Geometric Quantization & Hamiltonian Analysis in Field Theory

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PLAN

Quantum theory is defined as a unitary irreducible representation of the algebra of observables. Geometric quantization gives a way to realize this, elucidating the role of the geometry and topology of the phase space.

- Classical phase space dynamics
- Pre-quantum Hilbert space, operators, polarization
- Role of topology: $\mathcal{H}^1(M,\mathbb{R}),\ \mathcal{H}^2(M,\mathbb{R})$
- Quantizing S^2
- Configuration space for gauge fields
- Chern-Simons theory
- θ-vacua in gauge theories
- WZW action and the Dirac determinant
- Hamiltonian Analysis of Yang-Mills (2+1)

THE SYMPLECTIC STRUCTURE

Phase space = A smooth even dimensional manifold M endowed with a symplectic structure Ω

- Ω is a differential 2-form on M which is closed and nondegenerate.
  - Closed: $d\Omega = 0$
  - Nondegenerate: For any vector field ξ on M, $i_\xi \Omega = 0 \;\Rightarrow \xi = 0$

$$\Omega \;=\; \tfrac{1}{2}\, \Omega_{\mu\nu} dq^\mu \wedge dq^\nu$$

- The condition $d\Omega = 0$ becomes

$$\begin{aligned} d\Omega &= \frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= \frac{1}{3}\left[\frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} + \frac{\partial \Omega_{\alpha\mu}}{\partial q^\nu} + \frac{\partial \Omega_{\nu\alpha}}{\partial q^\mu}\right] dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= 0 \end{aligned}$$

- Interior contraction with $\xi = \xi^\mu (\partial/\partial q^\mu)$ is

$$i_\xi \Omega = \xi^\mu \Omega_{\mu\nu} dq^\nu$$

$i_\xi \Omega = 0 \Rightarrow \xi = 0$ $\equiv$ $\xi^\mu \Omega_{\mu\nu} = 0 \Rightarrow \xi^\mu = 0$; $\Longleftrightarrow$ Ω is nondegenerate as a matrix

THE SYMPLECTIC STRUCTURE (CONT'D.)

- Inverse of Ω can be defined by

$$\Omega_{\mu\nu}\,\Omega^{\nu\alpha} = \delta^{\alpha}_{\mu}$$

  (If Ω has zero modes, one has gauge symmetries.)

- Since $d\Omega = 0$, we can write

$$\Omega = d\mathcal{A} \qquad \Omega_{\mu\nu} = \frac{\partial}{\partial q^{\mu}}\mathcal{A}_{\nu} - \frac{\partial}{\partial q^{\nu}}\mathcal{A}_{\mu}$$

- What are the qualifications to this statement?
  - If there are noncontractible 2d-surfaces Σ such that

$$\int_{\Sigma} \Omega \neq 0$$

    then $\mathcal{A}$ cannot exist globally. (Equivalent to $\mathcal{H}^2(M) \neq 0$; e.g. CS, WZW theories)
  - Even if $\mathcal{H}^2(M) = 0$, one can have inequivalent $\mathcal{A}$'s. For example, $\mathcal{A}$ and $\mathcal{A} + A$ give same Ω if $dA = 0$.
    - Evidently $A = d\Lambda$ is one possibility (Canonical transformations)
    - One can have $A \neq d\Lambda$ with $dA = 0 \Longleftrightarrow \mathcal{H}^1(M) \neq 0$ (e.g. θ-vacua)

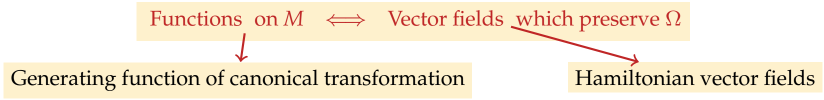
CANONICAL TRANSFORMATIONS

- Transformations of (phase space) coordinates which preserve Ω are canonical transformations.
- For infinitesimal transformations, $q^\mu \to q^\mu + \xi^\mu$, change in Ω is

$$\begin{aligned} \delta\Omega &= \left[\tfrac{1}{2}\Omega_{\mu\nu}(q+\xi)d(q^\mu+\xi^\mu)\wedge d(q^\nu+\xi^\nu) - \tfrac{1}{2}\Omega_{\mu\nu}(q)dq^\mu\wedge dq^\nu\right] \equiv L_\xi\Omega \\ &= d(i_\xi\Omega) + i_\xi d\Omega = d(i_\xi\Omega) \\ &= 0 \end{aligned}$$

  The solution is $i_\xi\Omega = -df \quad (\text{if } \mathcal{H}^1(M) = 0)$.

- Conversely, for any function f, one can define $\xi^\mu = \Omega^{\mu\nu}\partial_\nu f. \implies L_\xi\Omega = 0$.
- This leads to



Functions on M $\Longleftrightarrow$ Vector fields which preserve Ω

Generating function of canonical transformation

Hamiltonian vector fields

CANONICAL TRANSFORMATIONS (CONT'D.)

- If ξ and η preserve Ω, so does their Lie commutator

$$[\xi, \eta]^\mu = \xi^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu \xi^\mu$$

- If $\xi \leftrightarrow f$ and $\eta \leftrightarrow g$, then there is a function corresponding to $[\xi, \eta]$; this is called the Poisson bracket $-\{f, g\}$ and is defined by

$$\{f, g\} = i_\xi i_\eta \Omega = \eta^\mu \xi^\nu \Omega_{\mu\nu} = -i_\xi dg = i_\eta df = \Omega^{\mu\nu} \partial_\mu f \partial_\nu g$$

- The Poisson bracket obeys

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} &= 0 \end{aligned}$$

- Poisson brackets are important because the change in a function on phase space due to a canonical transformation is

$$\delta F = \xi^\mu \partial_\mu F = \{F, f\}$$

CANONICAL TRANSFORMATIONS (CONT'D.)

- The change in the canonical 1-form is given by

$$\delta\mathcal{A} = L_\xi \mathcal{A} = d(i_\xi \mathcal{A} - f) = d\Lambda$$

- Classical dynamics is given by

$$\Omega_{\mu\nu} \frac{\partial q^\nu}{\partial t} = \frac{\partial H}{\partial q^\mu}$$

- This can be obtained from an action

$$\mathcal{S} = \int_{t_i}^{t_f} dt \, \left(\mathcal{A}_\mu \frac{dq^\mu}{dt} \; - \; H \right)$$

- Variation of the action gives

$$\delta\mathcal{S} = i_\xi \mathcal{A}(t_f) - i_\xi \mathcal{A}(t_i) + \int dt \, \left(\Omega_{\mu\nu} \frac{dq^\nu}{dt} - \frac{\partial H}{\partial q^\mu} \right) \xi^\mu$$

- Given the action, the boundary term in its variation can be used to identify $\mathcal{A}$ and, hence, Ω.

QUANTIZATION

Quantum Theory = Unitary Irreducible Representation of the Algebra of Observables

- The problem of quantization is: How do we realize this explicitly?
  - Canonical transformations $\Longleftrightarrow$ Unitary transformations
  - (Poisson bracket) classical algebra of observables $\Longleftrightarrow$ Commutator algebra of operators
  - Ensure irreducibility
- Geometric quantization provides a way to do this

STRATEGY:

1. Define pre-quantum wave functions and pre-quantum operators
2. Impose a polarization to achieve irreducibility

QUANTIZATION (CONT'D.)

- Since canonical transformations are $\mathcal{A} \to \mathcal{A} + d\Lambda$, we consider wave functions to have the property

$$\Psi(q) \to e^{i\Lambda}\, \Psi(q), \qquad \mathcal{A} \to \mathcal{A} + d\Lambda$$

- Ψ depends on all phase space coordinates. They are analogous to fields coupled to a $U(1)$ gauge field $\mathcal{A}$. (They are sections of a line bundle on M with curvature Ω.)
- The Ψ's are pre-quantum wave functions and form a (pre-quantum) Hilbert space with the inner product

$$(1|2) \;=\; \int d\sigma(M)\ \Psi_1^*\ \Psi_2$$

$d\sigma(M) = \Omega \wedge \Omega \cdots \wedge \Omega \sim \det(\Omega)\, d^{2n}q$.

- How does Ψ change under $q^\mu \to q^\mu + \xi^\mu$? Under such a change, $\mathcal{A} \to \mathcal{A} + i_\xi \mathcal{A} - f$, so that

$$\begin{aligned} \delta\Psi &= \xi^\mu \partial_\mu \Psi \;-\; i(\, i_\xi \mathcal{A} - f)\Psi \\ &= \xi^\mu\, (\partial_\mu - i\mathcal{A}_\mu)\, \Psi + i f\, \Psi \;=\; (\xi^\mu \mathcal{D}_\mu + i f)\ \Psi \end{aligned}$$

The first term gives change of Ψ as a function, the second compensates for the change of $\mathcal{A}$.

