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Renormalization of the Electroweak Standard Model

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Abstract

These lecture notes give an introduction to the algebraic renormalization of the Standard Model. We start with the construction of the tree approximation and give the classical action and its defining symmetries in functional form. These are the Slavnov-Taylor identity, Ward identities of rigid symmetry and the abelian local Ward identity. The abelian Ward identity ensures coupling of the electromagnetic current in higher orders of perturbation theory, and is the functional form of the Gell-Mann–Nishijima relation. In the second part of the lectures we present in simple examples the basic properties of renormalized perturbation theory: scheme dependence of counterterms and the quantum action principle. Together with an algebraic characterization of the defining symmetry transformations they are the ingredients for a scheme independent unique construction of Green's functions to all orders of perturbation theory.

Contents

1. Introduction

In these lectures we give an introduction to the algebraic renormalization of the Standard Model of electroweak interaction. The Standard Model of elementary particle physics is a renormalizable quantum field theory and allows consistent predictions of physical processes in terms of a few parameters, as masses and couplings, order by order in perturbation theory. The Standard Model includes electromagnetic, weak and strong interactions and the classical model is a non-abelian gauge theory with gauge group $U(1) \times SU(2) \times SU(3)$. The $U(1) \times SU(2)$ gauge group is spontaneously broken to the electromagnetic subgroup providing masses for the charged leptons and quarks and for the vector bosons of weak interactions via the Higgs mechanism, but leaving the photon as a massless particle [1, 2, 3]. Since the electromagnetic subgroup does not correspond to the abelian factor subgroup it turns out that weak interactions cannot be described consistently without the electromagnetic interactions, but we are able to split off the unbroken $SU(3)$-colour gauge group responsible for the strong interactions without destroying the physical structure of the theory. In these lectures we only consider the $SU(2) \times U(1)$-structure of electroweak interactions.

The Standard Model of electroweak interactions has been tested to high accuracy with the precision experiments at the Z-resonance at LEP [4]. The degree of precision enforces to take into account also contributions beyond the tree approximation in the perturbative formulation. For this reason an extensive calculation of 1-loop processes and also 2-loop processes has been carried out in the past years and compared to the experimental results. (For reviews see [5, 6] and references therein; for a recent review see [7].) A careful analysis shows that the theoretical predictions and the experiments are in excellent agreement with each other [8].

A necessary prerequisite for carrying out precision tests of the Standard Model is the consistent mathematical and physical formulation of the Standard Model in the framework of its perturbative construction. Explicitly one has to prove the following properties in order to bring it into the predictive power, which the Standard Model is expected to have:

- The Green's functions of the theory are *uniquely* determined as functions of a finite (small) number of free parameters to all orders of perturbation theory. This property is called renormalizability.
- The physical scattering matrix constructed from the Green's functions is unitary and gauge parameter independent. In particular these properties ensure a probability interpretation of S-matrix elements and guarantee at the same time that unphysical

particles are cancelled in physical scattering processes. Only then the theory has indeed a physical interpretation.

- It has to be shown, that the theory is in agreement with the experiments by calculating different processes as accurately as possible.

In the present lectures we only treat the first point, the unique construction of the Green's functions to all orders of perturbation theory. We want to point out, that unitarity and gauge parameter independence of the S-matrix are not rigorously derived in the Standard Model by now, but are commonly assumed to hold. However, its analysis includes the important problem of unstable particles, whose solution will have far reaching consequences in phenomenological applications (see for example [9]).

Renormalizability of gauge theories has been first shown in the framework of dimensional regularization [10, 11]. One has used that dimensional regularization is an invariant scheme for gauge and BRS invariance, respectively, as long as parity is conserved. In this scheme it has been proven that all the divergencies can be absorbed into invariant counter terms to the coupling, the field redefinitions and the masses of the classical action. This method implies the unique construction of the Green's functions. These proofs are not applicable to the Standard Model, since there parity is broken. It is also well-known, that the group structure of the Standard Model allows the presence of anomalies. For this reason an invariant scheme is very likely not to exist. The algebraic method of renormalization provides a proof of renormalizability also in such cases where an invariant scheme does not exist. It gives in a scheme-independent way the symmetry relations of finite Green's functions to all orders.

The algebraic method has been applied to gauge theories with semi-simple gauge groups [12, 13]. Necessary prerequisite for the algebraic method to work was the discovery of the BRS symmetry [12, 14] named after Becchi, Rouet and Stora. In its functional form BRS symmetry is called the Slavnov-Taylor identity. This identity is the defining symmetry of gauge theories in renormalizable and Lorentz invariant gauges and includes the gauge-fixing action and the action of the Faddeev-Popov ghosts.

To gauge theories with non-semisimle groups the algebraic method has been applied in [15]. In particular this paper includes an investigation of the anomaly structure and an investigation of the instability of abelian factor groups, but the authors do not consider massless particles and do not care about physical normalization conditions. The Green's functions of the electroweak Standard Model and its defining symmetry transformations are constructed in [16] by algebraic renormalization to all orders. In this paper we have given also special attention to on-shell normalization conditions and to a careful analysis of free parameters. In the present lecture we will give an introduction to this construction: In the first lectures, section 2, we construct the classical action as an $SU(2) \times U(1)$ gauge theory. Special attention is paid to the uniqueness of the action and transformations and their algebraic characterization. In section 3 we introduce the renormalizable gauges, BRS symmetry and Faddeev-Popov ghosts. Finally we summarize the defining symmetry transformations of the tree approximation in functional form, the Slavnov Taylor identity, the Ward identities of rigid symmetry and the local abelian Ward identity. In the last lecture, section 4, the construction to all orders by the algebraic method is outlined. In particular we present the basic ingredients of the algebraic method, namely scheme dependence of counterterms and the action principle. In Appendix A we collect the important formulae of the tree approximation: the classical action and the symmetry transformations of the Standard Model. The exercises that were given during the lectures can be found in Appendix B.

Since we assume in these lectures, that the reader has a basic knowledge about quantum field theory and renormalization, we give a few books and reviews separated from the usual references, which introduce the foundations of perturbative quantization and renormalization. The books and reviews that we have selected are mostly close to our presentation and these lectures continue the methods presented therein to the Standard Model of electroweak interactions.

2. The classical limit of the Standard Model

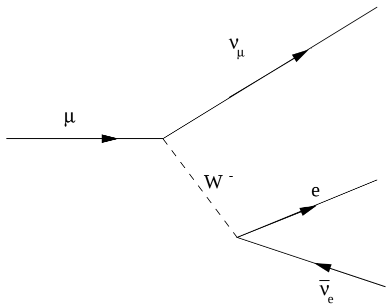
2.1. Particle content of the Standard Model

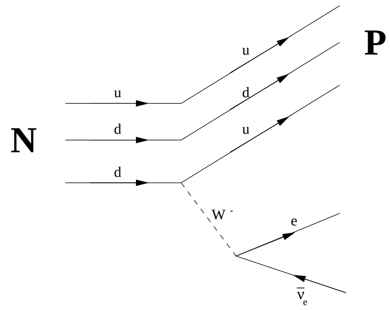
The particles of the Standard Model are divided into groups according to their particle properties, as spin and electric charge. The first group consists of particles with spin $\frac{1}{2}$, the fermions. The group of fermions has two subgroups, the leptons and quarks. Whereas leptons only participate in weak interactions, the quarks interact by weak and strong interactions. Accordingly all quarks are colour vectors. Strong interaction is described by $SU(3)$ colour gauge theory, weak and electromagnetic by a $SU(2) \times U(1)$ gauge theory, so that the complete Standard Model is a $SU(3) \times SU(2) \times U(1)$ gauge theory. In these lectures strong interaction will not be taken into account, so we restrict ourselves in treating the $SU(2) \times U(1)$ gauge theory of electroweak interactions and consider colour $SU(3)$ as a global symmetry. (We come back to this point at the end of this subsection.) Quarks and leptons are also distinguished by their electric charge: There exist two types leptons, charged leptons e, μ, τ with electric charge $Q_e = -1$ and the neutral neutrinos ν_e, ν_μ, ν_τ. Up-type quarks, the *u*p, *c*harm, and *t*op, have charge $Q_u = \frac{2}{3}$, down type quarks, the *d*own, *s*trange and *b*ottom have electric charge $Q_d = -\frac{1}{3}$. fermions in the Standard Model are furthermore arranged into three families according to the following scheme:

$$\begin{array}{ccc} \nu_e & \nu_\mu & \nu_\tau \\ e & \mu & \tau \\ u & c & t \\ d & s & b \end{array} \tag{2.1}$$

In the following we only consider the first generation of fermions (e, ν_e, u, d). In particular we disregard any mixing effects between different generations. In generality mixing between three families leads to CP violation via the Cabibbo-Kobayashi-Maskawa matrix [17], which makes proving the renormalizability more difficult.

