From Super-Yang-Mills to QCD

Christoph Lüdeling*
Deutsches Elektronensynchrotron DESY
Notkestr. 85
22603 Hamburg
Germany

December 16, 2004

Abstract

This article contains lecture notes of M. Shifman from the Saalburg Summer School 2004. The topic is supersymmetric Yang-Mills theory, in particular the gluino condensate in pure SUSY gluodynamics.

1 Introduction

These are Lecture Notes from the Saalburg Summer School 2004 Lecture of M. Shifman, William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA. They deal with supersymmetric gauge theories, in particular with the calculation of the gluino condensate. A more in-depth treatment can be found in the review [1] and the book [2]. A concise and very clearly written introduction to supersymmetry can be found in the book [10] which I also recommend. There are various older papers on the subject, see e.g. [3, 4].

2 SUSY Preliminaries

2.1 SUSY Algebra

In contrast to the generators of the Lorentz group, the generators of supersymmetry transformations ("supercharges") are spinors and obey anticommutation relations. In this lectures we only consider $\mathcal{N} = 1$ supersymmetry in four spacetime

*christoph.luedeling@desy.de

dimensions. Then the SUSY generators are a left-handed Weyl spinor $\mathcal{Q}_\alpha$, $\alpha = 1, 2$, and its Hermitean conjugate $\overline{\mathcal{Q}}_{\dot\alpha}$, $\dot\alpha = 1, 2$ which satisfy the (anti)commutation relations [6]

$$\left\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\right\} = \left\{\overline{\mathcal{Q}}_{\dot\alpha}, \overline{\mathcal{Q}}_{\dot\beta}\right\} = 0 \tag{1a}$$

$$\left\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_{\dot\alpha}\right\} = 2\sigma^m_{\alpha\dot\alpha}\mathcal{P}_m \tag{1b}$$

$$\left[\mathcal{Q}_\alpha, \mathcal{P}_m\right] = \left[\overline{\mathcal{Q}}_{\dot\alpha}, \mathcal{P}_m\right] = 0 \tag{1c}$$

From Eq. (1b), wee see that the mass dimension of $\mathcal{Q}$ is $[\mathcal{Q}_\alpha] = \frac{1}{2}[\mathcal{P}_m] = \frac{1}{2}$. The last line tells us that $[\mathcal{Q}_\alpha, \mathcal{P}^2] = 0$, which means that supersymmetry transformations do not change the mass of a state.

Poincaré and supersymmetry basically exhaust the possible symmetries of the S matrix of a sensible QFT, apart from a compact internal symmetry[1]. The part of the internal symetry which does not commute with the generators of supersymmetry transformations is called R symmetry, which in the $\mathcal{N} = 1$ case is at most a $\mathsf{U}(1)$ rotating the supercharges.

2.2 Superspace

Just as the momentum operator is realised as translations $-\mathrm{i}\partial_m$ in Minkowski space, the supersymmetry generators can be represented by differential operators Q_α, $\overline{Q}_{\dot\alpha}$ on a larger space which is (of course) called superspace (it was introduced in [8]). In addition to the usual Minkowski coordinates x^m, this space contains two additional Grassmann valued (i.e. anticommuting) Weyl spinors θ^α, $\bar\theta_{\dot\alpha}$. Points in this space are thus labeled by coordinates $z^M = (x^m, \theta^\alpha, \bar\theta_{\dot\alpha})$. (Our conventions regarding spinors and σ-matrices are collected in the appendix.) Note that the θ's have mass dimension $[\theta] = \left[\bar\theta\right] = -\frac{1}{2}$. If an R symmetry is present, the θ's are rotated as well. The supersymmetry generators are then represented by the operators

$$\begin{aligned} Q_\alpha &= \partial_\alpha - \mathrm{i}\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\,, \\ \overline{Q}_{\dot\alpha} &= -\bar\partial_{\dot\alpha} + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m. \end{aligned} \tag{2}$$

Other important operators are the supersymmetry-covariant derivatives

$$\begin{aligned} D_\alpha &= \partial_\alpha + \mathrm{i}\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \\ \overline{D}_{\dot\alpha} &= -\bar\partial_{\dot\alpha} - \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m \end{aligned} \tag{3}$$

which fulfill

$$\left\{D_\alpha, \overline{D}_{\dot\alpha}\right\} = -2\mathrm{i}\sigma^m_{\alpha\dot\alpha}\partial_m\,, \qquad \left\{D_\alpha, D_\beta\right\} = \left\{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\right\} = 0\,,$$
$$\left\{D_\alpha, Q_\beta\right\} = \left\{D_\alpha, \overline{Q}_{\dot\beta}\right\} = \left\{\overline{D}_{\dot\alpha}, Q_\beta\right\} = \left\{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0.$$

[1]This is roughly the Haag–Łopuszanski–Sohnuius theorem, see [7].

2.3 Superfields

Superfields (which were introduced in the same work of Salam and Strathdee) are functions of the superspace coordinates which are defined by their expansion in the anticommuting coordinates. Since $\theta^3 = \bar{\theta}^3 = 0$, the expansion can contain only terms with at most two θ's and two $\bar{\theta}$'s:

$$F(x, \theta, \bar{\theta}) = b + \theta\chi + \bar{\theta}\bar{\phi} + \theta^2 m + \bar{\theta}^2 n + \theta\sigma^m\bar{\theta}a_m + \theta^2\bar{\theta}\bar{\psi} + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}^2 d, \qquad (4)$$

where the component fields b, m, n and d are scalars, a_m is a vector and χ, λ, $\bar{\psi}$ and $\bar{\phi}$ are left- or right-handed spinors, all functions of x^m. Moreover,

$$\theta^2 \equiv \theta^\alpha\theta_\alpha\,, \qquad \bar{\theta}^2 \equiv \theta_{\dot{\alpha}}\theta^{\dot{\alpha}}\,.$$

One may or may not impose the condition of reality, $F^\dagger = F$. The superfield F is usually referred to as the vector superfield. The reason will become clear shortly. The component fields in F form a representation of the supersymmetry algebra. They will be involved in construction of the gauge sector. For constructing supersymmetric Yang-Mills theories with matter, we will also use shorter superfields as generalisations of matter fermions. The matter fields will be components of the (irreducible) chiral multiplet ϕ, defined as

$$\overline{D}_{\dot{\alpha}}\phi = 0\,. \qquad (5a)$$

The chiral superfield can be regarded as a function defined not on the full superspace, but, rather, on the left-handed subspace parametrised by the coordinates $x_L^m = x^m + \mathrm{i}\theta\sigma^m\bar{\theta}$ and θ, with no explicit dependence on $\bar{\theta}$, since in this basis, $\overline{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}$, and thus Eq. (5a) means that ϕ does not depend on $\bar{\theta}$ explicitly. The possibility of introducing such superfield is due to the fact that the supertransformations of x_L and θ are closed, namely,

$$\delta_\eta x_L^m = 2\mathrm{i}\theta\sigma^m\bar{\eta} \qquad\qquad \delta_\eta\theta^\alpha = \eta^\alpha,$$

where η is the (Weyl spinor valued) transformation parameter.

The component expansion of the chiral superfield is particularly simple. The expansion in terms of the usual coordinates can be obtained by the Taylor expansion in the Grassmann variables,

$$\begin{aligned}\phi &= A(x_L) + \sqrt{2}\,\theta\chi(x_L) + \theta^2 F(x_L)\\ &= A(x) + \mathrm{i}\theta\sigma^m\bar{\theta}\partial_m A(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A(x)\\ &\quad + \sqrt{2}\,\theta\chi(x) - \tfrac{\mathrm{i}}{\sqrt{2}}\theta^2\partial_m\chi(x)\sigma^m\bar{\theta} + \theta^2 F(x)\,.\end{aligned} \qquad (5b)$$

Analogously, antichiral fiels $\overline{\phi}$ are defined as

$$D_\alpha \overline{\phi} = 0, \tag{5c}$$

and their expansion is best expressed in terms of $x^m_R = x^m - \mathrm{i}\theta\sigma^m\bar{\theta}$ and $\bar{\theta}$:

$$\begin{aligned}\overline{\phi} &= \overline{A}(x_R) + \sqrt{2}\,\bar{\theta}\bar{\chi}(x_R) + \theta^2\overline{F}(x_R) \\ &= \overline{A}(x) - \mathrm{i}\theta\sigma^m\bar{\theta}\partial_m\overline{A}(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box\overline{A}(x) \\ &\quad + \sqrt{2}\,\bar{\theta}\bar{\chi}(x) + \tfrac{\mathrm{i}}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\chi}(x) + \theta^2\overline{F}(x)\,.\end{aligned} \tag{5d}$$

Conjugates of chiral fields are antichiral, and a field that is both chiral and antichiral is constant. Products of chiral fields are again chiral, a product of a chiral and an antichiral field is neither chiral nor antichiral.

If we want the scalar A to have canonical mass dimension $[A] = 1$, we must assign to ϕ the dimension $[\phi] = 1$. Since $[\theta] = -\frac{1}{2}$, the spinor field χ also has the correct dimension. The field F, however, has mass dimension two. This signifies that it is an auxiliary field, appearing in the Lagrangean without derivatives, and that it can be eliminated by its equations of motion.