QUANTIZATION (CONT'D.)

- Define the pre-quantum operator corresponding to f as

$$\mathcal{P}(f) = -i(\xi \cdot \mathcal{D} + if)$$

- In terms of Hamiltonian vector fields, $f \leftrightarrow \xi$, $g \leftrightarrow \eta$, $\{f, g\} \leftrightarrow -[\xi, \eta]$; this gives

$$\begin{aligned}
[\mathcal{P}(f), \mathcal{P}(g)] &= [-i\xi \cdot \mathcal{D} + f, -i\eta \cdot \mathcal{D} + g] \\
&= -[\xi^\mu \mathcal{D}_\mu, \eta^\nu \mathcal{D}_\nu] - i\xi^\mu [\mathcal{D}_\mu, g] + i\eta^\mu [\mathcal{D}_\mu, f] \\
&= i\xi^\mu \eta^\nu \Omega_{\mu\nu} - (\xi^\mu \partial_\mu \eta^\nu)\mathcal{D}_\nu + (\eta^\mu \partial_\mu \xi^\nu)\mathcal{D}_\nu - i\xi^\mu \partial_\mu g + i\eta^\mu \partial_\mu f \\
&= i\,(-\xi^\mu \eta^\nu \Omega_{\mu\nu} + i[\xi, \eta] \cdot \mathcal{D}) \\
&= i\left(-i\,(i_{[\eta,\xi]}\mathcal{D}) + \{f, g\}\right) \\
&= i\mathcal{P}(\{f, g\})
\end{aligned}$$

- The pre-quantum operators form a representation of the Poisson bracket algebra of functions on the phase space, with $[A, B] \sim i\{A, B\}$.

QUANTIZATION (CONT'D.)

- We get a representation, but this is reducible in general, since Ψ depends on all phase space variables.
- Illustrate by example: Point-particle in one space dimension

$$\Omega = dp \wedge dx, \qquad \mathcal{A} = pdx$$

- Hamiltonian vector fields and pre-quantum operators for q and p are

$$x \leftrightarrow -\frac{\partial}{\partial p}, \qquad p \leftrightarrow \frac{\partial}{\partial x}$$

$$\mathcal{P}(x) = i\frac{\partial}{\partial p} + x, \qquad \mathcal{P}(p) = -i\left(\frac{\partial}{\partial x} - ip\right) + p = -i\frac{\partial}{\partial x}$$

  $[\mathcal{P}(x), \mathcal{P}(p)] = i$, so that we have a representation of the Poisson bracket algebra.

- Consider a subset of wave functions obeying

$$\frac{\partial \Psi}{\partial p} = 0$$

  In this case, $\mathcal{P}(x) = x$, $\mathcal{P}(p) = -i\frac{\partial}{\partial x}$, which still obey $[\mathcal{P}(x), \mathcal{P}(p)] = i$.

  We have a representation on a subspace $\Longrightarrow$ previous representation is reducible.

QUANTIZATION (CONT'D.)

- Choose subsidiary conditions on Ψ which restrict its dependence to half the number of variables (Choice of polarization).
- Choose n vector fields $P_i = P_i^\mu (\partial/\partial q^\mu)$, obeying

$$\Omega_{\mu\nu} P_i^\mu P_j^\nu = 0$$

and impose

$$P_i^\mu \mathcal{D}_\mu \Psi = 0$$

The vectors P_i define the polarization. The restricted wave functions are the true wave functions of the theory.

- Inner product on the true wave functions? Generally difficult, no natural volume measure on restricted subspace of phase space.
- One case where this is possible: M is a Kähler space, Ω is the Kähler form.

QUANTIZATION (CONT'D.)

- For a Kähler space,

$$\begin{aligned} \Omega &= \Omega_{a\bar{a}} dx^a \wedge d\bar{x}^{\bar{a}} = \frac{i}{2} \partial_a \partial_{\bar{a}} K \;\; dx^a \wedge d\bar{x}^{\bar{a}} = d\mathcal{A} \\ \mathcal{A}_a &= -\frac{i}{2} \partial_a K, \qquad \mathcal{A}_{\bar{a}} \;=\; \frac{i}{2} \partial_{\bar{a}} K \\ \text{Metric } g_{a\bar{a}} &= \partial_a \partial_{\bar{a}} K \end{aligned}$$

- Since $\Omega_{ab} = 0$, choose the (holomorphic or Bargmann) polarization condition

$$\begin{aligned} \mathcal{D}_{\bar{a}} \Psi &= \left(\partial_{\bar{a}} + \tfrac{1}{2} \partial_{\bar{a}} K \right) \Psi = 0 \\ \Psi &= \exp(-\tfrac{1}{2} K) \; F \end{aligned}$$

  F is holomorphic, with $\partial_{\bar{a}} F = 0$.

- The inner product is

$$\langle 1|2 \rangle = \int d\sigma(M) \; e^{-K} \; F_1^* F_2$$

- Operator = Pre-quantum operator subject to polarization if it preserves polarization; otherwise construct matrix element directly.

TOPOLOGICAL FEATURES: $\mathcal{H}^1(M, \mathbb{R})$

- Consider $\mathcal{A}$ and $\mathcal{A} + A$ which lead to same Ω,

$$d\mathcal{A} = \Omega, \quad d(\mathcal{A} + A) = \Omega \qquad \Longrightarrow \; dA = 0$$

- $A = d\Lambda \Longrightarrow$ remove it by canonical (unitary) transformation, $\Psi \Longrightarrow e^{i\Lambda}\Psi$.
- We can have $dA = 0$ with $A \neq d\Lambda$; this means $\mathcal{H}^1(M, \mathbb{R}) \neq 0$.
- We can try $\Psi = \exp\left(i \int_0^q A\right) \Phi$.

- The path-dependence of the phase factor:
  - $\int_C A - \int_{C'} A = \oint A = \int_S dA = 0$
  - If the path $C - C'$ is noncontractible with no surface S whose boundary is $C - C'$, then $\oint A$ can be nonzero.

TOPOLOGICAL FEATURES: $\mathcal{H}^1(M,\mathbb{R})$

- Using $\Psi = \exp\left(i\int_0^q A\right)\Phi$ eliminates A but Φ need not be single-valued.
- Let $A = \theta\alpha$ where θ is a constant and $\int \alpha = 1$ for a single traversal of the basic noncontractible path corresponding to $C - C'$ (once around the red dot).
- Then for n traversals of the path, $\oint A = \theta n$.
- We can eliminate A and use Φ; but Φ is not single-valued and changes by $\exp(i\theta n)$ going around the noncontractible path n times.
- We have an extra constant θ required to define the quantum theory.
- Examples:
    - Fractional statistics in two spatial dimensions
    - Theta vacua in quantum chromodynamics

TOPOLOGICAL FEATURES: $\mathcal{H}^2(M,\mathbb{R})$

- This occurs when we have closed 2-forms which are not exact; i.e., $d\Omega = 0$, but $\Omega \neq d\mathcal{A}$ for any globally defined $\mathcal{A}$.
- Correspondingly, there are two-surfaces which are closed but are not boundaries of any 3-volumes
- If $\Omega = d\mathcal{A}$, with $\mathcal{A}$ well-defined globally, for a closed surface Σ,

$$\int_\Sigma \Omega = \int_{\partial\Sigma} \mathcal{A} = 0$$

- If $\Omega \neq d\mathcal{A}$, the integral of Ω over a closed noncontractible 2-surface can be nonzero.

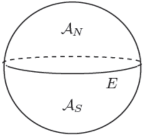
$$I(\Sigma) = \int_\Sigma \Omega$$

$$I(\Sigma) - I(\Sigma') = \int_{\Sigma-\Sigma'} \Omega = \int_V d\Omega = 0$$

- The integral of Ω over any closed two-surface is a topological invariant, invariant under small deformations of the surface.
- If Σ is contractible, deform Σ to zero $\Longrightarrow \int_\Sigma \Omega = 0$.
- Otherwise, $I(\Sigma)$ can be nonzero.

TOPOLOGICAL FEATURES: $\mathcal{H}^2(M, \mathbb{R})$

- Example of Σ as a two-sphere:
  - Cover the surface with two patches, a northern hemisphere and a southern hemisphere, with $\Omega = d\mathcal{A}_N$ and $\Omega = d\mathcal{A}_S$ on corresponding patches
  - On the overlap region, the equator E,



$$\begin{aligned} \mathcal{A}_N &= \mathcal{A}_S + d\Lambda \\ \Psi_N &= \exp(i\Lambda)\,\Psi_S \end{aligned}$$

$$\Delta\Lambda = \oint_E d\Lambda = \int_E \mathcal{A}_N - \mathcal{A}_S = \int_{\partial N} \mathcal{A}_N + \int_{\partial S} \mathcal{A}_S = \int_N \Omega + \int_S \Omega = \int_\Sigma \Omega$$

- Λ is not single-valued on the equator; but Ψ must be. Thus $\exp(i\Delta\Lambda) = 1$, or

$$\int_\Sigma \Omega = 2\pi n, \qquad (\textit{Dirac; Generalized Bohr–Sommerfeld condition})$$

- Examples of this are:
  - Charged particle in a magnetic monopole background
  - Chern-Simons and WZW theories

SUMMARIZING QUANTIZATION

We will consider quantization with the holomorphic polarization.