The second group of particles consists of the vector bosons, which are particles with spin 1. The gauge bosons of electroweak interactions are the photon (A_μ), the Z-boson (Z_μ) and the $W^\pm$-bosons $(W^\pm_\mu)$. The photon and Z-boson are neutral, $W^\pm$-bosons have electric charge $+1$ and -1, respectively. The full Standard Model in addition contains eight gluons of strong interactions, which are not considered in the course of these lectures. The photon is massless and couples to all the electric charged particles, in particular it couples also to the charged bosons of electroweak interactions. The other three bosons are massive, which makes the weak interactions important only at small distance scales. The weak force is responsible for the decay of the muon and the β-decay of the neutron:

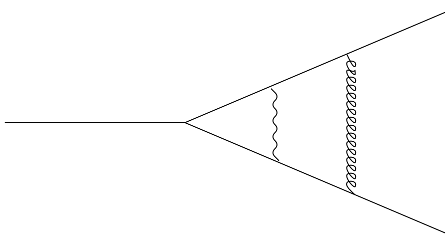


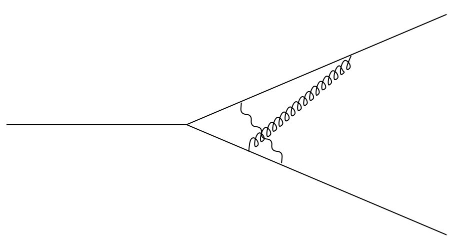


$\mu \longrightarrow e + \bar{\nu}_e + \nu_\mu,$ $\qquad$ $n \longrightarrow p + e + \bar{\nu}_e.$

In the Standard Model there is one scalar particle, which has spin 0 and is neutral with respect to electric charge. It is called the Higgs boson. In the theoretical prescription of electroweak interactions it is needed to give masses to the vector bosons and to the fermions in agreement with $SU(2) \times U(1)$ gauge symmetry [18, 19]. However, the Higgs particle has not been observed until now. All the particles of the Standard Model with their properties are listed in table 1.

The following remarks should be made about the exclusion of the strong interactions in these notes. The QCD coupling constant is by far the largest coupling in the full Standard Model for energy scales that are reached in experiments now and in the near future. This means that the QCD corrections are more important in phenomenological applications than the electroweak corrections: for the precision tests of the physics at the Z resonance at LEP1 the following calculations were needed: 3 loop QCD corrections, 1 loop electroweak corrections with 1 loop QCD corrections on top of that; 2 loop electroweak was of minor importance. In the lectures we disregard QCD corrections, but this does not mean that QCD corrections factor out of the calculations of scattering matrix elements. Indeed this is not at all the case, as can been seen from the following two diagrams:





The diagram on the left can be understood as first an electroweak correction is applied and then a QCD correction, so this diagram is factorable. But this analysis can't clearly be done to the diagram on the right.

Disregarding QCD corrections in the proof of renormalizability is justified, since the colour group $SU(3)$ is unbroken in the Standard Model and its generators of global symmetry do not mix with the one of $SU(2) \times U(1)$ symmetry. In contrast to this renormalizing only $SU(2)$ instead of the $SU(2) \times U(1)$ symmetry means to treat a different theory, since the symmetry of the electroweak model is spontaneously broken in such a way that the abelian subgroup cannot be factorized out anymore. (The electromagnetic charge operator is a linear combination of a genuine abelian operator Y and the third component of weak isospin.) So if we understand the renormalization there, then the inclusion of QCD requires just the addition of an unbroken local symmetry, whose global symmetry is conserved by construction.

2.2. The construction of gauge theories

2.2.1. The free Dirac equation

We start our discussion with the Dirac equation of free fermions:

$$(i\gamma^\mu \partial_\mu - m_f)f = 0, \tag{2.2}$$

$$\bar{f}(i\gamma^\mu \overleftarrow{\partial}_\mu + m_f) = 0. \tag{2.3}$$

Here f is a four component Dirac spinor and $\bar{f} = f^\dagger \gamma^0$ the adjoint spinor. γ^μ are the Dirac matrices which form a Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}, \tag{2.4}$$

with the metric $g^{\mu\nu} = (1, -1, -1, -1)$. For a set of fermions $\{f\}$ the equations of motion can be derived from the action

$$\Gamma^{bil}_{Dirac} = \sum_f \int d^4x \bar{f}(i\partial\!\!\!/ - m_f)f \tag{2.5}$$

by the classical principle of least action, i.e.

$$\delta \Gamma^{bil}_{Dirac} = 0. \tag{2.6}$$

Here we have defined $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ and The summation is understood over all fermions in question, as for example $f = \nu, e, u, d$, if we include the first fermion generation of the Standard Model.

Type	Spin	Particle	Charge	Mass(MeV)	
Vector	1	g	0	0	
		γ	0	0	
		Z	0	$91.1884GeV$	$\pm 0.022GeV$
		$W^{\pm}$	± 1	$80.26GeV$	± 0.16
Leptons	$\frac{1}{2}$	ν_e	0	$< 7.3eV$	$CL = 90\%$
		ν_μ		< 0.17	$CL = 90\%$
		ν_τ		< 24	$CL = 95\%$
		e	-1	0.51099907	± 0.0000015
		μ		105.658389	± 0.000034
		τ		1777.0	$+3.0, -2.7$
Quarks	$\frac{1}{2}$	u	$\frac{2}{3}$	5.6	± 1.1
		c		1350	± 50
		t		$180GeV$	$\pm 12GeV$
		d	$-\frac{1}{3}$	9.9	± 1.1
		s		199	± 33
		b		$5GeV$	$\pm 1GeV$
Higgs	0	H	0	$> 58.4GeV$	$CL = 95\%$

Table 1: Properties of the particles which make up the electroweak Standard Model [20].

To evaluate the variation of a functional we introduce – for later use – the functional derivative. A functional F assigns to functions u^i of some function space B a complex number: $F : u^i \in B \longrightarrow F[u] \in \mathbb{C}$. In generalization of ordinary variations of functions the variation δF of the functional is given by:

$$\delta F = \sum_j \int d^4x \frac{\delta F}{\delta u^j(x)} \delta u^j(x). \tag{2.7}$$

The $\frac{\delta}{\delta u^j(x)}$ denotes the functional derivative with respect to u^j at x, which is defined by the usual properties of a derivative together with:

$$\frac{\delta u^i(x)}{\delta u^j(y)} = \delta_{ij}\delta^4(x-y). \tag{2.8}$$

If we apply functional variation for determining the variation of the Dirac action Γ^{bil}_{Dirac} (2.5) we get:

$$\begin{aligned} \delta\Gamma^{bil}_{Dirac} &= \sum_f \int d^4x \left\{ \delta\bar{f}(x) \frac{\delta\Gamma^{bil}_{Dirac}}{\delta\bar{f}(x)} + \frac{\delta\Gamma^{bil}_{Dirac}}{\delta f(x)} \delta f(x) \right\} \\ &= \sum_f \int d^4x \left\{ \delta\bar{f}(x)(i\partial\!\!\!/ - m_f) f(x) + \bar{f}(x)(i\overleftarrow{\partial\!\!\!/} + m_f)\delta f(x) \right\} \end{aligned} \tag{2.9}$$

Since $\delta\bar{f}$ and δf are independent variations, the Dirac equation of the fermions and adjoint fermions follow from the principle of least action (2.6). Note that in (2.9) spinor variation is applied from the right and variation with respect to the adjoint spinor from the left for consistency.

2.2.2. The electromagnetic interaction

Noether's theorem tells that current conservation and charge conservation is connected with the symmetries of the action. For this reason we now want to look for symmetries of the Dirac action.

Of course the Dirac action is invariant,

$$\Gamma^{bil}_{Dirac}(\bar{f}, f) = \Gamma^{bil}_{Dirac}(\bar{f}', f'), \tag{2.10}$$

if we redefine all fermions by a single phase factor:

$$f \longrightarrow f' = e^{-i\varepsilon q_f} f \text{ and } \bar{f} \longrightarrow \bar{f}' = e^{i\varepsilon q_f}\bar{f}. \tag{2.11}$$

Here ε denotes a real parameter, q_f are numbers associated to the different fermions. These transformations form an abelian group for arbitrary q_f as long as no further symmetries are considered. Assigning to q_f the electric charge Q_f of the respective fermion the transformation (2.11) is related to electric current and finally charge conservation. We could also assign

$$q_f = \begin{cases} 1, & \text{if} \quad f = \nu, e \\ 0, & \text{if} \quad f = u, d \end{cases} \qquad \text{or} \qquad q_f = \begin{cases} 0, & \text{if} \quad f = \nu, e \\ 1, & \text{if} \quad f = u, d \end{cases} \tag{2.12}$$

Then the transformation corresponds to lepton or baryon family number conservation. In the following we restrict ourselves to electromagnetic transformations ($q_f \equiv Q_f$), since – in contrast to lepton and baryon number symmetry – electromagnetic symmetry is gauged in the electroweak Standard Model.

From now on we do not consider the group transformations, but expand the exponential function for small ε and only consider the corresponding infinitesimal transformations. (So we restrict ourselves to the Lie algebra of the Lie group.). The infinitesimal transformations of (2.11) have the form:

$$\delta^{em}(\varepsilon) f = \varepsilon \delta^{em} f = -i\varepsilon Q_f f \qquad \text{and} \qquad \delta^{em}(\varepsilon)\bar{f} = \varepsilon\delta^{em}\bar{f} = i\varepsilon Q_f \bar{f}. \tag{2.13}$$

If we apply these transformations to the bilinear action (2.5), we find the infinitesimal version of (2.10)

$$\delta^{em}(\varepsilon)\Gamma^{bil}_{Dirac} = \varepsilon\mathcal{W}_{em}\Gamma^{bil}_{Dirac} = \varepsilon \int d^4x \mathbf{w}_{em}(x)\Gamma^{bil}_{Dirac} = 0. \tag{2.14}$$

In this equation we have introduced the functional operators which correspond to electromagnetic transformations: $\mathbf{w}_{em}$ is the functional operator of the infinitesimal *local* electromagnetic transformations,

$$\begin{aligned} \varepsilon\mathbf{w}_{em}(x) &= \sum_f \left(\delta^{em}\bar{f}(x)\frac{\delta}{\delta\bar{f}(x)} + \frac{\delta}{\delta f(x)}\delta^{em} f(x) \right) \\ &= \sum_f \left(i\varepsilon Q_f \bar{f}(x)\frac{\delta}{\delta\bar{f}(x)} - i\varepsilon Q_f \frac{\delta}{\delta f(x)} f(x) \right), \end{aligned} \tag{2.15}$$

and $\mathcal{W}^{em}$ is the one of *global* or *rigid* electromagnetic transformations:

$$\mathcal{W}_{em} = \int d^4x \; \mathbf{w}_{em}(x). \tag{2.16}$$

Now we are able to derive immediately Noether's first theorem: Since the Dirac action is invariant under global transformations, the local transformations can be only broken by a total derivative:

$$\mathbf{w}_{em}(x)\Gamma^{bil}_{Dirac} = -\partial^\mu j^{em}_\mu(x). \tag{2.17}$$

