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Topological Solitons

Lecture notes for the

XIII Saalburg summer school

Wolfersdorf, Thüringen,

Germany

September 2007

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1 Introduction

Solitons are exact solutions to classical non-linear field equations. They are localised and have a finite energy. In this sense, they behave like ordinary particles. Solitons and multi-solitons are stable because they carry a topological charge N, which is an integer and equals the net number of particles. Because the topological charge is a conserved quantity, a single soliton cannot decay. Notice, that the conservation of N is not due to a Noether theorem, but to the topological structure of the soliton. A soliton is described by its collective coordinates, like its centre and orientation. The space of collective coordinates is referred to as moduli space. Quantisation of a soliton proceeds by quantising its collective coordinates, but this is only mentioned briefly here.

To find solutions to a field theory one usually solves the Euler-Lagrange equations, which are of second order. However, Bogomolny [5] has shown that in certain field theories the energy of a soliton is bounded from below by a multiple of the topological charge and that equality is reached if the field satisfies a *first* order PDE. Because the Bogomolny equation does not involve time its solutions are static. Furthermore, the solutions are stable since they minimise the energy in a given topological sector.

If a single soliton has k collective coordinates, then N solitons have a kN-dimensional moduli space. This kN-dimensional manifold has a metric structure, which tells something about the interactions between solitons. Sometimes a potential is also defined on the moduli-space. In the case when there is no potential, there are no forces between static solitons and interactions are governed by the geometry of the moduli space, while the energy is simply proportional to the number of solitons.

Multi-soliton dynamics is tricky, but interesting. Usually solitons merge when they approach each other, and this distinguishes them from point-like particles. If one is interested in the adiabatic dynamics of solitons, the dynamics can be approximated by the dynamics on moduli space. As said above, the moduli space possesses a (curved) metric which can be determined from the kinetic energy of the field. The classical motion in the moduli space is along a geodesic. Therefore, no law for the force between moving solitons needs to be postulated, but the force naturally arises from the curved moduli space.

Unfortunately, there are few systems in nature which exhibit topological solitons and experimental tests of the mathematical results are limited. There are vortices in superconductors [11], but they either attract (Type I) or repel (Type II) each other, and hence the static solutions of the Bogomolny equation are unimportant, except perhaps close to the Type I/II boundary. Skyrmions are solitons which may describe the shape of nuclei, and are covered in Sec. 4. Supersymmetric field theories also have solitons, but so far SUSY has not been found in nature. Solitons in superstring theory [17] are known as "branes". Also, Derrick's theorem [6] prevents solitons in some simple field theories, e.g. pure gauge theories in 3 dimensions.

Examples of solitons are:

- kinks in one dimension;
- two-dimensional vortices in gauge theories with a Higgs field [1];
- lumps in two-dimensional non-linear scalar field theories (σ-models) [28];

- monopoles in three-dimensional gauge/Higgs theories [23, 18];
- solitons in three-dimensional σ-models (where they are known as Skyrmions) [21, 22];
- instantons in pure gauge theories of dimension four [4].

Lumps are the subject of Sec. 2. Sec. 3 discusses monopoles, while Sec. 4 is on Skyrmions and their relations to nuclei. The last section includes the exercises, with the example of the kink in 5.1 and 5.2.

When discussing these solitons the focus will be on the mathematics of rational maps. They give exact or approximate descriptions of the solitons. Furthermore, they show the symmetries of the solitons.

These lecture notes necessarily cover only a small amount of the subject on topological solitons. A thorough discussion and many references can be found in [14].

2 Lumps

We start with the S^2 σ-model. It is a two-dimensional scalar field theory with S^2 as the target space:

$$\Phi : \mathbb{R}^{2,1} \to S^2 \,. \tag{1}$$

Here, we use the complex coordinate $z = x + \mathrm{i}y$ on $\mathbb{R}^2$. On S^2 with polar coordinates (θ, ϕ) we introduce a complex coordinate R by stereographic projection $R = \tan\frac{\theta}{2}\mathrm{e}^{i\phi}$ with $R = 0, \infty$ for the north- and south pole respectively (see Fig. 1). Hence, in coordinates Φ is given by $\theta(x, y, t)$ and $\phi(x, y, t)$ or equivalently by $R(x, y, t)$. The differentials are

$$\begin{aligned} d\,R &= \frac{1}{2}\sec^2\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}d\theta + \mathrm{i}\tan\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}d\phi & (2)\\ d\,\bar{R} &= \frac{1}{2}\sec^2\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}d\theta - \mathrm{i}\tan\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}d\phi & (3)\\ \Rightarrow d\,R\,d\,\bar{R} &= \frac{1}{4}\sec^4\frac{\theta}{2}(d\theta^2 + \sin^2\theta d\phi^2) & (4)\\ \Leftrightarrow d\theta^2 + \sin^2\theta d\phi^2 &= 4\frac{d\,R\,d\,\bar{R}}{(1+|R|^2)^2}\,. & (5) \end{aligned}$$

The Lagrangian of the S^2 σ-model

$$\mathcal{L} = \int \frac{\frac{\partial R}{\partial t}\frac{\partial \bar{R}}{\partial t} - \frac{\partial R}{\partial x}\frac{\partial \bar{R}}{\partial x} - \frac{\partial R}{\partial y}\frac{\partial \bar{R}}{\partial y}}{(1+|R|^2)^2}\,dx\,dy \tag{6}$$

has Lorentz symmetry (with metric $\eta = \mathrm{diag}(1, -1, -1)$) and an internal $SO(3)$ symmetry (rotations on the target S^2). The potential energy E is the sum of the spatial derivative terms. Using $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ the energy of a static field configuration in terms of $R(z, \bar{z})$ reads

$$E = 2\int \frac{|\partial_z R|^2 + |\partial_{\bar{z}} R|^2}{(1+|R|^2)^2}\,dx\,dy\,. \tag{7}$$

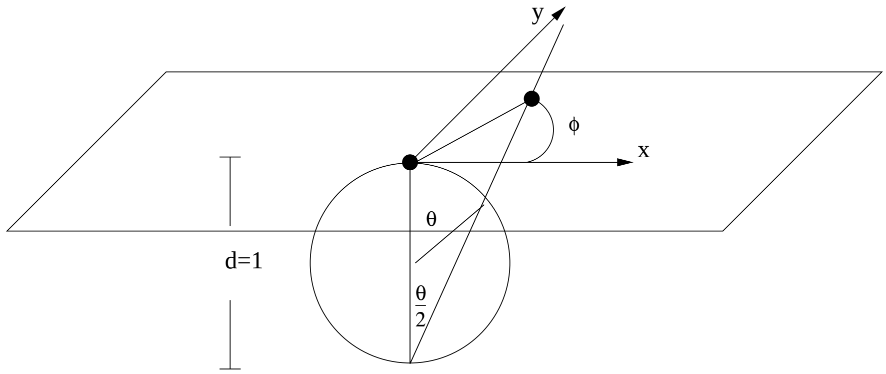

Figure 1: Definition of the Riemann projection.