- A phase space which is also Kähler; the symplectic two-form must be a multiple of the Kähler form.
- The polarization condition is chosen as $\mathcal{D}_{\bar{a}} \, \Psi \; = \; 0$.
- The inner product of the prequantum Hilbert space = Square integrability on the phase space ⇒ Inner product on the true Hilbert space in the holomorphic polarization.
- $f(q)$ which preserves the polarization ⇒ Prequantum operator $\mathcal{P}(f)$ restricted to the true (polarized) wave functions.
- For observables which do not preserve the polarization, one has to construct infinitesimal unitary transformations whose classical limits are the required canonical transformations.
- If the phase space M has noncontractible two-surfaces, then the integral of Ω over any of these surfaces must be quantized in units of 2π.
- If $\mathcal{H}^1(M, \mathbb{R})$ is not zero, then there are inequivalent $\mathcal{A}$'s for the same Ω and we need extra angular parameters to specify the quantum theory completely.

QUANTIZING THE TWO-SPHERE

- Take the phase space as the two-sphere $S^2 \sim \mathbb{C}P^1 \sim SU(2)/U(1)$.
- This is a Kähler manifold; basic parameters are:

Coordinates	$z = x + iy, \qquad \bar{z} = x - iy$
Kähler two-form	$\omega = i\, dz \wedge d\bar{z}/(1 + z\bar{z})^2$
Metric	$ds^2 = dz\, d\bar{z}/(1 + z\bar{z})^2$
Riemannian curvature	$R_{1\,2} = 4\, dx \wedge dy/(1 + z\bar{z})^2$
Euler number	$\chi = \int (R_{12}/2\pi) = 2$

- S^2 has nontrivial $\mathcal{H}^2(S^2, \mathbb{R})$ given by ω.
- The symplectic two-form is taken as

$$\Omega = n\,\omega = i\,n\,\frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}$$

where n is an integer, in agreement with Dirac-Bohr-Sommerfeld condition.

QUANTIZING THE TWO-SPHERE (CONT'D.)

- The symplectic potential is

$$\begin{aligned}\mathcal{A} &= \frac{in}{2}\left[\frac{z\,d\bar{z} - \bar{z}\,dz}{(1+z\bar{z})}\right] = \frac{i}{2}\partial_{\bar{z}}K\,d\bar{z} - \frac{i}{2}\partial_z K\,dz \\ K &= n\log(1+z\bar{z})\end{aligned}$$

- Choose the polarization condition as

$$(\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}})\;\Psi = \left[\partial_{\bar{z}} + \frac{n}{2}\frac{z}{1+z\bar{z}}\right]\Psi = 0$$

- This has the solution

$$\Psi = \exp\left(-\frac{n}{2}\log(1+z\bar{z})\right)\,f(z)$$

  with the inner product

$$\langle 1|2\rangle = i(n+1)\int \frac{dz\wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}\;{f_1}^{*}f_2$$

- Normalizable states correspond to linear combinations of $f(z) = 1,\ z,\ z^2, \cdots, z^n$;

  dimension of Hilbert space $= n+1$. (Inner product normalized so that $\mathrm{Tr}(\mathbf{1}) = n+1$.)

QUANTIZING THE TWO-SPHERE (CONT'D.)

- There are three independent vector fields on S^2 which preserve the metric and ω (Hamiltonian vector fields).

Vector field	Function on phase space
$\xi_+ = i\left(\frac{\partial}{\partial\bar{z}} + z^2\frac{\partial}{\partial z}\right)$	$J_+ = -n\,\frac{z}{1+z\bar{z}}$
$\xi_- = i\left(\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial\bar{z}}\right)$	$J_- = -n\,\frac{\bar{z}}{1+z\bar{z}}$
$\xi_3 = i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right)$	$J_3 = -\frac{n}{2}\left(\frac{1-z\bar{z}}{1+z\bar{z}}\right)$

- Check one case:

$$\begin{aligned} i_{\xi_+}\Omega &= i(\partial_{\bar{z}} + z^2\partial_z) \lrcorner\, in\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2} \\ &= -n\left[-\frac{dz}{(1+z\bar{z})^2} + \frac{z^2 d\bar{z}}{(1+z\bar{z})^2}\right] \\ &= -d\left[-\frac{nz}{(1+z\bar{z})}\right] \end{aligned}$$

QUANTIZING THE TWO-SPHERE (CONT'D.)

- The pre-quantum operators are

$$\begin{aligned}
\mathcal{P}(J_+) &= \left(z^2 \partial_z - \frac{n\, z}{2} \frac{2 + z\bar{z}}{1 + z\bar{z}} \right) \; - i\xi_+^{\bar{z}} \mathcal{D}_{\bar{z}} \\
\mathcal{P}(J_-) &= \left(-\partial_z - \frac{n}{2} \frac{\bar{z}}{1 + z\bar{z}} \right) - i\xi_-^{\bar{z}} \mathcal{D}_{\bar{z}} \\
\mathcal{P}(J_3) &= \left(z\partial_z - \frac{n}{2} \frac{1}{1 + z\bar{z}} \right) - i\xi_3^{\bar{z}} \mathcal{D}_{\bar{z}}
\end{aligned}$$

- On the polarized wave functions, $\mathcal{D}_{\bar{z}}\Psi = 0$, giving the quantum operators acting on $f(z)$,

$$\begin{aligned}
\hat{J}_+ &= z^2 \partial_z - n\, z \\
\hat{J}_- &= -\partial_z \\
\hat{J}_3 &= z\partial_z - \tfrac{1}{2}\, n
\end{aligned}$$

- These obey $SU(2)$ algebra.
- The full Hilbert space corresponds to one UIR of $SU(2)$ with $j = n/2$.

QUANTIZING THE TWO-SPHERE (CONT'D.)

- The form of the action is

$$\begin{aligned} \mathcal{S} &= \int dt\, \mathcal{A}_\mu \frac{dq^\mu}{dt} = i\frac{n}{2}\int dt\, \frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{1+z\bar{z}} \\ &= i\frac{n}{2}\int dt\ \mathrm{Tr}(\sigma_3\, g^{-1}\dot{g}) \end{aligned}$$

$g \in SU(2)$; explicitly

$$g = \frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 & z \\ -\bar{z} & 1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

- More generally, one can take, for $g \in G$,

$$\mathcal{S} = i\sum_a w_a \int dt\, \mathrm{Tr}(\, t^a\, g^{-1}\dot{g}), \qquad \mathcal{A}(g) = i\sum_a w_a \mathrm{Tr}(t^a g^{-1} dg)$$

Weights of a UIR

Diagonal Generators

Ω on G/H, H = maximal subgroup of G commuting with $\sum_a w_a t^a$.

- Hilbert space will give one UIR of G, highest weights given by w_a

CONFIGURATION SPACE FOR GAUGE FIELDS

Analyze topology and geometry of the space of gauge fields in a Hamiltonian description

- Choose $A_0 = 0$ gauge; we are then left with the spatial components $A_i(x)$ which are Lie-algebra-valued vector fields on space.
- A gauge transformation acts on A_i as $A_i \rightarrow A_i^g = g^{-1} A_i g + g^{-1} \partial_i g, \;\; g \in G$.
- Define

$$
\begin{aligned}
\tilde{\mathcal{A}} \quad &\equiv \quad \{\text{Set of all gauge potentials } A_i\} \\
&\equiv \quad \{\text{Set of all Lie} - \text{algebra} - \text{valued vector fields on space } \mathbb{R}^d\} \\
\mathcal{G} \quad &\equiv \quad \{\text{Set of all } g(\vec{x}) : \mathbb{R}^d \rightarrow G, \text{ such that } g(\vec{x}) \longrightarrow \text{ constant} \in G \text{ as } |\vec{x}| \longrightarrow \infty\} \\
\mathcal{G}_* \quad &\equiv \quad \{\text{Set of all } g(\vec{x}) : \mathbb{R}^d \rightarrow G, \text{ such that } g(\vec{x}) \longrightarrow 1 \text{ as } |\vec{x}| \longrightarrow \infty\}
\end{aligned}
$$

- Evidently $\mathcal{G}/\mathcal{G}_* = G$. This acts as a Noether symmetry classifying charged states in the theory.
- $\mathcal{G}_*$ is the true gauge symmetry, with A_i and A_i^g physically equivalent for $g(x) \in \mathcal{G}_*$.

