Introduction to Noncommutative Field Theory

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Outline

- Spacetime Quantization
- Noncommutative Spaces and Star-Products
- Noncommutative Perturbation Theory and Renormalization
- Noncommutative Yang-Mills Theory
- Reduced Models and Emergent Phenomena

1 Spacetime Quantization

Spacetime quantization requires to promote spacetime coordinates $\left(x^i\right) \in \Re^d$ to hermitean operators $\hat{x}^i$, which do not commute:

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij} \tag{1}$$

where θ^{ij} is a real-valued and antisymmetric matrix, of dimensions (length)^2. In the case θ^{ij} is constant, the commutators essentially define a Heisenberg algebra and imply spacetime uncertainty:

$$\Delta x^i \Delta x^j \geq \frac{1}{2}|\theta^{ij}| \tag{2}$$

Then, a spacetime point is replaced by some "cell", and thus, the spacetime becomes "fuzzy" at very short distances (the "fuzzyness" of size can be "measured" by the noncommutativity length scale $l = \sqrt{\theta}$). This is completely analogous to what happens in quantum mechanics: if $\theta^{ij} \backsim \hbar$, in the quantum phase space points no longer exist and as $\left[\hat{x}^i, \hat{p}^j\right] = i\hbar\delta^i_j$, they are replaced by "Planck cells" of size $\hbar$.

This spacetime quantization poses a real problem about the geometry of the spacetime. The first person who analysed it was Jon von Neumann, who began studying the geometry of quantum mechanics -in his own words, "a pointless geometry": a new branch of mathematics, noncommutative geometry, was born.

But what is it exactly? Let us motivate the problem in a mathematical way: consider the Gel'fand-Naimark duality theorem, a mathematical theorem which establishes a one-to-one correspondence between topological (Hausdorff) spaces X and commutative C^*-algebras A=C(X), where A is the algebra of continuous complex-valued functions $f : X \to \mathbb{C}$ with the pointwise multiplication $(f.g)(x) = f(x).g(x)$. Generalizing this result to non-commutative C^*-algebras, we obtain a relation between noncommutative algebras and noncommutative spaces. All geometry we know (differential, Riemannian, etc.) is done in a purely operator algebraic framework (originally by Connes, Woronowicz, Drinfel'd, etc., in the 1980's) which enables one to define field theories (in particular Yang-Mills gauge theories) on large classes of noncommutative spaces.

Note that in general, the tensor θ^{ij} can depend on spacetime coordinates x, and even on momenta p (defining algebras of pseudo-differential operators).

Why should we do such a radical thing? Let us now to motivate the problem physically: much of theoretical physics is based on length scales, and we know that physical phenomena happen in a huge range of distances, coming from down at the fundamental Planck scale $l_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.6.10^{-33}$ cm all the way up to the radius of the observable universe, $l_{un} \simeq 4.4.10^{14}$ cm $\sim 10^{61} l_p$. We know that quantum field theory (QFT) works well at least down to the LHC scale $l_{LHC} \simeq 2.10^{-18}$ cm. What happens for $l_p < l < l_{LHC}$? Experimentally, it won't be known for quite some time, so we may speculate on the types of interactions that may be pertinent at these length scales. More precisely, there are two inmmediate motivations:

1. The problem of renormalization in QFT: the notion of spacetime noncommutativity (quantization) is in fact very old and due to Heisenberg (1930's), who suggested it as a means to tame ultraviolet (UV, short distances) divergences of QFT-just like the Heisenberg uncertainty principle avoids the ultraviolet catastrophe of quantum mechanics, replacing points with spacetime cells could be an elegant alternative to lattice or cutoff regulatizations (this is subtle, as we shall see later on), hence UV divergences can be tamed by spacetime noncommutativity. This idea passed successively from Heisenberg to Peierls, Pauli, Oppenheimer and finally Snyder, who first wrote about it in 1947 (Physical Review). [1]

2. Quantum gravity: classical general relativity breaks down at Planck scale l_p, where quantum gravitational effects become important, and the classical Riemannian geometry of spacetime must be replaced by some other mathematical framework. Einstein's theory implies that gravity is equivalent to spacetime geometry, hence quantum gravity should quantize spacetime. In fact, some very simple semi-classical arguments, combining only fundamental postulates of general relativity with quantum mechanics, suggest that spacetime quantization (and noncommutative geometry) is expected to be a generic feature of any theory of quantum gravity: if we probe physics at Planck scale ($l = l_p$), we notice that the Compton wavelength is less than or equal to the Planck length, which creates a large mass $m \geq \frac{\hbar}{l_p c}$ in a tiny volume l_p^3. Then the energy density is large enough to form a black hole with a huge event horizon, which hides the information sent out by the probe. This problem is resolved by postulating spacetime uncertainty relations of the form (2).

Besides these speculative reasonings, there is a a deeper interest in spacetime quantization-it governs the effective dynamics of certain systems in strong background fields. The prototypical example in fact arises in a rudimentary quantum mechanical example, the *Landau problem*: it deals with a system of N non-relativistic, interacting electrons moving in two-dimensions. The system is subjected to a constant, external perpendicularly applied magnetic field $\vec{B} = B\hat{z}$. The Lagrangian governing the motion of the system is:

$$L = \frac{m}{2}\vec{v}^2 + \frac{e}{c}\vec{v}.\vec{A} \tag{3}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{A}(x,y) = \frac{B}{2}(-y, x)$. Canonical quantization of this system gives us the following commutation relations for the position and canonical momenta:

$$[x, p_x] = [y, p_y] = i\hbar \quad [x, y] = [p_x, p_y] = 0 \tag{4}$$

and so on. The Hamiltonian of the system is given by: $H = \frac{\vec{\pi}^2}{2m}$, where $\vec{\pi} = m\vec{v} = \vec{p} - \frac{e}{c}\vec{A}$ is the gauge-invariant kinematical momenta (non-canonical) and $\vec{p}$ is the canonical momenta (non-physical). It shows some non-vanishing quantum commutators, $[\pi_x, \pi_y] = i\hbar\frac{eB}{c}$ which tells us that the physical (gauge invariant) kinematical momenta of electrons in the background magnetic field live in a noncommutative space. If we rewrite the quantum momenta $\vec{\pi}$ in terms of the harmonic oscillator creation and destruction operators, the energy eigenvalues of normal-ordered Hamiltonian are those of Landau levels $E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$, where $n = 0, 1, 2...$ and $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The mass gap between Landau levels is $\Delta \sim \hbar\omega_c$, which tends to infinity if B increases too (when all excited levels decouple from ground state $n = 0$).

When this last situation happens, all the levels collapse onto the lowest Landau level (i.e. all the electrons have quantum numbers $n = 0$). Let us do a more accurate analysis to see how a noncommutative space arises: for $B \gg m$ ($B \to \infty$), L tends

[1] Snyder algebra (phase space commutation relations) are far more complex than what we will study here. In particular, they are Lorentz invariant.

to $L \to L_0 = -\frac{eB}{2c}(\dot{x}y - \dot{y}x)$, a Lagrangian which is of first order in time derivatives: it turns the coordinate space into a phase space, where $\left(\frac{eB}{c}x, y\right)$ is a canonical pair which sets $[x, y] = i\theta$, $\theta = \frac{\hbar c}{2B}$. We can conclude that noncommuting coordinates arise in electronic systems constrained to lie in the lowest Landau level. The present context is, in fact, the one in which the Peierls substitution was originally carried out in 1933. If one introduces an impurity, described by a potential energy function $V(x, y)$, into the electronic system, one can compute the first order energy shift in perturbation theory, due to the impurity, of the lowest Landau level by taking x and y as noncommuting coordinates.

The Landau problem has some analogue in string theory with D-branes in background "magnetic fields". Consider a pair of D-branes with open string excitations which may start and end on the same brane, or stretch between the two of them. The low energy limit gives the description of D-brane dynamics as supersymmetric Yang-Mills theory. The target space geometry is given by closed string supergravity fields – metric g_{ij}, Neveu-Schwarz (NS) 2-form B_{ij} (nondegenerate), ... The worldsheet field theory for open strings attached to D-branes is a σ-model for fields y^i on the worldsheet Σ. The action is:

$$S_\Sigma = \frac{1}{4\pi l_s^2}\int_\Sigma \mathrm{d}^2\xi (g_{ij}\partial^a y^i \partial_a y^j - 2\pi i l_s^2 B_{ij}\epsilon^{ab}\partial_a y^i \partial_b y^j) \tag{5}$$

where B_{ij} is the magnetic field on D-branes, $\partial_a = \frac{\partial}{\partial \xi^a}$ and l_s is the intrinsic string length scale. When B_{ij} =constant, we can integrate the second term from (5) by parts to get boundary action:

$$S_{\partial\Sigma} = -\frac{i}{2}\oint_{\partial\sigma} \mathrm{d}t B_{ij} y^i(t)\, \dot{y}^j(t) \tag{6}$$

where $\dot{y}^i = \frac{\partial y^i}{\partial t}$

Point particles here are really endpoints of strings, but there is a consistent low-energy limit which decouples all massive string modes and scales away bulk part of string worldsheet dynamics (closed string sector) from boundary: $g_{ij} \sim l_s^4 \sim \varepsilon \to 0$, when B_{ij} is fixed. Then, the worldsheet field theory is described solely by $S_{\partial\Sigma}$ above, and canonical quantization of open string endpoint coordinates gives: $[y^i, y^j] = i\theta^{ij}$, $\theta = B^{-1}$ on $\partial\Sigma$, just like in Landau problem. D-brane worldvolume becomes a noncommutative space, and since point particle limit $l_s \to 0$ is taken here, we get a low-energy effective field theory (in particular, a Yang-Mills gauge theory) on this noncommutative space. This is known as a noncommutative field theory, and the purpose of these lectures is to introduce the basic ideas, techniques and usual results of these special field theories.

Note: as for the Landau problem, these noncommutative field theories emerge here as a effective description of the string dynamics-nevertheless, the noncommutative setting is very natural and both conceptually and computationally useful, and it is from this formulation that the true Planck scale physics of string theory (a most promising candidate of a unified field theory, including quantum gravity) may be captured by quantum field theory. As we will see, field theories on noncommutative spaces seem to retain some of the nonlocality of string theory. They also emerge in many other scenarios related to quantum gravity, such as doubly special relativity and spin foam models of 2+1-dimensional quantum gravity.

2 Noncommutative Spaces and Star-Products

We will start by introducing in detail the noncommutative spaces that we shall deal with, and thoroughly develop the necessary tools for dealing with field theories defined thereon.

2.1 Moyal spaces

A *Groenewold-Moyal space* (or simply *Moyal space*) is a deformation of spacetime where coordinates $x = (x^i) \in \Re^d$ are promoted to hermitian operators $\hat{x}^i$ which obey Heisenberg commutation relations. Define the algebra $\Re^d_\theta$ as the associative algebra generated by $\hat{x}^i$: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$, where $\theta = (\theta^{ij})$ is a positive constant, invertible, antisymmetric $d \times d$ matrix (thus $d = 2n$, since θ is of maximal rank). We always should think of these coordinates as operators acting on some separable, necessarily infinite-dimensional Hilbert space (like we do in quantum mechanics).

Translations in $\hat{x}^i$ are generated by outer derivations $\hat{\partial}_i$, following:

$$\left[\hat{\partial}_i, \hat{x}_i\right] = \delta_i^j \quad (7)$$

(like conjugate momenta). We do not require $\hat{\partial}_i$ to be commuting operators - we will need a more general case where:

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = -iB_{ij} \quad (8)$$

where $B = (B_{ij})$ is a constant, antisymmetric matrix of maximal rank. If we regard B as a "background" magnetic field (but which we allow to be freely varying) then this defines the algebra of magnetic translation operators (like $\vec{\pi}$ in the Landau problem).