Now we consider the topological charge. For a map $R(x, y)$ the topological charge N is the topological degree of the map $R : \mathbb{R}^2 \to S^2$. To calculate it we start with the (normalised) area element on S^2:

$$\frac{1}{4\pi} \sin\theta d\theta \wedge d\phi = \frac{\mathrm{i}}{2\pi} \frac{d\,R \wedge d\bar{R}}{(1 + |R|^2)^2} \,. \tag{8}$$

The degree of R is then given by

$$N = \frac{\mathrm{i}}{2\pi} \int \begin{vmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} \\ \frac{\partial \bar{R}}{\partial x} & \frac{\partial \bar{R}}{\partial y} \end{vmatrix} \frac{dx\,dy}{(1 + |R|^2)^2} \,, \tag{9}$$

where $|\cdots|$ denotes the Jacobian of R. In terms of ∂_z and $\partial_{\bar{z}}$ Eq. (9) yields

$$N = \frac{1}{\pi} \int \frac{|\partial_z R|^2 - |\partial_{\bar{z}} R|^2}{(1 + |R|^2)^2} dx\,dy \,. \tag{10}$$

Provided $R \to \mathrm{const}$ as $|z| \to \infty$, N is an integer. A proof of this theorem can be found in [14]. The interpretation of N is that it gives the number of times that R winds around S^2. Comparing Eq. (7) and Eq. (10) we find

$$E = 2\pi\,N + 4 \int \frac{|\partial_{\bar{z}} R|^2}{(1 + |R|^2)^2} dx\,dy \,. \tag{11}$$

Thus, the energy is the sum of the topological charge and a positive correction, i.e.

$$E \geq 2\pi\,N \,, \tag{12}$$

with equality only if

$$\partial_{\bar{z}} R = 0 \,. \tag{13}$$

This equation is called a Bogomolny equation [5]. Its importance lies in the fact that it is a first order equation whose solutions give field configurations with minimal energy within a topological sector (fixed N). In contrast, the Euler-Lagrange equations are of second order. Here Eq. (13) also shows that R is a holomorphic function of z only. Notice, that R is allowed to have a pole at any point z_0 because its image on the target S^2 is just the south pole. The requirement that the total energy is finite, together with the boundary condition that R has a definite limit as $|z| \to \infty$ forces R to be a rational map:

$$R(z) = \frac{P(z)}{Q(z)}\,, \tag{14}$$

where P and Q are polynomials in z with no common factors. If P and Q have the same degree, than R goes to a constant as $|z| \to \infty$. The case of R being just a polynomial of degree n is also allowed, since $R(\infty) = \infty$ is a regular point (south pole) and the energy density

$$\frac{|\partial_z R|^2}{(1+|R|^2)^2} \sim \frac{|z|^{2n-2}}{|z|^{4n}} \tag{15}$$

falls off sufficiently fast.

What is the topological charge of a map given by Eq. (14)? One way to answer this would be to evaluate the integral Eq. (10) directly. However, there is an alternative way. Namely, the degree N of a map R is also given by the number of distinct preimages of a generic point c counted with multiplicity, the multiplicity being 1 if R preserves the orientation and -1 if the orientation is reversed. Also, the Jacobian needs to be non-zero at these points, which is true for almost all points. Again, the proof can be found in [14]. Because of Eq. (13) R is holomorphic and therefore preserves orientation locally. Hence, N is given by the number of preimages of a generic point c.

In the case where $R(z) = P(z)$ we have to solve $P(z) = c$. This equation has n distinct solutions, and the topological degree N equals the algebraic degree. In the case of $R(z) = P(z)/Q(z)$ we have to solve $P(z) - c\,Q(z) = 0$. The number of roots is given by the greater of the two algebraic degrees of P and Q and is also the topological degree of the map. Recall that the energy E of a rational map is $2\pi N$. Therefore, rational maps with degree $N > 1$ are called N-lump solutions.

The simplest example is one lump. It has $N = 1$, $E = 2\pi$ and in a certain orientation is given by $R(z) = \mu z$ with $\mu \in \mathbb{R}$. The energy density according to Eq. (15) is

$$\frac{2\mu^2}{(1+\mu^2|z|^2)^2}\,. \tag{16}$$

It is centred at the origin and has a width of $1/\mu$. Strictly speaking, the energy density is not confined within a finite region. The general 1-lump solution is parameterised by three complex parameters:

$$R(z) = \frac{\alpha z + \beta}{z + \gamma}\,. \tag{17}$$

Two real parameters specify the (two-dimensional) position of the lump, one real parameter gives its width, while the remaining three give the orientation on the target space. Using the boundary condition $R(\infty) = 0$ requires $\alpha = 0$, and the solution is given by four real parameters (two for position, one width, and one orientation). A rational map of degree N has $2N + 1$ complex parameters, or $2N$ if boundary conditions are specified.

We now consider the case where space is curved. The target space is S^2 as before. The flat spatial metric δ_{ij} is replaced by $g_{ij} = \Omega(x,y)\delta_{ij}$. Ω is called a conformal factor, and must be positive. Then the field energy is

$$E = \int \frac{g^{ij}\partial_i R \partial_j \bar{R}}{(1+|R|^2)^2} \sqrt{\det g}\, dx\, dy\,. \tag{18}$$

Notice, that we get one factor of $1/\Omega$ from g^{ij} which is cancelled by a factor of Ω from $\sqrt{\det g}$. Thus the energy is conformally invariant and Bogomolny's equation for lumps remains unchanged. If we choose $\Omega = 1/(1+x^2+y^2)^2$ we get the spatial metric of the two-sphere. So lumps are now maps from the spatial S^2 to the target S^2. To have a smooth holomorphic map ($\partial_{\bar{z}} R = 0$) $R(z)$ must be a rational map. Again, an N lump solution is a map of algebraic degree N. Well separated lumps with boundary condition $R(\infty) = 1$ are given by

$$R(z) = \frac{(z-\alpha_1)\dots(z-\alpha_N)}{(z-\beta_1)\dots(z-\beta_N)} \tag{19}$$

with $\alpha_1 \cong \beta_1$, ..., $\alpha_N \cong \beta_N$. The lumps are localised around the α_is and their sizes are roughly given by the distance between α_i and β_i.

Earlier we identified $\mathbb{R}^2 \cong \mathbb{C}$ with the spatial S^2 of infinite radius. This required that the field had a definite value at $|z| = \infty$, therefore reducing the symmetry from $SO(3)$ to $SO(2)$. If we have a map from unit S^2 to S^2 we maintain the full $SO(3)$ symmetry of the theory. In this case the energy E is given by

$$E = \frac{1}{2} \int \frac{|\partial_z R|^2 (1+|z|^2)^2}{(1+|R|^2)^2} \frac{2\mathrm{i} dz d\bar{z}}{(1+|z|^2)^2}\,. \tag{20}$$

Generically, a particular map need not be symmetric, but for example $R(z) = z$ has the full $SO(3)$ symmetry and uniform energy density on the spatial S^2. Similarly, $R(z) = z^N$ ($N > 1$) has axial symmetry and energy density concentrated around the equator. There are also very interesting examples of lumps with Platonic symmetries, which we consider now.

First, a definition: A map $R(z)$ has a symmetry element if there exists a pair of Möbius transformations (m, M) in $SU(2)$ such that

$$R(m(z)) = MR(z)\,, \tag{21}$$

that is, a spatial Möbius transformation m is compensated by a Möbius transformation M on target space. A symmetry group is a set of pairs $(m_1, M_1), \dots,$ (m_G, M_G) where $(m_1, \dots, m_G)$ form a symmetry group of rotations and where $m \to M$ is a homomorphism, i.e. if $m_1\, m_2 = m_3$ then $M_1\, M_2 = M_3$. Recall the definition of a Möbius transformation:

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}\,, \tag{22}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$.