CONFIGURATION SPACE FOR GAUGE FIELDS (CONT'D.)

- The physical configuration space is $\mathcal{C} = \tilde{\mathcal{A}}/\mathcal{G}_*$
- Consider $2+1$ dimensions

$$\Pi_2(\mathcal{C}) = \Pi_1(\mathcal{G}_*) = \Pi_3(G) = \begin{cases} \mathbb{Z} & \text{All compact } G \neq SO(4) \\ \mathbb{Z}\times\mathbb{Z} & G = SO(4) \end{cases}$$

- How does this arise?
  - An element of $\mathcal{G}_*$ is $g(\vec{x})$ with $g \to 1$ at spatial infinity $\Rightarrow$ $\Pi_0(\mathcal{G}_*) = \Pi_2(G) = 0$.
  - For connectivity, examine closed paths starting and ending at $g(\vec{x}) = 1$. Such a path is given by $g(\vec{x}, \lambda)$; $0 \leq \lambda \leq 1$ parametrizes path, with $g(\vec{x}, 0) = g(\vec{x}, 1) = 1$.
  - $g(\vec{x}, \lambda) : \mathbb{R}^3 \to G$ with $g \to 1$ at the 'boundary'. This is equivalent to a map from S^3 to G, classified by $\Pi_3(G)$.
- There are noncontractible two-surfaces in $\mathcal{C}$ and hence in the phase space.

Gauge theories in $2+1$ dimensions have $\mathcal{H}^2(M, \mathbb{R}) \neq 0$; they can show Dirac quantization conditions (depending on choice of Ω)

CONFIGURATION SPACE FOR GAUGE FIELDS (CONT'D.)

- Consider $3+1$ dimensions

$$\Pi_1(\mathcal{C}) = \Pi_0(\mathcal{G}_*) = \Pi_3(G) = \begin{cases} \mathbb{Z} & \text{All compact simple } G \neq SO(4) \\ \mathbb{Z}\times\mathbb{Z} & G = SO(4) \end{cases}$$

- How does this arise? Similar reasoning as for $2+1$ dimensions
- There are noncontractible paths in $\mathcal{C}$ and hence in phase space.
- The phase space is multiply connected with connectivity given by $\mathbb{Z}$ (or $\mathbb{Z}\times\mathbb{Z}$ for $SO(4)$).

Gauge theories in 3+1 dimensions have $\mathcal{H}^1(M, \mathbb{R}) \neq 0$; the quantum theory will require additional vacuum angles (θ-vacua) to characterize it.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS

- The action is given by

$$\begin{aligned}\mathcal{S} &= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} \mathrm{Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]\\ &= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} d^3x\,\epsilon^{\mu\nu\alpha}\,\mathrm{Tr}\left[A_\mu\partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right]\end{aligned}$$

  Σ is usually taken as a Riemann surface.

- Choose $A_0 = 0$ as a gauge condition; then

$$\mathcal{S} = -\frac{ik}{\pi}\int dt d\mu_\Sigma\,\mathrm{Tr}(A_{\bar{z}}\partial_0 A_z) \qquad \Longrightarrow \qquad \mathcal{A} = -\frac{ik}{\pi}\int_\Sigma \mathrm{Tr}\big(A_{\bar{z}}\delta A_z\big) + \delta\rho[A]$$

- The symplectic two-form is

$$\Omega = -\frac{ik}{\pi}\int_\Sigma d\mu_\Sigma\,\mathrm{Tr}\big(\delta A_{\bar{z}}\delta A_z\big) = \frac{ik}{2\pi}\int_\Sigma d\mu_\Sigma\,\delta A^a_{\bar{z}}\delta A^a_z$$

- The space of 2-d gauge potentials is Kähler with the Kähler potential

$$K = \frac{k}{2\pi}\int_\Sigma A^a_{\bar{z}}A^a_z$$

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (CONT'D.)

- (Time-independent) gauge transformations act on the potentials as

$$A^g = gAg^{-1} - dgg^{-1} \approx A - D\theta \qquad \text{infinitesimally}$$

- The infinitesimal transformations are generated by the vector field

$$\xi = -\int_\Sigma \Big((D_z\theta)^a \frac{\delta}{\delta A_z^a} + (D_{\bar{z}}\theta)^a \frac{\delta}{\delta A_{\bar{z}}^a}\Big)$$

  Acting on Ω we get

$$\begin{aligned} i_\xi\Omega &= -\int \Big((D_z\theta)^a \frac{\delta}{\delta A_z^a} + (D_{\bar{z}}\theta)^a \frac{\delta}{\delta A_{\bar{z}}^a}\Big) \lrcorner \frac{ik}{2\pi}\int_\Sigma d\mu_\Sigma\, \delta A_{\bar{z}}^a \delta A_z^a \\ &= -\frac{ik}{2\pi}\int \left[((\bar{D}\theta)^a \delta A_z^a - (D\theta)^a \delta A_{\bar{z}}^a\right] = \frac{ik}{2\pi}\int \theta^a (\bar{D}\delta A_z - D\delta A_{\bar{z}})^a \\ &= \frac{ik}{2\pi}\int \theta^a \delta F_{\bar{z}z}^a = -\delta\left[\int \theta^a \frac{ik}{2\pi} F_{z\bar{z}}^a\right] \end{aligned}$$

- The generator of gauge transformations is

$$G^a = \frac{ik}{2\pi} F_{z\bar{z}}^a$$

  This has to vanish on wave functions, $G^a\Psi = 0$.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (CONT'D.)

- The prequantum wave functions have the inner product

$$(1|2) \;=\; \int d\mu(A_z, A_{\bar{z}})\; \Psi_1^*[A_z, A_{\bar{z}}]\; \Psi_2[A_z, A_{\bar{z}}]$$

- The symplectic potential is

$$\mathcal{A} \;=\; -\frac{ik}{2\pi}\int_\Sigma \mathrm{Tr}\big(A_{\bar{z}}\delta A_z \;-\; A_z\delta A_{\bar{z}}\big) \;=\; \frac{ik}{4\pi}\int_\Sigma \big(A^a_{\bar{z}}\delta A^a_z - A^a_z\delta A^a_{\bar{z}}\big)$$

- The covariant derivatives with $\mathcal{A}$ as the potential are

$$\nabla \;=\; \frac{\delta}{\delta A^a_z} + \frac{k}{4\pi}A^a_{\bar{z}}, \qquad \overline{\nabla} \;=\; \frac{\delta}{\delta A^a_{\bar{z}}} - \frac{k}{4\pi}A^a_z$$

- The Bargmann polarization condition is $\nabla\,\Psi = 0$, with the solution

$$\Psi \;=\; \exp\left(-\frac{k}{4\pi}\int A^a_{\bar{z}}A^a_z\right)\;\psi[A^a_{\bar{z}}] \;=\; e^{-\frac{1}{2}K}\;\psi[A^a_{\bar{z}}]$$

ψ's are antiholomorphic, depend only on $A_{\bar{z}}$'s.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (CONT'D.)

- The inner product is now

  $$\langle 1|2\rangle \;=\; \int [dA^a_{\bar z} dA^a_z]\; e^{-K(A^a_{\bar z}, A^a_z)}\; \psi_1^*\; \psi_2$$

  On the (anti)holomorphic wave functionals ψ's

  $$A^a_z\; \psi[A^a_{\bar z}] = \frac{2\pi}{k} \frac{\delta}{\delta A^a_{\bar z}}\; \psi[A^a_{\bar z}]$$

  and the condition of $G^a \Psi = 0$ becomes

  $$\left(D_{\bar z}\; \frac{\delta}{\delta A^a_{\bar z}} \;-\; \frac{k}{2\pi} \partial_z A^a_{\bar z} \right)\; \psi[A^a_{\bar z}] \;= 0.$$

- Construct a noncontracible two-surface in the configuration space. Start with the loop of gauge transformations

  $$C = g(x,\lambda), \qquad 0 \le \lambda \le 1, \qquad g(x,0) = g(x,1) \;=\; 1$$

  $$A(x,\lambda,\sigma) \;=\; (gAg^{-1} - dgg^{-1})\,\sigma \;+\; (1-\sigma)A$$

  where $0 \le \sigma \le 1$.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (CONT'D.)