What is the meaning of B_{ij}? Let us define $\hat{\partial}_i' = \hat{\partial}_i + i(\theta^{-1})_{ij}\hat{x}^j$. The commutation relations now become:

$$\left[\hat{\partial}_i', \hat{x}_j\right] = 0 \quad \left[\hat{\partial}_i', \hat{\partial}_j'\right] = -i(B_{ij} + \left(\theta^{-1}\right)_{ij}) \quad (9)$$

and here we can consider three special cases:

- $\theta = 0$: we get a commutative theory. We cannot define $\hat{\partial}_i'$ like above but we can define $\hat{\partial}_i' = \hat{\partial}_i - \frac{i}{2}B_{ij}\hat{x}^j$, and then, $[\hat{\partial}_i, \hat{x}_j] = \delta_i^j$, $[\hat{\partial}_i, \hat{\partial}_j] = 0$, being $\hat{\partial}_j$ proportional to canonical momenta. This is the familiar case of charged particles in background magnetic field B (as the Landau problem before), with $\hat{\partial}_i$ the ordinary derivative and $\hat{\partial}_i'$ the gauge-covariant derivative.

- $B = 0$: we get an ordinary noncommutative theory with commutative derivatives.

- $B = -\theta^{-1}$: this is the "self-dual point" in parameter space. In this case, $\hat{\partial}_i'$ commutes with both coordinates $\hat{x}^j$ and with themselves. Thus it is a constant, that we can take null. Then, $\hat{x}^i = i\theta^{ij}\hat{\partial}_j$. The "phase space" algebra generated by $\hat{x}^i$, $\hat{\partial}_j$ becomes degenerate. If we consider a quantum system based on this algebra (phase space), then the wave functions are not functions of all $\hat{x}$'s, but only of half of them. This choice arises in the Landau problem of charged particles constrained to the lowest Landau level, where the wave functions depend on only half of the position coordinates. It will be used later on in the reformulation of noncommutative field theories as matrix models.

Now, let us set up a map: there exists a one-to-one correspondence between a noncommutative algebra of operators and the algebra of fields $f(x)$ on $\Re^d$. We restrict ourselves to fields on $\Re^d$ in an appropiate Schwartz space of functions of sufficiently rapid decrease at the infinity. Then, any function (or field) $f(x)$ may be described by its Fourier transform:[2]

$$\tilde{f}(k) = \int \mathrm{d}^d x f(x) e^{-ik.x} \quad (10)$$

Given such a function $f(x)$ and its corresponding Fourier transforms, we introduce its *Weyl symbol* by

$$W[f] = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{f}(k) e^{ik.x} \quad (11)$$

where we have chosen the symmetric ordering prescription. If $f(x)$ is real-valued, the Weyl symbol is hermitean, and it is also possible to write (11) as

$$W[f] = \int \mathrm{d}^d x f(x) \hat{\Delta}(x) \quad (12)$$

where

$$\hat{\Delta}(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ik.(\hat{x}-x)} \quad (13)$$

is the "quantizer". Note that in the commutative case $\theta = 0$, (13) reduces to a delta-function: $\delta^D(\hat{x} - x)$. It is straightforward to show that:

$$\left[\hat{\partial}_i, \hat{\Delta}(x)\right] = -\partial_i \hat{\Delta}(x) \quad (14)$$

with $\partial_i = \frac{\partial}{\partial x^i}$, which leads to

$$\left[\hat{\partial}_i, W[f]\right] = W[\partial_i f] \quad (15)$$

It also follows that translation generators can be represented by unitary operators $e^{v^i \hat{\partial}_i}$, $v \in \Re^d$, with

$$e^{v^i \hat{\partial}_i} \hat{\Delta}(x) e^{-v^i \hat{\partial}_i} = \hat{\Delta}(x + v) \quad (16)$$

(16) implies that any cyclic trace Tr defined on the algebra of Weyl operators has the feature that

[2] Notation: $k = (k_i) \in (\Re^d)^* \cong \Re^d$, $k.x = \sum_{i=1}^d k_i x^i$, $k.\theta.k' = \sum_{i,j=1}^d k_i \theta^{ij} k_j'$.

$\mathrm{Tr}\left(\hat{\Delta}(x)\right)$ is independent of $x \in \Re^d$. Let us set the normalization

$$\mathrm{Tr}\left(\hat{\Delta}(x)\right) = 1 \tag{17}$$

Hence,

$$\mathrm{Tr}\left(W[f]\right) = \int \mathrm{d}^d x f(x) \tag{18}$$

It can be easily derived that

$$\hat{\Delta}(x)\hat{\Delta}(y) = \frac{1}{\pi^d \mid det\theta \mid} \int \mathrm{d}^d z \hat{\Delta}(z) e^{-2i(x-z).\theta^{-1}.(y-z)} \tag{19}$$

with the help of the Baker-Campbell-Hausdorff formula,

$$e^{ik_i\hat{x}^i} e^{ik'_i\hat{x}^i} = e^{-\frac{i}{2}\theta^{ij}k_i k'_j} e^{i\left(k+k'\right)_i \hat{x}^i} \tag{20}$$

This product of "quantizers" shows us that

$$\mathrm{Tr}\left(\hat{\Delta}(x)\hat{\Delta}(y)\right) = \delta^d(x-y) \tag{21}$$

i.e. that the "quantizers" form an orthonormal set. (21) and (11) imply that the transformation $f(x) \to W[f]$ is invertible with inverse given by the so-called *Wigner distribution function*:

$$\begin{aligned} f(x) &= W^{-1}\left[\hat{f}\right] \\ &= \mathrm{Tr}\left(W[f]\hat{\Delta}(x)\right) \\ &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik.x} \mathrm{Tr}\left(W[f] e^{ik.\hat{x}}\right) \end{aligned} \tag{22}$$

The product of two Weyl operators defines the *(Groenewold-Moyal) star product* of functions (fields):

$$W[f]W[g] = W[f \star g] \tag{23}$$

with

$$\begin{aligned} (f \star g)(x) &= \mathrm{Tr}\left(W[f]W[g]\hat{\Delta}(x)\right) \\ &= \frac{1}{\pi^d |\det\theta|} \int\int \mathrm{d}^d y \mathrm{d}^d z f(y) g(z) e^{-2i(x-y)\theta^{-1}(x-z)} \\ &= \int\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} \tilde{f}(k)\tilde{g}\left(k-k'\right) e^{-\frac{i}{2}k.\theta.k'} e^{ik'.x} \\ &= f(x) \exp\left(\frac{i}{2}\overleftarrow{\partial}.\theta.\overrightarrow{\partial}\right) g(x) \\ &= f(x)g(x) \\ &+ \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{i_1 j_1 \dots i_n j_n} \partial_{i_1} \dots \partial_{i_n} f(x) \partial_{j_1} \dots \partial_{j_n} g(x) \end{aligned} \tag{24}$$

Notice that for $\theta = 0$, we reobtain the ordinary product of fields. The star product is associative, but noncommutative:

$$x^i \star x^j - x^j \star x^i = i\theta^{ij} \tag{25}$$

A useful extension of the star product formula is:

$$f_1(x_1) \star \dots \star f_N(x_N) = \prod_{a<b} \exp\left(\frac{i}{2}\frac{\partial}{\partial x_a^i}\theta^{ij}\frac{\partial}{\partial x_b^j}\right) f_1(x_1) \dots f_N(x_N) \tag{26}$$

Note that by cyclicity of the operator trace, the integral

$$\mathrm{Tr}\left(W[f_1] \dots W[f_N]\right) = \int \mathrm{d}^d x f_1 \star \dots \star f_N \tag{27}$$

is invariant under cyclic (but not arbitrary) permutations of the functions f_a. In particular,

$$\int \mathrm{d}^d x f(x) \star g(x) = \int \mathrm{d}^d x f(x) g(x) \tag{28}$$

which follows upon integration by parts over $\Re^d$ for Schwartz functions.

Thus, we can see that spacetime noncommutativity is encoded in the ordinary product of noncommuting Weyl operators, or equivalently through deformation of product on commutative algebra of functions on $\Re^d$ to the noncommutative star-product.

2.2 Noncommutative torus

Consider now a "compactification" of $\Re^d$ to a d-dimensional torus $T^d = \frac{\Re^d}{\Gamma}$ with $\Gamma \simeq \mathbb{Z}^d$ a lattice or rank d acting by (integer) translations in $\Re^d$. The "tilting" of T^d is specified in terms of its period matrix, $(\Sigma : \mathbb{Z}^d \to \Gamma) \in GL(d, \Re)$, with $\{v_i\}$ the canonical basis of $\mathbb{Z}^d$. The fields on this lattice are periodic, i.e. $f\left(x^i + \Sigma^{ai} v_a\right) = f\left(x^i\right)$, $i = 1, ..., d$. Smooth functions on the torus must be single valued, which implies that the corresponding Fourier momenta k are quantized as $k = 2\pi\left(\Sigma^{-1}\right) m$, where $m = (m_1, ..., m_d) \in \mathbb{Z}^d$. The quantizer is given in this case as:

$$\hat{\Delta}(x) = \frac{1}{|\det\Sigma|} \sum_{m \in Z^d} e^{-2\pi i m \Sigma^{-1}.x} \prod_{i=1}^{d} \left(\hat{Z}_i\right)^{m_i} \prod_{i<j} e^{-\pi i m_i \theta^{ij} m_j} \tag{29}$$

where $\hat{Z}^i$ are the unitary operators of the *Weyl basis*,

$$\hat{Z}^a = e^{2\pi i \left(\Sigma^{-1}\right)_{ai} \hat{x}^i} \tag{30}$$

which generate the algebra of functions

$$\hat{Z}^i \hat{Z}^j = e^{-2\pi i \Theta^{ij}} \hat{Z}^j \hat{Z}^i \tag{31}$$

where

$$\Theta = 2\pi \left(\Sigma^{-1}\right)^T .\theta.\Sigma^{-1} \tag{32}$$

is the corresponding dimensionless noncommutativity parameter matrix. The derivations $\hat{\partial}_i$ obey,

$$\left[\hat{\partial}_i, \hat{Z}_j\right] = 2\pi i \left(\Sigma^{-1}\right)_{ji} \hat{Z}_j \tag{33}$$

This basis has the required properties: first, the periodicity of the quantizer operator,

$$\hat{\Delta}\left(x^i + \Sigma^{ai} v_a\right) = \hat{\Delta}(x^i) \tag{34}$$

and second,

$$\left[\hat{\partial}_i, \hat{\Delta}(x)\right] = -\partial_i \hat{\Delta}(x) \tag{35}$$

Notice that the torus has $SL(d, Z)$ modular invariance, and thus, if $H \in SL(d, \Re)$, the following transformations are immediate:

$$\begin{aligned} &\Sigma \to \Sigma.H^{-1} \\ &\hat{Z}_i \to \prod_{j=1}^{d} \left(\hat{Z}_j\right)^{H_{ij}} \\ &\Theta \to H.\Theta.H^T \end{aligned} \tag{36}$$

3 Noncommutative Perturbation Theory and Renormalization

We will take now a look at the perturbative expansion of noncommutative quantum field theory, including a discussion of renormalization. We will throughout these lectures work only in Euclidean spacetime signature, ignoring the complications arising in the Minkowski case. Some other fundamental problems, e.g. the loss of Lorentz invariance, will also not be dealt with here.