As a first example consider the map $R(z) = z^N$. It has an axial symmetry about $z = 0$, i.e. $R(\mathrm{e}^{\mathrm{i}\alpha} z) = \mathrm{e}^{\mathrm{i}N\alpha} R(z)$. The pairs $(\alpha, N\alpha) \mod 2\pi$ define the symmetry group $SO(2)$. The general symmetry group is D_∞, because, in addition, $R(1/z) = 1/R(z)$.

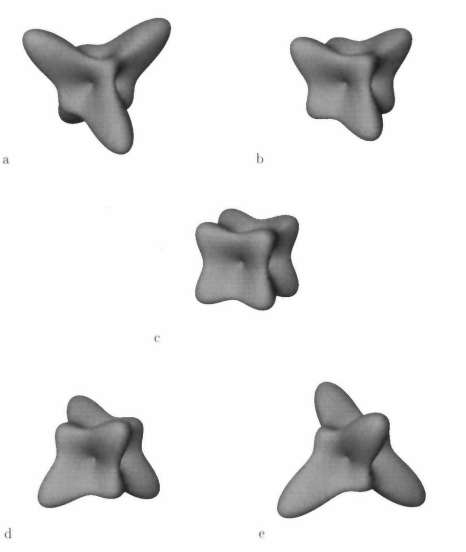

Figure 2: Surfaces displaying the energy density for lumps with $c = \frac{4}{5}, \frac{9}{10}, 1, \frac{10}{9}, \frac{5}{4}$ in Eq. (23) (taken from [14]).

Another example is the $N = 4$ lump solution with tetrahedral symmetry:

$$R(z) = c\frac{z^4 + 2\sqrt{3}\mathrm{i}z^2 + 1}{z^4 - 2\sqrt{3}\mathrm{i}z^2 + 1}\,, \tag{23}$$

where c is a real parameter, if reflection symmetry is also imposed. The map has the symmetries (see Sec. 5.3)

$$R(\mathrm{i}z) = \frac{1}{R(z)} \quad , \quad R\left(\frac{\mathrm{i}z + 1}{-\mathrm{i}z + 1}\right) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} R(z)\,. \tag{24}$$

Its energy density is visualised in Fig. 2 for some values of c by plotting surfaces whose height above the unit sphere is proportional to the energy density at that point on the sphere. Notice, that the energy density Eq. (15) vanishes at the points $0, -1, 1, \mathrm{i}, -\mathrm{i}$, and ∞. There is a cubic symmetry when $c = 1$.

By Eq. (20) the general N lump solution $R(z) = P(z)/Q(z)$ has zero energy density wherever the Wronskian $W(z) = P'(z)Q(z) - P(z)Q'(z)$ vanishes. Because $W(z)$ is a polynomial of degree $2N - 2$ the energy density vanishes at $2N - 2$ points. These zeros reflect the symmetry of the rational map. In special cases such as Eq. (23) the Wronskian is only a 5th degree polynomial, and we must consider the point ∞ as another formal zero.

The Wronskians of rational maps we have found so far are in fact polynomials known as Klein polynomials. They are one-dimensional representations of the Platonic group [12]. The Platonic solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. Out of these five solids the cube is dual to the octahedron while the dodecahedron is dual to the icosahedron. This means that the vertices of one are the face centres of the other. The Klein polynomials for the cube are constructed as follows: A regular cube is scaled in such a way that the six face centres lie on the unit sphere. Via the Riemann projection, these points correspond to six complex numbers. The Klein polynomial for the face centres is the polynomial which has these six points as zeros. Orientating the cube in a particular way, the Klein polynomial for the face centres is

$$O_f = z(1 - z^4)\,. \tag{25}$$

Notice that we have orientated the cube such that one face centre corresponds to the south pole of the Riemann sphere, and ∞ is considered as a further zero, making Eq. (25) a sixth order polynomial. By a similar construction (with the same orientation), the Klein polynomials for the eight vertices and twelve edge centres are

$$O_v = \left(z - \frac{1+\mathrm{i}}{\sqrt{3}+1}\right)\ldots = z^8 + 14z^4 + 1 \tag{26}$$

$$O_e = z^{12} - 33z^8 - 33z^4 + 1 \tag{27}$$

For example, the vertex $\frac{1}{\sqrt{3}}(1,1,1)$ corresponds to $z = \frac{1+\mathrm{i}}{\sqrt{3}+1}$. The Klein polynomials for the tetrahedron and icosahedron are

$$T_f = z^4 - 2\sqrt{3}\mathrm{i}z^2 + 1 \tag{28}$$

$$T_v = z^4 + 2\sqrt{3}\mathrm{i}z^2 + 1 \tag{29}$$

$$T_e = z^5 - z \tag{30}$$

$$Y_f = z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1 \tag{31}$$

$$Y_v = z^{11} + 11z^6 - z \tag{32}$$

$$Y_e = z^{30} + 522z^{25} - 10\,005z^{20} - 10\,005z^{10} - 522z^5 + 1 \tag{33}$$

Although the individual points are nontrivial complex numbers, most Klein polynomials only have integer coefficients! Comparing Eq. (28) and Eq. (29) with Eq. (23) we see that the rational map is just the quotient of two tetrahedral Klein polynomials and hence possesses tetrahedral symmetry.

In this last section on lumps we consider reduced soliton dynamics. The name refers to the idea that we make the parameters of a stationary soliton time dependent and get a moving soliton as a result. An explicit example of this idea can be found in Sec. 5.1.

First consider a time dependent position parameter $a(t)$ with a kinetic Lagrangian $\mathcal{L} = \frac{1}{2}M\dot{a}^2$ and constant potential. The moduli space is one dimensional and has constant metric $g_{aa} = M$ $(const)$: $\mathcal{L} = \frac{1}{2}g_{aa}\dot{a}\dot{a}$. Of course the equation of motion is just $\ddot{a} = 0$, with a straight line as a solution.

The same strategy is now applied to N lumps. We take the parameterised static solution Eq. (19) and let the parameters be functions of time:

$$R(z,t) = \frac{(z - \alpha_1(t))\ldots(z - \alpha_N(t))}{(z - \beta_1(t))\ldots(z - \beta_N(t))}\,. \tag{34}$$

Substituting Eq. (34) into the expression for the kinetic energy $\int |\dot{R}|^2/(1 + |R|^2)^2 dz d\bar{z}$ we obtain a Lagrangian quadratic in $\dot{\alpha}_1, \dot{\beta}_1, \ldots, \dot{\bar{\alpha}}_1, \dot{\bar{\beta}}_1, \ldots$. We collectively call these $\dot{\gamma}$, and the metric structure on the moduli space is $g_{lm}\dot{\gamma}_l\dot{\bar{\gamma}}_m$. The g_{lm} are not constant, and there are no $\dot{\gamma}_l^2$ terms. For the kinetic energy to be real, g must be a hermitian matrix. Furthermore, g is Kähler, i.e. there exists a closed form ω ($d\omega = 0$), such that

$$\omega = \mathrm{i} g_{lm} d\gamma_l \wedge d\bar{\gamma}_m \,, \tag{35}$$

and a Kähler potential $\mathcal{K}(\gamma, \bar{\gamma})$ with

$$g_{lm} = \partial_l \bar{\partial}_m \mathcal{K}(\gamma, \bar{\gamma}) \,. \tag{36}$$

Here, $\mathcal{K}$ is given by

$$\mathcal{K} = \int \log(|P(z)|^2 + |Q(z)|^2) dx\, dy \,. \tag{37}$$

Generically, for a purely kinematic Lagrangian, the equations of motion say that the motion is along a geodesic at constant speed. Note that this motion occurs in moduli space, but from this we get the lump motion and interaction in physical space ($\mathbb{R}^2$ or S^2).