- For simplicity, take the starting point as $A = 0$ to get

$$A(x, \lambda, \sigma) \;=\; -\; dg\, g^{-1}\, \sigma$$

$$\delta A(x, \lambda, \sigma) = g\, d(g^{-1}\delta g)\, g^{-1}\sigma \;+\; dg\, g^{-1} d\sigma$$

- The integral of Ω over this surface is

$$\begin{aligned}
\int \Omega \quad &= \quad \frac{k}{4\pi} \int \mathrm{Tr}(\delta A \wedge \delta A) \\
&= \quad \frac{k}{4\pi} 2 \int \mathrm{Tr}\left[d(g^{-1}\delta g) g^{-1} dg \right] \int \sigma d\sigma \\
&= \quad -2\pi\, k\, Q[g] \\
Q[g] \quad &= \quad \frac{1}{24\pi^2} \int \mathrm{Tr}(dgg^{-1})^3
\end{aligned}$$

$Q[g] =$ Winding number of the map $g : S^3 \rightarrow G \;\in \mathbb{Z}$

Dirac condition $\quad\Longrightarrow\quad$ k must be an integer.

θ-VACUA IN 3+1 DIMENSIONS

- Start with the Yang-Mills action and choose $A_0 = 0$,

$$\mathcal{S} = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a\mu\nu} \;=\; \frac{1}{2}\int d^4x\; \underbrace{\partial_0 A^a_i}_{E^a_i}\; \partial_0 A^a_i \;+\; \cdots$$

- The symplectic potential is $\mathcal{A} = \int d^3x\, E^a_i\, \delta A^a_i$ and

$$\Omega = \int d^3x\, \delta E^a_i\, \delta A^a_i \;=\; -2\int d^3x\, \mathrm{Tr}\,(\delta E_i\, \delta A_i)$$

  The condition of gauge invariance (under $g \approx 1 + \varphi$) is the Gauss law given by

$$G(\varphi)\Psi = \int d^3x\, \varphi^a (D_i E_i)^a\; \Psi = 0$$

- An element of $\mathcal{G}_*$ is a map $g(x) : \mathbb{R}^3 \to G$ with the condition $g \to 1$ at spatial infinity. These are equivalent to maps $S^3 \to G$ and are characterized by the winding number $Q[g]$.

$$\mathcal{G}_* = \sum_{Q=-\infty}^{+\infty} \oplus \mathcal{G}_{*Q}$$

  This leads to $\Pi_1(\mathcal{C}) = \mathbb{Z}$.

θ-VACUA IN 3+1 DIMENSIONS (CONT'D.)

- Construct a one-form on $\mathcal{C}$ which is closed but not exact.

$$K[A] = -\frac{1}{4\pi^2}\int \mathrm{Tr}(F \wedge \delta A) = \frac{1}{16\pi^2}\int d^3x\, \epsilon^{ijk}\, F^a_{jk}\, \delta A^a_i$$

  - Closure: $K[A] = \delta(\mathcal{S}_{CS}/2\pi)$, so using $\delta^2 = 0$, $\delta K = 0$
  - But K is not exact, even though $K = \delta(\mathcal{S}_{CS}/2\pi)$, because $\mathcal{S}_{CS}$ is not gauge-invariant. It is not a function on $\mathcal{C}$.
  - $K[A]$ is the generating element of $\mathcal{H}^1(\mathcal{C}, \mathbb{R})$.

- An example of the noncontractible loop:

$$A_i(x,\tau) = (gA_ig^{-1} - \partial_i g\, g^{-1})\tau + A_i(x)(1-\tau), \qquad 0 \leq \tau \leq 1$$

This is an open path in $\tilde{\mathcal{A}}$; the end-points are gauge transforms of each other, so it is closed in $\mathcal{C}$. If the path is contractible, it is deformable to

$$A_i(x,\tau) = A(x)^{g(x,\tau)}, \qquad g(x,0) = 1, \quad g(x,1) = g(x)$$

$g(x,\tau)$ makes $g(x)$ homotopic to $g = 1$. This is not possible if $Q[g] \neq 0$.

θ-VACUA IN 3+1 DIMENSIONS (CONT'D.)

- Integrate K along such a curve,

$$\begin{aligned}\oint K[A] &= \frac{1}{2\pi}\Big(\mathcal{S}_{CS}[A^g] - \mathcal{S}_{CS}[A]\Big)\\ &= -\frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F) \qquad\qquad \text{(Instanton number)}\\ &= -\frac{1}{32\pi^2}\int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta})\epsilon^{\mu\nu\alpha\beta}\end{aligned}$$

- Since $\delta K = 0$, we get the same Ω for $\mathcal{A}$ and $\mathcal{A} + \theta K$.

$$\mathcal{A} = \int d^3x\, E_i^a \delta A_i^a \;+\; \theta\, K[A]$$

  We need an additional parameter θ to characterize the quantum theory.

- $\oint K$ is an integer, so we can take $0 \leq \theta \leq 2\pi$.
- This is equivalent to using

$$\mathcal{S} = \mathcal{S}_{YM} \;+\; \theta\left[-\frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F)\right]$$

THE WESS-ZUMINO-WITTEN THEORY

- This is defined by an action functional in 2 Euclidean (or $1+1$) dimensions,

$$\begin{aligned}\mathcal{S}_{WZW} &= \frac{1}{8\pi}\int_{\mathcal{M}^2} d^2x\sqrt{g}\, g^{ab}\,\mathrm{Tr}(\partial_a M\,\partial_b M^{-1}) \;+\; \Gamma[M] \\ \Gamma[M] &= \frac{i}{12\pi}\int_{\mathcal{M}^3} d^3x\,\epsilon^{\mu\nu\alpha}\,\mathrm{Tr}(M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\, M^{-1}\partial_\alpha M) \\ &= \frac{i}{12\pi}\int_{\mathcal{M}^3}\mathrm{Tr}(M^{-1}dM)^3\end{aligned}$$

  $M(x) \in GL(N,\mathbb{C})$ (or suitable subgroups)

- $\Gamma[M]$ = Wess-Zumino term, defined by integration over $\mathcal{M}^3$ with $\partial\mathcal{M}^3 = \mathcal{M}^2$.
- Many $\mathcal{M}^3$'s with the same boundary $\mathcal{M}^2$ possible $\equiv$ Different ways to extend $M(x)$ to $\mathcal{M}^3$.
- If M and M' are two different extensions of the same field, then $M' = MN$, with $N = 1$ on $\mathcal{M}^2$,

$$\Gamma[MN] = \Gamma[M] + \Gamma[N] - \frac{i}{4\pi}\int_{\mathcal{M}^2} d^2x\,\epsilon^{ab}\mathrm{Tr}\,\underbrace{(M^{-1}\partial_a M\,\partial_b N N^{-1})}_{=0}$$

  $N = 1$ on $\partial\mathcal{M}^3 \Longrightarrow N$ is (equivalent to) a map $N:\; S^3 \to G$, classified by $\Pi_3(G)$ (or $Q[N]$).

THE WESS-ZUMINO-WITTEN THEORY (CONT'D)

- Independence of the extension follows from:

  1. $\Gamma[N] \;=\; 0$ for $N \approx 1$ (to linear order in $\partial N N^{-1}$).

     By successive transformations, $\Gamma[M]$ is independent of the extension to $\mathcal{M}^3$ for all N connected to identity.

  2. If N is homotopically nontrivial, $\Gamma[N] = 2\pi i\, Q[N]$

     ($\exp(-k\,\Gamma[M])$ is independent of the extension, if $k \in \mathbb{Z}$. So $\mathcal{S} = k\,\mathcal{S}_{WZW}$ can be used as the action for a theory, the WZW theory with level number k.)

- In complex coordinates

$$\mathcal{S}_{WZW} = \frac{1}{2\pi}\int_{\mathcal{M}^2} \mathrm{Tr}(\partial_z M \partial_{\bar{z}} M^{-1}) \;+ \Gamma[M]$$

$$\mathcal{S}_{WZW}[M\,h] = \mathcal{S}_{WZW}[M] + \mathcal{S}_{WZW}[h] - \frac{1}{\pi}\int_{\mathcal{M}^2} \mathrm{Tr}(M^{-1}\partial_{\bar{z}} M\,\partial_z h\,h^{-1})$$

(Polyakov-Wiegmann identity)

- Chiral splitting: Antiholomorphic derivative of M, holomorphic derivative of h

THE WESS-ZUMINO-WITTEN THEORY (CONT'D)

- Another important property $M \longrightarrow M + \delta M = (1+\theta)M, \;\; \theta = \delta M \, M^{-1}$ infinitesimal.

$$\begin{aligned} \delta \mathcal{S}_{WZW} &= -\frac{1}{\pi}\int \mathrm{Tr}\left(\partial_{\bar{z}}(\delta M M^{-1})\partial_z M M^{-1}\right) \\ &= -\frac{1}{\pi}\int \mathrm{Tr}(\delta M M^{-1}\partial_{\bar{z}} A_z) \\ &= -\frac{1}{\pi}\int \mathrm{Tr}(\delta M M^{-1} D_z \bar{C}) \\ &= -\frac{1}{\pi}\int \mathrm{Tr}(\bar{C}\,\delta A_z) = \frac{1}{2\pi}\bar{C}^a \delta A_z^a \end{aligned}$$

$A_z = -\partial_z M M^{-1}, \quad \bar{C} = -\partial_{\bar{z}} M \, M^{-1}$

$$D_z \bar{C} = \partial_z \bar{C} + [A_z, \bar{C}]$$

- A_z and $\bar{C}$ obey the equation

$$\partial_{\bar{z}} A_z - \partial_z \bar{C} \; + \; [\bar{C}, A_z] = 0, \qquad D_z\left[\frac{\delta \mathcal{S}_{WZW}}{\delta A_z}\right] = \frac{1}{2\pi}\partial_{\bar{z}} A_z$$

This will be useful for evaluating Dirac determinants.