The electromagnetic current,

$$j_\mu^{em}(x) = \sum_f Q_f \bar{f}(x)\gamma_\mu f(x), \tag{2.18}$$

is seen to be conserved by applying the equations of motions

$$\frac{\delta \Gamma_{Dirac}^{bil}}{\delta f} = 0 \qquad \text{and} \qquad \frac{\delta \Gamma_{Dirac}^{bil}}{\delta \bar{f}} = 0 \tag{2.19}$$

on the left-hand-side of eq. (2.17). According to Noether's second theorem we are able to gauge the symmetry by coupling the electromagnetic current to a vector field A_μ:

$$\Gamma_{matter} = \Gamma_{Dirac}^{bil} - \int d^4x\, e\, j_\mu^{em} A^\mu. \tag{2.20}$$

Here e is the electromagnetic coupling constant and we may interpret A_μ as the electromagnetic vector potential. The action (2.20) is indeed invariant under local gauge transformation as it stands if we assign the transformation

$$\delta^{em}(\varepsilon(x))A_\mu = \frac{1}{e}\partial_\mu \varepsilon(x) \tag{2.21}$$

as abelian gauge transformation to A_μ. The local gauge invariance of the new action Γ_{matter} can be expressed in functional form:

$$\left(\mathbf{w}_{em} - \frac{1}{e}\partial^\mu \frac{\delta}{\delta A_\mu}\right) \Gamma_{matter} = 0. \tag{2.22}$$

To interpret A_μ as a dynamical physical field, namely the photon, it needs to have a kinetic action as well: The free field action

$$\Gamma_{gauge} = -\frac{1}{4}\int d^4x F^{\mu\nu} F_{\mu\nu} \tag{2.23}$$

with the antisymmetric field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.24}$$

is invariant under the local transformation (2.22).

If we put the invariant actions (2.20) and (2.23) together, we arrive at the action of classical electrodynamics:

$$\Gamma_{em} = \int d^4x \left(\sum_f \bar{f}(i\partial\!\!\!/ - m_f)f - e j_\mu^{em} A^\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right). \tag{2.25}$$

The field equations that follow from the classical action (2.25) are:

$$\partial^\mu F_{\mu\nu} = e j_\nu^{em}, \tag{2.26}$$
$$(i\partial\!\!\!/ - m_f) f = e Q_f \gamma^\mu A_\mu f.$$

Gauge invariance of the electromagnetic action can be expressed by the functional identity:

$$\left(\mathbf{w}_{em} - \frac{1}{e}\partial^\mu \frac{\delta}{\delta A^\mu}\right)\Gamma_{em} = 0. \tag{2.27}$$

In perturbation theory this equation will be continued to the electromagnetic Ward identity, which plays an important role for the definition of Green's functions in higher orders (see [Q4,R5]). For this reason one has to note that the most general solution of the Ward identity for local actions with dimension less than or equal four is given by (2.25) up to the field and coupling redefinitions:

$$\begin{aligned} f &\to z_f f \qquad & \bar{f} &\to z_f \bar{f} \\ A &\to z_A A \qquad & e &\to z_A^{-1} e \end{aligned} \tag{2.28}$$

Note that these redefinitions leave the operator in (2.27) invariant.

One final remark about the dimensions of the fields [Q3]: If one scales the coordinates by: $x^\mu \longrightarrow e^{-\lambda} x^\mu$, then a field B may scale as $B \longrightarrow e^{\alpha\lambda} B$. The number α is called the (naive scale or mass) dimension of a field B. This leads to the following table.

Field	x^μ	∂_μ	f	$\bar{f}$	A_μ
Dim	-1	1	$\frac{3}{2}$	$\frac{3}{2}$	1

2.2.3. Beyond the Fermi model

In the previous subsection the electromagnetic interaction was discussed, we now turn to the weak interactions. In this discussion we take all fermions to be massless to start with. Low energy experiments, like the decay of neutrons or muons, suggested the existence of charged currents:

$$J_\mu^+ = \frac{1}{2\sqrt{2}}\Big(\bar{e}(1\!\!1 - \gamma_5)\nu + \bar{d}(1\!\!1 - \gamma_5)u\Big), \tag{2.29}$$
$$J_\mu^- = (J_\mu^+)^\dagger = \frac{1}{2\sqrt{2}}\Big(\bar{\nu}(1\!\!1 - \gamma_5)e + \bar{u}(1\!\!1 - \gamma_5)d\Big).$$

Their interaction could be described by the effective Lagrangian:

$$\mathcal{L}_{eff} = -4\sqrt{2}G_\mu J^{+\mu} J^-_\mu, \tag{2.30}$$

where the coupling constant G_μ is called the Fermi constant. This is the Fermi model of weak interactions, which worked phenomenologically quite well for describing the low energy processes of weak interactions. In the charged currents (2.29) we have introduced the γ_5 matrix, which is defined by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.31}$$

and has the following properties:

$$\{\gamma^\mu, \gamma_5\} = 0, \qquad \text{and} \qquad (\gamma_5)^2 = \mathbb{1}. \tag{2.32}$$

Out of the γ_5-matrix two projection operators can be constructed,

$$P^L = \frac{\mathbb{1} - \gamma_5}{2}, \qquad P^R = \frac{\mathbb{1} + \gamma_5}{2}, \tag{2.33}$$

with the properties:

$$\begin{aligned} P^L + P^R &= \mathbb{1}, \qquad & P^R P^L &= P^L P^R = 0 \\ P^L P^L &= P^L, \qquad & P^R P^R &= P^R. \end{aligned} \tag{2.34}$$

Next we introduce the notation of left- and right-handed fermions:

$$f^L = P^L f = \frac{\mathbb{1} - \gamma_5}{2} f, \qquad f^R = P^R f = \frac{\mathbb{1} + \gamma_5}{2} f, \tag{2.35}$$

with the Dirac conjugates $\overline{f^R} = \left(f^L\right)^\dagger \gamma_0$ and $\overline{f^L} = \left(f^R\right)^\dagger \gamma_0$. To the charged currents only the left-handed fermions contribute. The left-handed fermions can be combined into doublets, one doublet for the leptons and one doublet for the quarks:

$$F^L_l = \begin{pmatrix} \nu^L \\ e^L \end{pmatrix} \text{ and } F^L_q = \begin{pmatrix} u^L \\ d^L \end{pmatrix}. \tag{2.36}$$

The charged currents (2.29) can then be cast in the explicit $SU(2)$ form:

$$\begin{aligned} J^+_\mu &= \overline{F^L_l}\gamma_\mu \frac{\tau_-}{2} F^L_l + \overline{F^L_q}\gamma_\mu \frac{\tau_-}{2} F^L_q, \\ J^-_\mu &= \overline{F^L_l}\gamma_\mu \frac{\tau_+}{2} F^L_l + \overline{F^L_q}\gamma_\mu \frac{\tau_+}{2} F^L_q, \end{aligned} \tag{2.37}$$

where

$$\tau_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \qquad \text{and} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.38}$$

These matrices form a representation of the $SU(2)$ algebra with the commutation relation:

$$\left[\tau_\alpha, \tau_\beta\right] = 2i\epsilon_{\alpha\beta\gamma}\tau_\gamma^T \tag{2.39}$$

The structure constants $\epsilon_{\alpha\beta\gamma}$ are completely antisymmetric in all three indices and $\epsilon_{+-3} = -i$.

So we see a $SU(2)$ representation structure emerging for the charged currents of weak interactions. From current algebra one also expects the existence of a neutral current J_μ^3 which corresponds to the generator τ_3:

$$J_\mu^3 = \bar{F_l^L}\gamma_\mu \frac{\tau_3}{2} F_l^L + \bar{F_q^L}\gamma_\mu \frac{\tau_3}{2} F_q^L. \tag{2.40}$$

(Since in J_3^μ only left-handed fermions occur, it is not possible to identify this current with the electromagnetic current.) As in the case of electromagnetic interactions, also the weak currents can be identified as conserved currents when acting with the following functional $SU(2)$-generators

$$\mathbf{w}_\alpha(x) = i\sum_{\delta=l,q}\left(\overline{F_\delta^L}(x)\frac{\tau_\alpha^T}{2}\frac{\delta}{\delta \overline{F_\delta^L}(x)} - \frac{\delta}{\delta F_\delta^L(x)}\frac{\tau_\alpha^T}{2}F_\delta^L(x)\right) \tag{2.41}$$

on the massless Dirac action Γ_{Dirac}^{bil} (2.5) with $f = e, \nu, u, d$:

$$\mathbf{w}_\alpha(x)\ \Gamma_{Dirac}^{bil}\Big|_{m_f=0} = -\partial^\mu J_\mu^\alpha(x). \tag{2.42}$$

Indeed we see that the bilinear action (2.5) is invariant under rigid $SU(2)$ transformations as long as all fermion masses vanish:

$$\mathcal{W}_\alpha\ \Gamma_{Dirac}^{bil}\Big|_{m_f=0} = 0, \qquad \mathcal{W}_\alpha = \int d^4x \mathbf{w}_\alpha(x). \tag{2.43}$$

The functional generators satisfy the local and global $SU(2)$ algebra:

$$\left[\mathbf{w}_\alpha(x), \mathbf{w}_\beta(y)\right] = \delta^4(x-y)\varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathbf{w}_{\gamma'}(x) \tag{2.44}$$

$$\left[\mathcal{W}_\alpha, \mathcal{W}_\beta\right] = \varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathcal{W}_{\gamma'} \tag{2.45}$$

The charge conjugation matrix $\tilde{I}_{\alpha\alpha'}$

$$\tilde{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.46}$$

removes various transpositions from the formulae.