Regarding lumps, there are two further remarks:

1. The metric is infinite in certain directions. This means that lump motion in these directions is not really possible. For example, a single lump cannot expand. In the case of a N lump solution for which we can make an expansion $R = 1 + A_1/z + A_2/z + \ldots$ for large z, here, A_1 cannot depend on time.

2. On $\mathbb{R}^2$ or S^2 geodesic motion is incomplete. Lumps can collapse to zero size within a finite time. This happens if $\beta_r \to \alpha_r$ and the pole and the zero cancel. In this case scattering can still happen if the lumps approach each other fast enough, that is, before they shrink to a point. A numerical study of this has been carried out by Ward [25].

The most interesting phenomenon is 90° scattering of two lumps, which is discussed in detail by Zakrzewski [29]. A very similar situation occurs in two vortices scattering [20], but there, a vortex cannot collapse.

3 Monopoles

After the discovery by t'Hooft [23] and Polyakov [18] in 1974 that non-abelian gauge theories can have magnetic monopole solutions free of singularities, monopoles have become a large subject. In this lecture we will therefore concentrate on one aspect: monopoles and rational maps. For a recent review consult [26].

Here, we consider a Yang-Mills theory with gauge group $SU(2)$ together with an adjoint Higgs field Φ in $3+1$ dimensions. The Lagrangian for the BPS monopole reads

$$\mathcal{L} = -\frac{1}{8} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \mathrm{tr} D_\mu \Phi D^\mu \Phi \,, \tag{38}$$

with the following definitions: tr denotes the trace, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the covariant derivative D_μ acts on the adjoint Higgs field Φ by $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$.

As in the previous section, we first look for static solutions. To find these we introduce the Yang-Mills analog to the magnetic field:

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}\,. \tag{39}$$

With this, the Yang-Mills energy E is given by

$$E = -\frac{1}{4}\int \mathrm{tr}(B_i\,B_i) + \mathrm{tr}(D_i\Phi\,D_i\Phi)\,d^3x\,. \tag{40}$$

Completing the square in Eq. (40) we again arrive at a first order Bogomolny equation

$$B_i = -D_i\Phi\,, \tag{41}$$

and we impose the boundary condition that $|\Phi| \to 1$ as $|\mathbf{x}| \to \infty$ with $|\Phi|^2 = -\frac{1}{2}\mathrm{tr}(\Phi\Phi)$. This boundary condition breaks the $SU(2)$ gauge symmetry down to a $U(1)$ gauge symmetry. As a consequence, one obtains two massive gauge bosons $W^{\pm}$ together with a massless "photon". Before the experimental verification of the Glashow-Weinberg-Salam theory, the Lagrangian Eq. (38) was one contestant for the weak interaction. However, as this theory does not possess a massive Z^0 and the two W bosons carry electric charge 2, this model was ruled out.

The boundary condition implies for the field at spatial infinity Φ_∞ that it is a map from the boundary two-sphere into the unit vectors in the Lie algebra of $SU(2)$ which is also a two-sphere. The degree of the map Φ_∞ is the monopole number N, and we have the Bogomolny inequality

$$E \geq 2\pi N\,. \tag{42}$$

As with lumps, we have equality if the Bogomolny equation (41) is satisfied. It turns out, that the energy of monopoles is independent of where they are, that is, there are no static forces.

The basic BPS monopole solution with spherical symmetry can be found with a hedgehog ansatz for Φ and A_i [19]:

$$\Phi = h(r)\frac{x^a}{r}t^a \tag{43}$$

$$A_i = -\frac{1}{2}(1-k(r))\epsilon_{ija}\frac{x^j}{r^2}t^a\,. \tag{44}$$

As a basis of the Lie algebra we have chosen the three Pauli matrices multiplied by i, and thus $[t^a,t^b] = 2\epsilon^{abc}t^c$. Inserting this ansatz into Eq. (41) we obtain the two coupled equations

$$\frac{dh}{dr} = \frac{1}{2r^2}(1-k^2) \tag{45}$$

$$\frac{dk}{dr} = -2hk\,. \tag{46}$$

Using $H = h + \frac{1}{2r}$ and $K = k/2r$ these equations can be simplified and solved (see Sec. 5.4):

$$h(r) = \coth 2r - \frac{1}{2r} \tag{47}$$

$$k(r) = \frac{2r}{\sinh 2r}\,. \tag{48}$$

The energy density simplifies to $\frac{1}{2}\nabla^2(|\Phi|^2)$ if the Bogomolny equation is satisfied. This convenient formula is due to Ward [24] and states that the energy density of a BPS monopole depends only on the Higgs field.

In the asymptotic field only the projection of B_i onto Φ is not exponentially decaying. The projection $b_i = -\mathrm{tr}(B_i\Phi)$ is the true magnetic field, which, however, is only valid asymptotically. $\mathbf{b}$ gives information about the number of magnetic monopoles and their positions. One monopole ($N = 1$) has four physical significant parameters: three for the position and one phase. Time depending parameters lead to a monopole with momentum and electric charge (an object with both electric and magnetic charge is called a dyon). N monopoles have $4N$ real collective coordinates: N positions in $\mathbb{R}^3$ and each monopole has its individual phase. The resultant moduli space is a $4N$ dimensional manifold which is hyperkähler, in which the positions and phases are intricately linked as the monopoles get closer together.

We now discuss the relation between rational maps and monopoles. There are two methods which relate a monopole to a rational map. The first is due to Donaldson [7], the second due to Jarvis [10].

To define the Donaldson map, we have to single out a direction in $\mathbb{R}^3$ by choosing the (x_1, x_2)-plane and the x_3 direction. The Hitchin operator acts on a fundamental field v via $(D_3 - \mathrm{i}\Phi)v \equiv (\partial_3 + A_3 - \mathrm{i}\Phi)v$. Scattering data are obtained by solving

$$(D_3 - \mathrm{i}\Phi)v = 0\,, \tag{49}$$

along the x_3 direction. Here v is just a function of x_3, but the solutions of Eq. (49) are labeled by $z = x_1 + \mathrm{i}x_2$, which is the complex coordinate where the line intersects the (x_1, x_2)-plane. In the limit $x_3 \to -\infty$ we have $A_\mu \to 0$, $|\mathrm{i}\Phi| \to 1$ and Eq. (49) asymptotically yields (we are simplifying a bit here)

$$\begin{pmatrix} \partial_3 - 1 & 0 \\ 0 & \partial_3 + 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \tag{50}$$

which has the simple solution

$$v = a \begin{pmatrix} \mathrm{e}^{x_3} \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ \mathrm{e}^{-x_3} \end{pmatrix}. \tag{51}$$

Here, we choose the solution which looks like $v = (\mathrm{e}^{x_3}\,,\,0)^t$ as $x_3 \to -\infty$. With this initial condition we obtain the solution of Eq. (49) as $x_3 \to \infty$:

$$v = \tilde{a} \begin{pmatrix} \mathrm{e}^{x_3} \\ 0 \end{pmatrix} + \tilde{b} \begin{pmatrix} 0 \\ \mathrm{e}^{-x_3} \end{pmatrix}. \tag{52}$$

The coefficient $R = \tilde{a}/\tilde{b}$ is the scattering data on the line and it depends on whether the line is close to the monopole or not. In the case when the line is far away from the monopole $R \to 0$.