The physical components of the chiral superfields are scalar or spinor. To incorporate the gauge field we need to turn to Eq. (4). The vector superfield V, which is a generalisation of the gauge vector fields in non-supersymmetric theories, contains a vector field, a Weyl fermion and its conjugate (the gaugino, equivalent to a Majorana fermion) and a few other fields which can be eliminated by a supergauge transformation or are auxiliary. The reality condition

$$V^\dagger = V \tag{6a}$$

is implied. The component expansion of the vector superfield is given in Eq. (4). Supergauge transformations acting on this field are inhomogeneous. They involve chiral superfields as their parameters, ina way such that lower components of V can be gauged away. This is the famous Wess–Zumino gauge which leaves us with the following decomposition:

$$V = -\theta\sigma^m\bar{\theta}v_m + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D. \tag{6b}$$

The vector field v_m still has the usual gauge symmetry $v_m \to v_m + \partial_m\beta$ (in the Abelian case). The fields λ and D are gauge invariant in the Abelian case. In the non-Abelian case they transform homogeneously. The Wess–Zumino gauge is the one most frequently used in dealing with supersymmetric gauge theories.

On the other hand, if V becomes massive by some kind of the Higgs effect, the most convenient gauge is unitary, in which the lower components are no longer gauged away; rather, they represent additional degrees of freedom, a real scalar C and another Weyl fermion χ, as we will see later.

The vector superfield contains an auxiliary field as well, the real scalar D. Just as the F field in the chiral superfield, it has mass dimension two (since we want v_m to have canonical dimension). The auxiliary fields are needed to make the algebra close off-shell and will be useful when we consider the scalar potential of supersymmetric Yang-Mills theories.

2.4 Lagrangeans

Finally, we need a way to write down invariant actions from superfields. This can be achived by constructing Lagrangeans which vary by a total derivative under supersymmetry transformations. The highest (auxiliary) components of chiral and real superfields have exactly this property. They can be projected out by the rules of intergration over the Grassmann variables (Berezin integral [2]), which for one anticommuting variable θ are[3]

$$\int \mathrm{d}\theta\, 1 = 0 \qquad\qquad \int \mathrm{d}\theta\, \theta = 1. \tag{7}$$

This means that $[\mathrm{d}\theta] = -[\theta]$. Note also that integration and differentiation give the same result for the Grassmann variables, so a $\mathrm{d}^4\theta$-integration over a chiral field vanishes (actually, it is a total derivative, so its spacetime integral vanishes and it does not contribute to the action).

For superspace with two Grassmann coordinates, we define

$$\mathrm{d}^2\theta = -\tfrac{1}{4}\varepsilon_{\alpha\beta}\mathrm{d}\theta^\alpha \mathrm{d}\theta^\beta\,, \qquad \mathrm{d}^2\bar{\theta} = -\tfrac{1}{4}\varepsilon^{\dot{\alpha}\dot{\beta}}\mathrm{d}\bar{\theta}_{\dot{\alpha}}\mathrm{d}\bar{\theta}_{\dot{\beta}}\,, \qquad \mathrm{d}^4\theta = \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}, \tag{8}$$

with coefficients chosen such that integration over the superspace coordinates gives

$$\int \mathrm{d}^2\theta\,\theta^2 = \int \mathrm{d}^2\bar{\theta}\,\bar{\theta}^2 = \int \mathrm{d}^4\theta\,\theta^2\bar{\theta}^2 = 1.$$

So, the integral over the Grassmann coordinates projects out the F-components of chiral superfields and the D-components of real superfields. We can for example consider a theory with just a chiral superfield ϕ, i.e. a scalar and a spinor. For the

[2] Grassmann variable calculus, with applications in quantum field theory, was developed by Felix Berezin in a series of papers which were summarized in the book [9]. The book was published in Russian in 1965, and was translated immediately.

[3] This completely fixes the integration, since any function $f(\theta)$ can be expanded as $f(\theta) = a + b\theta$. This definition furthermore is unique up to a factor if we require translational invariance of the integral.

Lagrangean we need a kinetic term and, maybe, mass or interaction terms. The kinetic term is provided just by

$$\mathscr{L}_{\mathrm{kin}} = \int \mathrm{d}^4\theta \, \bar{\phi}\phi. \tag{9a}$$

The surviving terms need to have two θ's and two $\bar{\theta}$'s, so we have to collect the possible combinations from Eqs. (5b) and (5d). This gives

$$\begin{aligned}\mathscr{L}_{\mathrm{kin}} &= \tfrac{1}{4}\left(\overline{A}\Box A + \Box\overline{A}A\right) - \tfrac{1}{2}\partial_m\overline{A}\,\partial^m A + \tfrac{\mathrm{i}}{2}\left(\partial_m\bar{\chi}\bar{\sigma}^m\chi - \bar{\chi}\bar{\sigma}^m\partial_m\chi\right) + \overline{F}F \\ &= -\partial_m\overline{A}\partial^m A - \mathrm{i}\bar{\chi}\bar{\sigma}^m\partial_m\chi + \overline{F}F\end{aligned} \tag{9b}$$

and we see that indeed F appears without derivatives. In going from the first to the second line we have performed an integration by parts and dropped the total derivative terms.

Mass and interaction terms appear as purely chiral combinations of fields in the so-called superpotential,

$$\mathscr{L}_{\mathrm{SP}} = \int \mathrm{d}^2\theta \left(\underbrace{\tfrac{1}{2}m\phi^2 + \tfrac{1}{3}\lambda\phi^3}_{:= W(\phi)}\right) + \text{H.c.}, \tag{10a}$$

where m and λ are in general complex coefficients. In components, this gives

$$\mathscr{L}_{\mathrm{SP}} = mFA - \tfrac{1}{2}m\chi\chi + \lambda FA^2 - A\chi\chi + \text{H.c.} \tag{10b}$$

In Eq. (10a), a term linear in ϕ can be absorbed by a redefinition of the field.

If we now combine the two parts of the Lagrangean to $\mathscr{L} = \mathscr{L}_{\mathrm{kin}} + \mathscr{L}_{\mathrm{SP}}$, we can eliminate F and $\overline{F}$ by their equations of motion,

$$0 = \partial_m \frac{\delta\mathscr{L}}{\delta\left(\partial_m\overline{F}\right)} = \frac{\delta\mathscr{L}}{\delta\overline{F}} = F + \overline{m}\overline{A} + \bar{\lambda}\overline{A}^2. \tag{11}$$

This means that the part of the Lagrangean containing F and $\overline{F}$ reduces to

$$\mathscr{L}_F = -\left(\overline{m}\overline{A} + \bar{\lambda}\overline{A}^2\right)\left(mA + \lambda A^2\right) = -\left|\frac{\partial W}{\partial\phi}\right|^2 \equiv -V(A), \tag{12}$$

the potential for the scalar field which is always nonnegative.

3 Super-Yang-Mills Theory

3.1 Super-QED

Super-QED (SQED) is the supersymmetric generalisation of QED, so it must contain an electron, and a photon, together with their superpartners, selectron

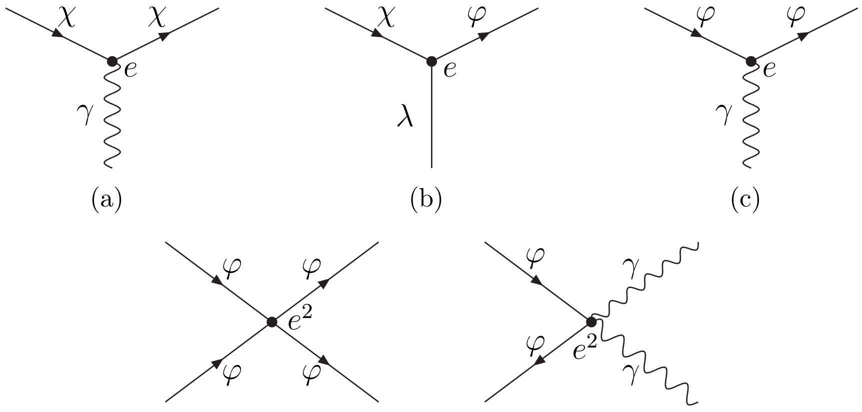


Figure 1: *Vertices of SQED. Vertex (a) is the ordinary QED photon-electron-electron vertex. Exchanging particles and their superpartners gives vertices (b) and (c). Additionally, there are the quartic scalar and scalar-photon vertices (d) and (e).*

and photino. What kinds of interactions can we expect? First of all, there should be the ordinary electron-electron-photon vertex from QED 1(a). But in this vertex, we can exchange two of the fields for their superpartners, resulting in an electron-selectron-photino vertex 1(b) and a selectron-selectron-photon vertex 1(c). We will see that there are also two quartic vertices, namely a four-selectron vertex 1(d) and a selectron-selectron-photon-photon vertex 1(e). The form of these vertices is fixed by supersymmetry, as can most convieniently be seen in superfields:

The electron and positron together form a Dirac spinor Ψ which can be decomposed into two Weyl spinors of opposite chirality, $\Psi = (\chi, \bar{\eta})^T$. Both χ and $\bar{\eta}$ have electric charge $+1$. The complex conjugate field $\bar{\Psi} = (\eta, \bar{\chi})$ also contains a left- and a right-handed spinor, both with electric charge -1. So we can describe the electron by two left handed Weyl spinors χ and η with electric charges $+1$ and -1, respectively. The conjugate fields $\bar{\chi}$ and $\bar{\eta}$ have opposite charges. We arrange χ and η in two chiral superfields:

$$Q = \varphi + \theta\chi + \theta^2 F \qquad \text{charge } +1 \tag{13a}$$

$$\widetilde{Q} = \tilde{\varphi} + \theta\eta + \theta^2 \widetilde{F} \qquad \text{charge } -1 \tag{13b}$$

They are accompanied by scalar fields φ and $\tilde{\varphi}$, the selectrons (or selectron and spositron) and by two auxiliary fields F and $\widetilde{F}$.