2.2.4. $SU(2) \times U(1)$ GAUGE THEORY

Now we note the following remarkable fact: If one subtracts the electromagnetic charge operator $\mathbf{w}_{em}$ (2.15) from the third component of the weak isospin $\mathbf{w}_3$, one finds a generator, denoted by $\mathbf{w}_4^Q$, which commutes with all $SU(2)$ operators and consequently with the charge operator:

$$[\mathbf{w}_\alpha, \mathbf{w}_4^Q] = 0 \quad \text{with} \quad \mathbf{w}_4^Q = \mathbf{w}_{em} - \mathbf{w}_3 \tag{2.47}$$

Therefore the symmetry operators $\mathbf{w}_\alpha$ and $\mathbf{w}_4^Q$ build a closed $SU(2) \times U(1)$ algebra and imply current conservation of weak and electromagnetic currents, when applied to the massless Dirac action:

$$\left(\mathbf{w}_{em} - \mathbf{w}_3\right) \Gamma^{bil}_{Dirac}\Big|_{m_f=0} = -\partial^\mu (j^{em}_\mu - J^3_\mu). \tag{2.48}$$

(The electromagnetic current j^{em}_μ is defined in (2.18).) For the procedure of quantization it is important to note that $\mathbf{w}_4^Q$ is not uniquely determined by the characterization that it commutes with the $SU(2)$ operators: any generator $\mathbf{w}_4$ is abelian with respect to $\mathbf{w}_\alpha$ when it has the form:

$$\begin{aligned}\mathbf{w}_4(x) = i\frac{Y_W^l}{2}\left(\overline{F_l^L}(x)\frac{\delta}{\delta \overline{F_l^L}(x)} - \frac{\delta}{\delta F_l^L(x)}F_l^L(x)\right) + i\frac{Y_W^q}{2}\left(\overline{F_q^L}(x)\frac{\delta}{\delta \overline{F_q^L}(x)} - \frac{\delta}{\delta F_q^L(x)}F_q^L(x)\right)\\ + \sum_f iQ_f\left(\overline{f^R}(x)\frac{\delta}{\delta \overline{f^R}(x)} - \frac{\delta}{\delta f^R(x)}f^R(x)\right)\end{aligned} \tag{2.49}$$

with arbitrary values of Y_W^l, Y_W^q and Q_f. (This means there are 5 linearly independent abelian operators in $\mathbf{w}_4$.) Applying $\mathbf{w}_4(x)$ to the massless Dirac equation all these symmetry operators are connected with classically conserved currents. Since only the electromagnetic symmetry is gauged, the parameters Y_W^l, Y_W^q and Q_f are determined by the relation (2.47), which is the functional form equivalent to the well-known Gell-Mann–Nishijima relation:

$$Q = \frac{Y_W}{2} + T_3. \tag{2.50}$$

From (2.47) one derives the following values for the weak hypercharge of leptons and quarks

$$Y_W^l = -1 \qquad Y_W^q = \frac{1}{3} \tag{2.51}$$

and identifies Q_f with the electric charge of the respective fermions:

$$Q_e = -1 \qquad Q_u = \frac{2}{3} \qquad Q_d = -\frac{1}{3}. \tag{2.52}$$

Having constructed the relevant symmetry transformations we are able to proceed as in the case of abelian gauge theories, when we want to construct the gauge theory belonging to the conserved currents. We couple the currents $J^{\pm}_\mu$, J^3_μ and $J^4_\mu \equiv j^{em}_\mu - J^3_\mu$ to the vector fields $W^\mu_\pm$, W^μ_3 and W^μ_4 and enlarge the free field action by these terms:

$$\Gamma_{matter} = \Gamma^{bil}_{Dirac}\Big|_{m_f=0} - \int d^4x \left(g_1 J^4_\mu W^\mu_4 - g_2 \left(J^3_\mu W^\mu_3 + J^+_\mu W^\mu_- + J^-_\mu W^\mu_+\right)\right). \qquad (2.53)$$

Since the gauge group of the Standard Model is a direct product of two groups, the couplings (g_2 and g_1) of $SU(2)$ and $U(1)$ are independent from each other. From the $SU(2)$ algebra of the functional operators (2.45) as well as from global invariance

$$\mathcal{W}_\alpha \Gamma_{matter} \stackrel{!}{=} 0 \qquad (2.54)$$

it is derived that the $\mathbf{w}_\alpha$'s have to be extended to include the vector bosons. If we now indicate the $\mathbf{w}$ we had on the fermions explicitly by $\mathbf{w}^{fermion}$, we have now:

$$\mathbf{w}_\alpha \to \mathbf{w}_\alpha \equiv \mathbf{w}^{fermion}_\alpha + \mathbf{w}^{vector}_\alpha \qquad (2.55)$$

with

$$\mathbf{w}^{vector}_\alpha(x) = I_{\alpha\alpha'} W^\mu_\beta(x) \varepsilon_{\beta\gamma\alpha'} \tilde{I}_{\gamma\gamma'} \frac{\delta}{\delta W^\mu_{\gamma'}(x)} \qquad \text{and} \qquad \mathbf{w}^{vector}_4(x) = 0, \qquad (2.56)$$

where $\alpha = +, -, 3, 4$. The structure constants $\varepsilon_{\alpha\beta\gamma}$ are defined as in (2.39) but with $\varepsilon_{a\beta 4} = 0$. The matrix $\tilde{I}$ is defined in (2.46).

Since the $SU(2) \times U(1)$ algebra uniquely determines the abelian transformation of vectors to vanish, it is possible to determine the charge of the vector bosons by looking at

$$\mathcal{W}^{vector}_{em} = \mathcal{W}^{vector}_3 + \mathcal{W}^{vector}_4 = -i \int d^4x \left(W^\mu_+ \frac{\delta}{\delta W^\mu_+} - W^\mu_- \frac{\delta}{\delta W^\mu_-}\right), \qquad (2.57)$$

thus $W_\pm$ has got charge ± 1.

With the functional operators $\mathbf{w}_\alpha$ (2.55) gauge invariance of Γ_{matter} (2.53) is expressed in functional form by the identities:

$$\begin{aligned} \left(\mathbf{w}_\alpha + \frac{1}{g_2}\partial^\mu \tilde{I}_{\alpha\alpha'} \frac{\delta}{\delta W^\mu_{\alpha'}}\right)\Gamma_{matter} &= 0 \qquad \alpha = +, -, 3 \\ \left(\mathbf{w}^Q_4 - \frac{1}{g_1}\partial^\mu \frac{\delta}{\delta W^\mu_4}\right)\Gamma_{matter} &= 0. \end{aligned} \qquad (2.58)$$

By introducing the covariant derivatives

$$\begin{aligned} D_\mu F^L_\delta &= \left(\partial_\mu - i g_2 \frac{\tau_\alpha}{2} W_{\alpha\mu} + i g_1 \frac{Y^\delta_W}{2} W_{4\mu}\right) F^L_\delta, \qquad \delta = l, q, \\ D_\mu f^R &= \left(\partial_\mu + i g_1 Q_f W_{4\mu}\right) f^R, \end{aligned} \qquad (2.59)$$

the matter action of fermions can also be rewritten into the form:

$$\Gamma_{matter} = \int d^4x \left(\sum_{\delta=l,q} \overline{F^L_\delta} i\gamma^\mu D_\mu F^L_\delta + \sum_f \overline{f^R} i\gamma^\mu D_\mu f^R \right). \tag{2.60}$$

Finally we have to add kinetic terms for the gauge fields to the action in such a way that the gauge invariance remains. The Yang-Mills action

$$\Gamma_{YM} = -\frac{1}{4} \int d^4x \Big(G^{\mu\nu}_\alpha \tilde{I}_{\alpha\alpha'} G_{\mu\nu\alpha'} + F^{\mu\nu} F_{\mu\nu} \Big) \tag{2.61}$$

with the abelian and non-abelian field strength tensors

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu W^\nu_4 - \partial^\nu W^\mu_4 \\ G^{\mu\nu}_\alpha &= \partial^\mu W^\nu_\alpha - \partial^\nu W^\mu_\alpha + g_2 \tilde{I}_{\alpha\alpha'} \epsilon_{\alpha'\beta\gamma} W^\mu_\beta W^\nu_\gamma, \qquad \alpha, \beta, \gamma = +, -, 3 \end{aligned} \tag{2.62}$$

is the properly normalized solution of the functional identities (2.58) with dimension 4.