Notice the analogy to one-dimensional quantum mechanical scattering. Our asymptotic initial condition $v = (\mathrm{e}^{x_3}\,,\,0)^t$ corresponds to an incoming particle, whereas $\tilde{a}$ and $\tilde{b}$ play the role of the transmission and reflection coefficient.

From the scattering data R for each line labeled by z, we get a holomorphic map $R(z)$. The proof that it is holomorphic depends on the Bogomolny equation, which implies that $D_{\bar{z}}$ commutes with the Hitchin operator. In fact, it is a rational map of degree N which equals the monopole number:

$$R(z) = \frac{p_1 z^{N-1} + p_2 z^{N-2} + \ldots + p_N}{(z-q_1)\ldots(z-q_N)} = \frac{P(z)}{Q(z)} \,. \tag{53}$$

The numerator is a polynomial of degree $N-1$ due to boundary conditions, while the denominator has algebraic degree N. The importance of this Donaldson map is that one can obtain the positions and phases of the monopoles from the coefficients in Eq. (53), provided the monopoles are well separated. The complex numbers $q_1, \ldots q_N$ give the coordinates of the monopoles in the (x_1, x_2) plane. To find the height (x_3) of a monopole projected to q_r evaluate $P(q_r)$. The height is given by $\frac{1}{2}\log|P(q_r)|$ and $\frac{1}{2}\arg P(q_r)$ is the phase. Together this data is in $\frac{1}{2}\log P(q_r)$.

As a simple example of why this is so, consider a single monopole at the origin with phase α. The Donaldson map is then

$$R(z) = \frac{\exp(\frac{1}{2}\mathrm{i}\alpha)}{z} \,. \tag{54}$$

Translating the monopole by a distance s along the x_3 direction replaces e^{x_3} in Eq. (52) by $\mathrm{e}^{x_3 - s}$. Making the redefinitions $\tilde{a} \to \tilde{a}\mathrm{e}^{-s}$ and $\tilde{b} \to \tilde{b}\mathrm{e}^{s}$, we obtain $R \to R\mathrm{e}^{-2s}$.

Another relation between rational maps and monopoles was found by Jarvis [10]. We start by choosing a point in $\mathbb{R}^3$, and take it as the origin. Now consider the radial Hitchin operator $D_r - \mathrm{i}\Phi$ with $D_r = \frac{x^i}{r}D_i$. The direction of a radial line away from the origin is labeled by a point $z = \tan\theta/2\mathrm{e}^{i\phi}$ on the Riemann sphere. The scattering data along a fixed direction z are obtained by solving

$$(D_r - \mathrm{i}\Phi)v = 0 \,, \tag{55}$$

for the fundamental field v. Along a fixed line the solution is given by

$$v(r) = \begin{pmatrix} v_1(r) \\ v_2(r) \end{pmatrix} , \tag{56}$$

and we select the solution which decays exponentially as $r \to \infty$. The scattering data for this line is then defined by $R = \frac{v_1(0)}{v_2(0)}$. By considering all lines we again obtain a map $R(z, \bar{z})$. Because of the Bogomolny equation, in this context too $D_{\bar{z}}$ commutes with the Hitchin operator. Hence, we obtain a holomorphic map $R(z)$ which is a rational map of degree N. Therefore, this Jarvis map has monopole number N. It is a fact, that the $2N + 2$ dimensional set of rational maps is isomorphic to the space of unframed N-monopoles. This implies the existence of a $1 - 1$ correspondence between rational maps and monopoles. The advantage of the Jarvis map over the Donaldson map is that the rotational symmetry of the corresponding monopole configuration can be read off from the rational map. A rational map with Platonic symmetry describes a monopole with the same symmetries. For example, there exists a unique cubically symmetric rational map of degree four describing the unique 4-monopole with cubic symmetry. A further example can be seen in Fig. 3. There, the surfaces of constant energy density of a 3-monopole with triangular symmetry are shown. Note that there is no up/down symmetry, because we did not consider the symmetry under parity.

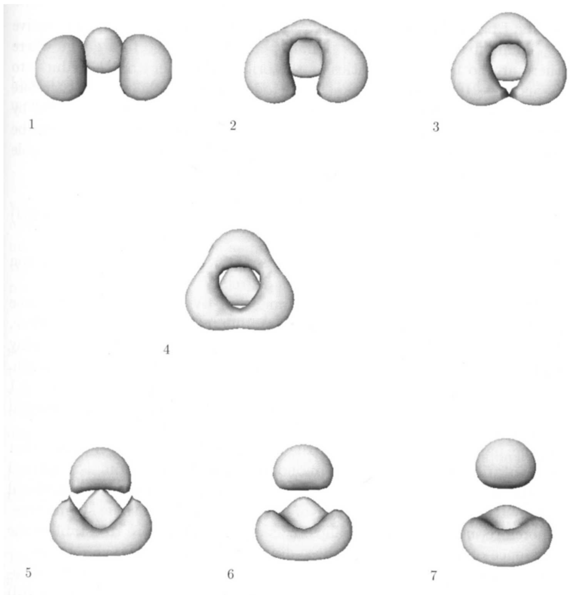

Figure 3: Energy density isosurfaces for a family of $N = 3$ monopoles with cyclic C_3 symmetry (taken from [14]).

4 Skyrmions

The Skyrme model [21, 22] is a soliton model of nucleons (protons and neutrons). The question is, whether it gives a satisfactory description of nuclei (bound states of several protons and neutrons). The Skyrme Lagrangian has only three parameters, a length scale, energy scale, and a dimensionless pion mass parameter. Recent progress has been achieved through a better treatment of the three parameters of the model, especially the pion mass [15]. Much mathematical insight of Skyrmion solutions have come from the mathematics of gauge theories. Notice, that the Skyrme model does not directly refer to quarks and gluons, but is regarded as a low energy limit of QCD with many quark colours [27]. For a deeper coverage of the topic consult the literature [2].

The ingredients of the Skyrme model are three nearly massless pion fields with an internal $SO(3)$ isospin symmetry. A spontaneous broken $SO(4)$ symmetry gives rise to this $SO(3)$ and the pion fields are the Goldstone fields which parametrise a 3-sphere. Lorentz invariance in 3+1 dimensions also holds, but in this lecture we will be interested mainly in static solutions. We will also ignore the electromagnetic and weak interaction, which is justified for nuclei smaller than ^{40}Ca. There exists a topological charge B, the baryon number, that ensures that protons, neutrons, and nuclei do not disperse away.

In the Skyrme model there are the three pion fields $\pi_1(x)$, $\pi_2(x)$, and $\pi_3(x)$. Together with a field $\sigma(x)$ they are subject to the constraint $\sigma^2+\pi_1^2+\pi_2^2+\pi_3^2 = 1$, which describes a 3-sphere. One defines the Skyrme field $U(x)$ as

$$U(x) = \begin{pmatrix} \sigma(x) + \mathrm{i}\pi_3(x) & \mathrm{i}\pi_1(x) + \pi_2(x) \\ \mathrm{i}\pi_1(x) - \pi_2(x) & \sigma(x) - \mathrm{i}\pi_3(x) \end{pmatrix} . \tag{57}$$

Due to the constraint above $U(x) \in SU(2)$. It is also useful to define the current $R_\mu = (\partial_\mu U)U^{-1} \in su(2)$. The Lagrangian of the Skyrme model in units of the Skyrme energy and length reads

$$\mathcal{L} = \int d^3x \left(-\frac{1}{2}\mathrm{tr}(R_\mu R^\mu) + \frac{1}{16}\mathrm{tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2\,\mathrm{tr}(U - \mathbb{1}) \right) . \quad (58)$$

The boundary condition of this non-linear σ-model is $U \to \mathbb{1}$ as $|\mathbf{x}| \to \infty$. The second term in Eq. (58) is called the Skyrme term. Also, the first two terms possess a chiral symmetry, while the third term breaks this symmetry and m equals the pion mass.

