4$, which requires N to be a multiple of 2, can the multiplet be CPT self-conjugate. It appears to be impossible to have interacting massless particles with helicities $|\lambda| > 2$, so we must impose the constraint

$$4 \geq \lambda_{\max} - \lambda_{\min} = N/2 \quad \Leftrightarrow \quad N \leq 8 \ . \tag{4.7}$$

In four dimensions one can have at most eight supersymmetries, corresponding to 32 real supercharges. This restriction actually applies in any number of spacetime dimensions.
Let us give some examples. The most important $N = 1$ multiplets are

$$\begin{aligned} &|\tfrac{1}{2}\rangle \, , \, |0\rangle \ \oplus \ |0\rangle \, , \, |-\tfrac{1}{2}\rangle \ , \\ &|1\rangle \, , \, |\tfrac{1}{2}\rangle \ \oplus \ |-\tfrac{1}{2}\rangle \, , \, |-1\rangle \ , \\ &|2\rangle \, , \, |\tfrac{3}{2}\rangle \ \oplus \ |-\tfrac{3}{2}\rangle \, , \, |-2\rangle \ , \end{aligned}$$

which correspond to the chiral, vector and supergravity multiplet, respectively. The construction of interacting quantum field theories for the first two will be the main subject of these lectures.
$N = 1$ chiral and vector multiplets can be combined into $N = 2$ vector multiplets:

$$|1\rangle \, , \, |\tfrac{1}{2}\rangle^2 \, , \, |0\rangle \ \oplus \ |0\rangle \, , \, |-\tfrac{1}{2}\rangle^2 \, , \, |-1\rangle \ ,$$

where superscripts denote the multiplicities of the states. However, at the field theory level this is possible only for very specific interactions between the $N = 1$ multiplets; in particular, all fields must transform in the adjoint representation of the gauge group.
The largest physically sensible, massless, irreducible representation is the $N = 8$ supergravity multiplet,

$$|2\rangle \, , \, |\tfrac{3}{2}\rangle^8 \, , \, |1\rangle^{28} \, , \, |\tfrac{1}{2}\rangle^{56} \, , \, |0\rangle^{70} \, , \, |-\tfrac{1}{2}\rangle^{56} \, , \, |-1\rangle^{28} \, , \, |-\tfrac{3}{2}\rangle^8 \, , \, |-2\rangle \ ,$$

which is CPT self-conjugate.

Exercise: *List the states contained in the massless $N = 3$ and $N = 4$ vector multiplets (with helicities $|\lambda| \leq 1$) and compare them.*

To close this section, let us briefly comment on massive multiplets. We can always choose a frame in which $p^\mu = (m, 0, 0, 0)$, whence

$$\{Q^i_\alpha \, , \, \bar{Q}_{\dot{\alpha} j}\} = 2m \, \delta^i_j \delta_{\alpha\dot{\alpha}} \ . \tag{4.8}$$

We now have $2N$ sets of fermionic creation and annihilation operators amd the multiplets are generically larger than in the massless case: for vanishing central charges they contain 2^{2N} states. In general the eigenvalues Z_r of the central charges are bounded by $|Z_r| \leq 2m$. The multiplets are shorter if for some r this so-called BPS bound is saturated.

5 Superspace and superfields

We now come to the construction of field theories that realize some of the above multiplets. We shall restrict ourselves to $N = 1$ supersymmetry for the remainder of these lectures; the R-symmetry index i can then be dropped. In particular, this means that there are no central charges.

Superfields $\phi(z)$ live in superspace $\mathbb{R}^{4|4}$, which extends ordinary Minkowski space $\mathbb{R}^4$ (with signature $\eta_{00} = -1$) by Grassmann-odd variables:

$$z^A = (x^\mu \,,\, \theta^\alpha \,,\, \bar{\theta}_{\dot{\alpha}}) \ . \tag{5.1}$$

The latter are related by complex conjugation via $(\theta^\alpha)^* = \bar{\theta}^{\dot{\alpha}}$. Note the position of the index of $z_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}$ as compared to $z^\alpha = \theta^\alpha$; this convention was chosen such that contractions like $z_1^A z_{2A}$ are real, cf. (2.12).

One can regard superspace as a coset space of the super-Poincaré group divided by the outer automorphism group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1)$. The variables z^A parametrize coset elements as

$$g(z) = \mathrm{e}^{-\mathrm{i}x^\mu P_\mu + \mathrm{i}\theta^\alpha Q_\alpha + \mathrm{i}\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} \ . \tag{5.2}$$

A supersymmetry transformation with Grassmann-odd parameters ξ^α, $\bar{\xi}_{\dot{\alpha}}$ corresponds to a motion in superspace induced by the left-multiplication

$$g(0, \xi, \bar{\xi})\, g(z) = g(z') \ , \tag{5.3}$$

which can be easily worked out using the Baker–Campbell–Hausdorff formula

$$\mathrm{e}^A\, \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[\,A\,,\,B\,]+\dots} \ . \tag{5.4}$$

Recall that the additional terms in the exponent on the right-hand side vanish if $[\,A\,,\,B\,]$ commutes with both A and B, which is the case here. Infinitesimally, we find for $\delta_\xi z^A = z'^A - z^A$ that

$$\delta_\xi x^\mu = \mathrm{i}(\theta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\theta}) \ , \quad \delta_\xi\theta^\alpha = \xi^\alpha \ , \quad \delta_\xi\bar{\theta}_{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}} \ . \tag{5.5}$$

Superfields $\phi(z) = g(z)\phi(0)g^{-1}(z)$ transform as scalars under such motions:

$$\phi'(z') = \phi(z) \ . \tag{5.6}$$

Infinitesimally, we then have

$$\delta_\xi\phi(z) = \phi'(z) - \phi(z) = \phi'(z' - \delta_\xi z) - \phi(z) = -\delta_\xi z^A \partial_A \phi(z) \ . \tag{5.7}$$

Plugging in the transformations of the coordinates and collecting terms proportional to ξ and $\bar{\xi}$, respectively, it follows that

$$\delta_\xi\phi(z) = \mathrm{i}(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})\, \phi(z) \tag{5.8}$$

with the supersymmetry charges being represented by differential operators

$$Q_\alpha = \mathrm{i}\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \qquad \bar{Q}_{\dot{\alpha}} = -\mathrm{i}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ . \tag{5.9}$$

They satisfy the anticommutation relations[5]

$$\{Q_\alpha\,,\,\bar{Q}_{\dot{\alpha}}\} = -2\mathrm{i}\,\partial_{\alpha\dot{\alpha}} \ , \quad \{Q_\alpha\,,\,Q_\beta\} = 0 = \{\bar{Q}_{\dot{\alpha}}\,,\,\bar{Q}_{\dot{\beta}}\} \ , \tag{5.10}$$

where the momentum operator is represented on superfields as $P_\mu = -\mathrm{i}\partial_\mu$. The commutator of two supersymmetry transformations yields a translation in x-direction:

$$[\,\delta_\xi\,,\,\delta_\zeta\,] = a^\mu\partial_\mu \quad \text{with} \quad a^\mu = 2\mathrm{i}(\xi\sigma^\mu\bar{\zeta} - \zeta\sigma^\mu\bar{\xi}) \ . \tag{5.11}$$

We will find that superfields in general describe reducible multiplets. In order to obtain irreducible representations, we have to impose suitable constraints on the superfields. These must neither restrict their spacetime dependence nor break supersymmetry. The latter is preserved if we formulate the constraints in terms of supercovariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \mathrm{i}(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \tag{5.12}$$

of the superfields. These are constructed such that they anticommute with Q_α and $\bar{Q}_{\dot{\alpha}}$ and therefore map superfields into superfields:

$$\delta_\xi D_\alpha\phi(z) = D_\alpha\delta_\xi\phi(z) = \mathrm{i}D_\alpha(\xi Q + \bar{\xi}\bar{Q})\phi(z) = \mathrm{i}(\xi Q + \bar{\xi}\bar{Q})\,D_\alpha\phi(z) \ .$$

They satisfy the same algebra as the supercharges:

$$\{D_\alpha\,,\,\bar{D}_{\dot{\alpha}}\} = -2\mathrm{i}\,\partial_{\alpha\dot{\alpha}} \ , \quad \{D_\alpha\,,\,D_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}\,,\,\bar{D}_{\dot{\beta}}\} \ . \tag{5.13}$$

Since the θ's anticommute and spinor indices take only two different values, third and higher powers vanish,

$$\theta^\alpha\theta^\beta\theta^\gamma = 0 \ , \quad \bar{\theta}_{\dot{\alpha}}\,\bar{\theta}_{\dot{\beta}}\,\bar{\theta}_{\dot{\gamma}} = 0 \ . \tag{5.14}$$

Thus, superfields are polynomials in these variables. Using the identities

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta^\gamma\theta_\gamma \equiv -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta^2 \ , \quad \bar{\theta}_{\dot{\alpha}}\,\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\theta}_{\dot{\gamma}}\bar{\theta}^{\dot{\gamma}} \equiv -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\theta}^2 \ , \tag{5.15}$$

we find that the most general superfield is of the form

$$\begin{aligned}\phi(z) = {} & \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta^2F(x) + \bar{\theta}^2G(x) \\ & + \theta^2\,\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\,\theta\eta(x) + \theta^2\,\bar{\theta}^2D(x) \ .\end{aligned} \tag{5.16}$$

Note that ϕ may carry unwritten spacetime, spinor, and gauge indices. The x-dependent coefficient functions in the θ-expansion of a superfield constitute a supersymmetry multiplet. They can be regarded as the lowest components of superfields obtained by applying

[5] We are using the notation $V_{\alpha\dot{\alpha}} \equiv \sigma^\mu_{\alpha\dot{\alpha}}V_\mu$ for arbitrary vectors V_μ here.

appropriate polynomials in supercovariant derivatives to $\phi(z)$. For $\phi(z)$ as above we have for instance

$$\varphi(x) = \phi(z)|_{\theta=\bar\theta=0} \ , \quad \psi_\alpha(x) = D_\alpha \phi(z)|_{\theta=\bar\theta=0} \ , \quad A_{\alpha\dot\alpha}(x) = \tfrac{1}{2}[\, D_\alpha \, , \, \bar D_{\dot\alpha} \,]\phi(z)|_{\theta=\bar\theta=0} \ . \tag{5.17}$$

Their supersymmetry transformations are obtained from the θ-expansion of the transformed superfield:[6]

$$\delta_\xi \phi(z) \equiv \delta_\xi \varphi(x) + \theta\, \delta_\xi \psi(x) + \bar\theta\, \delta_\xi \bar\chi(x) + \theta\sigma^\mu\bar\theta\, \delta_\xi A_\mu(x) + \dots \ . \tag{5.18}$$