The complete action containing massless vector bosons and massless fermions is the sum of the matter and Yang-Mills action:

$$\Gamma_{sym} = \Gamma_{YM} + \Gamma_{matter}. \tag{2.63}$$

In the same way as the electromagnetic action (2.25) is characterized by electromagnetic gauge invariance (2.27), Γ_{sym} is characterized up to field and coupling redefinitions by the functional identities of $SU(2) \times U(1)$ gauge symmetry

$$\Big(\mathbf{w}_\alpha + \frac{1}{g_2} \tilde{I}_{\alpha\alpha'} \partial^\mu \frac{\delta}{\delta W^\mu_{\alpha'}} \Big) \Gamma_{sym} = 0 \tag{2.64}$$

$$\Big(\mathbf{w}^Q_4 - \frac{1}{g_1} \partial^\mu \frac{\delta}{\delta W^\mu_4} \Big) \Gamma_{sym} = 0. \tag{2.65}$$

Here the operators $\mathbf{w}_\alpha$ are the sum of fermion and boson functional operators (2.55) defined in (2.41) and (2.56). The abelian operator $\mathbf{w}^Q_4$ is defined by the relation (2.47) and the electromagnetic charge operator includes fermions (2.15) and vector bosons (2.57):

$$\mathbf{w}^Q_4 = \mathbf{w}_{em} - \mathbf{w}_3 \qquad \mathbf{w}_{em} = \mathbf{w}^{fermion}_{em} + \mathbf{w}^{vector}_{em} \tag{2.66}$$

2.2.5. Higgs mechanism and masses

For deriving the $SU(2)\times U(1)$ gauge invariant action in the previous section we have assumed that all fermions are massless. In reality the charged leptons as well as the up-type and down-type quarks are massive, i.e. $m_e, m_u, m_d \neq 0$. The Dirac action (2.5) in terms of left- and right-handed fields (2.35) has the following form:

$$\Gamma^{bil}_{Dirac} = \int d^4x \left(\sum_f (\bar{f^L} i\partial\!\!\!/ f^L + \bar{f^R} i\partial\!\!\!/ f^R) + \sum_{f=e,u,d} m_f(\bar{f^R} f^L + \bar{f^L} f^R) \right) \tag{2.67}$$

Applying the $SU(2)$ transformations (2.41) on the free field action for massive fermions it is seen that the mass terms break the global $SU(2)$ symmetry:

$$\mathcal{W}_\alpha \Gamma^{bil}_{Dirac} = i\Delta_\alpha \equiv i\int d^4x Q_\alpha(x) \tag{2.68}$$

where

$$\begin{aligned}
\Delta_+ &= \int d^4x \frac{1}{\sqrt{2}} \Big(m_u \bar{d^L} u^R - m_d \bar{d^R} u^L - m_e \bar{e^R} \nu^L \Big) \qquad (2.69)\\
\Delta_- &= \int d^4x \frac{1}{\sqrt{2}} \Big(m_d \bar{u^L} d^R - m_u \bar{u^R} d^L + m_e \bar{\nu^L} e^R \Big)\\
\Delta_3 &= \int d^4x \frac{1}{2} \Big(m_u(\bar{u^L} u^R - \bar{u^R} u^L) - m_d(\bar{d^L} d^R - \bar{d^R} d^L) - m_e(\bar{e^L} e^R - \bar{e^R} e^L) \Big)
\end{aligned}$$

Electric charge invariance, of course, is not broken by the mass terms:

$$\mathcal{W}_{em} \Gamma^{bil}_{Dirac} = 0 \qquad \Longrightarrow \qquad \mathcal{W}^Q_4 \Gamma^{bil}_{Dirac} = -i\Delta_3 \tag{2.70}$$

For including fermion masses and vector boson masses in agreement with $SU(2)\times U(1)$ gauge symmetry into the Standard Model, the symmetry is spontaneously broken to the electromagnetic subgroup.

In these lectures we present a construction of spontaneous symmetry breaking which is purely algebraic and can be compared to the Noether construction of gauge theories, which we have carried out in the last sections. In contrast to the usual construction, which is presented in the books on quantum field theory (see for example [Q3]), it does not start from the symmetric theory, but from the bilinear massive Dirac action of free fermions. Eventually, if one carries out the algebraic characterization of the classical action in the course of algebraic renormalization (see section 4.3), the computation is equivalent to the analysis and construction presented here.

First we couple the breaking of $SU(2)$ transformations (2.69) to scalars $\phi^\pm$ and χ in such a way that the breaking terms can be expressed as field differentiations with respect to these scalars:

$$\Gamma^{mass}_{Dirac} \to \hat\Gamma^{mass}_{Dirac} \equiv \Gamma^{mass}_{Dirac} - \frac{2}{v}\int d^4x \big(\phi^+ Q_- - \phi^- Q_+ - i\chi Q_3\big). \tag{2.71}$$

and

$$\begin{aligned}\Big(\mathcal{W}_\pm \mp i\frac{v}{2}\int d^4x \frac{\delta}{\delta\phi^\mp}\Big)\hat\Gamma^{mass}_{Dirac}\Big|_{\substack{\phi^\pm=0\\ \chi=0}} &= 0 \\ \Big(\mathcal{W}_3 - \frac{v}{2}\int d^4x \frac{\delta}{\delta\chi}\Big)\hat\Gamma^{mass}_{Dirac}\Big|_{\substack{\phi^\pm=0\\ \chi=0}} &= 0\end{aligned} \tag{2.72}$$

Here Γ^{mass}_{Dirac} denotes the mass term of the Dirac action:

$$\Gamma^{mass}_{Dirac} = \sum_{f=e,u,d} \int d^4x\, m_f(\bar f^R f^L + \bar f^L f^R). \tag{2.73}$$

We assign quantum numbers to the scalar fields in such a way that the enlarged action $\hat\Gamma^{mass}_{Dirac}$ (2.71) is CP invariant, neutral with respect to electric charge and hermitian, i.e. the fields $\phi^\pm$ carry charge ± 1 and transform under CP according to $\phi^+ \stackrel{CP}{\to} -\phi^-$ and $\phi^- \stackrel{CP}{\to} -\phi^+$, and the field χ is a neutral field which is CP-odd. (Global signs and normalization in eq. (2.71) are chosen according to usual conventions.) It is seen that the transformation operators appearing in (2.72) are not yet algebraically closed, the commutation relations yield e.g.

$$\Big[\mathcal{W}_\pm \mp i\frac{v}{2}\int d^4x \frac{\delta}{\delta\phi^\mp}, \mathcal{W}_3 - \frac{v}{2}\int d^4x \frac{\delta}{\delta\chi}\Big] = \pm i\mathcal{W}_\pm, \tag{2.74}$$

and on the right-hand-side the inhomogeneous contributions of the shift are missing. It is also seen that $\hat\Gamma^{mass}_{Dirac}$ is not invariant under these transformations at $\phi^\pm, \chi \neq 0$. For this reason we have to enlarge the action as well as the transformation operators in such a way that the action is invariant under the enlarged transformations and that the algebra closes in presence of the inhomogeneous shifts.

For proceeding we note that the breaking terms $Q_\pm(x)$ and $Q_3(x)$ together with the mass term $Q_m(x)$

$$Q_m = \frac{1}{2}\sum_{f=e,u,d} m_f(\bar f^R f^L + \bar f^L f^R) \tag{2.75}$$

can be arranged into a $SU(2)$ doublet and its complex conjugate:

$$\begin{aligned}\mathcal{W}_\alpha \begin{pmatrix} -Q_+ \\ \frac{1}{\sqrt2}(Q_m + Q_3)\end{pmatrix} &= -i\frac{\tau^T_\alpha}{2}\begin{pmatrix} -Q_+ \\ \frac{1}{\sqrt2}(Q_m + Q_3)\end{pmatrix} \\ \mathcal{W}_\alpha \begin{pmatrix} Q_- \\ \frac{1}{\sqrt2}(Q_m - Q_3)\end{pmatrix} &= +i\frac{\tau_\alpha}{2}\begin{pmatrix} Q_- \\ \frac{1}{\sqrt2}(Q_m - Q_3)\end{pmatrix}\end{aligned} \tag{2.76}$$

$$\begin{pmatrix} -Q_+ \\ \frac{1}{\sqrt{2}}(Q_m + Q_3) \end{pmatrix}^* = \begin{pmatrix} Q_- \\ \frac{1}{\sqrt{2}}(Q_m - Q_3) \end{pmatrix} \tag{2.77}$$

From (2.76) one reads off that one has to introduce a further CP even scalar field H, when one wants to complete the enlarged mass action $\hat{\Gamma}^{mass}_{Dirac}$ in such a way, that it is invariant under $SU(2)$ transformations. The Yukawa action

$$\begin{aligned} \Gamma_{Yuk} &\equiv \Gamma^{mass}_{Dirac} - \frac{2}{v}\int d^4x(\phi^+Q_- - \phi^-Q_+ + HQ_m - i\chi Q_3) \qquad (2.78)\\ &= \Gamma^{mass}_{Dirac} - \frac{2}{v}\int d^4x\left(\begin{pmatrix} Q_- \\ \frac{1}{\sqrt{2}}(Q_m - Q_3) \end{pmatrix}^T \cdot \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(H + i\chi) \end{pmatrix}\right. \\ &\qquad \left. + \begin{pmatrix} \phi^- \\ \frac{1}{\sqrt{2}}(H - i\chi) \end{pmatrix}^T \cdot \begin{pmatrix} -Q_+ \\ \frac{1}{\sqrt{2}}(Q_m + Q_3) \end{pmatrix}\right) \end{aligned}$$

continues $\hat{\Gamma}^{mass}_{Dirac}$ (2.71) in a minimal way to a $SU(2)$ invariant action:

$$\mathcal{W}_\alpha \Gamma_{Yuk} = 0 \qquad \Big[\mathcal{W}_\alpha, \mathcal{W}_\beta\Big] = \epsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathcal{W}_{\gamma'}. \tag{2.79}$$

The transformation operators $\mathcal{W}_\alpha$ consist of the fermion, the vector and the scalar transformation operators. The latter ones are defined to include the shift which we have introduced for absorbing the breaking terms of the masses (2.72) and in this way they are uniquely determined by the construction:

$$\mathcal{W}_\alpha = \mathcal{W}^{fermion}_\alpha + \mathcal{W}^{vector}_\alpha + \mathcal{W}^{scalar}_\alpha \tag{2.80}$$

with

$$\begin{aligned} \mathcal{W}^{scalar}_\pm &= \int d^4x\Big(\Phi^\dagger(x)\frac{i\tau_\mp}{2}\frac{\delta}{\delta\Phi^\dagger(x)} - \frac{\delta}{\delta\Phi(x)}\frac{i\tau_\mp}{2}\Phi(x) \mp i\frac{v}{2}\frac{\delta}{\delta\phi^\mp}\Big), \qquad (2.81)\\ \mathcal{W}^{scalar}_3 &= \int d^4x\Big(\Phi^\dagger(x)\frac{i\tau_3}{2}\frac{\delta}{\delta\Phi^\dagger(x)} - \frac{\delta}{\delta\Phi(x)}\frac{i\tau_3}{2}\Phi(x) - \frac{v}{2}\frac{\delta}{\delta\chi}\Big). \end{aligned}$$