From Eq. (58) we obtain the expression for the energy E of a time-independent Skyrme field:

$$E = \int d^3x \left(-\frac{1}{2}\mathrm{tr}(R_i R_i) - \frac{1}{16}\mathrm{tr}([R_i, R_j][R_i, R_j]) - m^2\,\mathrm{tr}(U - \mathbb{1}) \right) . \quad (59)$$

This is invariant under translations and rotations of $U(\mathbf{x})$ as well as isospin rotations $U \to AUA^\dagger$ ($A \in SU(2)$). As in the previous sections the conserved topological charge B is given by the degree of the map $U : \mathbb{R}^3 \to S^3$, and it can be calculated via

$$B = \frac{-1}{24\pi^2} \int_{\mathbb{R}^3} \epsilon_{ijk}\mathrm{tr}(R_i R_j R_k) d^3x \,, \quad (60)$$

where the boundary condition $U \to \mathbb{1}$ as $|\mathbf{x}| \to \infty$ is important. By completing the square it is easy to see that

$$E \geq 12\pi^2 |B| \,. \quad (61)$$

Although we have again a Bogomolny equation (in the massless pion limit), there are no solutions which would satisfy the equality. The reason is, that equality in (61) is only possible if $U : \mathbb{R}^3 \to S^3$ were an isometry. But since $\mathbb{R}^3$ is flat and S^3 is curved an isometry between the two spaces cannot exist. Only numerical solutions are possible and they are obtained by numerically minimising the energy Eq. (59). Fig. 4 shows the surfaces of constant baryon density for Skyrmions with $1 \leq B \leq 8$. Notice the symmetries of the Skyrmions. A priori, it is not clear that the numerically found Skyrmions have Platonic symmetries. However, a rational map ansatz for Skyrmions exists which yields these symmetries. This only gives approximate solutions, but the Skyrmions thus obtained have the same symmetries and very similar shapes and energies compared to the true (numerical) solutions.

The rational map ansatz for Skyrmions was introduced in [9]. The first problem to overcome is that Skyrmions are maps from $\mathbb{R}^3 \to S^3$, whereas rational maps are functions from $S^2 \to S^2$. The idea is to identify the domain S^2 of the rational map with spheres in $\mathbb{R}^3$ of radius r and the target S^2 with spheres of constant latitude on S^3. For $\mathbb{R}^3$ we use polar coordinates, with r denoting the distance from the origin, and $z = \tan\frac{\theta}{2}\exp(\mathrm{i}\phi)$ gives the polar coordinates on a two-sphere, see Fig. 5. We now make the ansatz

$$U(r,z) = \begin{pmatrix} \cos f(r) + \mathrm{i}\sin f(r)\left(\frac{1-|R|^2}{1+|R|^2}\right) & \mathrm{i}\sin f(r)\left(\frac{2\bar{R}}{1+|R|^2}\right) \\ -\mathrm{i}\sin f(r)\left(\frac{2R}{1+|R|^2}\right) & \cos f(r) - \mathrm{i}\sin f(r)\left(\frac{1-|R|^2}{1+|R|^2}\right) \end{pmatrix} , \quad (62)$$

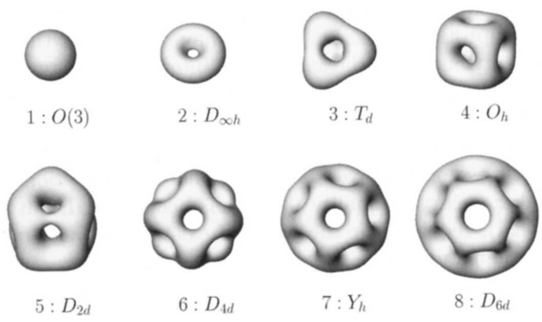

Figure 4: Baryon density isosurfaces of Skyrmions for $1 \leq B \leq 8$. The Baryon number and symmetry of each solution is shown (taken from [14]).

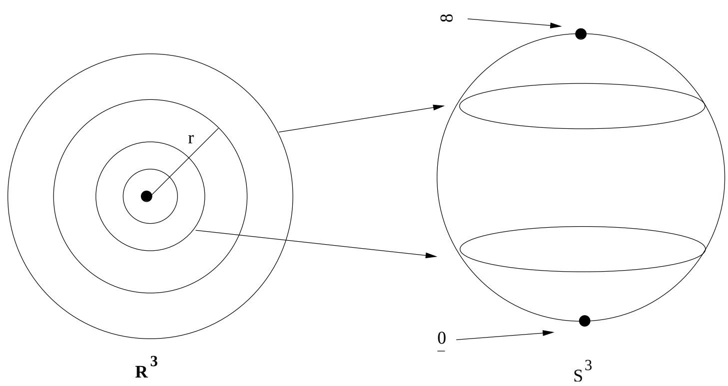

Figure 5: The mapping $\mathbb{R}^3 \rightarrow S^3$: each two-sphere in $\mathbb{R}^3$ at radius r is mapped to a two-sphere on S^3 with constant latitude.

B	R(z)	$\mathcal{I}$	Symmetry
1	z	1	$O(3)$
2	z^2	5.81	$O(2) \times \mathbb{Z}$
3	$\frac{\sqrt{3}iz^2-1}{z(z^2-\sqrt{3})}$	13.58	T_d
4	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$	20.65	O_h
5	$\frac{z(z^4+bz^2+a)}{az^4-bz^2+1}$, $a = 3.07$, $b = 3.94$	35.75	D_{2d}
6	$\frac{z^4+a}{z^2(az^4+1)}$, $a = 0.16\ldots0.19$	50.76	D_{4d}
7	$\frac{z^5-a}{z^2(az^5+1)}$, $a = \frac{1}{7}$	60.87	Y_h
8	$\frac{z^6-a}{z^2(az^6+1)}$, $a = 0.14$	85.65	D_{6d}

Table 1: Energy minimising rational maps: B baryon number, $R(z)$ rational map, $\mathcal{I}$ angular part of energy, and the corresponding Platonic symmetry.

where $f(r)$ is a radial profile function with $f(0) = \pi$, $f(\infty) = 0$, and $R(z) = \frac{P(z)}{Q(z)}$ is again a rational map of degree B. With this ansatz the energy of a static Skyrmion, given by Eq. (59), simplifies to

$$E = 4\pi \int_0^\infty r^2 \left(f'^2 + 2B \sin^2 f (f'^2 + 1) + \mathcal{I} \frac{\sin^4 f}{r^2} + 2m^2(\cos f - 1) \right) dr \,, \tag{63}$$

where $\mathcal{I}$ is the purely angular integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i dz d\bar{z}}{(1+|z|^2)^2} \,. \tag{64}$$

To minimise the energy E one first minimises $\mathcal{I}$ and then minimises Eq. (63). Table 1 lists Skyrmions which were found by this rational map ansatz. They were found by assuming the symmetries, and then minimising $\mathcal{I}$ with respect to one or two parameters. In [3] a systematic search for maps that minimise $\mathcal{I}$ has been performed, and results up to $B = 20$ were obtained.