The fact that the supercovariant derivatives differ from the supersymmetry generators $-\mathrm{i}Q_\alpha$ and $-\mathrm{i}\bar Q_{\dot\alpha}$ only by θ-dependent terms allows us to replace the latter by D_α and $\bar D_{\dot\alpha}$, respectively, under the projection to $\theta = \bar\theta = 0$. We can therefore determine the transformations of the components (5.17) by applying the operator $-(\xi D + \bar\xi\bar D)$ under the projection and then using the commutation relations to express the result in terms of the other components. For example, we find in this way that

$$\delta_\xi \varphi(x) = \mathrm{i}(\xi Q + \bar\xi \bar Q)\, \phi(z)|_{\theta=\bar\theta=0} = -(\xi D + \bar\xi\bar D)\, \phi(z)|_{\theta=\bar\theta=0} = -\xi\psi(x) - \bar\xi\bar\chi(x)$$

and

$$\begin{aligned} \delta_\xi \psi_\alpha(x) &= -(\xi D + \bar\xi\bar D)\, D_\alpha \phi(z)|_{\theta=\bar\theta=0} \\ &= \tfrac{1}{2}\big(\varepsilon_{\alpha\beta}\,\xi^\beta D^2 + \bar\xi^{\dot\alpha}[\, \bar D_{\dot\alpha} \, , \, D_\alpha \,] + \bar\xi^{\dot\alpha}\{\bar D_{\dot\alpha} \, , \, D_\alpha\}\big)\, \phi(z)|_{\theta=\bar\theta=0} \\ &= -2\xi_\alpha F(x) - (\sigma^\mu\bar\xi)_\alpha (A_\mu + \mathrm{i}\partial_\mu\varphi)(x) \ . \end{aligned}$$

From the lowest component $\varphi(x) = \phi(x, 0, 0)$ of a superfield the latter can be recovered by exponentiating component supersymmetry transformations with θ's as parameters:

$$\phi(z) = \exp(-\delta_\theta)\, \varphi(x) = \sum_{n=0}^{4} \frac{1}{n!}\, (-\delta_\theta)^n \varphi(x) \ . \tag{5.19}$$

More generally, from every supersymmetry multiplet superfields may be constructed by applying the operator $\exp(-\delta_\theta)$ to any component field.

6 Superspace integration

Just like translation-invariant actions can be constructed from spacetime integrals over scalar Lagrangians, we obtain supersymmetric actions from integrals of superfield Lagrangians over superspace. These will automatically be invariant if the superspace integral is invariant under supertranslations. This is the case for the Berezin integral: any

[6] Note the abuse of notation here: the operator δ_ξ on the left-hand side acts on superspace variables, whereas δ_ξ on the right-hand side does not. We refrain from using two different symbols in the hope that no confusion arises.

function of a single Grassmann variable θ can be expanded as $f(\theta) = f_0 + \theta f_1$, where the components may depend on additional bosonic and fermionic variables. We then define

$$\int d\theta\, f(\theta) \equiv f_1 = \frac{\partial}{\partial\theta} f(\theta) \ . \tag{6.1}$$

This integral has the curious property that it acts just like differentiation. It is indeed invariant under translations in θ:

$$\int d\theta\, f(\theta+\xi) = \int d\theta\, \big(f_0 + \theta f_1 + \xi f_1\big) = f_1 \ .$$

If we define measures $d^2\theta$ and $d^2\bar\theta$ such that

$$\int d^2\theta\ \theta^2 = 1\ , \quad \int d^2\bar\theta\ \bar\theta^2 = 1\ , \tag{6.2}$$

then the integration over θ^α and $\bar\theta_{\dot\alpha}$ projects to the highest component of a superfield. For $\phi(z)$ as in (5.16), we have

$$\int d^2\theta\, \phi(z) = \phi(z)|_{\theta^2} = F(x) + \bar\theta\bar\lambda(x) + \bar\theta^2 D(x)\ ,$$
$$\int d^2\theta\, d^2\bar\theta\, \phi(z) = \phi(z)|_{\theta^2\bar\theta^2} = D(x)\ . \tag{6.3}$$

The integral of an arbitrary superfield over the full superspace

$$\int d^8z\, \phi(z) \equiv \int d^4x\, d^2\theta\, d^2\bar\theta\, \phi(z) \tag{6.4}$$

is manifestly supersymmetric thanks to its translation invariance:

$$\delta_\xi \int d^8z\, \phi(z) = -\int d^8z\ \delta_\xi z^A \partial_A \phi(z) = -\int d^8z\, \Big(\delta_\xi x^\mu \partial_\mu + \xi^\alpha \frac{\partial}{\partial\theta^\alpha} + \bar\xi^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\Big)\phi(z) = 0\ .$$

Since the highest component of a real superfield integrand is usually denoted by D, this invariant is often called a D-term.

According to (5.12), integration over θ^α may be expressed in terms of the supercovariant derivatives:

$$\int d^2\theta = \frac{1}{2}\int d\theta^1 d\theta^2 = \frac{1}{4}\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} = -\frac{1}{4}D^\alpha D_\alpha + \dots\ , \tag{6.5}$$

and analogously for $\bar\theta_{\dot\alpha}$, where the ellipsis denotes terms that involve derivatives ∂_μ. Upon integration over spacetime, these can be neglected,

$$\int d^4x\, d^2\theta\, d^2\bar\theta = \frac{1}{16}\int d^4x\, D^2\bar D^2\ . \tag{6.6}$$

Note that the order of D^2 and $\bar D^2$ does not matter here, for their commutator is a total derivative:

$$[\, D^2\, ,\, \bar D^2\,] = -4\mathrm{i}\,\partial_\mu\,\bar\sigma^{\mu\dot\alpha\alpha}[\, D_\alpha\, ,\, \bar D_{\dot\alpha}\,]\ . \tag{6.7}$$

The above equations also imply that partial integration of a supercovariant derivative is always possible, since there is no “boundary term”:

$$\int d^4x\, d^2\theta\, D_\alpha\phi(z) = 0 = \int d^4x\, d^2\bar\theta\, \bar D_{\dot\alpha}\phi(z)\ . \tag{6.8}$$

7 Chiral superfields

An important example for an $N = 1$ supersymmetry multiplet is the so-called chiral superfield. It is a complex field $\phi(z)$ that satisfies the constraint

$$\bar{D}_{\dot{\alpha}}\phi(z) = 0 \ . \tag{7.1}$$

$\phi(z)$ need not be Lorentz-scalar, it may carry unwritten indices. Indeed, in the next section we shall encounter a chiral spinor superfield. In supersymmetric extensions of the standard model matter such as quarks and leptons is described by chiral scalar superfields, however.

If we introduce a complex composite bosonic coordinate

$$y^\mu = x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\theta} \tag{7.2}$$

and express the supercovariant derivatives in terms of the new coordinates $(y, \theta, \bar{\theta})$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2\mathrm{i}(\sigma^\mu\bar{\theta})_\alpha\partial_{y^\mu} \ , \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \ , \tag{7.3}$$

we see that the general solution to (7.1) is given by superfields which depend on $\bar{\theta}^{\dot{\alpha}}$ only via the combination y^μ:

$$\phi(z) = \phi(y, \theta) \ . \tag{7.4}$$

The space $\mathbb{C}^{4|2}$ parametrized by (y, θ) is being referred to as the "chiral superspace". Functions on this space have a shorter θ-expansion than full superfields:

$$\phi(y, \theta) = \varphi(y) + \theta\psi(y) + \theta^2 F(y) \ . \tag{7.5}$$

Thus, a chiral scalar multiplet comprises a complex scalar $\varphi(x)$, a left-handed Weyl spinor $\psi_\alpha(x)$, and what will turn out to be an auxiliary complex scalar $F(x)$. From (7.5) we obtain the expansion of $\phi(z)$; using the identity[7]

$$f(y) = \exp(\mathrm{i}\theta\sigma^\mu\bar{\theta}\,\partial_\mu)\, f(x) = \left(1 + \mathrm{i}\theta\sigma^\mu\bar{\theta}\,\partial_\mu + \tfrac{1}{4}\theta^2\bar{\theta}^2\,\Box\right) f(x) \ , \tag{7.6}$$

we find

$$\phi(z) = \varphi(x) + \theta\psi(x) + \theta^2 F(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\,\partial_\mu\varphi(x) + \tfrac{\mathrm{i}}{2}\theta^2\,\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\,\Box\varphi(x) \ . \tag{7.7}$$

Expressed in terms of supercovariant derivatives of $\phi(z)$, the components are given by

$$\varphi(x) = \phi(z)|_{\theta=\bar{\theta}=0} \ , \quad \psi_\alpha(x) = D_\alpha\phi(z)|_{\theta=\bar{\theta}=0} \ , \quad F(x) = -\tfrac{1}{4}D^2\phi(z)|_{\theta=\bar{\theta}=0} \ . \tag{7.8}$$

Their supersymmetry transformations can be determined as explained above:

$$\delta_\xi\varphi(x) = -(\xi D + \bar{\xi}\bar{D})\phi(z)|_{\theta=\bar{\theta}=0} = -\xi\psi(x)$$

[7]The σ-matrix identities imply that $\theta\sigma^\mu\bar{\theta}\,\theta\sigma^\nu\bar{\theta} = -\frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu}$.

$$\delta_\xi \psi_\alpha(x) = -(\xi D + \bar\xi \bar D) D_\alpha \phi(z)|_{\theta=\bar\theta=0} = -2\xi_\alpha F(x) - 2\mathrm{i}(\sigma^\mu \bar\xi)_\alpha \partial_\mu \varphi(x)$$
$$\delta_\xi F(x) = \tfrac{1}{4}(\xi D + \bar\xi \bar D) D^2 \phi(z)|_{\theta=\bar\theta=0} = -\mathrm{i}\, \bar\xi \bar\sigma^\mu \partial_\mu \psi(x) \ . \tag{7.9}$$

The first two we had already computed; the one of F follows simply by noting that third powers of D_α vanish and that on chiral superfields it is

$$\bar D_{\dot\alpha} D^2 \phi = [\, \bar D_{\dot\alpha} \, , \, D^2 \,] \phi = -2 D^\alpha \{ \bar D_{\dot\alpha} \, , \, D_\alpha \} \phi = 4\mathrm{i}\, \partial_{\alpha\dot\alpha} D^\alpha \phi \ .$$

Exercise: *Compute the commutator $[\, \delta_\xi \, , \, \delta_\zeta \,]$ of two supersymmetry transformations on the components of the chiral scalar multiplet.*

We note that since $\bar D_{\dot\alpha}$ is a derivation, sums and products of chiral superfields are again chiral. Moreover, since the $\bar D_{\dot\alpha}$ anticommute, $\bar D^2 \Sigma(z)$ is chiral for arbitrary superfields $\Sigma(z)$. Complex conjugation of a chiral superfield $\phi(z)$ gives an antichiral superfield $\bar\phi(z)$ satisfying the constraint $D_\alpha \bar\phi(z) = 0$.