Here we have arranged the scalars into $SU(2)$ doublets and have introduced the notation:

$$\Phi \equiv \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(H(x) + i\chi(x)) \end{pmatrix} \qquad \Phi^* = \begin{pmatrix} \phi^-(x) \\ \frac{1}{\sqrt{2}}(H(x) - i\chi(x)) \end{pmatrix}.$$

With $\tilde{\Phi} = i\tau_2\Phi^*$ it is straightforward to calculate that the Yukawa action (2.78) can be written in the conventional form

$$\begin{aligned} \Gamma_{Yuk} =& -\int d^4x \sum_{f=e,u,d} m_f(\bar{f}^R f^L + \bar{f}^L f^R) \qquad (2.82)\\ &- \frac{\sqrt{2}}{v}\int d^4x\Big(m_e\bar{F}^L_l\Phi e^R + m_d\bar{F}^L_q\Phi d^R + m_u\bar{F}^L_q\tilde{\Phi}u^R + \text{h.c.}\Big)\\ =& -\frac{\sqrt{2}}{v}\int d^4x\Big(m_e\bar{F}^L_l(\Phi + \mathrm{v})e^R + m_d\bar{F}^L_q(\Phi + \mathrm{v})d^R + m_u\bar{F}^L_q(\tilde{\Phi} + i\tau_2\mathrm{v})u^R + \text{h.c.}\Big). \end{aligned}$$

Here v denotes the shift in vector notation:

$$\mathrm{v} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{2.83}$$

The Yukawa interaction is invariant not only under spontaneously broken global $SU(2) \times U(1)$ transformations, but even under the local ones:

$$\Big(\mathbf{w}_\alpha + \frac{1}{g_2}\partial^\nu \tilde{I}_{\alpha\alpha'}\frac{\delta}{\delta W^\nu_{\alpha'}}\Big)\Gamma_{Yuk} = 0, \qquad \alpha = +,-,3, \tag{2.84}$$

$$\Big(\mathbf{w}^Q_4 - \frac{1}{g_1}\partial^\nu \frac{\delta}{\delta W^\nu_4}\Big)\Gamma_{Yuk} = 0. \tag{2.85}$$

The operators $\mathbf{w}_\alpha$ are the non-integrated version of (2.80):

$$\mathcal{W}_\alpha \equiv \int d^4x \mathbf{w}_\alpha(x) = \int d^4x \Big(\mathbf{w}^{fermion}_\alpha(x) + \mathbf{w}^{vector}_\alpha(x) + \mathbf{w}^{scalar}_\alpha(x)\Big) \tag{2.86}$$

with

$$\mathbf{w}^{scalar}_\alpha(x) = \Big(\Phi(x) + \mathrm{v}\Big)^\dagger \frac{i\tau^T_\alpha}{2}\frac{\delta}{\delta\Phi^\dagger(x)} - \frac{\delta}{\delta\Phi(x)}\frac{i\tau^T_\alpha}{2}\Big(\Phi(x) + \mathrm{v}\Big). \tag{2.87}$$

The abelian operator $\mathbf{w}^Q_4$ is defined by eq. (2.47) $\mathbf{w}^Q_4 = \mathbf{w}_{em} - \mathbf{w}_3$ and the electromagnetic operator includes also the charged scalars (see (A.49)). Explicitly we find

$$\mathbf{w}^{Q\,scalar}_4(x) = \Big(\Phi(x) + \mathrm{v}\Big)^\dagger \frac{Y^s_W i}{2}\frac{\delta}{\delta\Phi^\dagger(x)} - \frac{\delta}{\delta\Phi(x)}\frac{Y^s_W i}{2}\Big(\Phi(x) + \mathrm{v}\Big) \tag{2.88}$$

with

$$Y^s_W = 1. \tag{2.89}$$

Since the symmetric action (2.63) does not depend on scalars it is trivially invariant also with respect to the spontaneously broken $SU(2)\times U(1)$ gauge transformations (2.64) with the enlarged local operators (2.86).

Looking for the most general, local action invariant under the local spontaneously broken gauge transformations (2.84) with mass dimension less or equal 4 we find in addition the kinetic and potential terms of the scalars:

$$\Gamma_{scalar} = \Gamma_{kin\ scalar} + \Gamma_{pot\ scalar}. \tag{2.90}$$

They read in the conventional normalization:

$$\Gamma_{pot\ scalar} = -\int d^4x \lambda(\Phi^\dagger\Phi + \mathrm{v}^\dagger\Phi + \Phi^\dagger\mathrm{v})^2, \tag{2.91}$$

$$\Gamma_{kin\ scalar} = -\int d^4x \Big(D^\mu(\Phi + \mathrm{v})\Big)^\dagger D_\mu(\Phi + \mathrm{v}) \tag{2.92}$$

with the covariant derivative

$$D_\mu \Phi = \big(\partial_\mu - i(g_2 \frac{\tau_\alpha}{2} W_{\mu\alpha} - g_1 \frac{1}{2} W_{\mu 4})\big)\Phi. \tag{2.93}$$

(We have already omitted the invariant, which is linear in the Higgs field, $\Phi^\dagger\Phi + \mathrm{v}^\dagger\Phi + \Phi^\dagger\mathrm{v}$, from the action.)

The bilinear term of the potential gives the mass term for the scalars, i.e.

$$-\lambda(\Phi^\dagger \mathrm{v} + \mathrm{v}^\dagger\Phi)^2 = -\frac{\lambda}{2} v^2 (H - i\chi + H + i\chi)^2 = -\frac{1}{2} 4\lambda v^2 H^2. \tag{2.94}$$

So only the real, CP-even scalar H gets a mass $m_H^2 = 4\lambda v^2$, the scalars χ and $\phi^\pm$ are massless.

We have seen that the fermion masses are generated by Yukawa couplings to the Higgs doublets and the Higgs mass arises from the scalar potential. Eventually also gauge bosons get mass via the covariant derivative of the scalars. Evaluating (2.92) one gets for the gauge boson masses:

$$\Big(D^\mu(\mathrm{v})\Big)^\dagger D_\mu(\mathrm{v}) = \frac{g_2^2 v^2}{4} W_+^\mu W_{\mu -} + \frac{1}{2}\frac{v^2}{4}(g_2 W_3^\mu + g_1 W_4^\mu)^2. \tag{2.95}$$

Since the mass terms are non-diagonal in the vector fields, the fields W_α^μ are not the physical fields. Physical on-shell fields are constructed, if one rotates the fields W_3^μ, W_4^μ by an orthogonal matrix $O_{\alpha a}(\theta_W)$ over an angle $\tan\theta_W = \frac{g_1}{g_2}$:

$$W_\alpha^\mu = O_{\alpha a}(\theta_W) V_a^\mu \quad \text{with} \quad O_{\alpha a}(\theta_W) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_W & -\sin\theta_W \\ 0 & 0 & \sin\theta_W & \cos\theta_W \end{pmatrix}. \tag{2.96}$$

Then the mass term is diagonalized,

$$\frac{g_2^2 v^2}{4} W_+^\mu W_{\mu -} + \frac{1}{2}\frac{v^2}{4}(g_2 W_3^\mu + g_1 W_4^\mu)^2 = M_W^2 W_+^\mu W_{\mu -} + \frac{1}{2} M_Z^2 Z^\mu Z_\mu, \tag{2.97}$$

and the masses are determined to

$$M_W = \frac{g_2 v}{2}, \quad M_Z = \frac{g_2 v}{2\cos\theta_W} \quad \text{and} \quad M_A = 0. \tag{2.98}$$

It is important to note that the kinetic terms of the vector bosons remain diagonal after an orthogonal rotation. From now on we denote with $V_a = (W_+, W_-, Z, A)$ the physical on-shell vector fields of the Standard Model.

At this stage a few remarks on notation should be made. With the indices $\alpha, \beta, \ldots$ we denote the $SU(2)\times U(1)$-indices: $+, -, 3, 4$. On the other hand the indices $a, b, \ldots$ refer to the indices of the physical fields in the theory: $+, -, Z, A$.

Having given masses to the vectors the massless Goldstone bosons $\phi^{\pm}, \chi$ become unphysical fields. This can be understood qualitatively as follows: A massless vector boson has only 2 transverse polarizations. A massive vector boson has one more, the longitudinal polarization. Before the symmetry breaking there were 4 vector bosons, each with 2 degrees of freedom and one Higgs doublet with 4 degrees of freedom. After the symmetry breaking there is left one scalar, the Higgs boson. The other three degrees of freedom of the scalar doublet make the longitudinal polarization of the vector bosons physical. So the total number of physical degrees of freedom has not changed. Figuratively one says that the Goldstone bosons $(\phi^{\pm}, \chi)$ are eaten up by the vector bosons for giving them masses. These results are obvious in the unitary gauge, whose lowest component we construct in exercise 5.