The photon A_m is placed in a real superfield V together with its superpartner, the photino and the auxiliary field D. In the Wess–Zumino gauge V reads

$$V = -\theta\sigma^m\bar{\theta}A_m + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D. \tag{14}$$

The gauge transformations we consider are direct generalisations of normal QED, where the fields transform as

$$\Psi \to e^{-\mathrm{i}e\alpha(x)}\Psi \tag{15a}$$

$$A_m \to A_m + \partial_m \alpha(x). \tag{15b}$$

In SQED, the gauge parameter α becomes a chiral superfield, and the gauge transformations become

$$Q \to e^{-\mathrm{i}\alpha} Q \tag{16a}$$

$$\widetilde{Q} \to e^{\mathrm{i}\alpha} \widetilde{Q} \tag{16b}$$

$$V \to V + \tfrac{1}{2}\mathrm{i}\left(\alpha - \bar{\alpha}\right), \tag{16c}$$

so the real superfield V transforms inhomogeneously. This is precisely the property that allows us to take V to be in the Wess–Zumino gauge. This procedure not only eliminates the lower components of V, but also fixes all the additional gauge parameters in the superfield α apart from the imaginary part of the scalar component which plays the rôle of the usual gauge parameter of QED. The photino and the auxilary field are gauge invariant.

The kinetic term for the chiral multiplet as it stands in Eq. (9b), however, is not gauge invariant. It has to be modified to

$$\mathscr{L}_{\text{kin,electron}} = \int \mathrm{d}^4\theta \left(\overline{Q} e^{2V} Q + \overline{\widetilde{Q}} e^{-2V} \widetilde{Q} \right). \tag{17}$$

The exponential $\exp\{2V\}$ is is actually quite simple in the Wess–Zumino gauge, since $V^3 = 0$. In zeroth order it just reproduces the kinetic terms of Eq. (9b); the higher terms mainly amount to changing ordinary derivatives to covariant ones, but introduce one important additional term that includes the D-component of V and is $D\left(\bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi}\right)$.

We also need kinetic terms for the gauge superfield and, possibly, a superpotential. The kinetic term for V is usually written in terms of the field strength superfield (or gaugino superfield)

$$W_\alpha = -\tfrac{1}{4}\overline{D}^2 D_\alpha V$$

as

$$\mathscr{L}_{\text{kin,SQED}} = \frac{1}{4\mathfrak{e}^2} \int \mathrm{d}^2\theta \, W^\alpha W_\alpha + \text{H.c.}, \tag{18}$$

where $\mathfrak{e}^2$ is the complexified coupling constant,

$$\frac{1}{\mathfrak{e}^2} = \frac{1}{e^2} + \mathrm{i}\frac{\vartheta}{8\pi^2},$$

with ϑ the vacuum angle. We are using a complex parameter (just as in Eq. (10a)) because chiral quantities depend holomorphically on these parameters, which can be a powerful tool in many calculations.

In the Abelian case we deal with now W_α is a gauge invariant chiral superfield. Its component expansion is

$$W_\alpha = -\mathrm{i}\lambda_\alpha + \theta^\beta \left(\mathrm{i}F_{\alpha\beta} + \varepsilon_{\alpha\beta} D\right) + \theta^2 \sigma^m_{\alpha\dot\alpha} \partial_m \bar\lambda^{\dot\alpha}, \tag{19}$$

where $F_{\alpha\beta}$ is the photon field strength with spinor indices,

$$F_{\alpha\beta} = \left(\partial_m A_n - \partial_n A_m\right) \sigma^{mn}_{\alpha\beta} = F_{\beta\alpha}.$$

So $\mathscr{L}_{\text{kin,SQED}}$ takes the component form

$$\mathscr{L}_{\text{kin,SQED}} = \frac{1}{4e^2}\left[-4\mathrm{i}\lambda \partial\!\!\!/ \bar\lambda + 2D^2 - F_{mn}F^{mn}\right] + \frac{\vartheta}{64\pi^2}\varepsilon^{mnkl} F_{mn} F_{kl}. \tag{20}$$

The superpotential is strongly restricted by gauge invariance. Since we have two chiral superfields of opposite charge, the only possible term (restricting to renormalisable couplings) is

$$\mathscr{L}_{\text{SP}} = \int \mathrm{d}^2\theta \, m\widetilde{Q}Q + \text{H.c.}, \tag{21}$$

since the superpotential can only depend on the neutral product $\widetilde{Q}Q$, and a term $\left(\widetilde{Q}Q\right)^2$ would already require a coefficient with mass dimension -1. Again, m is a complex parameter whose absolute value will be the mass of the electron and selectron.

Now we have kinetic terms for matter and gauge fields as as well as a superpotential. What else can be present? Actually, there is one more term, which is possible only for Abelian theories, the so-called Fayet-Iliopoulos (or ξ) term,

$$\mathscr{L}_{\text{FI}} = 2\int \mathrm{d}^4\theta \, \xi V = \xi D \tag{22}$$

which is (super)gauge invariant by itself (this is clearly seen fron Eq. (16c); the $\mathrm{d}^4\theta$ integral of chiral and antichiral superfields vanishes). It looks rather innocent, but has important consequences as we will see shortly. The coefficient of the real superfield, ξ, is real too. The the dependence on ξ is not holomorphic.

Now we can collect all the terms of the Lagrangean,

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_{\text{kin,electron}} + \mathscr{L}_{\text{kin,SQED}} + \mathscr{L}_{\text{SP}} + \mathscr{L}_{\text{FI}} \\
&= \int \mathrm{d}^4\theta \left(\overline{Q} e^{2V} Q + \overline{\widetilde{Q}} e^{-2V} \widetilde{Q} + 2\xi V \right) \\
&\quad + \left\{ \int \mathrm{d}^2\theta \left(m\widetilde{Q}Q + \frac{1}{4\mathfrak{e}^2} W^\alpha W_\alpha \right) + \text{H.c.} \right\} \\
&= -D_m \bar{\tilde{\varphi}} D^m \tilde{\varphi} - D_m \bar{\varphi} D^m \varphi - m\chi\eta - \overline{m}\bar{\chi}\bar{\eta} \\
&\quad - \frac{\mathrm{i}}{2} \left(D_m \bar{\chi} \bar{\sigma}^m \chi - \bar{\chi} \bar{\sigma}^m D_m \chi \right) - \frac{\mathrm{i}}{2} \left(D_m \bar{\eta}\, \bar{\sigma}^m \eta - \bar{\eta}\, \bar{\sigma}^m D_m \eta \right) \\
&\quad - \frac{1}{e^2} \left(2\mathrm{i}\lambda \partial\!\!\!/ \bar{\lambda} + F_{mn} F^{mn} \right) + \frac{\vartheta}{8\pi} \varepsilon^{mnkl} F_{mn} F_{kl} \\
&\quad + \overline{F} F + \overline{\widetilde{F}} \widetilde{F} + m\varphi \widetilde{F} + \overline{m}\bar{\varphi} \overline{\widetilde{F}} + m\tilde{\varphi} F + \overline{m}\bar{\tilde{\varphi}} \overline{F} \\
&\quad + \frac{2}{e^2} D^2 + \xi D + D \left(\bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi} \right) .
\end{aligned} \tag{23}$$

Here, $D_m\alpha = \partial_m\alpha \pm \mathrm{i}A_m\alpha$, where $\alpha = (\varphi, \tilde{\varphi}, \chi, \eta)$ and their conjugates and the sign depends on the charge.

The first two lines contain the kinetic terms for electrons and selectrons and mass terms for the fermions, the third line contains the gauge sector kinetic terms and the last two lines are the auxiliary field contribution. To obtain mass terms for the scalars and exhibit other interesting effects, we have to eliminate these auxiliary fields via their equations of motion.

The equations of motion are

$$0 = F + \overline{m}\bar{\tilde{\varphi}} \,, \tag{24}$$

$$0 = \widetilde{F} + \overline{m}\bar{\varphi} \,, \tag{25}$$

$$0 = D + \frac{e^2}{4} \left(\xi + \bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi} \right) , \tag{26}$$

so the auxiliary field Lagrangean becomes the potential term for the scalar fields,

$$\mathscr{L}_{\text{aux}} = -\underbrace{\left(|m|^2 \bar{\tilde{\varphi}}\tilde{\varphi} + |m|^2 \bar{\varphi}\varphi \right)}_{V_F(\varphi,\tilde{\varphi})} - \underbrace{\frac{e^2}{8} \left(\xi + \bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi} \right)^2}_{V_D(\varphi,\tilde{\varphi})} = -V(\varphi, \tilde{\varphi}). \tag{27}$$

This is called the scalar potential; it consists of two parts: the F term part and the D term part.

The scalar potential allows us to analyse the vacua of the theory. We see that the potential is a sum of squares, so it can never be negative. That means that field configurations for which $V = 0$ are vacua (and supersymmetry is unbroken in these cases). However, the existence of supersymmetric vacua depends on the parameters ξ and m. Let us consider four cases:

Case $\boldsymbol{m = \xi = 0}$: The scalar potential reduces to

$$V = V_D = (\bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi})^2 \,, \tag{28}$$

which vanishes for any field configuration where $\varphi = \tilde{\varphi}$. Both fields, being equal, can take arbitrary complex value. Thus, the vacuum manifold, i.e. the set of minima of the scalar potential V, is a complex line. What is more important is that distinct vacua on this line are not physically equivalent. This is different from the degenerate vacua situation in ordinary QFT. Typically the degenerate vacua situation takes place when a global symmetry is spontaneously broken. In this case one deals with a compact set of physically equivalent vacua (e.g. the Mexican hat potential). In supersymmety non-compact continuous vacuum manifolds are typical. They are usually called flat directions. The fields corresponding to these directions are referred to as moduli fields.