3.1 Noncommutative Perturbation Theory

Consider a massive Euclidean ϕ^4 scalar field theory, in a space of $d = 2n$ dimensions. To transform standard commutative (or ordinary) scalar field theory into a noncommutative field theory, we may use our quantization prescription in terms of the Weyl symbols. The action is:

$$S[\phi] = \mathrm{Tr}\left(\frac{1}{2}\left[\hat{\partial}_i, W[\phi]\right]^2 + \frac{m^2}{2} W[\phi]^2 + \frac{g^2}{4} W[\phi]^4\right) \tag{37}$$

where ϕ is a real scalar field on $\Re^d$. The path integral measure is taken as $D\phi = \prod_{x \in \Re^d} d\phi(x)$ (it's the ordinary Feynman measure dictated by e.g. string theory applications- other choices possible!). We may rewrite this action in coordinate space as

$$S[\phi] = \int \mathrm{d}^d x \frac{(\partial_i \phi)^2}{2} + \frac{m^2}{2}\phi^2 + \frac{g^2}{4!} \phi(x) \star \phi(x) \star \phi(x) \star \phi(x) \tag{38}$$

We have used the property of the invariance under cyclic permutations, which implies that noncommutative field theory and ordinary field theory are identical at the level of free fields. The bare propagators are unchanged in the noncommutative case. We notice the changes in the interaction terms, which in our case can be written as:

$$\mathrm{Tr}\left(W[\phi]^4\right) = \prod_{a=1}^{4} \int \frac{\mathrm{d}^d k_a}{(2\pi)^d} \tilde{\phi}(k_a) (2\pi)^d \delta^d\left(\sum_a k_a\right) V(k_1, ..., k_4) \tag{39}$$

where the interaction vertex in the momentum space is

$$V\left(k_1, ..., k_4\right) = \prod_{a<b} e^{-\frac{i}{2}k_a.\theta.k_b} \tag{40}$$

The phase factor is non-local, but local to each fixed order in θ. The non-locality in terms of non-polynomial derivative interactions is responsible for many novel effects (including "stringy" ones). At tree-level, this reduces to standard ϕ^4 field theory in d-dimensions at $\theta = 0$. The noncommutative energy scale is $||\theta||^{-\frac{1}{2}}$, where $||\theta|| := \max_{1\leq i,j\leq d} |\theta^{ij}|$.

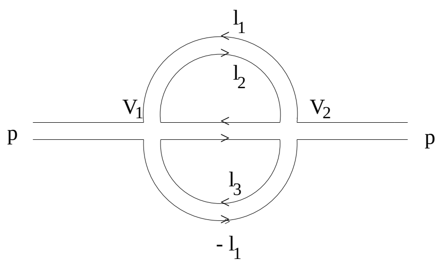
3.2 Planar Feynman-Filk diagrams

Due to momentum conservation, the interaction vertex is only invariant up to cyclic permutations of the momenta k_a, and because of this, one needs to carefully keep track of the cyclic order in which lines emanate from vertices in a given Feynman diagram. But there is an analogous situation in which we have some experience-large N expansion ('t Hooft) of $U(N)$ gauge theory. Indeed, noncommutative field theories can be formally regarded as matrix models, in the sense that the fields are operators on a separable Hilbert space. Later on we will derive this fact in a less formal way. We thus "fatten" lines in graphs and consider *ribbon graphs* that can be drawn on a Riemann surface of particular genus ("stringy")- these are called *noncommutative Feynman diagrams* or *Filk diagrams.*

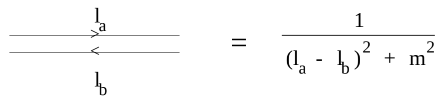
Let us consider first the structure of planar graphs - graphs which can be drawn on the surface of the plane or the sphere (for generic scalar field theory). For an L-loop (planar) graph G_L, let $k_1, ..., k_s$ be the cyclically ordered momenta entering a vertex V through s propagators. By introducing an oriented ribbon structure to the propagators of the diagram, we label the index lines of the ribbons by the "momenta" $l_1, ..., l_{s+1}$ such that $k_a = l_{m_a} - l_{m_{a+1}}$, where $m_a \in \{1, ..., s+1\}$ with $l_{m_{s+1}} = l_{m_1}$. Because adjacent edges in ribbon propagators have oppositely flowing momenta, this construction automatically enforces momentum conservation at each of the vertices. Given these decompositions, a noncommutative vertex V will decompose as

$$V = \prod_{a=1}^{s} e^{-\frac{i}{2}l_{m_a}.\theta.l_{m_{a+1}}} \tag{41}$$

i.e. it will decompose into a product of phases, one for each incoming propagator. However, the momenta associated to a given line will flow in the opposite direction at the other end of the propagator, so that the phase associated to any internal propagator is equal in magnitude and opposite in sign at its two ends. As an example, consider the 2-loop planar diagram:



(a)



(b)

The decomposed vertices are:

$$\begin{aligned} V_1 &= \exp\left(-\frac{i}{2}\left(l_2.\theta.l_3 + l_1.\theta.l_2 + l_3.\theta.l_1\right)\right) \\ V_2 &= \exp\left(-\frac{i}{2}\left(l_2.\theta.l_1 + l_1.\theta.l_3 - l_2.\theta.l_3\right)\right) = V_1^{-1} \end{aligned} \tag{42}$$

Thus, the total phase factor associated with any planar Feynman diagram is:

$$V_{G_L}\left(p_1, ..., p_s\right) = \prod_{a<b} \exp\left(-\frac{i}{2}p_a.\theta.p_b\right) \tag{43}$$

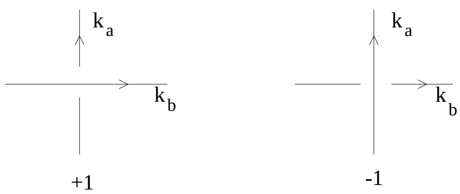
where $p_1, ..., p_s$ are the cyclically ordered external momenta of the graph. The global phase factor is completely independent of the details of the internal structure of the planar graph. We can see that the contribution of a planar graph to the noncommutative perturbation series is just the corresponding $\theta = 0$ contribution multiplied by the global phase factor.

At $\theta = 0$, divergent terms in the perturbation expansion are determined by products of local fields, and the global phase modifies these terms to the star-product of local fields. We conclude that *planar divergences at $\theta \neq 0$ may be absorbed into redefinitions of the bare parameters if and only if the corresponding commutative quantum field theory is renormalizable.* This dispells the naive expectation that the Feyman graphs of noncommutative quantum field theory would have better ultraviolet behaviour than the commutative ones. Note that here the renormalization procedure is not obtained by adding local counterterms, but rather the counterterms are of an identical non-local form as those of the bare Lagrangian. In any case, at the level of planar graphs for scalar fields, noncommutative quantum field theory has precisely the same renormalization properties as its noncommutative counterparts.

3.3 Non-Planar Feynman-Filk Diagrams

Non-planar graphs have propagators that cross each other in internal lines. Let us motivate the problem with an example of a 1-loop diagram in noncommutative ϕ^4:

Introduce positive and negative crossings in non-planar graphs G_L:



What is the global phase factor or this diagram? It can be easily proved that the total noncommutative phase factor for a general graph which generalizes the planar result is given by:

$$\tilde{V}_{G_L}\left(p_1, ..., p_s\right) = V_{G_L}\left(p_1, \ldots, p_s\right) \prod_{a<b} e^{-\frac{i}{2} I_{ab} p_a . \theta . p_b} \tag{44}$$

where I_{ab} is the signed intersection matrix of the graphs which counts the number of times that the a-th (internal or external) line crosses over the b-th line. Therefore, the θ dependence of non-planar graphs is much more complicated and we expect them to have a much different behaviour than their commutative counterparts. In particular, because of extra oscillatory phase factors which occur, we expect these diagrams to have an improved ultraviolet behaviour, at an energy scale of $\sim \|\theta\|^{-\frac{1}{2}}$: when internal lines cross in an otherwise divergent graph, the phase oscillations provide an effective cutoff $\Lambda_{eff} = \|\theta\|^{-\frac{1}{2}}$.

Now, let us illustrate these results with an explicit computation. Our example will be the 1-loop mass renormalization in the noncommutative ϕ^4 theory in four dimensions. For this, we will evaluate the one-particle irreducible two-point function

$$\Pi\left(p\right) = \langle \tilde{\phi}\left(p\right) \tilde{\phi}\left(-p\right) \rangle_{1PI} = \sum_{n \geq 0} g^{2n} \Pi^{(n)}\left(p\right) \tag{45}$$

to one loop order. The bare two-point function is $\Pi^{(0)}(p) = p^2 + m^2$. At the one-loop order there is (topologically) one planar and one non-planar Feynman-Filk graph, $\Pi^1\left(p\right) = \Pi^1_{pl} + \Pi^1_{npl}$. The symmetry factor for the planar graph is twice that of the non-planar graph, and they lead to the respective Feynman integrals

$$\Pi^{(1)}_{pl}(p) = \frac{1}{3} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \tag{46}$$

$$\Pi^{(1)}_{npl}(p) = \frac{1}{6} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{ik.\theta.p}}{k^2 + m^2} \tag{47}$$

The planar contribution (46) is proportional to the standard one-loop mass correction of commutative ϕ^4 theory, which is quadratically ultraviolet divergent in $d = 4$. The non-planar contribution is expected to be generically finite, because of the rapid oscillations of the phase factor $e^{ik.\theta.p}$ at high energies. In the case of $p = 0$, not only we got the relation $\Pi^{(1)}_p = 2\Pi^{(1)}_{np}(p = 0)$, but also the non-planar graph inherits the usual ultraviolet singularities, but now in the form of a long-distance divergence. The effective cutoff for 1-loop graph in momentum space is $(-p.\theta^2.p)^{-\frac{1}{2}}$-the non-planar graph inherits usual UV singularities, but now in form of long-distance divergences (i.e. at small momenta). So turning on noncommutativity θ^{ij} replaces standard UV divergence with a singular IR behaviour-this is the notorious UV/IR mixing problem of noncommutative field theory: an exotic mixing of the ultraviolet and infrared scales in noncommutative field theory.

Now we will try to quantify this phenomenon explicitly. To evaluate the Feynman-Filk integrals (46) and (47), let us introduce the Schwinger parametrization:

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}\alpha e^{-\alpha(k^2+m^2)} \tag{48}$$

Introducing this parametrization, and doing the Gaussian momentum integration, we get:

$$\begin{aligned}\Pi^{(1)}_{npl}(p) &= \frac{1}{6(4\pi)^n}\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^n} e^{-\alpha m^2 + \frac{p.\theta^2.p}{4\alpha} - \frac{1}{\Lambda^2\alpha}} \\ &= \frac{m^{n-1}}{6(2\pi)^n}\left(\frac{4}{\Lambda^2} - p.\theta^2.p\right)^{\frac{(1-n)}{2}} \\ & K_{n-1}\left(m\sqrt{\frac{4}{\Lambda^2} - p.\theta^2.p}\right)\end{aligned} \tag{49}$$

where $d = 2n$, $K_\gamma(x)$ is the irregular modified Bessel function of order γ and Λ is an ultraviolet cutoff on the $\alpha = 0$ singularity of the Schwinger integral. The complete 1-loop renormalized propagator is:

$$\Pi(p) = p^2 + m^2 + 2g^2\Pi^{(1)}_{npl}(0) + g^2\Pi^{(1)}_{npl}(p) + O\left(g^4\right) \tag{50}$$

Now $K_\nu(x) \simeq 2^{\nu-1}\Gamma(\nu)x^{\nu-1} + ...$ for $x \to 0$, $\nu \neq 0$ ($K_0(x) \simeq -\log x + ...$ for $x \to 0$), so in d=4 the $\frac{1}{\Lambda^2}$ expansion yields leading singular behaviour:

$$\Pi^{(1)}_{npl}(p) = \frac{1}{96\pi^2}\left(\Lambda^2_{eff} - m^2 \ln\frac{\Lambda^2_{eff}}{m^2}\right) + \dots \tag{51}$$

where $\Lambda^2_{eff} = \frac{1}{\frac{1}{\Lambda^2} - p.\theta^2.p}$ is the effective ultraviolet cutoff. Notice that in the limit $\Lambda \to \infty$ the non-planar one-loop graph remains finite, being effectively regulated by the noncommutativity of space-time; i.e. $\Lambda^2_{eff} \to -\frac{1}{p.\theta^2.p}$. Nevertheless, we have again the ultraviolet divergence when $p = 0$. In this zero momentum or infrared limit, we recover the standard mass renormalization of ϕ^4 theory in four dimensions. On the other hand, in the ultraviolet limit $\Lambda \to \infty$, the corrected propagator assumes a complicated, non-local form that cannot be attributed to any (mass) renormalization. Thus, we can conclude that the ultraviolet limit and the infrared limit do not commute, and noncommutative quantum field theory exhibits an intriguing mixing of the ultraviolet and infrared regimes. These effects of the ultraviolet modes on the infrared behaviour have no analogs in conventional quantum field theory.