The full classical action is combined from the single invariant 4-dimensional actions (2.60)(2.61) (2.82) and (2.90) and is called the Glashow-Salam-Weinberg model:

$$\Gamma_{GSW} = \Gamma_{YM} + \Gamma_{matter} + \Gamma_{scalar} + \Gamma_{Yuk}. \tag{2.99}$$

This action is uniquely determined up to field and parameter redefinitions by spontaneously broken $SU(2) \times U(1)$ gauge transformations:

$$\begin{aligned} \Big(\mathbf{w}_\alpha + \frac{1}{g_2}\partial^\mu \tilde{I}_{\alpha\alpha'}\frac{\delta}{\delta W^\mu_{\alpha'}}\Big)\Gamma_{GSW} &= 0 & (2.100)\\ \Big(\mathbf{w}^Q_4 - \frac{1}{g_1}\partial^\mu \frac{\delta}{\delta W^\mu_4}\Big)\Gamma_{GSW} &= 0. & (2.101) \end{aligned}$$

The local operators are defined in (2.86) as the sum of fermion, vector and (shifted) scalar operators.

By now a lot of parameters are introduced. But not all of these are independent for there were a couple of relations between them. So one question which one should ask, is which of these are taken to be fundamental. This fundamental set should be applicable in any order of perturbation theory and should also characterize the particle properties of the model. It is therefore natural to take physical on-shell parameters as fundamental.

The free parameters we choose are

$$M_W, M_Z, m_f, m_H \quad \text{and} \quad g_2 \tag{2.102}$$

and the vectors are expressed in physical on-shell fields V^μ_a. The weak mixing angle θ_W is not taken to be fundamental, but is defined by the relation [21]:

$$\cos\theta_W = \frac{M_W}{M_Z}. \tag{2.103}$$

As an illustration let us calculate the interaction of the photon to the electromagnetic current: If we apply the orthogonal rotation $O_{\alpha a}(\theta_W)$ to the interaction of the gauge fields with the currents Γ_{matter} (2.60), we get:

$$-g_2(J^+_\mu W^\mu_- + J^-_\mu W^\mu_+ + J^3_\mu W^\mu_3) - g_1 J^4_\mu W^\mu_4 = \tag{2.104}$$
$$-g_2\Big(J^-_\mu W^\mu_+ + J^+_\mu W^\mu_- + \frac{1}{\cos\theta_W}(J^3_\mu + \sin^2\theta_W j^{em}_\mu)Z^\mu\Big) - g_2 \sin\theta_W j^\mu_{em} A^\mu.$$

It is seen in an explicit form that the electromagnetic current couples to the massless vector boson A^μ which is identified by this property as the photon field. The same conclusion is derived by transforming the unphysical fields W^μ_3 and W^μ_4 into the physical on-shell fields Z^μ and A^μ in the functional operators of gauge transformations. There one reads off as well, that the photon couples to the electromagnetic current and is the massless field corresponding to the unbroken subgroup:

$$\Big(g_2 \sin\theta_W \mathbf{w}_{em} - \partial^\mu \frac{\delta}{\delta A^\mu}\Big)\Gamma_{GSW} = 0. \tag{2.105}$$

For this reason we introduce the electromagnetic coupling constant $e = g_2 \sin\theta_W$ as fundamental coupling of the electroweak Standard Model. The QED-like on-shell parameters are then given by

$$M_W, M_Z, m_f, m_H \quad \text{and} \quad e \tag{2.106}$$

(For fixing the coupling e to its experimental value a physical process has to be chosen, as it is for example Compton scattering at low energies or Bhabha scattering at LEP energies.)

The on-shell parameters have been used by several groups as fundamental parameters for calculating higher order processes in perturbation theory [22, 23, 21, 24, 25, 26, 27, 28, 29]. We want to mention already here, that in higher orders it is crucial for infrared existence of Green's functions to choose a parameterization, which ensures that the photon propagator has a pole at $p^2 = 0$. Unfortunately it turns out that the QED-Ward identity corresponding to the functional identity (2.105) cannot be proven in perturbation theory. So the photon will be characterized by the property of being the massless vector boson, and not by its property of coupling to the electromagnetic current.

2.2.6. Other (global) symmetries

In the previous section we have looked at the consequence of the local $SU(2) \times U(1)$ gauge invariance. We have built a phenomenologically acceptable model around this symmetry. It turns out that this model also has some extra global symmetries.

The un-quantized Standard Model action (2.99) is invariant under the combined transformation CP and under T. Parity is broken in the fermion sector, since only left-handed fermions contribute to the charged currents. (C denotes charge conjugation, P the parity reflection and T time reversal.) We should stress here that this not true in the Glashow-Salam-Weinberg model with three generations of fermions. In its most general form mixing between three families leads to CP violation via the Cabibbo-Kobayashi-Maskawa matrix. In that case the model is only invariant under the combined transformation CPT.

Two other symmetries of Γ_{GSW} are conservation of lepton and baryon numbers. For one generation the corresponding symmetry operators are

$$\mathcal{W}_l = i \int d^4x \Big(\bar{e} \frac{\delta}{\delta \bar{e}} + \bar{\nu^L} \frac{\delta}{\delta \bar{\nu^L}} - \frac{\delta}{\delta e} e - \frac{\delta}{\delta \nu^L} \nu^L \Big), \tag{2.107}$$

$$\mathcal{W}_q = i \int d^4x \Big(\bar{u} \frac{\delta}{\delta \bar{u}} + \bar{d} \frac{\delta}{\delta \bar{d}} - \frac{\delta}{\delta u} u - \frac{\delta}{\delta d} d \Big). \tag{2.108}$$

These operators are abelian operators and are included in the abelian operators we have found in generality in (2.49). These symmetries are not gauged in the Standard Model, but are global symmetries in the classical theory,

$$\mathcal{W}_l \Gamma_{GSW} = 0 \quad \text{and} \quad \mathcal{W}_q \Gamma_{GSW} = 0, \tag{2.109}$$

and in higher orders of perturbation theory. (Of course in principle these symmetries can also be made local in the classical theory, but then one needs extra $U(1)$ gauge fields as we demonstrate in exercise 7. In nature they are not observed, thus in the Standard Model the lepton and baryon numbers are globally conserved quantum numbers.)

3. Gauge fixing and BRS transformations

3.1. Free field propagators and gauge fixing

In the previous section we presented the $SU(2)\times U(1)$ gauge structure of the Glashow-Salam-Weinberg model. This section is devoted to the quantization of the theory in perturbation theory and in particular to the definition of the action in the tree approximation. First we want to review how the perturbative expansion of time ordered Green's functions is constructed (see e.g. [Q1] – [Q5]).

The basic formula for the perturbative construction is the Gell-Mann–Low formula, which relates time ordered expectation values of interacting fields φ_k to time ordered expectation values of free field $\varphi_k^{(0)}$:

$$\langle T\varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle = \mathcal{R}\langle T\varphi_{i_1}^{(0)}(x_1)\cdots\varphi_{i_n}^{(0)}(x_n)e^{i\Gamma_{int}(\varphi_k^{(0)})}\rangle. \tag{3.1}$$

Γ_{int} includes all the interaction polynomials appearing in the model, and is obtained by splitting off from the classical action the bilinear part:

$$\Gamma_{cl} = \Gamma_{bil} + \Gamma_{int} \tag{3.2}$$

After having expanded the exponential function in its Taylor series, the vacuum expectation values of free fields are decomposed into a sum of products of free field propagators and certain vertex factors according to Wick's theorem. The combinatoric and vertex factors are summarized in the Feynman rules. However, due to the well-known ultraviolet divergencies of the formal perturbative expansion the Gell-Mann–Low formula is not meaningful in higher loop orders of perturbation theory and has to be rendered meaningful in the course of renormalization. (This is the sense of $\mathcal{R}$ in eq. (3.1).) Let us now have a closer look to the free field propagators of the various particles.

The free scalar field obeys the Klein-Gordon equation:

$$(\Box + m^2)\varphi = 0. \tag{3.3}$$

The time ordered expectation value of the free scalar field is given by the solution of the inhomogeneous equation:

$$(\Box + m^2)\Delta_{\varphi\varphi}(x-y) = i\delta^4(x-y). \tag{3.4}$$

The solutions of such equations are the causal Green's functions, which are called the free field propagators

$$\Delta_{\varphi\varphi}(x) = \text{———} = \langle T\varphi^{(0)}(x)\varphi^{(0)}(0)\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - m^2 + i\epsilon}. \tag{3.5}$$

(Here we have also given the Feynman diagram corresponding to the propagator.) For fermions the free field propagators are calculated similarly by solving the inhomogeneous Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Delta_{\psi\bar{\psi}}(x-y) = i\delta^4(x-y). \tag{3.6}$$

From (3.6) the fermion propagator is determined:

$$\Delta_{\psi\bar{\psi}}(x) = \longrightarrow = \langle T\psi^{(0)}(x)\bar{\psi}^{(0)}(0)\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx} \frac{i(\gamma^\mu k_\mu + m)}{k^2 - m^2 + i\epsilon}. \tag{3.7}$$

In general, also in the case of several particles with non-diagonal bilinear parts, one can calculate the free field propagators in the following way:

$$\sum_\gamma \int \mathrm{d}^4z\ \Gamma^{(0)}_{\varphi_\alpha\varphi_\gamma}(x,z)\Delta_{\varphi_\gamma\varphi_\beta}(z,y) = i\delta_{\alpha\beta}\delta^4(x-y). \tag{3.8}$$

Here $\Delta_{\varphi_\alpha\varphi_\beta}(x,y) = \langle T\varphi^{(0)}_\alpha(x)\varphi^{(0)}_\beta(y)\rangle$ and $\Gamma^{(0)}_{\varphi_\alpha\varphi_b}(x,y)$ is derived from the bilinear part of the classical action (3.2):

$$\Gamma^{(0)}_{\varphi_\alpha\varphi_b}(x,y) = \frac{\delta^2\Gamma_{bil}}{\delta\varphi_\alpha(x)\delta\varphi_\beta(y)}. \tag{3.9}$$