THE WESS-ZUMINO-WITTEN THEORY (CONT'D)

- If we use $M^\dagger$, we get C rather than $\bar{C}$.

$$D_z \frac{\delta \mathcal{S}_{WZW}}{\delta A^a_{\bar{z}}} = \frac{1}{2\pi} \partial_z A_{\bar{z}}$$

- Comparing with wave function for CS theory,

$$\psi[\bar{A}] = \exp \left[k\, \mathcal{S}_{WZW}(M^\dagger) \right]$$

provided we can parametrize a general 2-dimensional gauge field as $A_z = -\partial_z M\, M^{-1}$.

THE DIRAC DETERMINANT IN TWO DIMENSIONS

Massless fermions in irreducible representation R of $U(N)$, coupled to a $U(N)$-gauge field.

- Dirac matrices: σ_i, $i = 1, 2$, $\quad \sigma_i \sigma_j + \sigma_j \sigma_i = 2\, \delta_{ij}$.

$$\mathcal{L} = \bar{\psi}(D_1 + iD_2)\psi + \bar{\chi}(D_1 - iD_2)\chi = 2\bar{\psi} D_z \psi + 2\bar{\chi} D_{\bar{z}} \chi$$

  ψ, χ: chiral components of $\Psi = (\psi, \chi)$

- A parametrization for gauge potentials

$$A_z = -\partial_z M\, M^{-1} \qquad\qquad A_{\bar{z}} = M^{\dagger\, -1} \partial_{\bar{z}} M^{\dagger}$$

  M is a complex matrix. ($\det M = 1$ if gauge group is $SU(N)$.)

- For $U(1)$, use elementary result $A_i = \partial_i \theta + \epsilon_{ij} \partial_j \phi$. $\qquad \Longrightarrow M = \exp(\phi + i\,\theta)$.

- One can invert ∂_z via

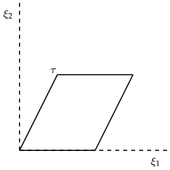
$$\left(\frac{1}{\partial_z} \right)_{xx'} = \frac{1}{\pi(\bar{z} - \bar{z}')}$$

THE DIRAC DETERMINANT IN TWO DIMENSIONS (CONT'D)

- Write $\partial_z M = -A_z M$,

$$M(x) = 1 - \int_{x'} \left(\frac{1}{\partial_z}\right)_{xx'} A_z(x') M(x')$$
$$= 1 - \int (\partial_z)^{-1} A_z + \int (\partial_z)^{-1} A_z (\partial_z)^{-1} A_z + \cdots$$

- $A \to A^g = gAg^{-1} - dg\, g^{-1} \implies M^g = gM$
- Comment: Space not simply connected $\implies \exists$ zero modes for $\partial_z \quad \implies \exists$ flat potentials a, not gauge equivalent to zero.
- Example: Torus $S^1 \times S^1$. Real coordinates $\xi_1,\ \xi_2$, $0 \le \xi_i \le 1$, with $\xi_1 = 0 \ \sim \ \xi_1 = 1$, same for ξ_2.



$z = \xi_1 + \tau \xi_2, \quad \tau =$ modular parameter

THE DIRAC DETERMINANT IN TWO DIMENSIONS (CONT'D)

- For the torus,the generalized parametrization is

$$A_z = M \left[\frac{i\pi \, a}{\text{Im } \tau} \right] M^{-1} - \partial_z M \, M^{-1}$$

- Ambiguity: M and $MV(\bar{z}) \Longrightarrow$ same A_z. (Must ensure this does not affect physical results)
- For determinant we need regularized version of $(D_z)^{-1}$

$$(\partial_z)^{-1}_{xx'} = G(x, x') = \frac{1}{\pi(\bar{x} - \bar{x}')}$$

$$D_z \phi = (\partial_z - \partial_z M M^{-1})\phi = M \partial_z (M^{-1}\phi) \quad \Longrightarrow D_z^{-1}(x, x') = \frac{M(x) M^{-1}(x')}{\pi(\bar{x} - \bar{x}')}$$

- Regularized version

$$\begin{aligned} D_z^{-1}(x, x')_{Reg} &\equiv \mathcal{G}(x, x') = \int d^2y \, \frac{M(x) M^{-1}(y)}{\pi(\bar{x} - \bar{y})} \, \sigma(x', y; \epsilon) \\ \sigma(x', y; \epsilon) &= \frac{1}{\pi\epsilon} \exp\left(-\frac{|x' - y|^2}{\epsilon} \right) \Longrightarrow \delta^{(2)}(x - x') \end{aligned}$$

The Dirac determinant in two dimensions (cont'd)

- The computation of the determinant:

$$S_{eff} \equiv \log\det D_z = \mathrm{Tr}\log D_z$$

$$\frac{\delta S_{eff}}{\delta A_z^a(x)} = \mathrm{Tr}\left[D_z^{-1}(x,x')(-it^a)\right]_{x'\to x} = \mathrm{Tr}\left[\mathcal{G}(x,x)(-it^a)\right]_{\epsilon\to 0}$$

$$\mathcal{G}(x,x) = \int d^2y\, \frac{\sigma(x,y)}{\pi}\left[\frac{1}{(\bar{x}-\bar{y})} - M\partial_z M^{-1}(x)\left(\frac{x-y}{\bar{x}-\bar{y}}\right) - M\partial_{\bar{z}}M^{-1} + \cdots\right]$$

$$\begin{aligned} \delta S_{eff} &= \int d^2x \mathrm{Tr}\left[\mathcal{G}(x,x)(-it^a)\right]_{\epsilon\to 0}\delta A_z^a(x) \\ &= \frac{1}{\pi}\int d^2x\ \mathrm{Tr}\left[\partial_{\bar{z}}MM^{-1}\delta A_z\right] = -\frac{1}{\pi}\int d^2x\ \mathrm{Tr}(\bar{C}\,\delta A_z) \end{aligned}$$

$\mathrm{Tr}(t^at^b)_R = A_R\ \mathrm{Tr}(t^at^b)_F$, A_R = index of the representation R.

$$\begin{aligned} \delta S_{eff} &= -\frac{A_R}{\pi}\int d^2x\ \mathrm{Tr}(\bar{C}\,\delta A_z)_F \\ &= A_R\ \delta\mathcal{S}_{WZW}(M) \end{aligned}$$

$$\Longrightarrow \quad \det D_z = \det(\partial_z)\ \exp\left[A_R\ \mathcal{S}_{WZW}(M)\right]$$

THE DIRAC DETERMINANT IN TWO DIMENSIONS (CONT'D)

- Our answer is not gauge-invariant; under $M \to g\,M \approx (1+\varphi)M$,

$$\delta S_{WZW} = -\frac{1}{\pi}\int d^2x\ \mathrm{Tr}(\partial_{\bar{z}} A_z\ \delta g\, g^{-1})$$

This is the two-dimensional gauge anomaly.