3.4 The Trouble with UV/IR Mixing

At energies $E \ll \|\theta\|^{-\frac{1}{2}}$, noncommutative quantum field theory is nothing like conventional (commutative) quantum field theory. This is due to their inherent non-locality-very low-energy processes can receive contributions from high energy virtual particles (e.g. pole at $p = 0$ in propagator for a ϕ field comes from high momentum region of integration ($\Lambda \to \infty$)). In particular, if we impose a UV cutoff Λ, this induces an effective IR cutoff $\Lambda_0 = \frac{1}{\Lambda|\theta|}$. This immediately casts a shadow of doubt on the renormalization of these models-standard renormalization schemes, such as the Wilsonian renormalization group (RG) approach, require a clear separation of energy scales. In fact, although the non-planar graphs are finite, their amplitude grows beyond any bound when internal momenta become exceptional, i.e. $p.\theta = 0$. When inserted as subgraphs into higher-loop graphs, these exceptional momenta are obtained in the loop integration and result in (horrible) divergences for any number of external legs.

For example, we saw above that the non-planar graph is well-defined. But when inserted into higher order graphs these subgraphs suddenly become ill-defined. In such graphs, the products of Fourier transforms of the usual QFT distributions (Δ_+, Θ, etc.) are well-defined. However, this is not true for:

$$\begin{aligned}u(x-y) &= \int\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{d^d p}{(2\pi)^d}\frac{e^{-ik.\theta.p}}{k^2+m^2}\frac{e^{ip.(x-y)}}{p^2+m^2} \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\Delta(\theta.p)\tilde{\Delta}(p)e^{ip.(x-y)}\end{aligned} \tag{52}$$

Thus, $\tilde{u}(p) = \Delta(\theta.p)\tilde{\Delta}(p)$ in momentum space, so Fourier transform of $u(x-y)$ contains both $\tilde{\Delta}$ and Δ itself. In d dimensions, products $\Delta(x)^n$ are ill-defined for $n \geq d-2$, due to the well-known singularity at $x = 0$- here it appears at $p = 0$ (IR). If u appears $\geq d-2$ times in a graph, we get an uncontrollable divergence. These divergences increase with the order of the perturbation theory, and all correlation functions are affected and diverge. The field theory cannot be renormalized.

Thankfully, there is a modification of noncommutative field theory which gives a way out. To help motivate this modification, it is useful to take a look at the physical origin of UV/IR mixing: recall from the Baker-Campbell-Hausdorff formula that

$$e^{ik.x} \star e^{iq.x} \star e^{-ik.x} = e^{-\frac{i}{2}k.\theta.q} e^{i(k+q).x} \star e^{-ik.x} = e^{iq.(x-\theta.k)} \tag{53}$$

By Fourier transformation, this gives for arbitrary fields $f(x)$ on $\Re^d$:

$$e^{ik.x} \star f(x) \star e^{-ik.x} = f(x - \theta.k) \tag{54}$$

Multiplication by a plane wave generates a nonlocal spacetime translation of fields by $x \Rightarrow x - \theta.k$. This exhibits the nonlocality of the theory- large momenta lead to large nonlocality. Here a plane wave $e^{ip.x}$ corresponds not to a particle (as in commutative QFT) but rather to a "dipole": an extended, oriented, rigid rod whose length or electric dipole moment Δx^i grows with its center of mass momentum p_j as:

$$\Delta x^i = \theta^{ij} p_j \tag{55}$$

(Analogous to electron-hole bound states in a strong magnetic field (e.g. in Landau problem)). Dipoles interact by joining at their ends.

With usual QFT relation $p = \hbar k$ between wave number and momentum, translation rule $x \Rightarrow x - \theta.x$ above follows.

Whence the IR dynamics are governed by quanta which behave like non-local dipoles (for $E \ll ||\theta||^{-\frac{1}{2}}$). On the other hand, the UV dynamics (for $E \gg ||\theta||^{-\frac{1}{2}}$ where effects of noncommutativity are negligible) are governed by quanta created by the elementary quantum fields ϕ themselves, with pointlike momenta k_i. Consequently, we can interpret the UV/IR mixing as an asymmetry between UV and IR quanta.

From here, we can have the following idea: there is a UV/IR duality suggested which relates dynamics in the 2 regimes- UV/IR mixing is then due to the asymmetry between supports of fields on extended and pointlike degrees of freedom in the different regimes. We seek a "covariant" version of QFT model which makes UV and IR regimes indistinguishable, and hence makes this UV/IR duality into a true symmetry. Covariantization turns UV degrees of freedom into extended objects by replacing their (pointlike) momenta with "Landau" momenta:

$$k_i \to K_i = k_i + B_{ij} x^j \tag{56}$$

where B_{ij} is the "magnetic" background (constant, nondegenerate, generically independent of θ^{ij}), generating a "noncommutative momentum space"

$$[K_i, K_j] = 2iB_{ij} \tag{57}$$

(assuming canonical pairs $\left[x^i, k_j\right] = i\delta^i_j$). This then restores the desired symmetry. As we'll see, this will also restore the exponential decay of correlation functions (for $m \neq 0$), which is spoilt by the long-range position space correlations which decay algebraically for small g due to UV/IR mixing.

3.5 Duality covariant noncommutative field theory

To make the above formulation more precise, consider now charged scalar fields $\phi(x) \in \mathbb{C}$ on Euclidean $\Re^d$:

$$S[\phi] = \int \mathrm{d}^d x \left(\phi^\dagger \left(D_i^2 + m^2\right)\phi + g^2 \phi^\dagger \star \phi \star \phi^\dagger \star \phi\right) \tag{58}$$

where

$$D_i := \frac{1}{\sqrt{2}}(-i\partial_j + B_{ij}x^i) \tag{59}$$

and $d = 2n$. A lengthy but straightforward computation shows that this action is invariant under a duality transformation of order 2 (generating cyclic $\mathbb{Z}_2$):

$$\begin{aligned} \phi(x) \Rightarrow \hat{\phi}(x) &= \sqrt{\det(B)}\tilde{\phi}(B.x) \\ \theta \Rightarrow \hat{\theta} &= -4B^{-1}\theta^{-1}B^{-1} \\ g \Rightarrow \hat{g} &= 2^n |\det(B\theta)|^{-\frac{1}{2}} g \end{aligned} \tag{60}$$

$\theta = 2B^{-1}$ is a "self-dual point" ($\hat{\theta} = \theta$, $\hat{g} = g$).

Essentially, there is a symmetry $k_i \Leftrightarrow B_{ij}x^j$ between position and momentum spaces, [3] hence no distinction between what is meant by UV or IR.

[3]For $\theta = 0$, the interaction vertices in position and momentum spaces are very different:

- Position: $\delta^d(x_1 - x_2 + x_3 - x_4)\,\delta^d(x_1 - x_4)\,\delta^d(x_2 - x_3)$ (fully) local

This establishes the duality symmetry at the classical level. The quantum field theory is defined formally by the generating functional for connected Green's functions:

$$\mathbb{G}(J) = -\log\left(\frac{Z[J]}{Z[0]}\right) \tag{61}$$

where

$$Z[J] = \int \mathbb{D}\phi\mathbb{D}\phi^\dagger \exp\left(-S[\phi] - \int \mathrm{d}^d x\left(\phi^\dagger J + \phi J^\dagger\right)\right) \tag{62}$$

Path integral measure $\mathbb{D}\phi\mathbb{D}\phi^\dagger$ is defined so that:

$$\mathbb{G}(J)\,|_{g=0} = \int\int \mathrm{d}^d x d^d y J^\dagger(x)\, C(x,y)\, J(y) \tag{63}$$

where $C(x,y) = \langle x|\frac{1}{D_i^2+m^2}|y\rangle$ is a free 2-point function. Using Parseval identity:

$$\int \mathrm{d}^d x \phi^\dagger(x)\, J(x) = \int \mathrm{d}^d k \tilde{\phi}^\dagger(k)\, \tilde{J}(k) \tag{64}$$

and formal invariance of measure under $\phi \Rightarrow \hat{\phi}$, we get the quantum duality:

$$\mathbb{G}(J;B,g,\theta) = \mathbb{G}\left(\hat{J};B,\hat{g},\hat{\theta}\right) \tag{65}$$

To substantiate this, we need to make sense of the functional integral, by finding a duality invariant regularization $\mathbb{G} \Rightarrow \mathbb{G}_\Lambda$ which cures all possible divergences.

For this, rather than expanding in the usual plane wave basis of momentum space (not convenient here because of the explicit x-dependence in the propagator), we expand the fields in a convenient "matrix basis" $f_{k,l} \in L^2\left(\Re^2\right)$, $k,l = 0,1,...$ of Landau wavefunctions (here $d=2$ for simplicity):

$$\phi(x) = \sum_{k,l} f_{k,l}(x)\, \phi_{k,l} \tag{66}$$

with $E_k f_{k,l} := D_i^2 f_{k,l} = 2B\left(k+\frac{1}{2}\right) f_{k,l}$, $D_i^2|_{B\Rightarrow -B} f_{k,l} = E_l f_{k,l}$ [4].

- Momentum: $\delta^d(k_1 - k_2 + k_3 - k_4)$ nonlocal

This is a novel property of interaction vertices in noncommutative QFT.

[4] $D_i^2 + D_i^2|_{B\Rightarrow -B} = -\partial_i^2 + (B.x)_i^2$, where $-\partial_i^2$ correspond to high-momentum modes and $(B.x)_i^2$ to long distance modes.

Pick a smooth cutoff function $F(s)$ such that $F(s) = 1$ for $0 \le s \le 1$ and $F(s) = 0$ for $s > 0$. Then replace the free propagator:

$$\begin{aligned} C(k,l) &= \frac{1}{E_k + m^2} \Rightarrow C_\Lambda(k,l) \\ &= \frac{1}{E_k+m^2} F\left(\frac{1}{\Lambda^2}(E_k + E_l)\right) \end{aligned} \tag{67}$$

Symbolically, any Feynman diagram is of the form:

$$\sum_{k_1,l_1,\ldots,k_s,l_s} \prod_{j=1}^{s} C_\Lambda(k_j,l_j)\,(\text{interaction vertices}) \tag{68}$$

Here, this quantity is different from zero only for $E_{k_j} + E_{l_j} = 2B(k_j + l_j + 1) < 2\Lambda^2$ (notice that there are finitely many k_j, l_j for Λ finite).

All Feynman graphs are represented by finite sums in this basis (hence so are all position space Green's functions by multiplying with Landau basis wavefunctions $f_{k_j,l_j}(x_j), \bar{f}_{k_j,l_j}(y_j)$ and summing over k_j, l_j).