Actually, the requirement $\varphi = \tilde{\varphi}$ is not necessary, since only the absolute value enters the potential, and it might seem that there is a larger vacuum manifold. However, the relative phase of φ and $\tilde{\varphi}$ is a gauge artefact, and one can use the product $\tilde{\varphi}\varphi$ as a gauge-independent parametrisation of the vacua (sometimes called a composite modulus).

All vacua in this case have vanishing potential, so supersymmetry is unbroken, but for $\langle\varphi\rangle \neq 0$ or $\langle\tilde{\varphi}\rangle \neq 0$, the gauge symmetry is spontaneously broken. It is clear that the vacuum $\langle\varphi\rangle = 0$ *and* $\langle\tilde{\varphi}\rangle = 0$, with unbroken gauge symmetry, is special.

Case $\boldsymbol{m = 0,\ \xi \neq 0}$: In this case $V_F = 0$ still holds, but V_D gets modified,

$$V = V_D = (\bar{\varphi}\varphi - \bar{\tilde{\varphi}}\tilde{\varphi} + \xi)^2 \,. \tag{29}$$

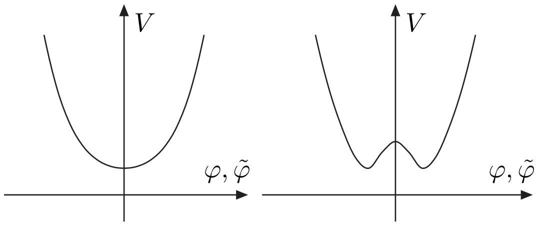
If we minimise the potential (assuming[4] $\xi > 0$), we arrive at the condition

$$\bar{\tilde{\varphi}}\tilde{\varphi} - \bar{\varphi}\varphi = \xi \,.$$

This equation has solutions which preserve supersymmetry (i.e. $V = 0$), but the gauge symmetry is always broken. Even if we put $\varphi = 0$, still $|\tilde{\varphi}|^2 = \xi$. Since in this case the absolute values of the fields are fixed, all that is left of the vacuum manifold is a compact $\mathsf{U}(1)$-circle, with all vacua on this circle being equivalent. Since φ need not vanish and its absolute value can be arbitrary, the full vacuum manifold is non-compact.

Case $\boldsymbol{m \neq 0,\ \xi = 0}$: This case is quite simple, since the potential has just one minimum at $\varphi = \tilde{\varphi} = 0$, so neither gauge nor supersymmetry is broken.

[4] If $\xi < 0$, just exchange φ and $\tilde{\varphi}$ in the following discussion.



(a) $\xi < \frac{1}{2}|m|^2$ (b) $\xi > \frac{1}{2}|m|^2$

Figure 2: *Sketch of the scalar potential for SQED with Fayet-Iliopoulos term. If $\xi < \frac{1}{2}|m|^2$, the minimum of the potential is at the origin, so gauge symmetry is unbroken (left figure, case (i)). If $\xi > \frac{1}{2}|m|^2$, the minima are at a nonzero value of some of the fields, breaking gauge symmetry (right figure, cases (ii) and (iii)). Since $V > 0$, supersymmetry is always broken for nonzero ξ and m.*

Case $m \neq 0$, $\xi \neq 0$: In the full potential, it is obvious that there will be no supersymmetric vacua, since V_F vanishes only at the origin, where V_D does not, and hence $V > 0$ always. So supersymmetry is broken. Now we proceed to find the minima:

$$0 = \frac{\partial V}{\partial \bar\varphi} = 2\left[(\bar\varphi\varphi - \bar{\tilde\varphi}\tilde\varphi + \xi) + \tfrac{1}{2}|m|^2\right]\varphi \tag{30a}$$

$$0 = \frac{\partial V}{\partial \bar{\tilde\varphi}} = 2\left[-(\bar\varphi\varphi - \bar{\tilde\varphi}\tilde\varphi + \xi) + \tfrac{1}{2}|m|^2\right]\tilde\varphi \tag{30b}$$

We distinguish four possibilities:

(i) $\varphi = \tilde\varphi = 0$: This is an extremal point of the potential where $V = \xi^2$. Whether is is a maximum or minimum depends on the magnitudes of m and ξ. This can be checked directly, but we will see that when we consider the other cases. The potential for the case where this is the minumum is sketched in Figure 2(a).

(ii) $\varphi = 0$, $\tilde\varphi \neq 0$: This means that $\frac{1}{2}|m|^2 - \xi + \bar{\tilde\varphi}\tilde\varphi = 0$, which is only possible if $\xi > \frac{1}{2}|m|^2$. The value of the potential is $V = |m|^2\left(\xi - \frac{1}{4}|m|^2\right)$, which is smaller than the value at the origin, $V(0) = \xi^2$, if the above condition holds.

(iii) $\varphi \neq 0$, $\tilde\varphi = 0$: Now we need $\frac{1}{2}|m|^2 + \xi + \bar{\tilde\varphi}\tilde\varphi = 0$, which is only possible if $\xi < -\frac{1}{2}|m|^2$. The potential is in this case $V = |m|^2\left(-\xi - \frac{1}{4}|m|^2\right) < \xi^2$. So in this case as in the one before, the extremum at the origin is actually not a minimum. The mimima occur at finite values of the fields, so gauge symmetry is broken in the vacuum. The potential in these cases is sketched in Figure 2(b).

(iv) $\varphi \neq 0$, $\tilde\varphi \neq 0$: In this case, there is no extremal point at all for nonzero mass. If $m = 0$, it reduces to the case considered before where gauge symmetry is broken, but supersymmetry is not.

3.2 SUSY Gluodynamics

After SQED, we will now consider non-Abelian gauge theories. The simplest example is a pure Super-Yang-Mills theory (SYM) without matter, sometimes called SUSY gluodynamics. This theory contains a vector field $A_m = A_m^a T^a$ and a Majorana spinor $\lambda_\alpha = \lambda_\alpha^a T^a$, both in the adjoint representation of the gauge group with generators T^a. They are again grouped in a vector superfield $V = V^a T^a$ which we can take to be in the Wess–Zumino gauge. The action looks rather simple,

$$\begin{aligned} S &= \frac{1}{2\mathfrak{g}^2} \int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha + \text{H.c.} \\ &= \int \mathrm{d}^4x \left\{ -\frac{1}{4g^2} G_{mn}^a G^{a\,mn} + \frac{\vartheta}{32\pi^2} G_{mn}^a \widetilde{G}^{a,mn} + \frac{1}{g^2} \bar{\lambda}^a \not{D} \lambda^a \right\}, \end{aligned} \tag{31}$$

where $D_m = \partial_m - \mathrm{i}A_m$ is the covariant derivative and $\mathfrak{g}$ is the complexified coupling constant,

$$\frac{1}{\mathfrak{g}^2} = \frac{1}{g^2} + \mathrm{i}\frac{\vartheta}{8\pi^2}.$$

Moreover,

$$\widetilde{G}^{a,mn} = \frac{1}{2}\,\epsilon^{mnkl}\, G_{kl}^a\,.$$

This theory is believed to be confining and is asymptotically free, i.e. the first coefficient of the β-function (Gell-Mann–Low function) is negative. For a $\mathsf{SU}(N)$ gauge group, $\beta_0 = -3N$.[5]

3.2.1 Gluino Condensate

The aspect we will focus on is the gluino condensate $\langle \lambda^a \lambda^a \rangle$, which vanishes perturbatively, but does not vanish in the nonperturbative treatment and can be computed *exactly* [1]. We will calculate its value in Chapter 3.3. Now we give the result and argue for its plausibility. The result is

$$\langle \lambda^{a,\alpha} \lambda_\alpha^a \rangle = -6N\Lambda^3 \exp\left\{ \mathrm{i}\left(\frac{2\pi k}{N} + \frac{\vartheta}{N} \right) \right\}, \qquad k = 0, 1, ..., N-1. \tag{32}$$

[5] This can be understood from the β function of a non-supersymmetric $\mathsf{SU}(N_\mathrm{C})$ gauge theory with N_F flavours, where $\beta_0 = -\frac{11}{3}N_\mathrm{C} + \frac{2}{3}N_\mathrm{F}$. The supersymmetric theory can be thought of as a usual theory with one gaugino, so $N_\mathrm{F} = 1$. There is an additional factor of N_C for the gaugino contribution because it is in the adjoint representation of the gauge group rather than in the fundamental. The graphs contributing to this term contain a Casimir operator for the representation, and the ratio of the Casimirs in the adjoint and fundamental representation is $2N_\mathrm{C}$. Furthermore, the gaugino is a Majorana fermion with only half as many degrees of freedom as the Dirac one, so there is another factor of $\frac{1}{2}$, resulting in $\beta_0 = -\frac{11}{3}N_\mathrm{C} + \frac{2}{3}N_\mathrm{C} = -3N_\mathrm{C}$.

In this expression, Λ is the scale where the theory becomes strongly coupled. It is the only dimensionful parameter, so it has to appear to the third power. The pre-factor N can be understood since the sum over group indices involves N^2 terms, but a factor of g is included in λ, and $g^2 \sim N^{-1}$. In the exponential, the parameter k, $k = 0, \ldots, N-1$ labels distinct supersymmetric vacua, reflecting the N-fold vacuum degeneracy.