- The gauge-invariant expression is given by

$$\det(D_z D_{\bar{z}}) = \det(\partial_z \partial_{\bar{z}})\ \exp\left[A_R\ \mathcal{S}_{WZW}(M^\dagger M)\right]$$

$$\mathcal{S}_{WZW}(M^\dagger M) = \mathcal{S}_{WZW}(M) + \mathcal{S}_{WZW}(M^\dagger) + \underbrace{\frac{1}{\pi}\int d^2x\ \mathrm{Tr}(A_{\bar{z}} A_z)}_{\text{local counterterm}}$$

- Abelian version: This corresponds to 2-dim. QED (the Schwinger model) $\Longrightarrow$ mass term for gauge field.

$$\begin{aligned}
\det(D_z D_{\bar{z}}) &= \det(\partial_z \partial_{\bar{z}})\ \exp\left[-\frac{1}{4\pi}\int_{x,y} F_{\mu\nu}(x) G(x-y) F_{\mu\nu}(y)\right] \\
G(x-y) &= \int \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2}\ \exp[ip\cdot(x-y)]
\end{aligned}$$

HAMILTONIAN IN FIELD THEORY

- In a Hamiltonian approach, we deal with the Hamiltonian operator and wave functions.
- In field theory, these have to be regularized and renormalized.
- The wave functional is defined on a time-slice; needs counterterms defined at fixed time, in addition to the familiar counterterms from the Hamiltonian/action.
  - For the usual φ^4-theory, upon integrating modes, of $\mu_1 < k \leq \mu$,

$$\begin{aligned} \Psi(\varphi) &= U \int [d\chi]\, \Psi_0(\chi)^* \Psi(\varphi, \chi) \\ U &\simeq 1 - i\frac{3\lambda^{(0)}}{16\pi^2} \log(\mu^2/\mu_1^2) \int d^3x\, (\varphi\pi + \pi\varphi) + \ldots \end{aligned}$$

- In the Hamiltonian, T and V cannot be independently regularized.
  - Their regularizations are correlated by Lorentz symmetry.
  - One has to check the Dirac-Schwinger condition

$$[T^{00}(x),\ T^{00}(y)] = i(T^{0i}(x) + T^{0i}(y))\partial_x^i \delta(x-y)$$

This can be a delicate task.

YANG-MILLS THEORY IN $2 + 1$ DIMENSIONS

An example which illustrates many features of what we discussed is the YM theory in $2 + 1$ dimensions.

Why is this theory interesting?

- Interesting in its own right
  - YM(1+1) is exactly solvable, but has no propagating degrees of freedom
  - YM(3+1) is highly nontrivial and difficult
  - YM(2+1) has propagating degrees of freedom, it is nontrivial. Can be amenable to a Hamiltonian analysis.
  - It has a dimensional coupling constant and is super-renormalizable. This helps to simplify it.
- A real physical context for YM(2+1)
  - Mass gap of YM(2+1) $\approx$ Magnetic screening mass of YM(3+1) at high temperatures

• We will use a Hamiltonian approach because some exact calculations are possible

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HAMILTONIAN ANALYSIS OF YM(2+1)

- We choose $A_0 = 0$ and use complex coordinates $z = x_1 - ix_2$ with

$$\tfrac{1}{2}(A_1 + iA_2) = -\partial M\, M^{-1}, \quad \tfrac{1}{2}(A_1 - iA_2) = M^{\dagger\,-1}\bar{\partial} M^{\dagger}$$

  $M \in SL(N, \mathbb{C})$, for gauge group $SU(N)$. (More generally, $G \Rightarrow G^{\mathbb{C}}$.)

- Since $M \to g\, M$ under a gauge transformation, $g \in SU(N)$, $H = M^{\dagger} M \in SL(N, \mathbb{C})/SU(N)$ is gauge-invariant.

- We first calculate the volume of the gauge-invariant subspace:

$$\begin{aligned} \delta A &= -\partial(\delta M M^{-1}) + [\partial M M^{-1}, \delta M M^{-1}] \\ &= -D(\delta M M^{-1}) \\ \delta \bar{A} &= \bar{D}(M^{\dagger\,-1} \delta M^{\dagger}) \\ ds^2_{\tilde{\mathcal{A}}} &= \int d^2x\, \mathrm{Tr}(\delta A\, \delta \bar{A}) \\ &= \int \mathrm{Tr}\left[(M^{\dagger\,-1} \delta M^{\dagger})(-\bar{D} D)(\delta M M^{-1}) \right] \\ ds^2_{SL(N,\mathbb{C})} &= \int \mathrm{Tr}(M^{\dagger\,-1} \delta M^{\dagger}\; \delta M M^{-1}) \end{aligned}$$

HAMILTONIAN ANALYSIS OF YM(2+1) (CONT'D.)

- From the structure of the metric

$$d\mu_{\tilde{\mathcal{A}}} = \det(-\bar{D}D)\, d\mu_{SL(N,\mathbb{C})}(M, M^\dagger)$$

- One can do a polar decomposition $M = U\,\rho$, U unitary and ρ hermitian.

$$\begin{aligned} d\mu_{SL(N,\mathbb{C})} &= \underbrace{(dMM^{-1}) \wedge \cdots \wedge (dMM^{-1})}_{(N^2-1)\ times} \wedge \underbrace{(M^{\dagger -1}dM^\dagger) \wedge \cdots \wedge (M^{\dagger -1}dM^\dagger)}_{(N^2-1)\ times} \\ &= d\mu(H) \quad d\mu(U) \end{aligned}$$

Volume of $SL(N,\mathbb{C})/SU(N)$

Volume of $SU(N)$

- Thus the volume the gauge-invariant configuration space is

$$\begin{aligned} d\mu(\mathcal{C}) &= d\mu(H)\ \det(-\bar{D}D) = d\mu(H) \exp[2\, c_A\, S_{WZW}(H)] \\ d\mu(H) &= [d\varphi] \det R \end{aligned}$$

$H = e^{t^a \varphi^a}, \quad H^{-1}\delta H = \delta\varphi^a R_{ab}(\varphi) t^b.$

HAMILTONIAN ANALYSIS OF YM(2+1) (CONT'D.)

- Wave functions are gauge-invariant (Gauss law), depend on $H = M^\dagger M$ with the inner product

$$\langle 1|2\rangle = \int d\mu(H)\, \exp[2\, c_A\, S_{WZW}(H)]\; \Psi_1^* \Psi_2$$

- This leads to an intuitive argument for a nonzero mass gap:
  - The Hamiltonian has the form

$$\mathcal{H} = \int \tfrac{1}{2} \left[e^2 E^2 + B^2/e^2 \right]$$

$[E, B] \sim p$ (in momentum space) $\Longrightarrow \Delta E\, \Delta B \sim p$, or $\Delta E \sim p/\Delta B$

$$\mathcal{E} = \langle \mathcal{H} \rangle \approx \frac{1}{2} \left[e^2 \frac{p^2}{(\Delta B)^2} + \frac{(\Delta B)^2}{e^2} \right]$$

Minimize with respect to $\Delta B \;\Longrightarrow (\Delta B)^2 \sim p \Longrightarrow\; \mathcal{E} \sim p$. This is the photon.

  - For us

$$\langle \mathcal{H} \rangle = \int d\mu(H)\, \exp\left[2\, c_A S_{WZW}(H)\right]\; \int \tfrac{1}{2} \left[e^2 E^2 + B^2/e^2 \right]$$

HAMILTONIAN ANALYSIS OF YM(2+1) (CONT'D.)

- Expanding the WZW action

$$\langle \mathcal{H} \rangle \approx \int d\mu(H) \exp\left[-\frac{c_A}{2\pi} \int B \frac{1}{p^2} B + \ldots \right] \int \tfrac{1}{2} \left[e^2 E^2 + \tfrac{B^2}{e^2} \right]$$

Gaussian $\Longrightarrow$ $(\Delta B)^2 \sim \pi p^2 / c_A \Longrightarrow$ mass gap $\sim e^2 c_A / 2\pi$.

- More detailed analysis confirms $m = e^2 c_A / 2\pi$.
- The Wilson loop operator is given by

$$W(C) = \mathrm{Tr}\, \mathcal{P} e^{-\oint A} = \mathrm{Tr}\, \mathcal{P} \exp\left(\frac{e}{2} \oint J \right)$$

All gauge-invariant quantities can be made from the current $J = (2/e) \partial H \, H^{-1}$.

- The Hamiltonian and the wave functions can be expressed as functions of the current $J = (2/e) \partial H \, H^{-1}$.

THE HAMILTONIAN OPERATOR

- The Hamiltonian is given by

$$\mathcal{H} = \underbrace{\frac{e^2}{2} \int E^a E^a}_{} + \underbrace{\frac{1}{2e^2} \int B^a B^a}_{}$$

$$\equiv \qquad T \qquad + \qquad V$$

- The potential energy is easy to simplify,

$$V = \frac{1}{2e^2} \int B^a B^a = \frac{1}{2} \int_x : \bar{\partial} J^a(x)\, \bar{\partial} J^a(x) :$$

- The kinetic term is simplified via the chain rule

$$\begin{aligned} T\,\Psi &= -\frac{e^2}{2} \int_x \frac{\delta^2}{\delta A(x) \delta \bar{A}(x)}\, \Psi \\ &= -\frac{e^2}{2} \left[\int \underbrace{\frac{\delta J(u)}{\delta A(x)}\, \frac{\delta J(v)}{\delta \bar{A}(x)}}_{\color{red}{\Omega}}\, \frac{\delta^2 \Psi}{\delta J(u) \delta J(v)} + \int \underbrace{\frac{\delta^2 J(u)}{\delta A(x) \delta \bar{A}(x)}}_{\color{red}{\omega}}\, \frac{\delta \Psi}{\delta J(u)} \right] \\ &= \int \Omega_{ab}(u,v) \frac{\delta^2 \Psi}{\delta J^a(u) \delta J^b(v)} + \int \omega^a(u) \frac{\delta \Psi}{\delta J^a(u)} \end{aligned}$$

THE HAMILTONIAN OPERATOR (cont'd.)