To determine an invariant action for the chiral scalar multiplet, let us consider the highest component of the real composite superfield $\bar\phi\phi$ build from ϕ and its complex conjugate:

$$\bar\phi\phi = \frac{1}{4}\, \theta^2 \bar\theta^2 \big(- 2\, \partial_\mu \bar\varphi\, \partial^\mu \varphi + \bar\varphi\, \Box \varphi + \varphi\, \Box \bar\varphi - \mathrm{i} \psi \sigma^\mu \overleftrightarrow{\partial}_\mu \bar\psi + 4 \bar F F \big) + \ldots \ . \tag{7.10}$$

Thus, we obtain the free field Lagrangian by projection to the $\theta^2\bar\theta^2$ component of this expression, the D-term, and the manifestly supersymmetric action is simply

$$S_0 = \int d^8 z\, \bar\phi \phi = \int d^4 x \, \big(- \partial_\mu \bar\varphi\, \partial^\mu \varphi - \tfrac{\mathrm{i}}{2} \psi \sigma^\mu \partial_\mu \bar\psi + \bar F F \big) \ . \tag{7.11}$$

We find that there is no kinetic term for the complex scalar F — it is an auxiliary field, whose purpose it is to equalize the numbers of bosonic and fermionic components and allow for a linear representation of the supersymmetry algebra that closes off-shell, see below. That no derivatives of F can occur in a renormalizable action also follows from the fact that F has mass dimension 2 if we assign mass dimension 1 to φ.[8] F has an algebraic equation of motion and can therefore be eliminated by inserting its solution back into the action. In the free case, this simply sets $F = 0$.

There are two kinds of self-interactions for chiral scalar multiplets. First, we may replace the integrand $\bar\phi\phi$ of S_0 by a more general real function $K(\phi, \bar\phi)$ of several[9] chiral and antichiral superfields ϕ^i and $\bar\phi^i$ without spoiling supersymmetry. This then gives rise to supersymmetric non-linear sigma models (NLSM):

$$\int d^8 z\, K(\phi, \bar\phi) = \int d^4 x \, \Big(- \frac{\partial^2 K(\varphi, \bar\varphi)}{\partial \bar\varphi^i\, \partial \varphi^j}\, \partial_\mu \bar\varphi^i\, \partial^\mu \varphi^j + \ldots \Big) \ .$$

[8] From (5.12) and (5.13) it immediately follows that supercovariant derivatives and θ's carry mass dimension 1/2 and $-1/2$, respectively.

[9] The index i that counts chiral scalar fields is not to be confused with the R-symmetry index in N-extended supersymmetry.

$K(\phi,\bar\phi)$ is called a Kähler potential, because the target space geometry of the NLSM is that of a complex Kähler manifold. We shall not consider target spaces other than $\mathbb{C}^n$, as these would correspond to non-renormalizable theories.

The second way of introducing interactions is via a so-called superpotential. This method relies on the fact that we can build supersymmetry invariants by integrating a (composite) chiral superfield over chiral superspace $\mathbb{C}^{4|2}$: For any chiral superfield $\phi(z)$ we have

$$\delta_\xi \int d^4x\, d^2\theta\ \phi = -\int d^4x\, d^2\theta\ (\xi D + \bar\xi \bar D)\,\phi = -\int d^4x\, d^2\theta\ \xi^\alpha \frac{\partial}{\partial\theta^\alpha}\,\phi = 0\ .$$

Here we have used that Q and $\mathrm{i}D$ differ only by a spacetime derivative and that ϕ satisfies the constraint (7.1). This provides a means of introducing supersymmetric mass terms and interactions by integrating over chiral superspace a polynomial $W(\phi)$ of the chiral scalar superfields in the theory:

$$S_W = \int d^6z\, W(\phi) + \text{c.c.}\ . \tag{7.12}$$

$d^6z \equiv d^4x\, d^2\theta$ denotes the measure on $\mathbb{C}^{4|2}$. Note that $W(\phi)$ can only depend on the ϕ^i, not on their antichiral complex conjugates — in this sense the superpotential is a holomorphic function. By the chain rule satisfied by the supercovariant derivatives, it is itself chiral:

$$\bar D_{\dot\alpha} W(\phi) = \bar D_{\dot\alpha}\phi^i\, \frac{\partial W}{\partial\phi^i} = 0\ . \tag{7.13}$$

As the θ-integration picks out the F-component of the chiral integrand, this action is often being referred to as an F-term.

To determine the component Lagrangian of S_W, we epress the chiral superfields as functions of (y,θ), expand $W(\phi)$ about the lowest component $\varphi(y)$ of $\phi(y,\theta)$, and pick out the part proportional to θ^2. A translation in spacetime, under which the integral is invariant, then takes us back from y to x:

$$\begin{aligned}\int d^6z\, W(\phi) &= \int d^6z\, W(\varphi + \theta\psi + \theta^2 F)\\ &= \int d^6z\,\left[W(\varphi) + (\theta\psi^i + \theta^2 F^i)W_i(\varphi) + \tfrac12 \theta\psi^i\theta\psi^j W_{ij}(\varphi)\right]\\ &= \int d^4x\,\left[F^i W_i(\varphi) - \tfrac14 \psi^i\psi^j W_{ij}(\varphi)\right]\ .\end{aligned} \tag{7.14}$$

Here we use the notation

$$W_{i_1\ldots i_n}(\varphi) = \frac{\partial^n W(\varphi)}{\partial\varphi^{i_1}\ldots\partial\varphi^{i_n}}\ . \tag{7.15}$$

If we add S_W to the free action S_0,

$$\begin{aligned}\mathcal{L} = &-\partial_\mu\bar\varphi^i\,\partial^\mu\varphi^i - \tfrac{\mathrm{i}}{2}\psi^i\sigma^\mu\partial_\mu\bar\psi^i - \tfrac14\psi^i\psi^j W_{ij}(\varphi) - \tfrac14\bar\psi^i\bar\psi^j \bar W_{ij}(\bar\varphi)\\ &+ \bar F^i F^i + F^i W_i(\varphi) + \bar F^i \bar W_i(\bar\varphi)\ ,\end{aligned} \tag{7.16}$$

which we can do since both are independently supersymmetric, the equations of motion for the auxiliary fields are no longer trivial:

$$\frac{\delta S}{\delta F^i} = 0 \;\Leftrightarrow\; \bar{F}^i = -W_i(\varphi)\ . \tag{7.17}$$

Plugging the solutions back into the action, we obtain a salar potential ($\mathcal{L} = \ldots - V$)

$$V(\varphi, \bar{\varphi}) = \bar{F}^i F^i = \sum_i |W_i(\varphi)|^2\ . \tag{7.18}$$

Being a sum of squares, the potential is non-negative, which will turn out to play an important role in the discussion of spontaneous supersymmetry breaking.

If we require the theory to be power-counting renormalizable, then the superpotential $W(\phi)$ can be an at most cubic polynomial in ϕ. This follows from a dimensional analysis: The mass dimension of d^6z is -3, therefore the one of $W(\phi)$ is 3. As a chiral scalar superfield has mass dimension 1, coupling constants of negative dimension would occur for quartic and higher powers of ϕ. Thus, the most general renormalizable superpotential reads

$$W(\phi) = \lambda_i \phi^i + \tfrac{1}{2} m_{ij}\phi^i\phi^j + \tfrac{1}{3} g_{ijk}\phi^i\phi^j\phi^k \tag{7.19}$$

with in general complex totally symmetric parameters λ_i, m_{ij}, and g_{ijk} of mass dimension 2, 1, and 0, respectively. The action then takes the form

$$\begin{aligned}\mathcal{L} = &-\partial_\mu\bar{\varphi}^i\,\partial^\mu\varphi^i - \tfrac{\mathrm{i}}{2}\psi^i\sigma^\mu\partial_\mu\bar{\psi}^i - \tfrac{1}{4}m_{ij}\psi^i\psi^j - \tfrac{1}{4}\bar{m}_{ij}\bar{\psi}^i\bar{\psi}^j \\ &- \tfrac{1}{2}g_{ijk}\varphi^i\psi^j\psi^k - \tfrac{1}{2}\bar{g}_{ijk}\bar{\varphi}^i\bar{\psi}^j\bar{\psi}^k - \sum_i |\lambda_i + m_{ij}\varphi^j + g_{ijk}\varphi^j\varphi^k|^2\ .\end{aligned} \tag{7.20}$$

This is the Wess–Zumino model. As long as the scalars φ^i have vanishing vacuum expectation values, the parameters m_{ij} set the masses of the fields while the g_{ijk} correspond to Yukawa couplings.

Exercise: *Show that the supersymmetry Noether current of the Lagrangian (7.16) is given by the spinorial vector*

$$J^\mu_\alpha = -\partial_\nu\bar{\varphi}^i(\sigma^\nu\bar{\sigma}^\mu\psi^i)_\alpha + \mathrm{i}(\sigma^\mu\bar{\psi}^i)_\alpha\bar{W}_i(\bar{\varphi})\ , \tag{7.21}$$

and compute the action of the corresponding supercharge Q_α on φ^i using the canonical commutation relations. Hint: the current can be read off from the result $\delta_{\xi(x)}\mathcal{L} = \partial_\mu K^\mu + \partial_\mu\xi\, J^\mu + \partial_\mu\bar{\xi}\,\bar{J}^\mu$ of a local supersymmetry transformation of the Lagrangian.