The above general form can be derived from holomorphicity in conjunction with renormalization group (RG) arguments. The only RG invariant expression of dimension of mass which can be built from g and the renomalisation scale μ is $\mu \exp\{-8\pi^2(|\beta_0|g)^{-1}\}$, so

$$\begin{aligned} \langle\lambda\lambda\rangle &= c\left[\mu \exp\left\{-\frac{8\pi^2}{|\beta_0|}\frac{1}{g^2}\right\}\right]^3 \\ &= c\,\mu^3 \exp\left\{-\frac{8\pi^2}{N}\frac{1}{g^2}\right\} \end{aligned} \tag{33}$$

where c is a constant and the second equality holds for $\mathsf{SU}(N)$ gauge groups ($\beta_0 = -3N$). On the other hand, $\langle\lambda\lambda\rangle$ is a chiral quantity, so it must depend analytically on $\mathfrak{g}$, not just on g, see [1],

$$\langle\lambda\lambda\rangle = c'\mu^3 \exp\left\{-\frac{8\pi^2}{N}\left(\frac{1}{g^2} + \mathrm{i}\frac{\vartheta}{8\pi^2}\right)\right\} = c'\mu^3 \exp\left\{-\frac{8\pi^2}{N}\frac{1}{g^2}\right\} \exp\left\{\mathrm{i}\frac{\vartheta}{N}\right\} \tag{34}$$

with a possibly different constant c'. Next we notice that physical results must be 2π-periodic in the vacuum angle ϑ. The above result is not, so we have to change the last factor in Eq. (34) to

$$\exp\left\{\mathrm{i}\left(\frac{2\pi k}{N} + \frac{\vartheta}{N}\right)\right\}.$$

The parameter k labels the distinct (but degenerate) vacua, of which there are N This is exactly the value of the Witten index I_W in $\mathsf{SU}(N)$ theories [5], which in fact counts the number of vacua. If there are no matter fields, supersymmetry is always unbroken.

3.2.2 Existence of N Vacua

We can also deduce the number of vacua by considering a global symmetry of the Lagrangean (31) (besides supersymmetry). There is a global chiral $\mathsf{U}(1)$ symmetry which acts as

$$\lambda \to e^{\mathrm{i}\alpha}\lambda\,, \tag{35a}$$

$$\bar{\lambda} \to e^{-\mathrm{i}\alpha}\bar{\lambda}\,. \tag{35b}$$

This global symmetry corresponds to the current $J^m = \bar{\lambda}\bar{\sigma}^m\lambda$ which is classically conserved, $\partial_m J^m = 0$.

However, in fact, this current is not conserved because of an anomaly originating from the triangle diagram in Figure (3). This means that the current is no longer conserved on the quantum level. Its divergence can be calculated exactly,

$$\partial_m J^m = \frac{N}{16\pi^2} G^a_{mn} \widetilde{G}^{a,mn} \tag{36}$$

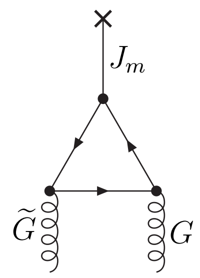


Figure 3: *Triangle anomaly graph.*

Thus, the chiral $\mathsf{U}(1)$ is not a symmetry of the theory. The anomaly explicitly breaks the chiral $\mathsf{U}(1)$ down to a discrete subgroup, namely the $\mathbb{Z}_{2N}$ corresponding to $\alpha = \pi k/N$, $k = 1, \ldots, 2N$.

This is not the end of the story. A nonvanishing $\langle\lambda\lambda\rangle$ is not invariant even under this $\mathbb{Z}_{2N} \simeq \mathbb{Z}_N \times \mathbb{Z}_2$, but breaks it further down to $\mathbb{Z}_2$. This spontaneous breaking of the discrete symmetry implies that we are left with N distinct (discrete) vacua.

3.3 Calculation of the Gluino Condensate

Now we will actually calculate the gluino condensate. To do this head on in the strong coupling regime is, however, impossible. To reach the result, we will have to take a rather complicated detour to stay with theories that are calculable, like like in crossing an ocean by jumping from small island to small island. One false step and we will be lost in the sea of noncalculability. We will go through the following steps:

- Restrict to $\mathsf{SU}(2)$ gauge group: This is mainly for technical simplicity, and since $\langle\lambda\lambda\rangle \sim N$, the result generalises to any $\mathsf{SU}(N)$.
- Introduce one flavour of matter fields (fermions and sfermions): They will generically break the gauge symmetry, making the gauge fields massive. The remaining (massless) fields can ge assembled in one chiral superfield.
- Go to the low energy effective theory: Since we are interested in the gluino condensate and hence in the vacuum structure, we can integrate out the heavy modes. We will find that instantons generate a nonperturbative superpotential. The resulting scalar potential is a runaway potential, i.e. the minima are at infinite field values. We will cure this by adding a small mass term for the formerly massless fields.
- At the next step, we will relate the vacuum expectation value of the light scalar fields to the gluino condensate via the Konishi anomaly. The result will still depend on the mass introduced before.

- Finally, we take the mass to infinity (to get rid of the matter fields we introduced in the beginning). Luckily, we have full control over the extrapolation procedure, the gluino condensate stays finite in this limit, and eventually we arrive at the result given in Eq. (32).

3.3.1 One-Flavour SQCD

From now on, we will consider only $\mathsf{SU}(2)$ as gauge group. The generators are given by the Pauli matrices, $T^a = \frac{1}{2}\tau^a$, $a = 1, 2, 3$. We also add one flavour of matter, i.e. a Dirac spinor in the fundamental representation and its scalar superpartners. The fields can be assembled in two chiral superfields Q^i, $\widetilde{Q}^j$ transforming as doublets under $\mathsf{SU}(2)$.[6] For notational brevity, we will write them as Q^i_f with a "subflavour" index $f = 1, 2$ to denote both $Q^i \equiv Q^i_1$ and $\widetilde{Q}^i \equiv Q^i_2$. The component fields are called ϕ_f (scalars), ψ_f (fermions) and F_f (auxiliaries). The matter Lagrangean is just the kinetic term, we will add a mass term only later,

$$\begin{aligned}\mathscr{L}_{\text{matter}} &= \int \mathrm{d}^4\theta \left\{ \overline{Q} e^{2V} Q + \overline{\widetilde{Q}} e^{2V} \widetilde{Q} \right\} = \int \mathrm{d}^4\theta \, \overline{Q}_f e^{2V} Q_f \\ &= -\bar{\psi}_f \mathrm{i} \not{D} \psi_f - D^m \overline{\phi}_f D_m \phi_f + \tfrac{\mathrm{i}}{\sqrt{2}} \left[\overline{\phi}_f \left(\lambda \psi_f \right) - \text{H.c.} \right] \\ &\quad + D^a \overline{\phi}_f T^a \phi_f + \overline{F}_f F_f .\end{aligned} \tag{37}$$

Since there is no superpotential so far, we can forget about the F-terms. The D-term, however, gives a contribution to the scalar potential after elimination of D^a through its equation of motion (there is a D^2 term from the kinetic Lagrangean of the gauge field),

$$V = V_D = \frac{1}{g^2} \left(\bar{\phi}_f T^a \phi_f \right)^2 . \tag{38}$$

This potential vanishes for the following field values:

$$\phi_1 = v \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \phi_2 = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{39}$$

where v is a complex number. *A priori*, ϕ_1 and ϕ_2 could have different phases, but the relative phase can be gauged away. That the potential indeed vanishes can be seen from the explicit form of the generators:

- $a = 1, 2$: The generators are off-diagonal, and the product of the form

  $$(1, 0) \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  vanishes for any X, Y.

[6] Actually, they should transform as doublet and antidoublet, but for $\mathsf{SU}(2)$ the antidoublet is isomorphic to the doublet (via $\widetilde{Q}^i = \varepsilon^{ij} \widetilde{Q}_j$).

before				after				
SF	Component field	b	f	SF	Component field	b	f	M
V	3 gluons	6		V_m	3 W bosons	9		M_g
	3 gluinos		6		3 gluinos		6	
Q	2 complex scalars	4			3 gauge fermions		6	
	2 Weyl fermions		4		3 real scalars	3		
$\widetilde{Q}$	2 complex scalars	4		X	1 complex scalar	2		0
	2 Weyl fermions		4		1 Weyl fermion		2	

Table 1: *Degrees of freedom before and after gauge symmetry breaking. In the SF column, we give the superfield and in the next column the contained component fields. b and f denote bosonic and fermionic degrees of freedom, respectively, and M is the mass of the fields. Before symmetry breaking, all fields are massless. V_m denotes a massive vector superfield, including the additional fermion and scalar, and X is a chiral multiplet which includes the fields which are still massless after the Higgs mechanism.*

- $a = 3$: τ^3 is diagonal, but traceless, and hence

$$\phi_1 \tau^3 \phi_1 + \phi_2 \tau^3 \phi_2 = v^2 - v^2 = 0.$$

So there is a non-compact vacuum manifold parameterised by v. The definition of v in Eq. (39) seems to be gauge dependent. In fact, the degree of gauge dependence is minimal, since v^2 is gauge independent. This is clearly seen from

$$v^2 = \tfrac{1}{2} \phi^i_f \phi^j_g \varepsilon_{ij} \varepsilon^{fg} \equiv H^2. \tag{40}$$

Hence the flat direction is a one-dimensional complex line. For any value of $v^2 \neq 0$, the gauge symmetry is broken and the gauge bosons acquire the mass $M_g = gv$, proportional to the as yet undetermined parameter v.

We will later see that the theory actually wants to develop a large v. If then $M_g \gg 1$, the theory is weakly coupled and we can actually calculate everything. The gluons thus behave rather like W bosons.