The rational map ansatz works well for B up to 7 if the pion mass is taken to be non-zero, and up to 20 or 30 if the pion is taken to be massless. For $m \simeq 1$, more stable solutions are made from $B = 4$ cubic Skyrmions joined together. Here, a 2-layer version of the Rational Map Ansatz can be a useful approximation, capturing Skyrmion symmetries and shapes.

The quantisation proceeds by quantising the collective motion, i.e. translations, rotations, and isorotations. With some difficulty some vibrational modes can also be included. One can choose the momentum to be zero, so that the wavefunction is a function of $3+3$ Euler angles in space and isospace and the Hamiltonian is that of "coupled rigid bodies". For Skyrmions with symmetry the inertia tensor simplifies and the energy is schematically given as

$$E \sim M_{\text{classical}} + \frac{J(J+1)}{2I} \,, \tag{65}$$

where I is the moment of inertia and J is the quantised spin.

Another aspect of the quantisation was noticed by Finkelstein and Rubinstein [8]. They found that the wavefunction of the Skyrmion is single-valued only on the double cover of the space of collective coordinates. If a symmetry rotation is a non-contractible loop in configuration space then the wavefunction acquires a factor of -1. For B odd the 2π rotations in space and isospace are non-contractible, and therefore a Skyrmion with an odd B is a fermion. On the other hand, if B is even then rotations by 2π are contractible loops, and thus Skyrmions with even B are bosons. This is also important if one wants to build nuclei out of Skyrmions. The $B = 1$ Skyrmion is a fermion, as it should be if it is to describe a nucleon (proton or neutron). Further symmetries of the Skyrmion lead to further constraints. For example, the ground state of the toroidal $B = 2$ Skyrmion has spin 1 and isospin zero, as expected for the deuteron.

As another more detailed example, consider the quantum states for $B = 6$. The corresponding rational map is

$$R(z) = \frac{z^4 + a}{z^2(az^4 + 1)} \tag{66}$$

with $a = 0.16 \ldots 0.19$. The value $a = 0.16$ minimises $\mathcal{I}$, but $a = 0.19$ gives a quadrupole moment in agreement with experiment [15]. It has the symmetries

$$\begin{aligned} R(\mathrm{i}z) &= -R(z) \quad (67) \\ R\left(\frac{1}{z}\right) &= \frac{1}{R(z)}\,, \quad (68) \end{aligned}$$

generating the D_4 symmetry group. Comparing Eq. (67) with Eq. (21), we see that a spatial rotation by $\pi/2$ around the 3-axis corresponds to a rotation by π around the 3-axis in isospace. Similarly, Eq. (68) relates a rotation by π around the 1-axis in space to a rotation by π around the 1-axis in isospace. The Finkelstein-Rubinstein constraints on a state $|\psi\rangle$ then read

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}J_3}\mathrm{e}^{\mathrm{i}\pi I_3}|\psi\rangle &= |\psi\rangle \quad (69) \\ \mathrm{e}^{\mathrm{i}\pi J_1}\mathrm{e}^{\mathrm{i}\pi I_3}|\psi\rangle &= -|\psi\rangle\,. \quad (70) \end{aligned}$$

Here, the J_i and I_i are the generators of rotation in space and isospace respectively. The sign in Eqs. (69) and (70) is due to a formula by Krusch [13]:

$$\mathrm{sign} = (-1)^{\frac{B}{2\pi}(B\theta_{\mathrm{rot}} - \theta_{\mathrm{isorot}})}\,, \tag{71}$$

depending only on the angles of rotations θ_{rot} and θ_{isorot} and not on the rotation axis (e.g. $\theta_{\mathrm{rot}} = \frac{\pi}{2}$, $\theta_{\mathrm{isorot}} = \pi$, and $B = 6$ in Eq. (69)). The allowed states with isospin zero are those with $J_3 = 0 \mod 4$ and J odd: $|1,0\rangle_J \otimes |0,0\rangle_I$, $|3,0\rangle_J \otimes |0,0\rangle_I$, $|5,0\rangle_J \otimes |0,0\rangle_I$. States with isospin 1 have $J = 0$ or $J = 2$: $|0,0\rangle_J \otimes |1,0\rangle_I$, $|2,0\rangle_J \otimes |1,0\rangle_I$. Apart from the spin 5 state, which has rather high energy, all these states match low-lying states of the nuclei ^{6}He ^{6}Li and ^{6}Be with baryon number 6 [16].

To make detailed contact with experimental data, the three free parameters of the Skyrme model need to be calibrated. Traditionally this is done by using the masses of the pion, nucleon, and Δ-resonance. This led to nuclei which were too small and too tightly bound. A new calibration [15] uses the mass of the pion, and the ^{6}Li ground state energy and charge radius. This leads to new Skyrmion structure for $B \geq 8$. Nuclei with baryon number a multiple of four look like clusters of $B = 4$ cubes (α-particles).

As a conclusion, it is surprising that a pion field theory has solitons that can model nucleons and nuclei. It seems to give a sensible account of short-range nuclear structure. In particular, nucleons merge when they are close together. The recent recalibration gives better energy spectra and nuclear radii [16]. However, more work is needed on larger nuclei (e.g. $B = 10$), on Coulomb energies, on Skyrmion scattering and cross-sections, and on Skyrmion vibrations.

5 Exercises

5.1 Kinks

The Lagrangian of a ϕ^4 scalar field theory in $1+1$ dimensions with symmetry breaking is

$$L = \int_{-\infty}^{\infty} \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \lambda(m^2 - \phi^2)^2 \, dx \, .$$

Show that the energy of a time-independent field configuration $\phi(x)$ is

$$E = \int_{-\infty}^{\infty} \frac{1}{2}(\phi')^2 + \lambda(m^2 - \phi^2)^2 \, dx \, ,$$

where $\phi' = \frac{d\phi}{dx}$. Show that the only field configurations of zero energy are the two vacua $\phi = \pm m$.

Reexpress the energy as

$$E = \int_{-\infty}^{\infty} \frac{1}{2}\left(\phi' - \sqrt{2\lambda}(m^2 - \phi^2)\right)^2 + \text{further terms} \, dx \, ,$$

and show that the further terms can be integrated to give just boundary contributions at $\pm\infty$. Assume that the field configuration $\phi(x)$ interpolates between the two vacua, so that ϕ approaches $-m$ as $x \to -\infty$ and m as $x \to \infty$. For such configurations, show that the energy is at least $\frac{4}{3}m^3\sqrt{2\lambda}$. Show that the minimal energy is attained provided ϕ satisfies the (Bogomolny) equation

$$\phi' - \sqrt{2\lambda}(m^2 - \phi^2) = 0 \, ,$$

and show that this has the general solution with the right boundary conditions

$$\phi(x) = m \tanh\left(\sqrt{2\lambda}m(x - a)\right) \, .$$

Sketch this solution, and give the interpretation of a. This solution is called the ϕ^4 kink, and is interpreted as a particle in this theory, with mass equal to the energy already calculated. It only exists in the sector which interpolates between the two different vacua, and there is an antikink in the sector where the signs of ϕ at $\pm\infty$ are flipped. Sketch the antikink, and find the equation it satisfies.