In particular, this reformulates the QFT as the $N \Rightarrow \infty$ limit of an $N \times N$ complex matrix model in an external field:

$$Z_N[0] = \int \prod_{k,l=1}^{N} \mathrm{d}\phi_{k,l}\mathrm{d}\phi_{k,l}^\dagger e^{-S[\phi]}$$

There is no analog of such mappings and natural matrix regularizations in commutative QFT!

This matrix model representation is the key feature of the duality covariant model, and it is obtained from a remarkable "projector" property of the Landau wavefunctions. For simplicity, set $\theta = 2B^{-1} > 0$ (the self-dual point), $d = 2$, and consider the explicit construction of the basis set $f_{k,l}(x)$. Set:

$$a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{B}}\partial + \sqrt{B}\bar{z}\right) \tag{69}$$

$$b = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{B}}\bar{\partial} + \sqrt{B}z\right) \tag{70}$$

with $z = x^1 + ix^2$, $\partial = \frac{\partial}{\partial x^i} - i\frac{\partial}{\partial x^2} \equiv \partial_1 - i\partial_2$. These define 2 decoupled 1-D harmonic oscillators:

$$\left[a,a^\dagger\right] = \left[b,b^\dagger\right] = 1 \tag{71}$$

$$[a,b] = \left[a,b^\dagger\right] = 0 \tag{72}$$

such that

$$D_i^2 = \frac{1}{2}\left(aa^\dagger + a^\dagger a\right) \tag{73}$$

$$D_i^2 \mid_{B\Rightarrow -B} = \frac{1}{2}\left(bb^\dagger + b^\dagger b\right) \tag{74}$$

Then the Landau wavefunctions are given by $f_{k,l}(x) = \langle x|k,l\rangle$ where

$$|k,l\rangle = \frac{\left(a^\dagger\right)^k}{\sqrt{k!}}\frac{\left(b^\dagger\right)^l}{\sqrt{l!}}|0,0\rangle \tag{75}$$

$a|0,0\rangle = b|0,0\rangle = 0$. These are 2-particle number basis Fock space states.

These functions can be computed explicitly via their generating function:

$$F_{s,t}(x) := \sum_{k,l=0}^{\infty}\frac{s^k}{\sqrt{k!}}\frac{t^l}{\sqrt{l!}}f_{k,l}(x) \tag{76}$$

such that $f_{k,l} = \frac{\partial^k}{\partial s^k}\frac{\partial^l}{\partial t^l}F_{s,t}$. Standard oscillator algebra from quantum mechanics gives us:

$$F_{s,t}(x) = \sqrt{\frac{B}{\pi}}e^{-B|z|^2+\sqrt{B}(sz+t\bar{z})}e^{-st} \tag{77}$$

Basic formulas for star-products of Gaussian wavepackets yields:

$$F_{s,t}\star F_{s',t'} = \frac{e^{s't}}{\sqrt{4\pi\theta}}F_{s,t'} \tag{78}$$

which by differentiation implies

$$f_{k,l}\star f_{k',l'} = \frac{1}{\sqrt{4\pi\theta}}\delta_{l,k'}f_{k,l'} \tag{79}$$

i.e. $f_{k,l}$ can be identified with the Wigner distributions of the rank-1 operators $\hat{f}_{k,l} = |k\rangle\langle l|$ on the 1-particle Fock space $\mathbb{H}$ (generated by a, for example). One also has $f_{0,0}(x) = \frac{1}{\sqrt{\pi\theta}}e^{-\frac{|z|^2}{2\theta}}$ (that solves differential equations $af_{0,0} = bf_{0,0} = 0$) and

$$\int d^2x\bar{f}_{k,l}\star f_{k',l'} = \delta_{k,k'}\delta_{l,l'} \tag{80}$$

with $\bar{f}_{k,l} = f_{l,k}$. Moreover from the oscillator representation above, one can show:

$$f_{k,l}(x) = \frac{\left(x^1 - ix^2\right)^{\star k}}{\sqrt{k!\,(2\theta)^k}}\star\frac{2e^{-\frac{1}{\theta}\left(x^{1\,2}+x^{2\,2}\right)}}{\sqrt{4\pi\theta}}\star\frac{\left(x^1 + ix^2\right)^{\star l}}{\sqrt{l!\,(2\theta)^l}} \tag{81}$$

One then can show that the matrix model action above is given by:

$$S[\phi] = \mathrm{Tr}\left(\phi^\dagger B\phi + m^2\phi^\dagger\phi + g^2\left(\phi^\dagger\phi\right)^2\right) \tag{82}$$

with $\phi = (\phi_{k,l})$, $B_{k,l} = \theta^{-1}\left(k+\frac{1}{2}\right)\delta_{k,l}$.

3.6 The Grosse-Wulkenhaar Model

Let us now go back to the real noncommutative ϕ^4-theory that we started with. In this theory we can't couple real (uncharged) scalar fields ϕ to a magnetic background, but we can couple them to the operator $D_i^2 + D_i^2\mid_{B\Rightarrow -B}$, which just adds an oscillator potential to the standard kinetic term:

$$\partial_i^2 \Rightarrow \partial_i^2 + \frac{\omega^2}{2}\tilde{x}_i^2 \tag{83}$$

where $\tilde{x}_i := 2\theta_{ij}^{-1}x^j$ in $d = 4$ (or otherwise)- this is known as the Grosse-Wulkenhaar model and its main feature is the following:

Theorem: "The Euclidean quantum field theory with action containing an additional harmonic potential

$$S[\phi] = \int d^4x\Big(\frac{1}{2}\partial_i\phi\star\partial^i\phi + \frac{\omega^2}{2}\left(\tilde{x}_i\phi\right)\star\left(\tilde{x}^i\phi\right) + \frac{m^2}{2}\phi\star\phi + \frac{g^2}{4!}\phi\star\phi\star\phi\star\phi\Big) \tag{84}$$

where $\tilde{x}_i = 2\theta_{ij}^{-1}x^j$, is covariant under the position/momentum space duality $p_i \Leftrightarrow \tilde{x}_i$, and is renormalizable to all orders in g^2".

The confining harmonic oscillator potential serves as an IR cutoff, and is the unique one with the UV/IR duality symmetry that makes the field theory just renormalizable. The key feature is the structure of the new propagator, which involves the Mehler kernel in the Schwinger parametric representations (rather than the usual heat kernel), and is bounded by exponential decay $e^{-(\mathrm{const.})|x-y|^2}$. The corresponding new RG corresponds to a completely new mixture of standard UV and IR notions.

At $\omega = 1$, field strength renormalization compensates coupling constant renormalization such that $g^2\phi^4$ remains invariant. Thus the coupling constant RG flow is bounded, and the field theory is asymptotically safe (rather than asymptotically free). In particular, there is no Landau ghost (contrary to the usual ϕ^4 field theory), and hence a non-perturbative completion is believed to be possible. This can (and in fact already has) shed light on helping to tackle ordinary QFT problems with noncommutative QFT techniques-problems which look untractible in the ordinary geometry language may do so simply because they correspond in that geometry to non-perturbative and non-local effects (e.g., quark confinement or behaviour of systems under influence of strong magnetic fields).

4 Noncommutative Yang-Mills Theory

Now we will analyze the nature of gauge interactions on noncommutative spacetime. As we will see, gauge symmetries in noncommutative field theory are extremely rich and contain a beautiful mixture of spacetime and internal symmetries, with some rather surprising physical, technical and algebraic features.

4.1 Action and star-gauge symmetry

Let $A_i(x) = A_i^a \otimes t_a$ be a hermitean $U(N)$ gauge field on $\Re^d$, where $\mathrm{tr}(t_a t_b) = \delta_{ab}$, $a, b = 1, ..., N^2$, and $[t_a, t_b] = i f_{ab}^c t_c$ (this last relation gives us the Lie algebra of $U(N)$). An hermitean operator corresponding to $A_i(x)$ can be introduced by:

$$\hat{A}_i = \int \mathrm{d}^d x \hat{\Delta}(x) \otimes A_i(x) \tag{85}$$

If we introduce the corresponding gauge-covariant derivatives $\hat{\nabla}_i = \hat{\partial}_i - i\hat{A}_i$ whose curvature is given by:

$$\left[\hat{\nabla}_i, \hat{\nabla}_j\right] = -i\left(\hat{F}_{ij} + B_{ij}\right) \tag{86}$$

(where $\hat{F}_{ij} := \left[\hat{\partial}_i, \hat{A}_j\right] - \left[\hat{\partial}_j, \hat{A}_i\right] + i\left[\hat{A}_i, \hat{A}_j\right]$) the Yang-Mills action may be defined as:

$$\begin{aligned} S_{YM} &= -\frac{1}{4g^2} \mathrm{Tr} \sum_{i,j=1}^{d} \left[\hat{\nabla}_i, \hat{\nabla}_j\right]^2 \\ &= -\frac{1}{4g^2} \mathrm{Tr} \sum_{i,j=1}^{d} \left(\hat{F}_{ij} + B_{ij}\right)^2 \\ &= -\frac{1}{4g^2} \int \mathrm{d}^d x \mathrm{Tr} \sum_{i,j=1}^{d} \left(F_{ij}(x) + B_{ij}\right) \\ &\star \left(F_{ij}(x) + B_{ij}\right) \end{aligned} \tag{87}$$

with

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i - i\left(A_i \star A_j - A_j \star A_i\right) \\ &= \partial_i A_j - \partial_j A_i - i\left[A_i, A_j\right] + \\ &+ \frac{1}{2}\theta^{kl}\left(\partial_k A_i \partial_l A_j - \partial_k A_j \partial_l A_i\right) + O\left(\theta^2\right) \end{aligned} \tag{88}$$

which is the so-called noncommutative field strength tensor of gauge field $A_i(x)$.

Thus $A_i(x) \in \Re_\theta^d \otimes Mat_{N\times N}(\mathbb{C})$-note intertwining of "spacetime" and "colour" degrees of freedom here (this will be a running theme for the remainder of the present lectures). Notice also that "noncommutative electrodynamics"-the rank one case $N = 1$- is an interacting field theory. The global minimum is at $\left[\hat{\nabla}_i, \hat{\nabla}_j\right] = 0 \Leftrightarrow \hat{F}_{ij} = -B_{ij} \neq 0$ (in general) which is the (non-trivial) vacuum.

Concerning gauge symmetry, let us define

$$\hat{g} := \int \mathrm{d}^d x \hat{\Delta}(x) \otimes g(x) \in U(\infty) \tag{89}$$

with $\hat{g}\hat{g}^\dagger = \mathbb{I} = \hat{g}^\dagger\hat{g} \Leftrightarrow \left(g \star g^\dagger\right)(x) = \mathbb{I}_N = \left(g^\dagger \star g\right)(x)$. We say that the matrix field $g : \Re^d \to U(\infty)$ is star unitary. In general, $g^\dagger \neq g^{-1}$ for $\theta \neq 0$, but a relation order by order in θ can be worked out:

$$g^\dagger = g^{-1} + \frac{i}{2}\theta^{ij} g^{-1}\left(\partial_i g\right) g^{-1}\left(\partial_j g^{-1}\right) + O\left(\theta^2\right) \tag{90}$$

The normal gauge transformation of the Yang-Mills theory can now be defined as:

$$\hat{\nabla}_i \to \hat{g}\hat{\nabla}_i\hat{g}^\dagger \tag{91}$$

which leads to:

$$\hat{A}_i \to \hat{g}\hat{A}_i\hat{g}^{-1} - i\left[\hat{\partial}_i, \hat{g}\right] \tag{92}$$

$$\leftrightarrow A_i \longmapsto g \star A_i \star g^\dagger - ig \star \partial_i g^\dagger \qquad (93)$$

which is a star gauge transformation.