3.3.2 Low-Energy Theory and Nonperturbative Superpotential

The gauge symmetry breaking and Higgs mechanism reshuffle the degrees of freedom, since the vector bosons eat some part of the scalar fields acquiring longitudinal components. The degrees of freedom before and after gauge symmetry breaking are listed in Table 1.

To analyse the vacuum structure, we can integrate out the heavy fields and consider just the low-energy effective action. All we are left with is a chiral multiplet

X defined by the formula

$$X^2 = Q^i_f Q^j_g \varepsilon_{ij} \varepsilon^{fg}\,.$$

Its scalar component is the Higgs field $H \equiv v$, and there are superpartners,

$$X = H + \sqrt{2}\,\theta\psi + \theta^2 F_X\,. \tag{41}$$

For the resulting Lagrangean, we get the usual kinetic term and nothing else, at the perturbative level. However, instantons induce a nonperturbative contribution to the superpotential. This contribution is severely restriced by the non-anomalous R-symmetry of the theory, on which I will dwell momentarily. The only allowed form is

$$\mathscr{L} = \int \mathrm{d}^4\theta\, \overline{X} X + \left\{ c \int \mathrm{d}^2\theta \frac{\Lambda^5}{X^2} + \text{H.c.} \right\}. \tag{42}$$

The nonperturbative nature of the superpotential can be seen from its dependence on Λ^5, as opposed to perturbative terms which could contain only powers of $(\ln \Lambda)^{-1}$. Division by X is to be understood as an expansion in θ,

$$\begin{aligned} \frac{1}{X^2} &= \frac{1}{H^2 + 2\sqrt{2}\,\theta\chi H + \theta^2\,(2FH - \chi\chi)} \\ &= \frac{1}{H^2} - \frac{1}{H^3}\left(2\sqrt{2}\,\theta\chi + 2\theta^2 F\right) - \frac{1}{H^4}\theta^2\chi\chi. \end{aligned} \tag{43}$$

Now I return to the issue of the R-symmetry, which is a global (chiral) $\mathsf{U}(1)$ symmetry of the fundamental theory with respect to rotations of the fermions and sfermions. R-symmetry does not commute with supersymmetry, as it requires a rotation of the Grassmann parameters θ and $\bar\theta$. Assume that $\mathrm{d}\theta \to e^{\mathrm{i}\alpha}\mathrm{d}\theta$. (Naturally $\mathrm{d}\bar\theta \to e^{-\mathrm{i}\alpha}\mathrm{d}\bar\theta$.) Since the kinetic term of the gauge fields is proportional to $\int \mathrm{tr}\, W^2 \mathrm{d}^2\theta$ the invariance of the Lagrangean then requires that $W \to e^{-\mathrm{i}\alpha} W$ which entails, in turn that $\lambda \to e^{-\mathrm{i}\alpha}\lambda$.

The $\mathsf{U}(1)$ charges of the matter fermions are determined by anomaly cancellation: we want the R current to be conserved not only classically, but, rather, exactly conserved. The exactly conserved R current is

$$\bar\lambda \bar\sigma^m \lambda - 2 \sum_f \bar\psi^f \bar\sigma^m \psi_f\,.$$

The emergence of the factor 2 in front of the matter fermions is explained by the fact that for $\mathsf{SU}(2)$ the contributions to the anomaly of the gaugino is twice larger that that of the matter fermions, see footnote before Eq. (32). Therefore, from the law of the R rotation of λ above we deduce that $\psi_f \to e^{2\mathrm{i}\alpha}\psi_f$. What remains to be established is the R charge of the sfermion field ϕ_f. The easiest way is to examine the vertex $(\psi_f \lambda)\bar{\phi}^f$ in the Lagrangean. Its invariance requires that $\bar{\phi}^f \to e^{-\mathrm{i}\alpha}\bar{\phi}^f$, which implies that $\phi^f \to e^{\mathrm{i}\alpha}\phi^f$. Finally, let us note that if $\mathrm{d}\theta \to e^{\mathrm{i}\alpha}\mathrm{d}\theta$ then $\theta \to e^{-\mathrm{i}\alpha}\theta$. Combining this with the above results we conclude that the matter superfields are transformed as $Q^i_f \to e^{\mathrm{i}\alpha}Q^i_f$ while $V \to V$. The R-charges of the fields are collected in table 2. The R-charge n means that a field ϕ transforms as $\phi \to e^{\mathrm{i}n\alpha}\phi$.

ϕ	λ	φ_f	ψ_f	θ	$\mathrm{d}\theta$
n	-1	1	2	-1	1

Table 2: *R-charges of the fields.*

The non-anomalous R-symmetry must be conserved in the low-energy effective action which emerges after integration over the heavy fields. Note that $X \to e^{\mathrm{i}\alpha}X$; in other words, the R charge of X is unity. Since the kinetic terms are $\sim \overline{X}X$, there is no restriction from R-symmetry on the kinetic term. The superpotential is quite constrained, however. Indeed, if we impose analyticity ($W \sim X^k$), we find

$$\int \underbrace{\mathrm{d}^2\theta}_{\text{charge 2}} X^k \to e^{\mathrm{i}(2+k)\alpha} \int \mathrm{d}^2\theta \, X^k, \tag{44}$$

so the superpotential must be $\sim X^{-2}$. To guarantee the appropriate mass dimension, we then have to insert the fifth power of Λ, the only dimensionful parameter of the theory. Note that exactly Λ^5 emerges from the one-instanton measure.

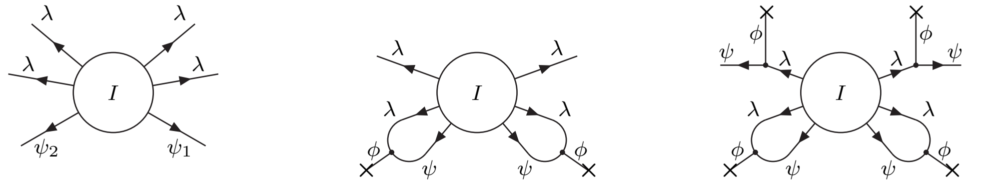
Now, the superpotential is completely fixed up to a coefficient. This coefficient, in principle, could be zero. Therefore, strictly speaking, we do not yet know whether a superpotential is generated nonperturbatively, but if it is, it must be of the form given in Eq. (42).

3.3.3 Instantons Generate this Superpotential

We will now check that instantons do generate this superpotential. Instantons (at weak coupling) are classical solutions to the gauge field equations of motion with non-trivial topology in field space and finite action $S_* \neq 0$. Their contributions are suppressed by $\exp\{-S_*\} = \exp\{-8\pi^2/g^2\}$, and so perturbative effects (corresponding to $S = 0$) are dominant. But in the case of the superpotential, there are no perturbative contributions (this is the famous non-renormalisation theorem for superpotentials), so the instanton effects decide the outcome.

Our model is constructed in such a way that at $v \neq 0$ the gauge symmetry of the model is completely broken and *all* gauge bosons become massive (in fact, very heavy if v is large). Why is it that the requirement of the complete breaking of the gauge symmetry is so important?

If all gauge bosons are heavy (much heavier than Λ) the theory is at weak coupling. This means that all calculations, including instanton calculations, are under complete control and are reliable. If, on the other hand, a non-Abelian



(a) *Generic Instanton with all zero modes*

(b) *Four zero modes eliminated*

(c) *Remaining λ zero modes converted to ψ's*

Figure 4: *Instanton contribution to the superpotential. Fermionic zero modes are convoluted and converted using the vacuum expaectation value of ϕ*

subgroup remains unbroken,the analysis is dragged, with necessity, into a strong coupling domain, where theoretical control is lost. Technically this loss manifests itself in the fact that, with unbroken non-Abelian gauge group, integrals over the instanton size become divergent, and there is no way one can make instanton calculation well-defined.

After this remark let us pass to determination of the coefficient c in the model at hand. Usually, instanton analyses are quite cumbersome. Supersymmetry makes it rather trivial, however.

The instanton suppression factor $\exp\{-8\pi^2/g^2\}$ quoted above is just the classical instanton action. To carry out the actual calculation we must integrate over fluctuations of all fields over the instanton solution. To this end one expands the field fluctuations in modes. There are a few zero modes, and an infinite number of non-zero modes. Handling non-zero modes is the most labour-intensive and time-consuming part of the problem. In supersymmetric theories one can prove, however, that all nonzero modes cancel each other, so the instanton analysis reduces to that of zero modes [11].

Pictorially, the instanton is represented by a circle with an I in it in Figure 4. In the model under consideration the instanton has four gaugino zero modes and two zero modes corresponding to two subflavors of the matter fermions (see Fig. a). In order to determine the number of the zero modes one does not have to actually find them. The anomaly relation (36), combined with

$$\int d^4x \frac{1}{16\pi^2} \left(G^a_{mn} \tilde{G}^{a,\,mn} \right)_{\text{inst}} = 2,$$

fixes the number of the gaugino zero modes, while a similar anomaly relation for the matter fermion axial current determines the number of the matter fermion zero modes. All fermion lines are directed outward, so the instanton breaks the chiral symmetry. It makes all anomalies explicit. The chiral currents of both gauginos and matter fermions are anomalous. As I have already mentioned, however, the combination $j_\lambda - 2j_\psi$ is not.

At first sight, it seems that the fact that there are six fermion zero modes in the instanton background forbids generation of the effective Lagrangean (42) (see also Eq. (43)). Indeed, this effective Lagrangean requires two and only two fermion zero modes, since the only term surviving after integration over $\mathrm{d}^2\theta$ is the last one in Eq. (43). Let us not rush to hasty conclusions, however.