The Lorentz boosted kink satisfies the full field equation of the theory. An approximation to this solution, valid for non-relativistic speeds v is

$$\phi(x,t) = m \tanh\left(\sqrt{2\lambda}m(x - vt)\right) \, .$$

Use the expression for the kinetic energy of a generic field configuration to calculate the kinetic energy for this moving kink.

Hint: recall, and check, that $1 - \tanh^2 x = \text{sech}^2 x$ and $(\tanh x)' = \text{sech}^2 x$.

5.2 Unstable Kink

The Lagrangian density for a complex scalar field ϕ in $1+1$ dimensions is

$$\mathcal{L} = \frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2 - \frac{1}{2}\left|\frac{\partial\phi}{\partial x}\right|^2 - \frac{1}{2}\lambda^2(a^2 - |\phi|^2)^2 .$$

By treating the real and imaginary parts of ϕ independently, or better, by treating ϕ and ϕ^* as independent, verify that the field equation is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - 2\lambda^2(a^2 - |\phi|^2)\phi = 0$$

and that it has the real kink $\phi_0(x) = a\tanh\lambda a x$ as a solution. Now consider a small pure imaginary perturbation $\phi(x,t) = \phi_0(x) + i\eta(x,t)$, with η real. Find the linear equation for η. By considering η of the form $\mathrm{sech}(\alpha x)e^{\omega t}$, show that the kink is unstable.

Is there a topological argument which suggests that the kink is either stable or unstable? (Consider the vacua of this theory.)

5.3 Rational Map Symmetry

Using $z = \tan(\frac{\theta}{2})e^{i\varphi}$, locate the points $z = 0, 1, i, -1, -i, \infty$ on the sphere, and show they can be identified with the face centres of a cube. Show that the transformation $z \to iz$ permutes these points and corresponds to a 90° rotation mapping the cube to itself. Similarly show that the transformation $z \to \frac{iz+1}{-iz+1}$ permutes these points and corresponds to a 120° rotation mapping the cube to itself.

Consider the map

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1} .$$

Calculate its Wronskian, and deduce that this gives evidence for cubic symmetry.

Show that the map has the symmetries

$$R(iz) = \frac{1}{R(z)} ,$$

$$R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1} .$$

Does this prove that the map is cubically symmetric? Why is the Wronskian argument insufficient?

5.4 BPS Monopole

Given the static, spherically symmetric ansatz

$$\begin{aligned}\Phi^a &= h(r)\frac{x^a}{r} \\ A_i^a &= -\varepsilon^{aij}\,\frac{x^j}{r^2}\,(1 - k(r))\end{aligned}$$

calculate $D_i\Phi$ and F_{ij}, and show that the Bogomolny equation $F_{ij} = \varepsilon_{ijk}D_k\Phi$ reduces to

$$\frac{dh}{dr} = \frac{1}{r^2}(1-k^2)\,, \quad \frac{dk}{dr} = -kh\,.$$

Use the change of variables $H = h + \frac{1}{r}$, $K = \frac{k}{r}$ to find the monopole solution of these equations. Show that the fields have no singularity at the origin.

References

[1] A. A. Abrikosov: On the magnetic properties of superconductors of the second group, *Sov. Phys. JETP* **5**, 1174 (1957)

[2] R. A. Battye, N. S. Manton and P. M. Sutcliffe: Skyrmions and the α-particle model of nuclei, *Proc. R. Soc.* **A463**, 261 (2007)

[3] R. A. Battye and P. M. Sutcliffe: Skyrmions, fullerenes and rational maps, *Rev. Math. Phys.* **14**, 29 (2002)

[4] A. A. Belavin, A. M. Polyakov, A. S. Schwarz and Yu. S. Tyupkin: Pseudoparticle solutions of the Yang-Mills equations, *Phys. Lett.* **B59**, 85 (1975)

[5] E. B. Bogomolny: The stability of classical solutions, *Sov. J. Nucl. Phys.* **24**, 449 (1976)

[6] G. H. Derrick: Comments on nonlinear wave equations as models for elementary particles, *J. Math. Phys.* **5**, 1252 (1964)

[7] S. K. Donaldson: Nahm's equations and the classification of monopoles, *Commun. Math. Phys.* **96**, 387 (1984)

[8] D. Finkelstein and J. Rubinstein: Connection between spin, statistics and kinks, *J. Math. Phys.* **9**, 1762 (1968)

[9] C. J. Houghton, N. S. Manton and P. M. Sutcliffe: Rational maps, monopoles and Skyrmions, *Nucl. Phys.* **B510**, 507 (1998)

[10] S. Jarvis: A rational map for Euclidean monopoles via radial scattering, *J. reine angew. Math.* **524**, 17 (2000)

[11] J. B. Ketterson and S. N. Song: *Superconductivity*, Cambridge University Press, 1999

[12] F. Klein: *Lectures on the Icosahedron*, London, Kegan Paul, 1913

[13] S. Krusch: Homotopy of rational maps and the quantization of Skyrmions, *Ann. Phys* **304**, 103 (2003)

[14] N. S. Manton and P. Sutcliffe: *Topological Solitons*, Cambridge University Press, 2004 (paperback edition, 2007)

[15] N. S. Manton and S. W. Wood: Reparametrizing the Skyrme model using the lithium-6 nucleus, *Phys. Rev.* **D74**, 125017 (2006)

[16] O. V. Manko, N. S. Manton and S. W. Wood: Light nuclei as quantized Skyrmions, *Phys. Rev.* **C76**, 055203 (2007)

[17] J. Polchinski: *String Theory, vol. 2*, Cambridge University Press, 1998

[18] A. M. Polyakov: Particle spectrum in quantum field theory, *JTEP Lett.* **20**, 194 (1974)

[19] M. K. Prasad and C. M. Sommerfield: Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon, *Phys. Rev. Lett.* **35**, 760 (1975)

[20] P. J. Ruback: Vortex string motion in the abelian Higgs model, *Nucl. Phys.* **B296**, 669 (1988)

[21] T. H. R. Skyrme: A nonlinear field theory, *Proc. R. Soc. Lond.* **A260**, 127 (1961)

[22] T. H. R. Skyrme: A unified field theory of mesons and baryons, *Nucl. Phys.* **31**, 556 (1962)

[23] G. 't Hooft: Magnetic monopoles in unified gauge theories, *Nucl. Phys.* **B79**, 276 (1974)

[24] R. S. Ward: A Yang-Mills-Higgs monople of charge 2, *Commun. Math. Phys.* **79**, 317 (1981)

[25] R. S. Ward: Slowly moving lumps in the $\mathbb{CP}^1$ model in (2+1) dimensions, *Phys. Lett.* **B158**, 424 (1985)

[26] E. Weinberg and P. Yi: Magnetic monopole dynamics, supersymmetry, and duality, *Phys. Rept.* **438**, 65-236 (2007)

[27] E. Witten: Global Aspects of of current algebra, *Nucl. Phys.* **B223**, 422 (1983); Current algebra, baryons, and quark confinement, *ibid* **B223**, 433 (1983)

[28] W. J. Zakrzewski: *Low Dimensional Sigma Models*, Bristol, Institute of Physics Publishing, 1989

[29] W. J. Zakrzewski: Soliton-like scattering in the $O(3)$ sigma model in (2+1) dimensions, *Nonlinearity* **4**, 429 (1991)