If we replace the F^i in the supersymmetry transformations (7.9) of ψ^i by their on-shell expressions,

$$\delta_\xi\psi^i = 2\xi\bar{W}_i(\bar{\varphi}) - 2\mathrm{i}\sigma^\mu\bar{\xi}\,\partial_\mu\varphi^i\ , \tag{7.22}$$

the commutator of two such transformations closes into a translation only for spinor fields that satisfy their equations of motion:

$$[\,\delta_\xi\,,\,\delta_\zeta\,]\,\psi^i_\alpha = a^\mu\partial_\mu\psi^i_\alpha + \mathcal{E}^{ij}_{\alpha\dot{\alpha}}\frac{\delta S}{\delta\bar{\psi}^j_{\dot{\alpha}}} \quad \text{with} \quad \mathcal{E}^{ij}_{\alpha\dot{\alpha}} = 4\delta^{ij}(\zeta_\alpha\bar{\xi}_{\dot{\alpha}} - \xi_\alpha\bar{\zeta}_{\dot{\alpha}})\ . \tag{7.23}$$

The right-hand side, being the commutator of two symmetry transformations, must be a linear combination of symmetry transformations even off-shell. Indeed, the extra term is a special case of a so-called trivial symmetry

$$\delta_{\mathrm{triv}}\phi^I = \mathcal{E}^{IJ}(\phi, x)\,\frac{\delta S}{\delta \phi^J} \quad \text{with} \quad \mathcal{E}^{IJ} = -(-)^{|\phi^I||\phi^J|}\mathcal{E}^{JI} \tag{7.24}$$

for an arbitrary set of fields ϕ^I that can be bosonic ($|\phi^I| = 0$) or fermionic ($|\phi^I| = 1$). These obviously leave the action S invariant thanks to the graded antisymmetry of $\mathcal{E}^{IJ}$:

$$\delta_{\mathrm{triv}} S[\phi] = \int \delta_{\mathrm{triv}}\phi^I\,\frac{\delta S}{\delta\phi^I} = \int \mathcal{E}^{IJ}(\phi)\,\frac{\delta S}{\delta\phi^J}\,\frac{\delta S}{\delta\phi^I} = 0\ .$$

Such trivial symmetries typically appear only for fermions, since supersymmetry transformations are at most linear in derivatives whereas field equations of bosons are of second order.

As can be seen from (7.22), after elimination of the auxiliary fields the free action and the superpotential terms are no longer separately invariant, only the sum is supersymmetric. It is one of the great advantages of an off-shell formulation with auxiliary fields that one can add invariants to obtain new ones as the symmetry transformations do not depend on the choice of the action.

8 Supersymmetric gauge theories

We shall now couple the matter fields contained in chiral multiplets to gauge fields. In order to preserve supersymmetry, the latter have to be embedded into supersymmetry multiplets as well. Recall that under gauge transformations with spacetime-dependent parameters $\varepsilon^a(x)$ matter fields φ transform as

$$\varphi(x) \to \varphi'(x) = \mathrm{e}^{\mathrm{i}\varepsilon^a(x)T_a}\varphi(x)\ , \tag{8.1}$$

where the T_a are hermitean representation matrices of the compact Lie algebra of internal symmetries with real structure constants:

$$[\,T_a\,,\,T_b\,] = \mathrm{i}f_{ab}{}^c\,T_c\ . \tag{8.2}$$

As long as the T_a commute with the supercharges, all components of a chiral superfield must carry the same gauge charge. It is therefore tempting to simply implement gauge transformations of chiral superfields by extending the above to $\phi'(z) = \mathrm{e}^{\mathrm{i}\varepsilon^a(x)T_a}\phi(z)$. But for arbitrary parameters $\varepsilon^a(x)$, $\phi'(z)$ is no longer chiral, $\bar{D}_{\dot\alpha}\phi'(z) \neq 0$. To preserve this constraint, we have to promote the gauge parameters to full chiral multiplets themselves:

$$\phi(z) \to \mathrm{e}^{\mathrm{i}\Lambda(z)}\phi(z)\ , \quad \Lambda(z) = \Lambda^a(z)T_a\ , \quad \bar{D}_{\dot\alpha}\Lambda^a(z) = 0\ . \tag{8.3}$$

We will later have to fix the gauge transformations involving the additional bosonic and fermionic parameters contained in $\Lambda(z)$.

The antichiral matter fields obtained by complex conjugation transform as

$$\phi^\dagger(z) \to \phi^\dagger(z)\,\mathrm{e}^{-\mathrm{i}\Lambda^\dagger(z)}\ , \tag{8.4}$$

where $\Lambda^\dagger(z) = \bar{\Lambda}^a(z)T_a$ is antichiral as well. As expected, the integrand $\phi^\dagger\phi$ of the free superfield action then is not gauge-invariant:

$$\phi^\dagger\phi \to \phi^\dagger\,\mathrm{e}^{-\mathrm{i}\Lambda^\dagger}\,\mathrm{e}^{\mathrm{i}\Lambda}\phi \neq \phi^\dagger\phi\ . \tag{8.5}$$

We need to couple to a gauge connection whose transformation compensates for those of the matter fields. The action

$$\int d^8z\,\phi^\dagger\,\mathrm{e}^{2V}\phi \qquad \text{with} \qquad V(z) = V^a(z)T_a = V^\dagger(z) \tag{8.6}$$

is supersymmetric *and* gauge-invariant if the real superfields $V^a(z)$ transform such that

$$\mathrm{e}^{2V(z)} \to \mathrm{e}^{\mathrm{i}\Lambda^\dagger(z)}\,\mathrm{e}^{2V(z)}\,\mathrm{e}^{-\mathrm{i}\Lambda(z)}\ . \tag{8.7}$$

This includes the inhomogeneous term characteristic of a connection:

$$V \to V - \tfrac{\mathrm{i}}{2}(\Lambda - \Lambda^\dagger) - \tfrac{\mathrm{i}}{2}[\,V\,,\,\Lambda + \Lambda^\dagger\,] + \ldots\ . \tag{8.8}$$

Hermiticity of $V(z)$ implies that its expansion in terms of Grassmann coordinates is given by

$$\begin{aligned} V(z) = C + \mathrm{i}\theta\chi - \mathrm{i}\bar{\theta}\chi^\dagger + \tfrac{\mathrm{i}}{2}\theta^2 M - \tfrac{\mathrm{i}}{2}\bar{\theta}^2 M^\dagger + \theta\sigma^\mu\bar{\theta}A_\mu \\ + \theta^2\bar{\theta}(\lambda^\dagger - \tfrac{\mathrm{i}}{2}\bar{\sigma}^\mu\partial_\mu\chi) + \bar{\theta}^2\theta(\lambda + \tfrac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\chi^\dagger) + \tfrac{1}{2}\theta^2\bar{\theta}^2(D + \tfrac{1}{2}\Box C)\ , \end{aligned} \tag{8.9}$$

where we have shifted λ and D by derivative terms for convenience. Comparison with

$$\begin{aligned} \tfrac{\mathrm{i}}{2}(\Lambda - \Lambda^\dagger)(z) = \tfrac{\mathrm{i}}{2}(\varphi - \varphi^\dagger) + \tfrac{\mathrm{i}}{2}\theta\psi - \tfrac{\mathrm{i}}{2}\bar{\theta}\psi^\dagger + \tfrac{\mathrm{i}}{2}\theta^2 F - \tfrac{\mathrm{i}}{2}\bar{\theta}^2 F^\dagger - \tfrac{1}{2}\theta\sigma^\mu\bar{\theta}\,\partial_\mu(\varphi + \varphi^\dagger) \\ - \tfrac{1}{4}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi + \tfrac{1}{4}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\psi^\dagger + \tfrac{\mathrm{i}}{8}\theta^2\bar{\theta}^2\,\Box(\varphi - \varphi^\dagger)\ , \end{aligned} \tag{8.10}$$

where for simplicity we denote the components of the chiral superfields Λ with the same symbols as for chiral matter, shows that in the abelian case

$$\begin{aligned} C^a \to C^a + \mathrm{Im}\,\varphi^a\ , \qquad \chi^a \to \chi^a - \tfrac{1}{2}\psi^a\ , \qquad M^a \to M^a - F^a\ , \\ A^a_\mu \to A^a_\mu + \partial_\mu\,\mathrm{Re}\,\varphi^a\ , \qquad \lambda^a \to \lambda^a\ , \qquad D^a \to D^a\ . \end{aligned} \tag{8.11}$$

In the non-abelian case these transformations are augmented by nonlinear (indeed non-polynomial) terms. It is now apparent that we can gauge away the fields C^a, χ^a, and M^a using the degrees of freedom contained in $\mathrm{Im}\,\varphi^a$, ψ^a, and F^a, which leaves us with the usual gauge transformation of A^a_μ involving the parameters $\mathrm{Re}\,\varphi^a$ that we identify with the above ε^a. This choice of gauge is called Wess–Zumino (WZ) gauge.

Apart from a minimal field content, consisting of gauge fields A_μ^a, so-called gauginos λ^a, and auxiliary scalar fields D^a, this gauge has the major advantage that the action (8.6) is polynomial in the fields. This follows immediately from the fact that in WZ gauge the lowest component of $V(z)$ already contains two θ's,

$$V_{\mathrm{WZ}}(z) = \theta\sigma^\mu\bar\theta A_\mu + \theta^2\bar\theta\bar\lambda + \bar\theta^2\theta\lambda + \tfrac{1}{2}\theta^2\bar\theta^2 D \ , \tag{8.12}$$

and therfore third and higher powers of V_{WZ} vanish identically:

$$V_{\mathrm{WZ}}^2(z) = -\tfrac{1}{2}\theta^2\bar\theta^2 A_\mu A^\mu \ , \qquad V_{\mathrm{WZ}}^n(z) = 0 \quad \text{for} \quad n \geq 3 \ . \tag{8.13}$$

Using gauge invariance of the action (8.6), we can evaluate it in WZ gauge and find

$$\begin{aligned}\int d^8z\, \phi^\dagger\, \mathrm{e}^{2V_{\mathrm{WZ}}}\phi &= \int d^8z\, \phi^\dagger\, (1 + 2V_{\mathrm{WZ}} + 2V_{\mathrm{WZ}}^2)\, \phi \\ &= \int d^4x\, \big[-D_\mu\varphi^\dagger D^\mu\varphi - \tfrac{\mathrm{i}}{2}\psi^\dagger\bar\sigma^\mu D_\mu\psi + F^\dagger F + D^a\varphi^\dagger T_a\varphi \\ &\qquad - \varphi^\dagger T_a\psi\lambda^a - \bar\lambda^a\psi^\dagger T_a\varphi\big] \ . \end{aligned} \tag{8.14}$$

Here, $D_\mu = \partial_\mu - \mathrm{i}A_\mu^a T_a$ denotes the gauge-covariant derivative, giving rise to couplings of the gauge fields to matter currents in the usual way. Moreover, we have obtained Yukawa interactions involving the gauginos. We will find below that the term linear in the auxiliary fields D^a contributes to the scalar potential.