The fundamental Lagrangean of our model contains the vertex $(\psi_f\lambda)\bar{\phi}^f$. It allows one to convolute a pair of the gaugino zero modes with those of the matter fields exploiting the non-vanishing vacuum value of $\bar{\phi}^f$, Fig. b. Next, we use the same vertex again to trade two remaining gluino zero modes into ψ^2 (Fig. c). Thus, the one-instanton contribution does generate a $\psi\psi/H^4$-term, and supersymmetry ensures that the F_X/H^3 term needed for the scalar potential is generated as well [4], although the emergence of this term is harder to see (an off-shell background field would be required). This contribution also has the Λ^5-factor needed for dimensional reasons. Higher instanton effects vanish. For instance, two instantons would produce Λ^{10} in the superpotential, which, as we know on general grounds, is impossible.

3.3.4 Runaway Vacuum and the Quark Mass Term

Now that we have established generation of the superpotential in Eq. (42), we can use the expansion in (43) and perform the $\mathrm{d}^2\theta$-integration. Note that the numerical constant c is fully determined by normalizations of the zero modes, which are very well known. For further technical details I refer to [11].

Eliminating the auxiliary field, we obtain the scalar potential

$$V(H) = \underbrace{V_D}_{=0} + V_F = 4c\bar{c}\frac{\Lambda^{10}}{|H|^6}, \tag{45}$$

which goes to zero at infinity and does not have a minimum at finite field values. Such a situation is usually referred to as a *runaway vacuum*.

We can, however, cure this disaster by adding a small quark mass to the superpotential of the fundamental theory (small for now, we will later take the limit $m \to \infty$, so the (s)quarks will drop out again). With the mass term switched on the superpotential will take the form

$$W(X) = mX^2 + \frac{c\Lambda^5}{X^2}\,. \tag{46}$$

Now $\partial W/\partial X$ (and hence $V(H)$) vanishes at finite values of X, namely, at

$$H^2 = \pm\sqrt{6c}\sqrt{\frac{\Lambda^5}{m}} \gg \Lambda^2 \,. \tag{47}$$

These are the minima of the scalar potential. As was expected, there are two vacua[7]. The vacuum expectation value of H^2 is stabilized far away from the origin. This justifies *a posteriori* our starting assumption that the gauge bosons are very heavy and can be integrated out.

3.3.5 The Konishi Anomaly

Thus, we found $\langle H^2 \rangle$. Remember, however, that our task was to determine the gluino condensate, $\langle \lambda^2 \rangle$. Since the gluon and gluino fields are integrated out, it might seem that information on $\langle \lambda^2 \rangle$ is lost. This is not the case — we can relate the vacuum expectation value of H^2 to the gluino condensate by virtue of the Konishi anomaly. The Konishi anomaly is a supersymmetric generalisation of the chiral anomaly, i.e. the nonconservation of the axial current which is proportional to $G_{mn}\widetilde{G}^{mn}$. This product is contained in the highest component of the superfield $\mathrm{tr}\, W^\alpha W_\alpha$, and the Konishi relation is

$$D^2\left(\widetilde{Q}Q\right) = \frac{1}{2\pi}\,\mathrm{tr}\, W^\alpha W_\alpha + Q\frac{\partial W}{\partial Q}, \tag{48}$$

where W^α is the field strength superfield and the W appearing in the last term is the superpotential. Spelled out explicitly, the last term is $4mQ^i_f Q^j_g \varepsilon^{fg}\varepsilon_{ij} \equiv 4mQ^2$. However, the left hand side of Eq. (48) is a total superspace derivative, so its expectation value in any supersymmetric vacuum must vanish. This, in turn, gives us the relation we desire, namely

$$\left\langle \frac{1}{2\pi}\,\mathrm{tr}\, W^\alpha W_\alpha + 4mQ^2 \right\rangle = 0,$$

or in particular its lowest component,

$$\left\langle \frac{1}{2\pi}\lambda^\alpha\lambda_\alpha + 4mH^2 \right\rangle = 0. \tag{49}$$

Inserting the expression for H^2 from Eq. (47), we get

$$\frac{1}{2\pi}\left\langle \lambda^\alpha\lambda_\alpha \right\rangle = \pm\frac{6}{\pi}\sqrt{\Lambda^5 m}. \tag{50}$$

[7] A natural question which one might ask is why I quote here the solutions for H^2 rather than H. The point is that H^2 is gauge invariant, see the definition before Eq. (41).

3.3.6 Limit $m \to \infty$: Gluino Condensate, Finally

The above result was computed at small m. However, in the very end we want to take the limit $m \to \infty$. How do we know that Eq. (50) is still valid?

Here we can exploit the fact that $\lambda\lambda$ is a chiral quantity which must depend *analytically* on the (complex) mass parameter m, so its functional dependence is determined by its singularity structure. Furthermore, we can consider the R-symmetry of the theory. Invariance of $\mathrm{d}^2\theta\, mX^2$ under the R transformation requires the R-charge of m to be -4. Since the R-charge of λ^2 is -2, the analytic m dependence of $\lambda\lambda$ implies that $\lambda\lambda \propto m^k$ with $k = \frac{1}{2}$, which means that Eq. (50) is exact at all values of m.

Still, the limit $m \to \infty$ of Eq. (50) seems not to make sense, since it would imply $\langle\lambda\lambda\rangle \to \infty$ as well. However, the scale $\Lambda \equiv \Lambda_{1\mathrm{fl}}$ is the scale of a theory with one flavour. In the $m \to \infty$ limit, the theory reduces again to pure SYM theory, where the relevant scale Λ_{SYM} may and does differ. Actually, there is an exact result relating the scales,

$$\Lambda^3_{\mathrm{SYM}} = \left(\Lambda^5_{1\mathrm{fl}} m\right)^{\frac{1}{2}}, \tag{51}$$

which follows from the matching of the gauge coupling running with appropriate β functions. We see that the right-hand side of this expression is precisely the right-hand side in Eq. (50). So we can now safely take the limit and arrive at

$$\langle\lambda^\alpha\lambda_\alpha\rangle = \pm 12\Lambda^3_{\mathrm{SYM}}. \tag{52}$$

This is exactly the expression advertised in Eq. (32) restricted to $\mathsf{SU}(2)$, since the exponential just reduces to ± 1.

Acknowledgements

I am very grateful to Misha Shifman for many comments and corrections of the manuscript. I also would like to thank the organisers of the Saalburg Summer School for two very interesting weeks in the forest.

A Convention and Notation

A.1 Metric

We use latin indices $m, n, \ldots$ for vectors and tensors, the metric and Levi-Civita symbols are

$$\eta_{mn} = \mathrm{diag}(+,-,-,-), \qquad \varepsilon_{0123} = -1. \tag{53}$$

A.2 Weyl Spinors

In this notes we are always using two component (Weyl) spinors in van-der-Waerden notation, i.e. with dotted and undotted greek letters as indices, indicating the $\left(\frac{1}{2},0\right)$ or $(0,\frac{1}{2})$ representation of $\mathsf{SL}(2,\mathbb{C})$[8]. These indices are raised and lowered with the ε-symbol,

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta, \qquad \chi_\alpha = \varepsilon_{\alpha\beta}\chi^\beta, \qquad \varepsilon^{12} = \varepsilon_{21} = 1, \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta. \tag{54}$$

In products of spinors, the convention is that undotted indices are contracted from upper left to lower right, and dotted ones from lower right to upper left:

$$\chi\eta \equiv \chi^\alpha\eta_\alpha \qquad \bar\chi\bar\eta \equiv \bar\chi_{\dot\alpha}\bar\eta^{\dot\alpha} \tag{55}$$

Since the spinors anticommute and raising and lowering of a contracted index also gives a factor of -1, we have

$$\chi\eta = \chi^\alpha\eta_\alpha = -\eta_\alpha\chi^\alpha = -\varepsilon_{\alpha\beta}\eta^\beta\chi^\alpha = \eta^\beta\chi_\beta = \eta\chi \tag{56}$$

und also $\bar\chi\bar\eta = \bar\eta\bar\chi$.

The rôle of the γ matrices is played by the σ matrices,

$$\sigma^m_{\alpha\dot\alpha} = \left(-\mathbb{1}_2, \sigma^i\right)_{\alpha\dot\alpha} \quad \text{and} \quad \bar\sigma^{m\,\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta} = \left(-\mathbb{1}_2, -\sigma^i\right)^{\dot\alpha\alpha} \tag{57}$$

with σ^i the Pauli matrices. We also need the “commutator” of the σ’s,

$$\left(\sigma^{mn}\right)_\alpha{}^\beta = \tfrac{1}{4}\left(\sigma^m\bar\sigma^n - \sigma^n\bar\sigma^m\right)_\alpha{}^\beta \; \left(\bar\sigma^{mn}\right)^{\dot\alpha}{}_{\dot\beta} = \tfrac{1}{4}\left(\bar\sigma^m\sigma^n - \bar\sigma^n\sigma^m\right)^{\dot\alpha}{}_{\dot\beta}, \tag{58}$$

which are antisymmetric in (mn). If one of the spinor indices is raised or lowered, they are also symmetric in $(\alpha\beta)$ or $(\dot\alpha\dot\beta)$, respectively.

Whenever possible, we omit the spinor indices and write e.g. $\chi\eta$ for $\chi^\alpha\eta_\alpha$, θ^2 for $\theta^\alpha\theta_\alpha$ and $\chi\sigma^m\bar\eta$ for $\chi^\alpha\sigma^m_{\alpha\dot\alpha}\bar\eta^{\dot\alpha}$.