In particular, noncommutative field strength transforms as:

$$F_{ij} \to g \star F_{ij} \star g^\dagger \qquad (94)$$

and noncommutative Yang-Mills action is invariant under star-gauge transformations (by cyclicity of matrix and operator traces).

The gauge symmetry in the noncommutative case is special-it unifies spacetime and ordinary gauge symmetry in a way we will explain in detail. This is evident from the non-trivial mixing between colour and spacetime transformations contained in star-gauge transformations- for $\theta \neq 0$ they cannot be disentangled. In particular, in general $\det(g \star h) \neq \det(g) \star \det(h)$; in contrast to the commutative case, $U(1)$ and $SU(N)$ sectors of $U(N) \sim U(1) \times SU(N)$ do not decouple, and $U(1)$ "photon" interacts with $SU(N)$ "gluons" (this problem can be resolved by using $SU(N)$ envelopping algebra valued gauge fields and gauge transformations, but that is beyond the scope of these lectures).

Another crucial point, which will essentially forbid the construction of local gauge invariant operators (observables), is that the gauge group in noncommutative Yang-Mills theory contains spacetime translations. For this, consider the basic plane wave fields for $N = 1$:

$$g_a(x) = e^{ia.\theta^{-1}.x} \qquad (95)$$

with $a \in \Re^d$. From the Baker-Campbell-Hausdorff formula one can check that they are star-unitary:

$$g_a \star g_a^\dagger = g_a^\dagger \star g_a = 1 \qquad (96)$$

From the previous "dipole" relation (54) it follows that they implement translations of fields by $a \in \Re^d$:

$$g_a(x) \star f(x) \star g_a^\dagger(x) = f(x+a) \qquad (97)$$

The corresponding star-gauge transformations are:

$$A_i(x) \longmapsto A_i(x+a) - \left(\theta^{-1}\right)_{ij} a^j \qquad (98)$$

The global transformation of A_i leaves field strength F_{ij} invariant. Therefore, *spacetime translations are equivalent to gauge transformations in noncommutative Yang-Mills theory.* The only other known theory with such a geometrical gauge symmetry is gravitation. Thus, *noncommutative gauge theory is a toy model of general relativity.*

Gauging translational symmetry gives gauge theories of gravity-so noncommutative Yang-Mills can provide an alternative approach to the problem of quantizing gravity (the biggest problem of modern theoretical physics). This all ties in nicely with the "origins" of noncommutative gauge theories as world-volume effective field theories on D-branes in string theory with background (magnetic) B-fields, as discussed before- some of the main interests in these non-local field theories.

4.2 Gauge-invariant observables

Let $C \subset \Re^d$ be an oriented, embedded curve with smooth parametrization $\xi(t) : [0,1] \to \Re^d$, and endpoints $\xi(0) = 0$, $\xi(1) = v$ in $\Re^d$. Introduce the noncommutative parallel transport operator:

$$\begin{aligned} U(x;C) &= P \exp_\star \left(i \int_C \mathrm{d}\xi^i A_i(x+\xi) \right) \\ &= 1 + \sum_{n=1}^{\infty} i^n \int_0^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \ldots \int_{t_{n-1}}^1 \mathrm{d}t_{n-1} \\ &\qquad \dot{\xi}^{i_1}(t_1) \ldots \dot{\xi}^{i_n}(t_n) \\ &\qquad A_{i_1}(x+\xi(t_1)) \star \ldots \star A_{i_n}(x+\xi(t_n)) \end{aligned} \qquad (99)$$

with extended star-product and where P is the path ordering. This operator is the holonomy of noncommutative gauge field $A_i(x)$.

Under gauge symmetry,

$$U(x;C) \longmapsto g(x) \star U(x;C) \star g(x+v)^\dagger \qquad (100)$$

and the operator $U(x;C)$ is an $N \times N$ star-unitary matrix field depending on the line C.

In the traditional commutative use, gauge invariance would force us to close the contour C, setting $v = 0$. But in the noncommutative case, gauge transformations can affect translations of spacetime:

$$e^{ik.x} \star g(x) \star e^{-ik.x} = g(x+v) \qquad (101)$$

where $k = \theta^{-1}.v$ is the "total momentum" of path C (c.f. via Fourier transformation from Baker-Campbell-Hausdorff formula). Thus a gauge-invariant observable associated to every (generic) contour C is given by:

$$O(C) := \int \mathrm{d}^d x \mathrm{Tr}\left(U(x;C)\right) \star e^{ik.x} \tag{102}$$

with $k = \theta^{-1}.v$ (we notice here that UV/IR mixing manifests itself in the property that $k \to \infty$ as $v \to \infty$). They are called open Wilson line operators. If $C = C_k$ is the straight line path:

$$\xi(t) = (\theta.k)\, t \tag{103}$$

then the open Wilson line operators $O(C_k)$ create and annihilate the weakly-interacting, non-local dipoles, of dipole moment $\theta.k$, describing the elementary quanta of noncommutative gauge theory in the IR regime.

Gauge-invariant operators, generalizing the standard local gauge theory operators in the commutative limit, are local in momentum space and are given by a Fourier-type transformation:

$$\hat{O}(k) = \int \mathrm{d}^d x O(x) \star U(x;C_k) \star e^{ik.x} \tag{104}$$

where $O(x)$ is any local, gauge invariant operator of ordinary Yang-Mills theory (e.g. $O(x) = F_{ij}(x)$). It attaches $O(x)$ at one end of Wilson line with non-vanishing momentum k.

In the commutative limit $\theta = 0$, $v = 0$; there are no gauge-invariant quantities associated with open lines in ordinary Yang-Mills theory. In that case, the total momentum of a closed loop is unrestricted, and we can replace $e^{ik.x}$ (the momentum eigenstate) by an arbitrary function $f(x)$. In particular, taking $f(x) = \delta^d(x-a)$ recovers the standard gauge-invariant Wilson loops of Yang-Mills theory. But for $\theta \neq 0$, closed loops have 0 momentum k, and only $e^{ik.x} = 1$ is permited in $O(C)$ above-there is no local star-gauge invariant dynamics, because everything has to be smeared out by the Weyl operator trace $\mathrm{Tr} \sim \int \mathrm{d}^d x$. Hence the gauge dynamics below the noncommutativity scale is quite different from the commutative case.

Using the Weyl transformation, the Open Wilson Line Operators can be expressed as:

$$O(C) = \mathrm{Tr}\left(\hat{U}(C)\,\hat{D}(C)^\dagger e^{ik.\hat{x}}\right) \tag{105}$$

where

$$\hat{U}(C) = P\exp\left(\int_C \mathrm{d}\xi.\hat{\nabla}\right) \tag{106}$$

$$\hat{D}(C) = P\exp\left(\int_C \mathrm{d}\xi.\hat{\partial}\right) \tag{107}$$

are the Weyl symbols, with

$$\hat{U}(C) \longmapsto \hat{g}\hat{U}(C)\hat{g}^\dagger \tag{108}$$

$$\hat{D}(C)\,\hat{\Delta}(x)\,\hat{D}^\dagger(C) = \hat{\Delta}(x+v) \tag{109}$$

$$\left(e^{ik.\hat{x}}\hat{\Delta}(x)\,e^{-ik.\hat{x}} = \hat{\Delta}(x+v)\right) \tag{110}$$

On a d-dimensional torus T^d, the dipole moments v are given by:

$$v = \theta.k + \Sigma.n \tag{111}$$

with $k \equiv 2\pi\Sigma^{-1}.m$ and $m,n \in \mathbb{Z}^d$ (for single-valued gauge fields $A_i(x)$ on T^d). Here n_i are "winding numbers" around cycles of T^d. Thus there is now a larger set of line momenta, due to the ambiguity in identifying the translation vector v in $e^{ik.x} \star g(x) \star e^{-ik.x} = g(x+v)$ up to integer translation of periods of T^d. Thus, *open Wilson lines generalize Polyakov lines* in noncommutative Yang-Mills theory (the only open line obervables at $\theta = 0$). This justifies their interpretation as creating electric dipoles, as it generalizes the creation of electric charges by Polyakov line operators. Here the electric field operator E, generating translations $A \longmapsto A + 2\pi\Sigma.n$, is modified as:

$$E = \frac{\delta}{\delta A} \longmapsto E + \theta.P \tag{112}$$

where P is the transverse momentum operator, due to the star-gauge transformations generated by $e^{2\pi i n.\Sigma.x}$.

5 Reduced Models and Emergent Phenomena

In this final part we will work out the nonperturbative, constructive definition of noncommutative Yang-Mills theory. Unlike the commutative case, this can be completely described in the language of matrix models (arising here as reduced models). This will also reveal some beautiful features of the vacuum structure of noncommutative gauge theories.

5.1 Background independence

Our goal is to remove derivative operators ∂_i or $\hat{\partial}_i$ entirely from the noncommutative gauge theory action. There is no analog of this manipulation in ordinary Yang-Mills theory.

Let us introduce covariant coordinates:

$$\hat{C}_i = \left(\theta^{-1}\right)_{ij} \hat{x}^j + \hat{A}_i \tag{113}$$

Then $\hat{C}_i \longmapsto \hat{g}\hat{C}_i\hat{g}^\dagger$ under gauge transformations (hence the name)- this follows from the very special property of the Heisenberg algebra which implies that derivative operators $\hat{\partial}_i$ can be represented via the adjoint actions $\left(\theta^{-1}\right)_{ij}\left[\hat{x}^j, -\right]$, i.e. $\hat{\partial}_i$ are inner derivations of the algebra $\Re^d_\theta$. Then the entire noncommutative gauge theory can be rewritten in terms of the $\hat{C}_i$, which absorb completely the derivatives $\hat{\partial}_i$. In particular, we may rewrite the covariant derivative as:

$$\hat{\nabla}_i = \hat{\partial}'_i - i\hat{C}_i \tag{114}$$

where $\hat{\partial}'_i = \hat{\partial}_i + i\left(\theta^{-1}\right)_{ij}\hat{x}^j$. We will momentarily set $B \equiv 0$ here. Then, using $\left[\hat{\partial}'_i, \hat{x}^j\right] = 0$, we compute:

$$\left[\hat{\nabla}_i, \hat{f}\right] = -i\left[\hat{C}_i, \hat{f}\right] \tag{115}$$

$$\hat{F}_{ij} = i\left[\hat{\nabla}_i, \hat{\nabla}_j\right] = -i\left[\hat{C}_i, \hat{C}_j\right] + \left(\theta^{-1}\right)_{ij} \tag{116}$$

and consequently,

$$S_{YM} = \frac{1}{4g^2}\mathrm{Tr}\sum_{i\neq j}\left(\left[\hat{C}_i, \hat{C}_j\right] + i\left(\theta^{-1}\right)_{ij}\right)^2 \tag{117}$$

$\hat{C}_j$ are elements of the abstract algebra $\Re^d_\theta$ (or more exactly, $\Re^d_\theta \otimes Mat_{N\times N}(\mathbb{C})$), so spacetime derivatives have completely disappeared in this rewriting of noncommutative Yang-Mills theory. Since $\hat{C}_j$ are formally space-independent, this is just an infinite-dimensional matrix model, with $\hat{C}_i \in Mat\left(\infty, \mathbb{C}\right)$.