This action is *not* invariant under the linear supersymmetry transformations generated by the Q's, as these are broken by adopting the WZ gauge. This may be seen for instance by inspecting the susy transformation of χ_α,

$$\delta_\xi\chi_\alpha = -\xi_\alpha M - (\sigma^\mu\bar\xi)_\alpha(\partial_\mu C - \mathrm{i}A_\mu) \ , \tag{8.15}$$

which does not vanish in WZ gauge where χ_α itself vanishes. We must augment the susy transformation by a compensating gauge transformation that restores the WZ gauge:

$$\delta_\xi^{\mathrm{WZ}} = \delta_\xi + \delta_\Lambda \quad \text{with} \quad \Lambda(z) = 2\mathrm{i}\,\theta\sigma^\mu\bar\xi A_\mu(y) + 2\mathrm{i}\,\theta^2\bar\xi\bar\lambda(y) \ . \tag{8.16}$$

Exercise: *Show that the WZ gauge-preserving susy transformations of a chiral multiplet are given by*

$$\delta_\xi^{\mathrm{WZ}}\varphi = -\xi\psi \ , \quad \delta_\xi^{\mathrm{WZ}}\psi = -2\xi F - 2\mathrm{i}\sigma^\mu\bar\xi D_\mu\varphi \ , \quad \delta_\xi^{\mathrm{WZ}} F = -\mathrm{i}\bar\xi\bar\sigma^\mu D_\mu\psi - 2\bar\xi\bar\lambda\varphi \ , \tag{8.17}$$

and that the commutator of two such transformations yields in addition to a translation a gauge transformation with field-dependent parameter:

$$[\,\delta_\xi^{\mathrm{WZ}}\,,\,\delta_\zeta^{\mathrm{WZ}}\,] = a^\mu\partial_\mu - (a^\mu A_\mu^a)\,\delta_a \ . \tag{8.18}$$

In order to make the gauge multiplets dynamical, we need to construct a supersymmetric extension of the Yang–Mills action. Recall that the gauge fields enter the latter through gauge-covariant field strengths

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f_{bc}{}^a \ . \tag{8.19}$$

A superfield constructed out of $V(z)$ that transforms covariantly under the gauge transformations (8.7) is given by

$$W_\alpha(z) = -\frac{1}{8}\bar{D}^2(\mathrm{e}^{-2V} D_\alpha \mathrm{e}^{2V}) \ . \tag{8.20}$$

Note that it is a chiral spinor: $\bar{D}_{\dot\alpha} W_\alpha = 0$.

Exercise: *Show that the gauge transformation of $W_\alpha(z)$ is given by*

$$W'_\alpha = \mathrm{e}^{\mathrm{i}\Lambda} W_\alpha \, \mathrm{e}^{-\mathrm{i}\Lambda} \ , \tag{8.21}$$

implying that $\mathrm{tr}(W^\alpha W_\alpha)$ *is gauge-invariant.*

Using the identity[10]

$$\mathrm{e}^{-X^a T_a} \partial \, \mathrm{e}^{X^b T_b} = \partial X^a \int_0^1 ds \, (\mathrm{e}^{\mathrm{i}sX\cdot f})_a{}^b \, T_b \quad \text{with} \quad (X\cdot f)_a{}^b \equiv X^c f_{ac}{}^b \ , \tag{8.22}$$

which holds for arbitrary fields X^a and derivations ∂ and shows that the expression on the left-hand side is Lie algebra-valued, we find that

$$\mathrm{e}^{-2V}[\, D_\alpha \, , \, \mathrm{e}^{2V} \,] = 2 \, D_\alpha V^a (\delta^b_a + \mathrm{i} V^c f_{ac}{}^b + \ldots) \, T_b \ . \tag{8.23}$$

Thus, in WZ gauge the field strength superfield (8.20) reads

$$W^a_\alpha = -\frac{1}{4}\bar{D}^2\big(D_\alpha V^a + \mathrm{i} D_\alpha V^b \, V^c f_{bc}{}^a\big)_{\mathrm{WZ}} \ . \tag{8.24}$$

The effect of the second, non-linear term is to turn the derivatives of A^a_μ and λ^a appearing in the θ expansion of W^a_α into gauge-covariant field strengths. The calculation is most easily performed in chiral superspace (y, θ), where the operator $-\bar{D}^2/4 = \int d^2\bar\theta$ simply picks out the $\bar\theta^2$ component. We obtain

$$W^a_\alpha(z) = \lambda^a_\alpha(y) + \theta_\alpha D^a(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - \mathrm{i}\theta^2(\sigma^\mu D_\mu \bar\lambda)^a_\alpha(y) \ , \tag{8.25}$$

where

$$D_\mu \bar\lambda^a = \partial_\mu \bar\lambda^a + A^b_\mu \bar\lambda^c f_{bc}{}^a \ . \tag{8.26}$$

We observe that the gauginos transform in the adjoint representation.

[10]This follows from the identities $f(X) = \int_0^1 ds \, \partial_s f(sX)$ for differentiable functions satisfying $f(0) = 0$ and $\mathrm{e}^{-X\cdot T} T_a \, \mathrm{e}^{X\cdot T} = (\mathrm{e}^{\mathrm{i}X\cdot f})_a{}^b \, T_b$.

Let us list the supersymmetry transformations of the physical component fields in WZ gauge:

$$\begin{aligned}
\delta_\xi^{\mathrm{WZ}} A_\mu &= \xi\sigma_\mu\bar\lambda + \lambda\sigma_\mu\bar\xi \ , \\
\delta_\xi^{\mathrm{WZ}} \lambda_\alpha &= -\xi_\alpha D - \mathrm{i}(\sigma^{\mu\nu}\xi)_\alpha F_{\mu\nu} \ , \\
\delta_\xi^{\mathrm{WZ}} D &= \mathrm{i} D_\mu\lambda\sigma^\mu\bar\xi - \mathrm{i}\xi\sigma^\mu D_\mu\bar\lambda \ .
\end{aligned} \tag{8.27}$$

The transformations of the covariant components of W_α can be inferred from those given in (8.17) by noting that only the chirality of ϕ was used in deriving the latter, a property shared by W_α. Comparing the expansions of W_α and ϕ suggests to make the identifications $\varphi_\alpha = \lambda_\alpha$, $\psi_{\alpha\beta} = \varepsilon_{\alpha\beta} D - \mathrm{i}\sigma^{\mu\nu}{}_{\alpha\beta} F_{\mu\nu}$, and $F_\alpha = -\mathrm{i}(\sigma^\mu D_\mu\bar\lambda)_\alpha$, which gives rise to the above equations and the transformation of the Yang–Mills field strength:

$$\delta_\xi^{\mathrm{WZ}} F_{\mu\nu} = 2D_{[\mu}\lambda\sigma_{\nu]}\bar\xi + 2\xi\sigma_{[\nu}D_{\mu]}\bar\lambda \ . \tag{8.28}$$

As shown in the previous section, integrating a chiral superfield over chiral superspace yields a supersymmetric action. In particular, this applies to the composite chiral scalar

$$\begin{aligned}
W^\alpha W_\alpha &= \theta\sigma^{\mu\nu}\sigma^{\varrho\sigma}\theta\, F_{\mu\nu}F_{\varrho\sigma} - 2\mathrm{i}\,\theta^2\lambda\sigma^\mu D_\mu\bar\lambda + \theta^2 D^2 + \ldots \\
&= 2\theta^2\big(-\tfrac14 F_{\mu\nu}F^{\mu\nu} - \tfrac{\mathrm{i}}{8}\varepsilon^{\mu\nu\varrho\sigma}F_{\mu\nu}F_{\varrho\sigma} - \mathrm{i}\lambda\sigma^\mu D_\mu\bar\lambda + \tfrac12 D^2\big) + \ldots \ .
\end{aligned} \tag{8.29}$$

We recognize the Yang–Mills term in this expression. We still need to divide by g^2, the coupling constant (the canonical normalization of the fields is restored by rescaling the vector multiplet by a factor g, which then also appears in the matter action). We would also like to keep the second term in the above expression, despite its being a total derivative and therefore of no consequence in perturbation theory:

$$\tfrac12\varepsilon^{\mu\nu\varrho\sigma}F_{\mu\nu}F_{\varrho\sigma} \equiv F_{\mu\nu}\tilde F^{\mu\nu} = \partial_\mu K^\mu \ . \tag{8.30}$$

It does contribute non-perturbatively through instanton configurations, though, for which the boundary term

$$-\frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde F^{\mu\nu a} = n \in \mathbb{Z} \tag{8.31}$$

is non-zero. We can introduce separate parameters for the two field strength bilinears through the complex coupling

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2} \ . \tag{8.32}$$

The correctly normalized super Yang–Mills action is then given by

$$\begin{aligned}
S_{\mathrm{sYM}} &= \frac{1}{8\pi}\,\mathrm{Im}\int d^6z\,\tau\, W^{\alpha a}W^a_\alpha \\
&= \int d^4x\,\Big[-\frac{1}{4g^2}\,F^a_{\mu\nu}F^{\mu\nu a} - \frac{\vartheta}{32\pi^2}\,F^a_{\mu\nu}\tilde F^{\mu\nu a} - \frac{\mathrm{i}}{g^2}\,\bar\lambda^a\bar\sigma^\mu D_\mu\lambda^a + \frac{1}{2g^2}\,D^aD^a\Big] \ .
\end{aligned} \tag{8.33}$$

There is another renormalizable supersymmetric action that can be constructed from vector multiplets, provided the gauge group contains abelian factors. In this case we infer from (8.27) that the corresponding D^a transform into total derivatives. Thus, the so-called Fayet–Iliopoulos term

$$S_{\mathrm{FI}} = \frac{2}{g}\int d^8z\,\kappa_a V^a = \frac{1}{g}\int d^4x\,\kappa_a D^a \tag{8.34}$$

is supersymmetric and gauge invariant for arbitrary real parameters κ_a that may be non-zero for U(1) generators T_a.
Finally, we can add a superpotential if $W(\phi)$ is gauge-invariant,

$$\delta_a W(\varphi) = -\mathrm{i}(T_a\varphi)_i\, W_i(\varphi) = 0\ . \tag{8.35}$$

The most general renormalizable, supersymmetric, and gauge-invariant action for chiral and vector multiplets is then given by

$$S = \frac{1}{8\pi}\,\mathrm{Im}\int d^6z\,\tau\, W^{\alpha a}W^a_\alpha + \frac{2}{g}\int d^8z\,\kappa_a V^a + \int d^8z\,\phi^\dagger \mathrm{e}^{2V^aT_a}\phi + 2\,\mathrm{Re}\int d^6z\,W(\phi)\ . \tag{8.36}$$