[8] This is the universal cover of the Lorentz group.

A.3 Relation to Dirac and Majorana Spinors

The transition to usual four-component Dirac spinors Ψ is most easily done in the chiral (or Weyl) basis for the γ-matrices,

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}. \tag{59}$$

In this basis, a Dirac spinor Ψ directly reduces to two Weyl ones,

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi} = \left(\eta^\alpha, \bar{\chi}_{\dot{\alpha}}\right), \tag{60}$$

and kinetic and mass terms become

$$\bar{\Psi}\gamma^m \partial_m \Psi = \left(\eta^\alpha, \bar{\chi}_{\dot{\alpha}}\right) \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \partial_m \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} = \bar{\chi}\sigma^m \partial_m \chi + \eta \sigma^m \partial_m \bar{\eta}, \tag{61}$$

$$m\bar{\Psi}\Psi = m\left(\eta^\alpha, \bar{\chi}_{\dot{\alpha}}\right) \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} = m\left(\eta\chi + \bar{\chi}\bar{\eta}\right). \tag{62}$$

Majorana spinors Ψ_M, on the other hand, correspond to just one Weyl spinor and decompose as

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{63}$$

A.4 Spinor Derivatives

Some care has to be taken when raising ore lowering indices on spinorial derivatives (there is an additional minus sign). They act in the following way (note the order of the indices on the ε's):

$$\frac{\partial}{\partial \theta^\alpha}\theta^\beta \equiv \partial_\alpha \theta^\beta = \delta^\alpha_\beta \qquad \partial_\alpha \theta_\beta = \varepsilon_{\beta\alpha} \qquad \partial^\alpha \theta_\beta = \delta^\alpha_\beta \qquad \partial^\alpha \theta^\beta = \varepsilon^{\beta\alpha} \tag{64}$$

$$\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} \equiv \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \qquad \bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\alpha}} \qquad \bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \qquad \bar{\partial}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \varepsilon^{\dot{\beta}\dot{\alpha}} \tag{65}$$

Rules for the action of products of θ's:

$$\partial_\alpha \theta^\beta \theta_\beta = \left(\partial_\alpha \theta^\beta\right)\theta_\beta - \theta^\beta \partial_\alpha \theta_\beta = 2\theta_\alpha \qquad \partial^2\theta^2 = \partial^\alpha \partial_\alpha \theta^\beta \theta_\beta = 2\partial^\alpha \theta_\alpha = 4 \tag{66}$$

$$\bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} = \left(\bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}}\right)\bar{\theta}^{\dot{\beta}} - \bar{\theta}_{\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = -2\bar{\theta}_{\dot{\alpha}} \qquad \bar{\partial}^2\bar{\theta}^2 = \bar{\partial}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} = 2\bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} = 4 \tag{67}$$

A.5 Vectors and Bispinors

Vector indices m can be converted to a pair of spinor ones $\alpha\dot{\alpha}$ by contraction with the σ matrices, specifically for vectors X_m

$$X_{\alpha\dot{\alpha}} = X_m \sigma^m_{\alpha\dot{\alpha}} \qquad X_m = -\tfrac{1}{2} X_{\alpha\dot{\alpha}} \bar{\sigma}^{\alpha\dot{\alpha}}_m \qquad X^m Y_m = -\tfrac{1}{2} X^{\alpha\dot{\alpha}} Y_{\alpha\dot{\alpha}}, \tag{68}$$

and for antisymmetric tensors F_{mn},

$$F_\alpha{}^\beta = F_{mn} \left(\sigma^{mn}\right)_\alpha{}^\beta \qquad F_\alpha{}^\beta \left(\sigma^{mn}\right)_\beta{}^\alpha = -F^{mn} - \tfrac{1}{2}\mathrm{i}\widetilde{F}^{mn}, \tag{69}$$

where the dual tensor appears as the imaginary part.

A.6 Useful Identities

Spinor Products:

$$\theta^\alpha \theta^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta\theta, \quad \theta_\alpha\theta_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \quad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \quad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \tag{70}$$

$$\theta_\alpha \theta^\beta = -\tfrac{1}{2}\theta\theta\delta^\beta_\alpha, \quad \theta^\alpha\theta_\beta = \tfrac{1}{2}\theta\theta\delta^\alpha_\beta \quad \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\bar{\theta}\bar{\theta}\delta^{\dot{\beta}}_{\dot{\alpha}}, \quad \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\bar{\theta}\bar{\theta}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{71}$$

Rules for σ matrices:

$$\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{n\,\dot{\alpha}\beta} = -\eta^{mn}\delta^\beta_\alpha + 2(\sigma^{mn})_\alpha{}^\beta \qquad \bar{\sigma}^{m\,\dot{\alpha}\alpha}\sigma^n_{\alpha\dot{\beta}} = -\eta^{mn}\delta^{\dot{\alpha}}_{\dot{\beta}} + 2(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \tag{72}$$

$$\left(\sigma^{mn}\right)_\alpha{}^\beta = \tfrac{1}{4}\left(\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{n\,\dot{\alpha}\beta} - \sigma^n_{\alpha\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\beta}\right) \quad \left(\bar{\sigma}^{mn}\right)^{\dot{\alpha}}{}_{\dot{\beta}} = \tfrac{1}{4}\left(\bar{\sigma}^{m\,\dot{\alpha}\alpha}\sigma^n_{\alpha\dot{\beta}} - \bar{\sigma}^{n\,\dot{\alpha}\alpha}\sigma^m_{\alpha\dot{\beta}}\right) \tag{73}$$

$$\left(\sigma^{mn}\right)_\alpha{}^\alpha = \left(\bar{\sigma}^{mn}\right)^{\dot{\alpha}}{}_{\dot{\alpha}} = 0 \qquad \mathrm{tr}(\sigma^m\bar{\sigma}^n) = -2\eta^{mn} \tag{74}$$

$$\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_m = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \qquad \sigma^m_{\alpha\dot{\alpha}}\sigma_{m\,\beta\dot{\beta}} = -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \tag{75}$$

$$\sigma^a\bar{\sigma}^b\sigma^c = \eta^{ac}\sigma^b - \eta^{bc}\sigma^a - \eta^{ab}\sigma^c + \mathrm{i}\varepsilon^{abcd}\sigma_d \tag{76}$$

$$\bar{\sigma}^a\sigma^b\bar{\sigma}^c = \eta^{ac}\bar{\sigma}^b - \eta^{bc}\bar{\sigma}^a - \eta^{ab}\bar{\sigma}^c - \mathrm{i}\varepsilon^{abcd}\bar{\sigma}_d \tag{77}$$

$$\sigma^{mn\,\beta}_\alpha\sigma^{kl\,\alpha}_\beta = -\tfrac{1}{2}\left(\eta^{mk}\eta^{nl} - \eta^{ml}\eta^{nk} + \mathrm{i}\varepsilon^{mnkl}\right) \tag{78}$$

$$\bar{\sigma}^{mn\,\dot{\beta}}{}_{\dot{\alpha}}\bar{\sigma}^{kl\,\dot{\alpha}}{}_{\dot{\beta}} = -\tfrac{1}{2}\left(\eta^{mk}\eta^{nl} - \eta^{ml}\eta^{nk} + \mathrm{i}\varepsilon^{mnkl}\right) \tag{79}$$

References

[1] M. A. Shifman and A. I. Vainshtein,"Instantons versus supersymmetry: Fifteen years later," in M. Shifman *ITEP Lectures on Particle Physics and Field theory* (World Scientific, Singapore, 1999), Vol. 2, p. 485, `hep-th/9902018`.

[2] M. A. Shifman (Ed.), *Instantons in Gauge Theories* (World Scientific, Singapore, 1994)

[3] M. A. Shifman and A. I. Vainshtein, "On holomorphic dependence and infrared effects in supersymmetric gauge theories," Nucl. Phys. B **359** (1991) 571.

M. A. Shifman and A. I. Vainshtein, "On Gluino Condensation In Supersymmetric Gauge Theories. SU(N) And O(N) Groups," Nucl. Phys. B **296** (1988) 445].

[4] I. Affleck, M. Dine and N. Seiberg,"Dynamical Supersymmetry Breaking In Supersymmetric QCD," Nucl. Phys. B **241**, 493 (1984).

[5] E. Witten,"Constraints On Supersymmetry Breaking," Nucl. Phys. B **202**, 253 (1982).

[6] Y. A. Golfand and E. P. Likhtman, " Extension Of The Algebra Of Poincare Group Generators And Violation Of P-Invariance," Pisma Zh. Eksp. Teor. Fiz. **13** (1971) 452 [JETP Lett. **13**, 323 (1971); reprinted in *Supersymmetry*, Ed. S. Ferrara, (North-Holland/World Scientific, 1987) Vol. 1, p. 7].

[7] R. Haag, J. T. Łopuszanski and M. Sohnius, "All Possible Generators Of Supersymmetries Of The S Matrix," Nucl. Phys. B **88**, 257 (1975) [Reprinted in *Supersymmetry*, Ed. S. Ferrara, (North-Holland/World Scientific, 1987) Vol. 1, p. 51].

[8] A. Salam and J. Strathdee, "Supergauge Transformations," Nucl. Phys. B **76**, 477 (1974).

[9] F. Berezin, *Methods of Second Quantization*, (Academic Press, New York, 1966).

[10] D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory*, (Institute of Physics Publishing, Bristol, 1994).

[11] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "Supersymmetric Instanton Calculus: Gauge Theories With Matter," Nucl. Phys. B **260**, 157 (1985).