Flat connections $\hat{F}_{ij} = 0$ give:

$$\left[\hat{C}_i, \hat{C}_j\right] = -i\left(\theta^{-1}\right)_{ij} \tag{118}$$

So formally, $\hat{C}_i$ are like the momentum operators- $i\hat{\partial}_i$ with $B = \theta^{-1}$- recall that this is the special point where derivatives and coordinates were degenerately related. In particular, $\hat{X}^i = \theta^{ij}\hat{C}_j$ formally represent noncommuting position operators in the ground state, wherein $\hat{X}^i = \hat{x}^i$ ($\hat{A}_i \equiv 0$). Then the noncommutative gauge degrees of freedom $\hat{C}_i$ are fluctuations around this canonical (Moyal) noncommutative spacetime. More generally noncommutative spacetimes $\left[\hat{X}^i, \hat{X}^j\right] = i\Theta^{ij}\left(\hat{X}\right)$ are obtained as non-vacuum solutions of Yang-Mills equations of motion:

$$\left[\hat{C}_i, \left[\hat{C}_i, \hat{C}_j\right]\right] = 0 \tag{119}$$

Thus (noncommutative) spacetime emerges as a dynamical effect in the matrix model. This is the essence of its relation to the so-called IKKT matrix model for the non-perturbative dynamics of type IIB superstrings, and also in the more recent models of emergent gravity which clarify the origin of gravity in noncommutative gauge theory. In this setting, gravity is related to quantum fluctuations $\hat{C}_i$ of spacetime at the Planck scale, while noncommutative field theory arises from field dependent fluctuations of spacetime geometry (determined via $\theta^{ij}(x)$). In particular, UV/IR mixing arises due to a non-renormalizable gravitational sector in the IR (with $G \sim \Lambda$)!!

This large N matrix model is called a twisted reduced model. The "twist" is $\left(\theta^{-1}\right)_{ij}$ (needed to cancel an infinite extra term from $\left[\left(\theta^{-1}.\hat{x}\right)_i, \left(\theta^{-1}.\hat{x}\right)_j\right]$). The noncommutative spacetime $\Re^d_\theta$ is effectively hidden in the infinitely-many degrees of freedom of the large N matrices $\hat{C}_i$, and it reappears from expanding the matrix model around its classical vacuum (this is a dynamical emergence of spacetime). This is just a special instance of the Eguchi-Kawai reduction of multi-colour field theories. It is formally gotten by reduction of ordinary Yangs-Mills theory (with background flux) to a point (i.e., by making fields constant)[5].

[5] The (straight) open Wilson line has a particularly simple form in this matrix model formulation:

$$O\left(C_k\right) := \int \mathrm{d}^d x \mathrm{tr}\left(U\left(x_j, C_k\right)\right) \star e^{ik.x} = \mathrm{Tr}e^{ik.\theta.\hat{C}} \tag{120}$$

which is manifestly gauge-invariant under $\hat{C}_i \longmapsto \hat{g}\hat{C}_i\hat{g}^{-1}$. This follows easily from:

$$O\left(C_k\right) = \mathrm{Tr}\left(\hat{U}\left(C_k\right)\hat{D}\left(C_k\right)^\dagger e^{ik.\hat{x}}\right) \tag{121}$$

5.2 Universal gauge symmetry

There are two remarkable consequences of the background independence noted above. First, it implies the universality of the noncommutative gauge group. For simplicity, here we will work in $d = 2$ spacetime dimensions- the general case follows by stitching together independent 2×2 blocks by means of an $SO(d)$ transformation of $\Re^d$. In this case:

$$S_{YM} = -\frac{1}{g^2}\mathrm{Tr}\left(\left[\hat{C}_z, \hat{C}_{\bar{z}}\right] + \frac{1}{2\theta}\right)^2 \tag{122}$$

where $\hat{C}_z = \frac{1}{2}\left(\hat{C}_1 + i\hat{C}_2\right)$, $\hat{C}_{\bar{z}} = \frac{1}{2}\left(\hat{C}_1 - i\hat{C}_2\right)$, and the classical vacuum is now:

$$\left[\hat{C}_{\bar{z}}, \hat{C}_z\right] = \frac{1}{2\theta} \tag{123}$$

($\theta > 0$). This is just the Heisenberg commutation relation. Up to unitary equivalence, by the Stone-von Neumann-Mackey theorem it has a unique irreducible representation, the Schrödinger representation on Fock space:

$$\hat{C}_z^{(1)} = -\frac{\hat{a}^\dagger}{\sqrt{2\theta}} \tag{124}$$

$$\hat{C}_{\bar{z}}^{(1)} = -\frac{\hat{a}}{\sqrt{2\theta}} \tag{125}$$

$\hat{a}^\dagger$, $\hat{a}$ are the Fock space creation and annihilation operators obeying $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

In order to get the general solution, let us make $N \geqslant 1$ copies of the Fock space, represented by operators $\hat{C}_z^{(N)}$, $\hat{C}_{\bar{z}}^{(N)}$ on the Hilbert space $\mathbb{H}^N \equiv \mathbb{H} \otimes \mathbb{C}^N$. Recall here the Hilbert hotel argument $\mathbb{H}^N \equiv \mathbb{H}$. Regroup the Fock space number basis states $\mid n\rangle$, with $n = 0, 1, 2, ...$ into basis vectors $|p, \mu\rangle$, $p = 0, 1, 2, ...$, $\mu = 0, 1, ..., N-1$ of $\mathbb{H}^N$ as:

$$|n\rangle = |pN + \mu\rangle \equiv |p, \mu\rangle \tag{126}$$

where p are the Hilbert space labels (spacetime), and μ the $\mathbb{C}^N$ "internal" symmetry of $U(N)$. Up to star-gauge transformation, vacuum state in this basis in sector of solution space labelled by N is:

$$\hat{C}_z^{(N)} = -\frac{\hat{a}^\dagger}{\sqrt{2\theta}} \otimes 1_N \tag{127}$$

$$\hat{C}_{\bar{z}}^{(N)} = -\frac{\hat{a}}{\sqrt{2\theta}} \otimes 1_N \tag{128}$$

with $\hat{U}(C_k) = Pe^{-k.\theta.\hat{\nabla}}$, $\hat{D}(C_k) = Pe^{-k.\theta.\hat{\partial}}$.

This configuration has two types of unitary gauge symmetries: the infinite-dimensional $U(\mathbb{H}) \equiv U(\infty)$ symmetry acting on Fock space labels (under which S_{YM} is invariant), and $U(N)$ symmetry acting by finite-dimensional rotations of μ labels.

The quantum field theory decomposes according to these vacuum configurations, in "topological" sectors labelled by N with "hidden" internal $U(N)$ gauge symmetry- notice the remarkable emergence of colour from $U(1)$ gauge fields. We mean "topological" in the sense that N cannot be changed by any local gauge transformation:

$$N = \mathrm{Tr}_{\mathbb{H}^N}\left[\hat{C}_z^{(N)}, \hat{C}_{\bar{z}}^{(N)}\right] = \dim\ker \hat{C}_z^{(N)}\hat{C}_{\bar{z}}^{(N)} \tag{129}$$

This analytical index is a topological invariant which detects differential operators hidden in $\hat{C}_i$ (identifies sectors with a higher-dimensional interpretation in the 0-dimensional matrix model).

Now, any path in field space connecting different vacua has infinite action, so the quantum theory about any of these vacua doesn't mix with the others. Evaluate path integral as (semi-classical, finite-action) sum over classical vacuum field configurations:

$$Z = \int \frac{\mathrm{d}C_z \mathrm{d}C_{\bar{z}}}{\mathrm{vol}\left(U(\mathbb{H})\right)} e^{-S_{YM}} = \sum_{N=0}^{\infty} Z_N \tag{130}$$

where the Z_N are the partition functions for each $U(N)$ theory. We can expand: $\hat{C}_i = \hat{C}_i^{(N)} + \hat{A}_i^{(N)}$ and then,

$$\begin{aligned} S_{YM} &= -\frac{1}{2g^2}\mathrm{Tr}_{\mathbb{H}}\hat{F}^2 = -\frac{1}{2g^2}\mathrm{Tr}_{\mathbb{H}^N}(\hat{F}^{(N)})^2 \\ &= -\frac{1}{2g^2}\mathrm{Tr}_{\mathbb{H}} \otimes \mathrm{tr}_N\big(\frac{1}{\sqrt{2\theta}}\big(\left[\hat{a}\otimes 1_N, \hat{A}_z\right] + \\ &\left[\hat{a}^\dagger \otimes 1_N, \hat{A}_{\bar{z}}\right]\big) + \left[\hat{A}_{\bar{z}}, \hat{A}_z\right]\big)^2 \end{aligned} \tag{131}$$

where $\left(\hat{A}_i\right)_{\mu\nu}$ is a $N \times N$ operator on $\mathbb{H}$ defined via

$$\langle p|\left(\hat{A}_i\right)_{\mu\nu}|q\rangle = \langle p, \mu|\hat{A}_i^{(N)}|q, \nu\rangle \tag{132}$$

But this is just the standard Weyl representation of noncommutative $U(N)$ gauge theory.

Thus $U(1)$ noncommutative Yang-Mills theory contains noncommutative $U(N)$ gauge theory for all values of N. Here the rank N of the gauge group emerges as a superselection parameter, labelling the individual star-gauge inequivalent vacuum sectors of the original quantum Hilbert space. Noncommutative Yang-Mills theory is a universal gauge theory, containing all Yang-Mills theories (including noncommutative ones).

5.3 The twisted Eguchi-Kawai model

We'll now see how those finite $U(N)$ gauge groups arise from $U(\infty)$ gauge symmetry in an alternative way, which brings us to the second striking consequence of the background independence of noncommutative gauge theory. The twisted reduced model we found above is intrinsically infinite-dimensional, because of its ground state (there are no finite-dimensional representations of the Heisenberg commutation relations). So of course the path integral is still formal and must be made sense of. We will now show that there is a finite-dimensional version of noncommutative gauge theory that provides a non-perturbative regularization, and thus establishes the existence of noncommutative Yang-Mills theory as quantum field theory within a rigorous framework. Again, there is no analog of this regularization in the ordinary, commutative case.

A regulated, $N \times N$ matrix model of $U(1)$ noncommutative gauge theory is provided by the *twisted Eguchi-Kawai model*:

$$S_{TEK} = -\frac{1}{4g^2}\sum_{i\neq j} z_{ij}\,\mathrm{tr}\left(V_i V_j V_i^\dagger V_j^\dagger\right) \qquad (133)$$

where $V_i \in U(N)$, $i = 1, ..., d$, $z_{ij} = e^{\frac{2\pi Q_{ij}}{N}}$ with $Q_{ij} = -Q_{ji} \in \mathbb{Z}$. Let ϵ be a dimensionful lattice spacing. Identifying $V_i = e^{i\epsilon \hat{C}_i}$, the action S_{TEK} becomes the reduced model action above for noncommutative gauge theory in the limit $\epsilon \to 0$, $N \to \infty$ with $\left(\theta^{-1}\right)_{ij} = \frac{2\pi Q_{ij}}{\epsilon^2 N}$. Thus, the twisted Eguchi-Kawai model is the natural non-perturbative version of noncommutative Yang-Mills theory. This standard "trick" is due to Weyl- by exponentiating the Heisenberg algebra, finite-dimensional representations are possible for certain (discrete) values of θ.

This unitary matrix model originates as the one-plaquette reduction of ordinary Wilson lattice gauge theory in d-dimensions with multivalued gauge fields (and background 't Hooft flux):

$$S_W = -\frac{1}{4g^2}\sum_x\sum_{i\neq j}\mathrm{tr}(U_i(x)\,U_j(x+\epsilon v_i) \times U_i(x+\epsilon v_j)^\dagger\, U_j(x)^\dagger) \qquad (134)$$

where the sum over x runs through a periodic hypercubic lattice, v_i are standard basis vectors of $\Re^d$ and $U_j(x) \in U(N)$. Periodicity here (and below) is a non-perturbative form of UV/IR mixing.