The corresponding Lagrangian for the component fields reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4g^2}\,F^a_{\mu\nu}F^{\mu\nu a} - \frac{\vartheta}{32\pi^2}\,F^a_{\mu\nu}\tilde{F}^{\mu\nu a} - D_\mu\varphi^\dagger D^\mu\varphi + \frac{1}{2g^2}D^aD^a + F^\dagger F\\ &-\frac{\mathrm{i}}{g^2}\,\bar\lambda^a\bar\sigma^\mu D_\mu\lambda^a - \frac{\mathrm{i}}{2}\,\psi^\dagger\bar\sigma^\mu D_\mu\psi - \varphi^\dagger T_a\psi\lambda^a - \bar\lambda^a\psi^\dagger T_a\varphi\\ &+ g^{-1}D^a(\kappa_a + g\varphi^\dagger T_a\varphi) + \big(F^iW_i(\varphi) - \frac{1}{4}\,\psi^i\psi^j W_{ij}(\varphi) + \text{c.c.}\big)\ .\end{aligned} \tag{8.37}$$

The F^i and D^a are auxiliary fields with algebraic equations of motion:

$$\frac{\delta S}{\delta F^i} = 0 \;\Leftrightarrow\; \bar{F}^i = -W_i(\varphi) \tag{8.38}$$

$$\frac{\delta S}{\delta D^a} = 0 \;\Leftrightarrow\; g^{-1}D^a = -\kappa_a - g\varphi^\dagger T_a\varphi\ . \tag{8.39}$$

Plugging the solutions back into the action, we obtain the salar potential ($\mathcal{L} = \ldots - V$)

$$V(\varphi,\bar\varphi) = F^\dagger F + \frac{1}{2g^2}D^aD^a = \sum_i |W_i(\varphi)|^2 + \frac{1}{2}\sum_a(\kappa_a + g\varphi^\dagger T_a\varphi)^2\ . \tag{8.40}$$

Being a sum of squares, the potential is non-negative. We shall study some properties of this theory in the next section.

9 Supersymmetry breaking

Unless supersymmetry is broken somehow, the masses of the fermions and bosons contained in each multiplet are degenerate. This is not what we observe in experiments, so we have to study the mechanisms that can lead to supersymmetry breaking. Due to time constraints we shall confine ourselves to spontaneous symmetry breakdown here, even though an explicit breaking that occurs softly, i.e., without introducing quadratic divergencies, is more attractive from a phenomenological point of view.

Let us first consider some general properties of spontaneously broken N-extended supersymmetry before specializing to the $N = 1$ theory we have constructed in the previous sections. The Hamiltonian in N-extended supersymmetry is given by

$$H = P^0 = -\frac{1}{4}\,\bar{\sigma}^{0\dot{\alpha}\alpha}\{Q^i_\alpha\,,\,\bar{Q}_{\dot{\alpha}i}\} = \frac{1}{4}\left(Q^i_1\bar{Q}_{\dot{1}i} + Q^i_2\bar{Q}_{\dot{2}i} + \bar{Q}_{\dot{1}i}Q^i_1 + \bar{Q}_{\dot{2}i}Q^i_2\right) \tag{9.1}$$

for each i, that is, we do not sum over i. It follows that its expectation value in any (normalized) state is a sum of squares and therefore non-negative:

$$\forall\, i:\; E_\Psi = \langle\Psi|H|\Psi\rangle = \frac{1}{4}\sum_\alpha\left(\|Q^i_\alpha|\Psi\rangle\|^2 + \|(Q^i_\alpha)^\dagger|\Psi\rangle\|^2\right) \geq 0\;. \tag{9.2}$$

Hence, normalizable states with vanishing energy are always ground states. Such states are supersymmetric, since they are annihilated by all supercharges:

$$E_\Psi = 0 \quad\Leftrightarrow\quad \forall\, i:\; Q^i_\alpha|\Psi\rangle = \bar{Q}_{\dot{\alpha}i}|\Psi\rangle = 0\;. \tag{9.3}$$

On the other hand, if a ground state has non-vanishing positive energy, then supersymmetry is spontaneously broken, as at least one supercharge for each i does not annihilate the state. This appears to imply that global supersymmetry cannot be partially broken — either all supersymmetries are broken, or none of them is. But there is a loop-hole to this argument: the supersymmetry algebra may contain additional bosonic terms that do not correspond to symmetries of the S-matrix (and therefore are not excluded by the Coleman–Mandula theorem) and which can indeed induce partial supersymmetry breaking.

Let us define a fermion-number operator:

$$(-)^{N_F} \equiv \mathrm{e}^{2\pi\mathrm{i}J_3}\;, \tag{9.4}$$

where $J_3 = M_{12}$ generates rotations about the 3-axis (any axis would do). As boson states $|B\rangle$ are mapped into themselves upon rotations about 2π, while fermion states $|F\rangle$ change sign, we have that

$$(-)^{N_F}|B\rangle = |B\rangle\;, \qquad (-)^{N_F}|F\rangle = -|F\rangle\;. \tag{9.5}$$

Hence

$$\{(-)^{N_F}\,,\,Q\} = 0\;, \tag{9.6}$$

where Q denotes any of the supercharges. Consider now the following trace over one-particle states with fixed momentum p^μ (which form a complete set):

$$\mathrm{tr}_p\left[(-)^{N_F}\{Q^i_\alpha\,,\,\bar{Q}_{\dot{\alpha}i}\}\right] = 2\,\mathrm{tr}_p\left[(-)^{N_F}P_{\alpha\dot{\alpha}}\right] = 2p_{\alpha\dot{\alpha}}\,\mathrm{tr}_p(-)^{N_F}\;. \tag{9.7}$$

The left-hand side of this equation vanishes thanks to (9.6) and the cyclicity of the trace. For non-zero momentum p^μ we therefore conclude that the trace of the fermion-number operator over the corresponding subspace vanishes, i.e., the number of bosonic and fermionic positive-energy states is equal:

$$\mathrm{tr}_{p\neq 0}\,(-)^{N_F} = n_B^{E>0} - n_F^{E>0} = 0\ . \tag{9.8}$$

Positive-energy states always come in boson-fermion pairs. If we now take the trace over the entire Hilbert space[11], only the zero-energy states contribute:

$$\mathrm{tr}_{\mathcal{H}}\,(-)^{N_F} = n_B^{E=0} - n_F^{E=0}\ . \tag{9.9}$$

This quantity is called the Witten index. If it is non-zero, then there are some states with vanishing energy, which, as we have just seen, implies that supersymmetry is unbroken. No statement can be made if the Witten index equals zero. What makes this index useful is that it is a non-perturbative quantity, independent of the values of coupling constants (as long as the variation of the parameters in the theory does not change the asymptotics in field space). It may be evaluated classically and the result will still be valid when quantum corrections are taken into account. Thus, if a classical calculation gives a non-vanishing Witten index, supersymmetry will not be broken by quantum effects. For example, pure super Yang–Mills theories without chiral multiplets all have non-vanishing Witten indices.

Let us now study spontaneous supersymmetry breaking in $N=1$ super Yang–Mills theories. A global continuous symmetry with hermitean Noether charge Q is spontaneously broken if there is at least one field $\phi(x)$ whose symmetry variation has a non-vanishing vacuum expectation value, $\langle 0|[\,Q\,,\,\phi(x)\,]|0\rangle \neq 0$, which is incompatible with $Q|0\rangle = 0$. We would like to preserve Poincaré symmetry, which is guaranteed if for all fields

$$\begin{aligned}\langle[\,\mathrm{i}P_\mu\,,\,\phi\,]\rangle &= \partial_\mu\langle\phi\rangle = 0\ ,\\ \langle[\,\mathrm{i}M_{\mu\nu}\,,\,\phi\,]\rangle &= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\langle\phi\rangle - \Sigma_{\mu\nu}\langle\phi\rangle = 0\ ,\end{aligned} \tag{9.10}$$

where $\Sigma_{\mu\nu}$ denotes the spin representation matrix for the field ϕ. Thus, only constant vevs of Lorentz scalar fields leave Poincaré symmetry unbroken. Inspection of the supersymmetry variations of chiral and vector multiplets shows that it is the vevs of the auxiliary scalars F^i and D^a that may break supersymmetry spontaneously, as only those can lead to a non-vanishing vev of a fermion variation:

$$\begin{aligned}\langle F^i\rangle = F_0^i \neq 0 \quad &\Rightarrow \quad \langle\delta_\xi\psi^i\rangle = -2\xi F_0^i \neq 0\ ,\\ \langle D^a\rangle = D_0^a \neq 0 \quad &\Rightarrow \quad \langle\delta_\xi\lambda^a\rangle = -\xi D_0^a \neq 0\ .\end{aligned} \tag{9.11}$$

[11]Strictly speaking, this trace is usually ill-defined and needs to be regularized. One may for instance include in the trace the operator $\mathrm{e}^{-\beta H}$, where β is a positive real number, and consider the limit $\beta \to 0$.

From (8.40), which expresses the potential in terms of a sum of squares of the auxiliary scalars, we infer that supersymmetry is spontaneously broken if and only if $V_{\rm min} > 0$. The potential is minimized for constant field values φ_0^i satisfying the equations

$$\frac{\partial V}{\partial \varphi^i}\Big|_{\varphi_0} = -W_{ij}(\varphi_0)F_0^j - (\varphi_0^\dagger T_a)_i D_0^a = 0 \ . \tag{9.12}$$

Note that in general non-zero φ_0^i break gauge symmetry. Vanishing φ_0^i will leave gauge symmetry unbroken but break supersymmetry if $V(\varphi_0) > 0$.
Let us expand the action about the vevs φ_0^i and consider the fermion bilinears:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\,\psi^i\psi^j W_{ij}(\varphi_0) - \varphi_0^\dagger T_a \psi \lambda^a + \ldots \\ &= -\frac{1}{2}\left(\lambda^a,\, \psi^i/\sqrt{2}\right)\begin{pmatrix} 0 & \sqrt{2}\,(\varphi_0^\dagger T_a)_j \\ \sqrt{2}\,(\varphi_0^\dagger T_b)_i & W_{ij}(\varphi_0)\end{pmatrix}\begin{pmatrix}\lambda^b \\ \psi^j/\sqrt{2}\end{pmatrix} + \ldots \\ &\equiv -\frac{1}{2}\,\chi^t M_F \chi + \ldots \ .\end{aligned} \tag{9.13}$$