- $\omega^a(u)$ needs regularization

$$\begin{aligned}\omega^a &= -\frac{e^2}{2}\int_x \frac{\delta^2 J^a(u)}{\delta A^b(x)\delta \bar{A}^b(x)} = \left(e^2 c_A/2\pi\right) M^\dagger_{am}(x)\,\mathrm{Tr}\left[t^m \bar{D}^{-1}_{reg}(y,x)\right]_{y\to x}\\ &= m\,J^a\end{aligned}$$

- The kinetic energy is thus given by

$$T = m\left[\int J^a \frac{\delta}{\delta J^a} \;+\; \int \Omega_{ab}(u,v)\frac{\delta^2}{\delta J^a(u)\delta J^b(v)}\right]$$

$$\Omega_{ab}(u,v) = \frac{c_A}{\pi^2}\,\frac{\delta_{ab}}{(u-v)^2} - i\,\frac{f_{abc}J^c(v)}{u-v} \;+\; \mathcal{O}(\epsilon)$$

- The Hamiltonian is $\mathcal{H} = \mathcal{H}_0 \;+\; \mathcal{H}_1$ with

$$\begin{aligned}\mathcal{H}_0 &= m\int_z J_a(\vec{z})\frac{\delta}{\delta J_a(\vec{z})} + \frac{2}{\pi}\int_{z,w}\frac{1}{(z-w)^2}\,\frac{\delta}{\delta J_a(\vec{w})}\,\frac{\delta}{\delta J_a(\vec{z})}\\ &\quad + \frac{1}{2}\int_x :\bar{\partial}J^a(x)\,\bar{\partial}J^a(x):\\ \mathcal{H}_1 &= i\,e\,f_{abc}\int_{z,w}\frac{J^c(\vec{w})}{\pi(z-w)}\;\frac{\delta}{\delta J_a(\vec{w})}\,\frac{\delta}{\delta J_b(\vec{z})}\end{aligned}$$

AN ASIDE ON REGULARIZATION

All calculations are done with proper regularization.

- Start with a regularization of the δ-function

$$\delta^{(2)}(u,w) \Longrightarrow \sigma(\vec{u},\vec{w},\epsilon) = \frac{1}{\pi\epsilon}\exp\left(-\frac{|u-w|^2}{\epsilon}\right)$$

- This is equivalent to

$$\begin{aligned} \bar{G}(\vec{x},\vec{y}) &= \frac{1}{\pi(x-y)} \\ \Longrightarrow \quad \bar{\mathcal{G}}(\vec{x},\vec{y}) &= \int_u \bar{G}(\vec{x},\vec{u})\,\sigma(\vec{u},\vec{y};\epsilon)H(u,\bar{y})H^{-1}(y,\bar{y}) \end{aligned}$$

- This simplifies as

$$\overline{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma} \; - \; e^{-\frac{(x-y)^2}{\epsilon}} \; [H(x,\bar{y})H^{-1}(y,\bar{y})]_{ma}\right]$$

One can check all results using regularized expressions, with a single regulator from beginning to end.

HAMILTONIAN ANALYSIS OF YM(2+1) (cont'd.)

- One can solve the Schrödinger equation to get the vacuum wave function as $\Psi_0 = \exp\left[\frac{1}{2}F(H)\right]$,

$$F(H) = -\int \bar{\partial}J_a \left[\frac{1}{(m+\sqrt{m^2-\nabla^2})}\right] \bar{\partial}J_a$$
$$+ 2\, f_{abc} \int f^{(3)}(\vec{x},\vec{y},\vec{z})\, J_a(\vec{x})\, J_b(\vec{y})\, J_c(\vec{z}) + \mathcal{O}(J^4)$$

- This has the correct expected limits

$$\Psi_0 \approx \exp\left[-\frac{1}{2e^2}\int B \frac{1}{\sqrt{-\nabla^2}} B\right] \qquad \frac{k}{m} \gg 1$$
$$\approx \exp\left[-\frac{1}{4e^2 m}\int B^2\right] \qquad \frac{k}{m} \ll 1$$

$\mathcal{O}(J^3, J^4)$ terms are small at $k \gg e^2$ and at $k \ll e^2$

- The high k limit agrees with perturbation theory

HAMILTONIAN ANALYSIS OF YM(2+1) (cont'd.)

- For quantities involving low momentum modes

$$\begin{aligned} \langle \mathcal{O} \rangle &= \int \Psi_0^* \Psi_0 \, \mathcal{O} = int \exp \left[-\frac{1}{4g^2} F_{ij}^a F^{aij} \right] \mathcal{O} \\ &= \langle \mathcal{O} \rangle_{2dYM} \end{aligned}$$

$g^2 = me^2 = e^4 c_A / 2\pi$.

- Since YM_{2d} confines

$$\langle W_R(C) \rangle = \exp \left[-\sigma_R \, \mathcal{A}_C \right]$$

$\mathcal{A}_C =$ area of the closed curve C.

- This gave values of string tension

$$\sqrt{\sigma}_R = e^2 \sqrt{\frac{c_A c_R}{4\pi}},$$

in agreement with lattice values to within $1 - 3\%$.

STRING TENSION

Group	Representations					
	k=1 Fund.	k=2 antisym	k=3 antisym	k=2 sym	k=3 sym	k=3 mixed
$SU(2)$	0.345 0.335					
$SU(3)$	0.564 0.553					
$SU(4)$	0.772 0.759	0.891 0.883		1.196 1.110		
$SU(5)$	0.977 0.966					
$SU(6)$	1.180 1.167	1.493 1.484	1.583 1.569	1.784 1.727	2.318 2.251	1.985 1.921
$SU(N)$ $N \to \infty$	0.1995 N 0.1976 N					

Comparison of $\sqrt{\sigma}/e^2$ with lattice estimates (lower entry, in red) from LUCINI &TEPER, BRINGOLTZ & TEPER. k is the rank of the representation.

THE WAVE FUNCTION: A DIFFERENT ARGUMENT

- Absorb $\exp(2c_A S_{WZW})$ from the inner product into the wave function by $\Psi = e^{-c_A S_{WZW}(H)} \Phi$. The Hamiltonian acting on Φ is

$$\mathcal{H} \to e^{-c_A S_{WZW}(H)}\, \mathcal{H}\, e^{-c_A S_{WZW}(H)}$$

- Consider $H = e^{t^a \varphi^a} \approx 1 + t^a \varphi^a + \cdots$, a small φ limit appropriate for a (resummed) perturbation theory. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \cdots \right]$$

where $\phi_a(\vec{k}) = \sqrt{c_A k\bar{k}/(2\pi m)}\, \varphi_a(\vec{k})$.

- The vacuum wave function is

$$\Phi_0 \approx \exp\left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2}\, \phi^a \right]$$

THE WAVE FUNCTION: A DIFFERENT ARGUMENT (cont'd.)

- Transforming back to Ψ,

$$\Psi_0 \approx \exp\left[-\frac{c_A}{\pi m} \int (\bar{\partial}\partial\varphi^a) \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right] (\bar{\partial}\partial\varphi^a) + \cdots \right]$$

- The full wave function must be a functional of J. The only form consistent with the above is

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2 c_A^2} \int \bar{\partial} J^a(x) \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right]_{x,y} \bar{\partial} J^a(y) \ + \cdots \right]$$

since $J \approx (c_A/\pi)\partial\varphi + \mathcal{O}(\varphi^2)$.

- This indicates the robustness of the Gaussian term in Ψ_0, since this argument only presumes
  1. Existence of a regulator, so that the transformation $\Psi \Longleftrightarrow \Phi$ can be carried out
  2. The two-dimensional anomaly calculation

STATUS REPORT

- Some things which can be calculated/understood
    - A clear gauge-invariant Hamiltonian formulation
    - Computation of the vacuum wave function
    - String tensions: lowest order,with a systematic expansion for higher order corrections (of the order a few percent)
    - Possibility of screening of $W_R(C)$ via string-breaking in some representations
    - Results on magnetic screening mass, glueballs
    - Extension to Yang-Mills-Chern-Simons Theory, Formulation on $\mathbb{R} \times S^2$
- But there are also unclear issues, more questions
    - Improving higher order corrections to string tension, better handle on glueballs
    - Calculations on the torus to understand the theory at finite temperature
    - Connecting the formulation on $\mathbb{R} \times S^2$ to the duality-matrix model approach
    - Fermions, supersymmetric cases
    - Geometrical properties of the configuration space $\tilde{\mathcal{A}}/\mathcal{G}_*$