Now dimensionally reduce this action to the point $x = 0$. Generate gauge fields at other corners of the plaquette, using multivaluedness, from $U_i \equiv U_i(0)$ via the twisted boundary conditions:

$$U_i(\epsilon v_j) = \Gamma_j U_i \Gamma_j^\dagger \qquad (135)$$

("large gauge transformations") where Γ_i are transition functions given by twist-eating solutions of the 't Hooft algebra:

$$\Gamma_i\Gamma_j = z_{ij}\Gamma_j\Gamma_i \qquad (136)$$

Substitute into reduced Wilson action at $x = 0$, use 't Hooft algebra and define $V_i := U_i\Gamma_i \in U(N)$. We obtain the action S_{TEK} from above.

In the 1980's, this was originally used as a matrix model which is equivalent to ordinary Yang-Mills gauge theory in the large N limit. We will demonstrate that for finite N the model admits another interpretation, which proves that noncommutative Yang-Mills theory is a twisted large N reduced model to all orders of perturbation theory. Some basic properties are:

- Gauge symmetry: $V_i \longmapsto \Omega V_i \Omega^\dagger$, $\Omega \in U(N)$
- Vacuum: $V_i^{(0)} = \Gamma_i$ (twist eaters for $SU(N)$)

Let us look for the irreducible representations of twist eaters: rotate 't Hooft matrix $Q \longmapsto S^T Q S$, $S \in SL(d, \mathbb{Z})$, $Q \equiv (Q_{ij})$ into canonical skew-diagonal form:

$$Q = \begin{pmatrix} 0 & -q_1 & \ldots & 0 & 0 \\ q_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -q_n \\ 0 & 0 & \ldots & q_n & 0 \end{pmatrix}$$

with $q_a \in \mathbb{Z}$, $a = 1, ..., n$. Define relatively prime sets of n integers: $N_a := \frac{N}{gcd(N,q_n)}$, $q_a' := \frac{q_a}{gcd(N,q_a)}$

and $N := N_0 (N_1 \dots N_n)$ (n is the dimension of the irreducible representation of Weyl-'t Hooft algebra). Then:

$$\Gamma_{2a-1} := 1_{N_1} \otimes \dots \otimes V_{N_a} \otimes \dots \otimes 1_{N_n} \otimes 1_{N_0} \tag{137}$$

$$\Gamma_{2a} := 1_{N_1} \otimes \dots \otimes (W_{N_a})^{q_a'} \otimes \dots \otimes 1_{N_n} \otimes 1_{N_0} \tag{138}$$

for $a = 1, \dots, n$ where:

$$V_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots 0 & 1 \\ 0 & 0 & 0 & \dots 0 & 0 \end{pmatrix}$$

is a $SU(N)$ shift matrix and

$$W_N = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{\frac{2\pi i}{N}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\frac{2\pi i (N-1)}{N}} \end{pmatrix}$$

a $SU(N)$ clock matrix, with $V_N W_N = e^{\frac{2\pi i}{N}} W_N V_N$.

For simplicity, we now assume that N is odd and $N = L^n$, $L \in \mathbf{N}$; $d = 2n$, $Q_{ij} = 2L^{n-1}\epsilon_{ij}$ and

$$\epsilon = (\epsilon_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathsf{I}_n$$

As $z_{ij} = e^{\frac{4\pi i \epsilon_{ij}}{L}}$ now, Γ_i can be constructed from $L \times L$ clock and shift matrices and hence:

$$(\Gamma_i)^L = \mathsf{I}_N \tag{139}$$

5.3.1 Matrix-Field Correspondence

Define $N \times N$ unitary unimodular matrices:

$$J_k = \prod_{i=1}^{d} (\Gamma_i)^{k_i} \prod_{i<j} e^{\frac{\pi i Q_{ij} k_i k_j}{N}} \tag{140}$$

$k \in \mathbb{Z}^d$ with

$$J_{L-k} = J_{-k} = J_k^\dagger \tag{141}$$

$$J_k J_q = e^{\frac{\pi i k.Q.q}{N}} J_{k+q} \tag{142}$$

By periodicity, there are only $N^2 = L^d$ independent J_k. We interpret k_i as momenta on a periodic lattice, restricted to a Brillouin zone $k \in \mathbb{Z}_L^d$. Below we'll see that J_k are finite (discrete) versions of the quantization maps $\hat{\Delta}(x)$ from Chapter 2.

Note that

$$[J_k, J_q] = 2i \sin\left(\frac{\pi}{N} \sum_{i<j} k_i Q_{ij} q_j\right) J_{k+q} \tag{143}$$

which is a trigonometric basis for $su(N)$ (or $gl(N, \mathbb{C})$). Taking $N \longmapsto \infty$ with k_i, $q_j \ll \sqrt{N}$ and rescaling J_k appropiately, we obtain the W_∞-algebra:

$$\left[J_k^\infty, J_q^\infty\right] = 2\pi i \,(k \wedge q)\, J_{k+q} \tag{144}$$

with $k \wedge q := \sum_{i<j} k_i \epsilon_{ij} q_j$. This is the Lie algebra of symplectomorphisms (canonical transformations) of $\Re^d$- this can be used to infer that the gauge group of noncommutative gauge theory is a certain "quantum deformation" of the symplectomorphism group of $\Re^d$ (with its standard symplectic structure here). This geometrical description of star-gauge transformations is important for gravitational applications.

Now we use orthonormality and completeness:

$$\frac{1}{N} \mathrm{tr}\left(J_k J_q^\dagger\right) = \delta_{k,q(\mathrm{mod} L)} \tag{145}$$

$$\frac{1}{N} \sum_{k \in \mathbb{Z}_L^d} (J_k)_{\mu\nu} (J_k)_{\lambda\rho} = \delta_{\mu\rho}\delta_{\nu\lambda} \tag{146}$$

The set $\{J_k\}_{k \in \mathbb{Z}_L^d}$ is called the Weyl basis for the linear space $gl(N, \mathbb{C})$. We can thus expand the matrices $V_i = U_i \Gamma_i$ of the twisted Eguchi-Kawai model about the classical vacuum as (just like $\hat{C}_i$ before):

$$U_i = \frac{1}{N^2} \sum_{k \in \mathbb{Z}_L^d} U_i(k) J_k \tag{147}$$

where $U_i(k) = N \mathrm{tr}\left(U_i J_k^\dagger\right)$ are c-number Fourier coefficients for expansion of a lattice field, describing dynamical degrees of freedom in the twisted Eguchi-Kawai model, on a fuzzy torus:

$$\begin{aligned} U_i(x) &= \frac{1}{N^2} \sum_{k \in \mathbb{Z}_L^d} U_i(k) e^{\frac{2\pi i k.x}{l}} \\ &= \frac{1}{N} \sum_{k \in \mathbb{Z}_L^d} e^{\frac{2\pi i k.x}{l}} \mathrm{tr}\left(U_i J_k^\dagger\right) \end{aligned} \tag{148}$$

Here $l = \epsilon L$ is a dimensionful extent of a periodic hypercubic lattice with $N^2 = L^d$ sites x^i.

In this way, the N^2 degrees of freedom of the unitary matrices U_i are transformed (absorbed) into the N^2 lattice points x^i:

$$U_i = \frac{1}{N^4}\sum_x U_i(x) \sum_{k\in\mathbb{Z}_L^d} e^{\frac{-2\pi i k.x}{l}} \quad (149)$$

which is a finite-dimensional version of the Weyl symbol in the continuum.

Because of unitarity,

$$U_i U_i^\dagger = U_i^\dagger U_i = 1_N \quad (150)$$

if and only if

$$U_i(x) \star U_i(x)^\dagger = U_i(x)^\dagger \star U_i(x) = 1 \quad (151)$$

where the lattice star-product is defined by:

$$f(x)\star g(x) = \frac{1}{N^2}\sum_{y,z} f(x+y)\, g(x+z)\, e^{2iy.\theta^{-1}.z} \quad (152)$$

with $\theta_{ij} = \frac{\epsilon^2 L}{\pi}\varepsilon_{ij}$.

We are finally ready to interpret the twisted Eguchi-Kawai model in terms of noncommutative gauge theory. Substitute $V_i = U_i\Gamma_i$, with U_i given above, into the action S_{TEK}. The key point is that Γ_i are lattice shift operators in this picture, i.e., discrete derivatives $e^{\epsilon\hat{\partial}_i}$ (with background $B = \frac{2\pi\epsilon Q}{N}$). Using the 't Hooft algebra, the definition of J_k, and the expansion of U_i above, we compute:

$$U_i(x+\epsilon v_j) = \frac{1}{N}\sum_{k\in\mathbb{Z}_L^d} \mathrm{tr}\left(\Gamma_j U_i \Gamma_j^\dagger J_k^\dagger\right) e^{\frac{-2\pi i k.x}{l}} \quad (153)$$

and then,

$$S_{TEK} = -\frac{1}{4\lambda^2}\sum_x\sum_{i\neq j} U_i(x)\star U_j(x+\epsilon v_i) \star U_i(x+\epsilon v_j)^\dagger \star U_j(x)^\dagger \quad (154)$$

where $\lambda = \sqrt{g^2 N}$ is the 't Hooft coupling constant. This is the noncommutative version of $U(1)$ Wilson lattice gauge theory.

$U(N)$ invariance of unitary matrix model becomes local noncommutative gauge invariance of lattice theory: $U_i(x) \longmapsto g(x)\star U_i(x)\star g(x+\epsilon v_i)^\dagger$ with $g(x)\star g(x)^\dagger = g(x)^\dagger \star g(x) = 1$. Thus, we can conclude that the twisted Eguchi-Kawai model is equivalent to the noncommutative $U(1)$ Yang-Mills theory.

5.3.2 Continuum limits

1. $N = L^n \to \infty$ first for finite ϵ ($\Rightarrow l \to \infty$), then $\epsilon \to 0$. It implies that $\theta \to \infty$, and only planar Feynman diagrams of noncommutative lattice gauge theory survive. This is the 't Hooft limit of ordinary large-N Yang-Mills theory on the continuum spacetime $\Re^d$. Note that in any noncommutative quantum field theory, one can show that:

$$\lim_{\theta\to\infty}\prod_{a<b} e^{\frac{i}{2}p_a.\theta.p_b} G_{conn}(p_1,\ldots,p_n;\theta) = G_{conn}^{planar}(p_1,\ldots,p_n) \quad (155)$$

   where G_{conn} is any n-point connected Green's function in momentum space. This follows from the results of sections 3.2 and 3.3, which can be used to show that nonplanar graphs tend to zero as $\theta \to \infty$ because of rapid oscillations of their Feynman integrands.

2. $N \to \infty$, $\epsilon \to 0$ with $L\epsilon^2$ finite. This implies that $l \sim \sqrt{L} = N^{\frac{1}{d}} \to \infty$ and θ is finite, hence we recover noncommutative gauge theory on flat, infinite space $\Re^d$. Thus there exists a well-defined, finite-dimensional matrix model representation of noncommutative Yang-Mills theory on $\Re^d$.

Let us give some remarks about these limits: in the first case, this result explains, e.g., the remarkable coincidence of perturbative β-functions in planar commutative and noncommutative gauge theories. In the second case, $l = \frac{\pi\theta}{\epsilon}$: finite noncommutativity in lattice regularization also requires finite-size of spacetime. This is just a non-perturbative manifestation of UV/IR mixing in noncommutative quantum field theory. In this formalism it is particularly evident that the limits $l \to \infty$ (noncommutative planar limit) and $\theta \to 0$ (commutative limit) do not commute. It also shows that the UV/IR mixing phenomenon is not merely a perturbative one.

The above construction can also be extended to get noncommutative $U(r)$ gauge theory with $r > 1$ (take $U_i \in U(r)\otimes U(N)$), and also continuum gauge theory on a noncommutative torus (take $L = n.m$, implying that $\Theta = \frac{n}{m}$, and having a finite extent ϵm as $N \to \infty$).

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7 Further Reading

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