The squared masses of the combined spinors χ are given by the eigenvalues of the hermitean matrix $M_F^\dagger M_F$. The conditions (8.35) and (9.12) now imply that this mass matrix has an eigenvector with eigenvalue zero:

$$M_F \begin{pmatrix} D_0^b/\sqrt{2} \\ F_0^j \end{pmatrix} = 0 \ . \tag{9.14}$$

Thus, if this vector doesn't vanish, which is just the condition for supersymmetry to be spontaneously broken, there exists a massless fermionic particle — the Goldstino:

$$\chi_G \sim (F_0^i\,\psi^i + D_0^a\,\lambda^a) \ . \tag{9.15}$$

We have seen that the necessary and sufficient condition for supersymmetry to be broken is that the equations

$$F_0^i = -\bar{W}_i(\varphi_0) = 0 \ , \qquad g^{-1}D_0^a = -\kappa_a - g\varphi_0^\dagger T_a \varphi_0 = 0 \tag{9.16}$$

admit no solution. In fact it can be shown that, as long as all Fayet–Iliopoulos constants κ_a vanish, if there is a solution to the F-equations, then there exits another solution that also solves the D-equations. It therefore suffices to study only the former.
We can always choose our field basis such that only F^1 has a non-vanishing vev:

$$\langle F^1 \rangle = -M^2 \ , \quad \langle F^I \rangle = 0 \ , \quad I = 2, \ldots, n \ . \tag{9.17}$$

It can then be shown that the most general renormalizable superpotential that leads to spontaneous supersymmetry breakdown is given by

$$W(\phi) = M^2\phi^1 + \tfrac{1}{2}\,m_{IJ}\phi^I\phi^J + \tfrac{1}{2}n_{IJ}\phi^1\phi^I\phi^J + \tfrac{1}{3}g_{IJK}\phi^I\phi^J\phi^K \ , \tag{9.18}$$

where the matrix $(m^{-1})^{IJ} n_{JK}$ must be nilpotent. This condition requires that the number of chiral multiplets $n \geq 3$. For vanishing D^a the scalar potential then reads

$$V(\varphi, \bar{\varphi}) = |M^2 + \tfrac{1}{2} n_{IJ} \varphi^I \varphi^J|^2 + \sum_{I=2}^{n} |m_{IJ} \varphi^J + n_{IJ} \varphi^1 \varphi^J + g_{IJK} \varphi^J \varphi^K|^2 . \tag{9.19}$$

The minimum $V_{\text{min}} = M^4 > 0$ is attained for $\varphi^I = 0$. We observe that the vev of φ^1 is arbitrary — the classical potential has a flat direction. In general the vevs of scalars whose F-partners break supersymmetry cannot be fixed by the tree-level potential.
The simplest example is obtained for three chiral multiplets with

$$m_{IJ} = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad n_{IJ} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , \qquad g_{IJK} = 0 , \tag{9.20}$$

which gives the superpotential of the O'Raifeartaigh model:

$$W(\phi) = M^2 \phi^1 + \mu \phi^2 \phi^3 + \tfrac{1}{2} \lambda \phi^1 (\phi^3)^2 . \tag{9.21}$$

Exercise: *For real parameters M, μ, λ with $\mu^2 > M^2 \lambda$, determine the tree-level masses of the six real bosonic scalars contained in $\phi^i = (A^i + \mathrm{i}B^i)/\sqrt{2}$ and of their superpartners (you can choose $\varphi^1_0 = 0$ for simplicity).*

10 Perturbative non-renormalization theorems

In the introduction we had observed the remarkable absence of one-loop corrections to the masses in the Wess–Zumino model. Here we want to outline, without going into all the details, a proof[12] that there is in fact no renormalization of the parameters in the superpotential of $N = 1$ super Yang–Mills theory to any order in perturbation theory. Moreover, gauge coupling constants receive no perturbative corrections beyond one loop. The arguments used in the proof do not exclude wave function renormalization or non-perturbative quantum corrections due to instantons, however, and these do occur in general.
A central ingredient in the non-renormalization theorems is the chiral U(1) R-symmetry of the $N = 1$ supersymmetry algebra, which acts on the supercharges as

$$Q_\alpha \to \mathrm{e}^{-\mathrm{i}\alpha} Q_\alpha , \quad \bar{Q}_{\dot{\alpha}} \to \mathrm{e}^{\mathrm{i}\alpha} \bar{Q}_{\dot{\alpha}} . \tag{10.1}$$

The representation (5.9) then implies that the Grassmann variables transform as $\theta \to \mathrm{e}^{\mathrm{i}\alpha}\theta$, $\bar{\theta} \to \mathrm{e}^{-\mathrm{i}\alpha}\bar{\theta}$. We say a superfield carries R-charge q if its transform is given by

$$\phi'(z) = \mathrm{e}^{\mathrm{i}q\alpha} \phi(x, \mathrm{e}^{-\mathrm{i}\alpha}\theta, \mathrm{e}^{\mathrm{i}\alpha}\bar{\theta}) . \tag{10.2}$$

[12] We follow Weinberg in [9], where more details can be found.

We will be concerned with the R-invariance of the chiral superspace integrals in (a modification of) the action (8.36). Recalling that Grassmann integration is the same as differentiation, we find that they are invariant if the superpotential $W(\phi)$ and the Yang–Mills super field strength W^a_α carry R-charges 2 and 1, respectively.

Let us now investigate the Wilsonian action $S(\mu)$, the local effective action describing the physics below an energy scale μ that is obtained by integrating out fluctuations of the fields with momenta above μ. The perturbative corrections to $S(\mu)$ are severly constrained by the following properties/conditions: First of all, as long as supersymmetry is preserved, the effective action can again be written as the sum of two integrals over the chiral and the full superspace, respectively. The integrand of the former must be holomorphic in the chiral superfields ϕ^i and W^a_α. Moreover, there are selection rules due to global bosonic symmetries like the R-symmetry above that are preserved in perturbation theory. These two properties will exclude most quantum corrections in the chiral sector of the effective action. Last but not least, one can assume that the theory behaves smoothly when going to weak coupling limits, which enables one to reduce certain calculations to tree-level, where they can be easily carried out.

Instead of the action (8.36) (without FI terms for simplicity) we consider a more general microscopic action that depends on two additional gauge-invariant chiral scalar superfields $Y(z)$ and $T(z)$:

$$\hat{\mathcal{L}} = \int d^4\theta\, \phi^\dagger \mathrm{e}^{2V^aT_a}\phi + 2\,\mathrm{Re}\int d^2\theta \left[YW(\phi) + \frac{T}{16\pi \mathrm{i}} W^{\alpha a} W^a_\alpha \right] . \tag{10.3}$$

This gives a new holomorphic superpotential $\hat{W}(Y,\phi) = YW(\phi)$ and a field-dependent gauge coupling. The original action is recovered if we replace the new fields by the constants $Y = 1$ and $T = \tau$, which are trivially chiral. Apart from super-Poincaré symmetry, $\hat{\mathcal{L}}$ has two global continuous symmetries that are preserved in perturbation theory: An R-symmetry with charge assignments

$$R(\phi^i) = R(V^a) = R(T) = 0\ , \quad R(Y) = 2 \tag{10.4}$$

(eq. (8.20) implies that $R(W^a_\alpha) = 1$), and the Peccei–Quinn shift symmetry

$$T \to T + \epsilon\ , \quad \epsilon \in \mathbb{R}\ , \tag{10.5}$$

which only changes the prefactor of the total derivative term (8.31). That the non-renormalization theorems hold only at the perturbative level is due to the fact that these two symmetries are broken by instantons: The chiral R-symmetry is anomalous, with the violation of R-charge conservation being proportional to the instanton number n defined in (8.31). Likewise, for $n \neq 0$ a shift of T produces a non-trivial phase factor $\mathrm{e}^{2\pi \mathrm{i} n\epsilon}$ in the path integral.

For the effective action of the theory with extended field content the most general supersymmetric Ansatz is of the form

$$\hat{\mathcal{L}}(\mu) = \int d^4\theta\, K(\phi,\bar{\phi},V,Y,\bar{Y},T,\bar{T}) + 2\,\mathrm{Re}\int d^2\theta\, f(\phi,W,Y,T)\ .$$

The functions K and f, which depend on the scale μ, are constrained by the above symmetries. In particular, since f is holomorphic in the chiral superfields, it cannot depend on any fields carrying negative R-charge. The perturbative R-symmetry and $\mathrm{SL}(2,\mathbb{C})$ invariance then imply that

$$f = Y h(\phi, T) + \frac{1}{16\pi \mathrm{i}} W^{\alpha a} W^b_\alpha \, k_{ab}(\phi, T) \; .$$

Furthermore, the shift symmetry (10.5) forbids any T dependence except for a linear term in the function k_{ab}, while holomorphy excludes a ϕ dependence of the latter since for $Y = 0$ each ϕ must be accompanied by a $\bar{\phi}$. Gauge invariance requires that k_{ab} be proportional to δ_{ab}, so we arrive at

$$f = Y h(\phi) + \frac{1}{16\pi \mathrm{i}} W^{\alpha a} W^a_\alpha \, (kT + 4\pi \mathrm{i} C) \; .$$

We can now employ the aforementioned weak coupling limit by first sending Y to zero. The usual counting of vertices, internal lines and loops in Feynman graphs then leads to the number of powers of the inverse gauge coupling $t = T|_{\theta=\bar{\theta}=0}$ being given by $N_t = 1 - N_{\text{loops}}$. Hence, the constants k and C correspond to tree-level and one-loop contributions to the effective action respectively; in particular $k = 1$. Letting in addition t approach infinity, only the tree-level superpotential contributes to the term in f proportional to Y, thus $h(\phi) = W(\phi)$.

Returning to the original field content by setting $Y = 1$ and $T = \tau$, we arrive at the effective action

$$\mathcal{L}(\mu) = \int d^4\theta \, K(\phi, \bar{\phi}, V; \mu) + 2 \,\mathrm{Re} \int d^2\theta \, W(\phi) + \frac{1}{8\pi} \,\mathrm{Im} \int d^2\theta \, \tau(\mu) \, W^{\alpha a} W^a_\alpha \; . \tag{10.6}$$

The superpotential is given by the tree-level expression, while the gauge coupling constant $g^{-2}(\mu) = g^{-2} + C(\mu)$ has received only a one-loop correction. Its cut-off dependence is given by

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \, \log\left(\Lambda/\mu\right) \; , \tag{10.7}$$

with intrinsic energy scale Λ that arises through dimensional transmutation. Here, b is the renormalization scheme-independent coefficient in the one-loop beta function:

$$\mu \, \frac{dg}{d\mu} = -\frac{b}{16\pi^2} \, g^3 \; . \tag{10.8}$$

One important conclusion that we can derive from this result is that if there exists a solution to the equations $W_i(\varphi) = 0$ (and all Fayet–Iliopoulos constants κ_a vanish), then supersymmetry is not broken in any finite order of perturbation theory.

Note that the non-renormalization theorems make no statement about the D-term of the effective action. In general it is corrected at any order in perturbation theory. Moreover, both the D- *and* F-terms receive non-perturbative corrections through instantons. In particular, the complex gauge coupling has the general form

$$\tau(\mu) = \frac{b}{2\pi \mathrm{i}} \, \log\left(\tilde{\Lambda}/\mu\right) + \sum_{n \geq 1} c_n \left(\tilde{\Lambda}/\mu\right)^{nb} \quad \text{with} \quad \tilde{\Lambda} = \Lambda \, \mathrm{e}^{\mathrm{i}\vartheta/b} \; . \tag{10.9}$$

A Sigma matrices

Minkowski metric: $\eta = \mathrm{diag}(-1,1,1,1)$.

$$(\varepsilon^{\alpha\beta}) = -(\varepsilon_{\alpha\beta}) = (\varepsilon^{\dot\alpha\dot\beta}) = -(\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.1}$$

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma \ , \qquad \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}_{\dot\gamma} \ , \qquad \varepsilon^{\dot\alpha\dot\beta} = (\varepsilon^{\alpha\beta})^* \tag{A.2}$$

$$\sigma^\mu = (-\mathbb{1}, \tau^i) \ , \quad \bar\sigma^\mu = (-\mathbb{1}, -\tau^i) \ , \qquad \bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} \tag{A.3}$$

$$(\sigma^\mu_{\alpha\dot\beta})^* = \sigma^\mu_{\beta\dot\alpha} \ , \qquad (\bar\sigma^{\mu\dot\alpha\beta})^* = \bar\sigma^{\mu\dot\beta\alpha} \tag{A.4}$$

$$\sigma^{\mu\nu} = \tfrac12 \sigma^{[\mu}\bar\sigma^{\nu]} \ , \qquad \bar\sigma^{\mu\nu} = \tfrac12 \bar\sigma^{[\mu}\sigma^{\nu]} \ , \qquad ({\sigma^{\mu\nu}}_\alpha{}^\beta)^* = -\bar\sigma^{\mu\nu\dot\beta}{}_{\dot\alpha} \tag{A.5}$$

$$\sigma^{0i} = -\bar\sigma^{0i} = \tfrac12 \tau^i \ , \qquad \sigma^{ij} = \bar\sigma^{ij} = -\tfrac{\mathrm{i}}{2}\varepsilon^{ijk}\tau^k \ , \qquad \mathrm{tr}\,\sigma^{\mu\nu} = \mathrm{tr}\,\bar\sigma^{\mu\nu} = 0 \tag{A.6}$$

$${\sigma^{\mu\nu}}_{\alpha\beta} = \varepsilon_{\beta\gamma}\,{\sigma^{\mu\nu}}_\alpha{}^\gamma = {\sigma^{\mu\nu}}_{\beta\alpha} \ , \qquad {\bar\sigma^{\mu\nu}}{}_{\dot\alpha\dot\beta} = \varepsilon_{\dot\alpha\dot\gamma}\,\bar\sigma^{\mu\nu\dot\gamma}{}_{\dot\beta} = {\bar\sigma^{\mu\nu}}{}_{\dot\beta\dot\alpha} \ . \tag{A.7}$$

Identities containing two σ-matrices ($\varepsilon^{0123} = 1$):

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = -2\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} \ , \qquad \sigma^\mu_{\alpha\dot\alpha}\,\bar\sigma_\mu^{\dot\beta\beta} = -2\,\delta^\beta_\alpha\,\delta^{\dot\beta}_{\dot\alpha} \tag{A.8}$$

$$(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta = -\eta^{\mu\nu}\delta^\beta_\alpha + 2{\sigma^{\mu\nu}}_\alpha{}^\beta \ , \qquad (\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta} = -\eta^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta} + 2\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} \tag{A.9}$$

$$\sigma^{[\mu}_{\alpha\dot\alpha}\sigma^{\nu]}_{\beta\dot\beta} = \varepsilon_{\alpha\beta}\,{\bar\sigma^{\mu\nu}}{}_{\dot\alpha\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,{\sigma^{\mu\nu}}_{\alpha\beta} \tag{A.10}$$

$$\tfrac12\varepsilon^{\mu\nu\varrho\sigma}\sigma_{\sigma\varrho} = -\mathrm{i}\sigma^{\mu\nu} \ , \qquad \tfrac12\varepsilon^{\mu\nu\varrho\sigma}\bar\sigma_{\varrho\sigma} = \mathrm{i}\bar\sigma^{\mu\nu} \tag{A.11}$$

$$\tfrac12\varepsilon^{\mu\nu\varrho\sigma}\sigma_{\varrho\alpha\dot\alpha}\sigma_{\sigma\beta\dot\beta} = \mathrm{i}(\varepsilon_{\alpha\beta}\,{\bar\sigma^{\mu\nu}}{}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,{\sigma^{\mu\nu}}_{\alpha\beta}) \ . \tag{A.12}$$

Identities containing three σ-matrices:

$$\sigma^{\mu\nu}\sigma^\varrho = \eta^{\varrho[\mu}\sigma^{\nu]} + \tfrac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\sigma}\sigma_\sigma \ , \qquad \bar\sigma^\varrho\sigma^{\mu\nu} = -\eta^{\varrho[\mu}\bar\sigma^{\nu]} - \tfrac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\sigma}\bar\sigma_\sigma \tag{A.13}$$

$$\bar\sigma^{\mu\nu}\bar\sigma^\varrho = \eta^{\varrho[\mu}\bar\sigma^{\nu]} - \tfrac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\sigma}\bar\sigma_\sigma \ , \qquad \sigma^\varrho\bar\sigma^{\mu\nu} = -\eta^{\varrho[\mu}\sigma^{\nu]} + \tfrac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\sigma}\sigma_\sigma \tag{A.14}$$

$${\sigma^{\mu\nu}}_{\alpha\beta}\,\sigma_{\nu\gamma\dot\alpha} = \varepsilon_{\gamma(\beta}\,\sigma^\mu_{\alpha)\dot\alpha} \ , \qquad {\bar\sigma^{\mu\nu}}{}_{\dot\alpha\dot\beta}\,\sigma_{\nu\alpha\dot\gamma} = \sigma^\mu_{\alpha(\dot\alpha}\,\varepsilon_{\dot\beta)\dot\gamma} \ . \tag{A.15}$$

Identities containing four σ-matrices:

$$\begin{aligned} \sigma^{\mu\nu}\sigma^{\varrho\sigma} &= -\tfrac12(\eta^{\mu\sigma}\sigma^{\nu\varrho} - \eta^{\mu\varrho}\sigma^{\nu\sigma} + \eta^{\nu\varrho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\varrho}) \\ &\quad + \tfrac14(\eta^{\mu\sigma}\eta^{\nu\varrho} - \eta^{\mu\varrho}\eta^{\nu\sigma} - \mathrm{i}\varepsilon^{\mu\nu\varrho\sigma})\mathbb{1} \end{aligned} \tag{A.16}$$

$$\begin{aligned} \bar\sigma^{\mu\nu}\bar\sigma^{\varrho\sigma} &= -\tfrac12(\eta^{\mu\sigma}\bar\sigma^{\nu\varrho} - \eta^{\mu\varrho}\bar\sigma^{\nu\sigma} + \eta^{\nu\varrho}\bar\sigma^{\mu\sigma} - \eta^{\nu\sigma}\bar\sigma^{\mu\varrho}) \\ &\quad + \tfrac14(\eta^{\mu\sigma}\eta^{\nu\varrho} - \eta^{\mu\varrho}\eta^{\nu\sigma} + \mathrm{i}\varepsilon^{\mu\nu\varrho\sigma})\mathbb{1} \end{aligned} \tag{A.17}$$

$${\sigma^{\mu\nu}}_\alpha{}^\beta\,{\sigma_{\mu\nu}}_\gamma{}^\delta = \delta^\beta_\alpha\,\delta^\delta_\gamma - 2\delta^\delta_\alpha\,\delta^\beta_\gamma \ , \qquad {\sigma^{\mu\nu}}_\alpha{}^\beta\,\bar\sigma_{\mu\nu}{}^{\dot\gamma}{}_{\dot\delta} = 0 \tag{A.18}$$

$${\sigma^{\mu\varrho}}_\alpha{}^\beta\,{\sigma_\varrho{}^\nu}_\gamma{}^\delta = -\tfrac12(\delta^\beta_\gamma{\sigma^{\mu\nu}}_\alpha{}^\delta - \delta^\delta_\alpha{\sigma^{\mu\nu}}_\gamma{}^\beta) + \tfrac14\eta^{\mu\nu}(\varepsilon_{\alpha\gamma}\,\varepsilon^{\beta\delta} + \delta^\delta_\alpha\,\delta^\beta_\gamma) \ . \tag{A.19}$$

A word on the literature: We chose not to cite original papers as all the material presented in these notes is covered in textbooks by now, many of which include extensive lists of references. Below we only list a collection of published review articles and books, many more lecture notes on supersymmetry can be found at [13].

References

[1] J. Wess and J. Bagger, *Supersymmetry and Supergravity.* Princeton Univ. Press, Princeton NJ, 1983 (2nd Edition 1992).

[2] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry.* Front. Phys. **58** (1983) 1, arXiv:hep-th/0108200.

[3] H.P. Nilles, *Supersymmetry, Supergravity and Particle Physics.* Phys. Rept. **110** (1984) 1.

[4] M. Sohnius, *Introducing Supersymmetry.* Phys. Rept. **128** (1985) 39.

[5] N. Dragon, U. Ellwanger and M.G. Schmidt, *Supersymmetry and Supergravity.* Prog. Part. Nucl. Phys. **18** (1987) 1.

[6] P.C. West, *Introduction to Supersymmetry and Supergravity.* World Scientific, Signapore, 1990.

[7] D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory.* Institute of Physics Publishing, Bristol UK, 1994.

[8] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity.* Institute of Physics Publishing, Bristol UK, 1998.

[9] S. Weinberg, *The Quantum Theory of Fields Volume III: Supersymmetry.* Cambridge Univ. Press, 2000.

[10] J. Terning, *Modern Supersymmetry – Dynamics and Duality.* Oxford Univ. Press, 2006.

[11] P. Binétruy, *Supersymmetry – Theory, Experiment, and Cosmology.* Oxford Univ. Press, 2007.

[12] I. Aitchison, *Supersymmetry in Particle Physics.* Cambridge Univ. Press, 2007.

[13] http://www.personal.uni-jena.de/~p5thul2/notes.html