If we try to apply the formula (3.8) for determining the photon propagator of the electromagnetic action (2.25), we get into trouble. The equations of motion for a free photon are given by

$$\partial^\mu F_{\mu\nu} = (\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu = 0 \tag{3.10}$$

and the respective inhomogeneous equations by

$$(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\Delta_{\nu\rho} = i\delta^4(x-y)\delta^\mu_\rho. \tag{3.11}$$

Since the operator which acts on A_μ is not invertible, the naive way of calculating the propagator does not work. The reason can be found in gauge invariance of the theory, which brings about, that the vector field A^μ is determined up to a gauge freedom by the classical equations of motion. In perturbation theory one usually adds a gauge-fixing action to the gauge invariant action:

$$\Gamma^{QED}_{cl} = \Gamma_{em} - \frac{1}{2\xi}\int d^4x(\partial^\mu A_\mu)^2, \tag{3.12}$$

with ξ, the gauge parameter. In this way it is possible to fix the gauge and to maintain at the same time Lorentz invariance and locality of the action. The propagator of the vector field is determined from (3.12) as solution of the inhomogeneous equations

$$\Big(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\Big)\Delta_{\nu\rho}(x-y) = i\delta^4(x-y)\delta^\mu_\rho. \tag{3.13}$$

It is given by

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{-i}{k^2+i\epsilon}\Big(P^T_{\mu\nu} + \xi P^L_{\mu\nu}\Big), \tag{3.14}$$

P^L and P^T are the projectors for longitudinal and transverse polarization:

$$P^L_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \quad \text{and} \quad P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{3.15}$$

The complete action of QED (3.12) is not invariant under gauge transformations, but gauge invariance is broken by the gauge-fixing action linearly

$$\Big(e\mathbf{w}_{em} - \partial^\mu\frac{\delta}{\delta A^\mu}\Big)\Gamma^{QED}_{cl} = -\frac{1}{\xi}\Box\partial A. \tag{3.16}$$

In fact one has introduced an unphysical scalar ∂A with spin 0 and negative norm into the theory. For making QED meaningful one has to prove that the S-matrix constructed from the action (3.12) indeed describes a physical theory with a spin 1 particle and that the resulting theory has a probability interpretation in the sense of quantum theory. In QED one is finally able to show that ∂A does not contribute to physical scattering processes and that the physical S-matrix indeed has positivity properties (see [Q4,R5]). The proof is based on the QED Ward identity

$$\Big(e\mathbf{w}_{em} - \partial^\mu\frac{\delta}{\delta A^\mu}\Big)\Gamma = -\frac{1}{\xi}(\Box + \xi m^2_{ph})\partial A. \tag{3.17}$$

This identity has to be proven for the Green's function of QED to all orders of perturbation in the course of renormalization. In our notations Γ denotes the generating functional of one-particle-irreducible (1PI) Green's functions. Its lowest order coincides with the classical action

$$\Gamma^{(0)} = \Gamma^{QED}_{cl}. \tag{3.18}$$

(For infrared definiteness we have introduced a photon mass term m_{ph} in addition, which breaks abelian gauge invariance not worse than the gauge fixing.) The final proof is then carried out by Legendre transforming the 1PI Green's functions to connected Green's functions and finally by applying the LSZ reduction formula (see [Q2,Q3]) on the Ward identity. Then the operator identity

$$\frac{1}{\xi}(\Box + \xi m_{ph}^2)\partial A^{op} = 0 \tag{3.19}$$

is deduced, i.e. ∂A^{op} satisfies the Klein-Gordon equation and does not interact. For the purpose of these lectures we only want to indicate how this result appears for the classical theory: Therefore we consider the Ward identity of QED (3.17) for the classical action Γ_{cl}. When we use the equations of motion for fermions and the vector bosons, the left-hand-side vanishes and we are left with the free field equation for the scalar part of the vector field

$$\frac{1}{\xi}(\Box + \xi m_{ph}^2)\partial A = 0. \tag{3.20}$$

This equation proves that ∂A does not interact in the classical theory.

In non-abelian gauge theories one fixes the gauge for the vectors as we have done it in QED and one gets the same expression for the free field propagators. But in contrast to QED a Ward identity as (3.17) does not exist, which would allow to draw conclusions for the physical interpretation. This role is taken over by BRS symmetry and by the Slavnov-Taylor identity. For this reason these symmetries are the basis for the definition of the non-abelian gauge theories in renormalized perturbation theory.

3.2. Gauge fixing in the Standard Model

For the massive vector bosons it is possible to determine the propagators without the difficulty described above. So in the unitary gauge the W propagator is given by:

$$\langle T W_+^\mu(x) W_-^\nu(0)\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx} \frac{-i}{k^2 - M_W^2}\Big(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{M_W^2}\Big). \tag{3.21}$$

(That (3.21) is a gauge choice will become clear below, as well as why it is called unitary.) However this propagator does not allow for naive power counting arguments of renormalizability to go through, since it behaves as a constant for asymptotically large momenta, i.e. when $k^2 \to -\infty$. If we want to apply the arguments of power counting renormalizability, the boson propagators have to behave as $1/k^2$ for asymptotic k^2. One-loop calculations within the Standard Model in the unitary gauge [26] have been carried out, but it is hard to see how these calculations are extended to higher orders. In order to have renormalizability by power counting one has to fix the gauge similarly as in QED by adding the gauge fixing part. For the purpose of algebraic renormalization we choose a (linearized) generalization of the usual R_ξ-gauges [30] and couple the gauge-fixing functions to the auxiliary field $B_a, a = +, -, Z, A$

$$\Gamma^{(B,\xi)}_{g.f.} = \int d^4x \left(\xi_W B_+ B_- + \frac{1}{2}\xi_Z B_Z^2 + \frac{1}{2}\xi_A B_A^2 + B_a \tilde{I}_{aa'} F_{a'} \right), \tag{3.22}$$

The gauge-fixing functions of the R_ξ-gauges fix the scalar part of vectors and introduce mass terms for the would-be Goldstone fields $\phi^\pm$ and χ

$$\begin{aligned} F_\pm &\equiv \partial_\mu W^\mu_\pm \mp i M_W \zeta_W \phi_\pm, \\ F_Z &\equiv \partial_\mu Z^\mu - M_Z \zeta_Z \chi, \\ F_A &\equiv \partial_\mu A^\mu. \end{aligned} \tag{3.23}$$

Various choices of the parameters ξ_a, ζ_a have special names: the Landau gauge has $\xi_W = \xi_Z = \xi_A = 0$ and $\zeta_W = \zeta_Z = 0$, the 't Hooft gauges have $\xi_Z = \zeta_Z, \quad \xi_W = \zeta_W$ and the 't Hooft-Feynman gauge has in addition $\xi_W = \xi_Z = \xi_A = 1$. (The unitary gauge is retrieved in the limit $\zeta_W = \xi_W^2 \to \infty$ and $\zeta_Z = \xi_Z^2 \to \infty$.) The B_a-fields can be eliminated from the action fields by their equations of motion and in this way one comes back to the usual R_ξ-gauges:

$$\frac{\delta \Gamma}{\delta B_a} = 0 \Longrightarrow \Gamma^\xi_{g.f.} = \int d^4x \left(-\frac{1}{\xi_W} F_+ F_- - \frac{1}{2\xi_Z} F_Z^2 - \frac{1}{2\xi_A} F_A^2 \right). \tag{3.24}$$

On a first sight the gauge-fixing with B_a-fields seems to be less practical than the R_ξ-gauges, since one introduces extra non-diagonal propagators, like $\langle T B_\pm(x) W_\mp(y) \rangle$, into the theory. But, as we discuss in section 3.3, in this formulation BRS transformations are nilpotent on all fields and the algebraic method is applied much easier as it is in the naive approach. One has already to note at this stage, that in the linear (B, ξ) gauges the gauge fixing part of the action does not get loop corrections and remains a local field polynomial as in the tree approximation. This observation is simply deduced from the observation that there are no interaction vertices of the B_a-fields with other propagating fields.

All the propagators now behave such that naive power counting is possible. In the 't Hooft gauges ($\xi_W = \zeta_W$) one finds for example:

$$\begin{aligned}
\langle TB_+(x)W_-^\mu(0)\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{-p^\mu}{p^2 - \xi_W M_W^2}, \\
\langle TB_+(x)\phi^-(0)\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{-M_W}{p^2 - \xi_W M_W^2}, \\
\langle TW_+^\mu(x)W_-^\nu(0)\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \left(P_T^{\mu\nu} \frac{-i}{p^2 - M_W^2} + P_L^{\mu\nu} \frac{-i\xi_W}{p^2 - \xi_W M_W^2} \right), \\
\langle T\phi^+(x)\phi^-(0)\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - \xi_W M_W^2}.
\end{aligned}$$

(A complete list of the free field propagators of the Standard Model in a general linear gauge can be found in [31].)

In section 2 we have constructed the $SU(2) \times U(1)$ gauge invariant part of the action of the electroweak Standard Model. We have to look how the gauge symmetries (2.100) act on the gauge-fixing part of the action (3.22). In the B_a-gauges we have to extend the symmetry transformations by the contributions of the auxiliary fields in a way that $\int d^4x B_a \tilde{I}_{ab} \partial V_b$ is invariant under *rigid* transformations:

$$\mathcal{W}_\alpha \int d^4x B_a \tilde{I}_{ab} \partial V_b = 0 \qquad \mathcal{W}_{em} \int d^4x B_a \tilde{I}_{ab} \partial V_b = 0. \tag{3.25}$$

By this requirement the transformation behaviour of B_a-fields is uniquely determined:

$$\mathcal{W}_\alpha^B = \tilde{I}_{\alpha\alpha'} \int d^4x B_b O^T_{b\beta}(\theta_W) \epsilon_{\beta\gamma\alpha'} O_{\gamma c}(\theta_W) \tilde{I}_{cc'} \frac{\delta}{\delta B_{c'}}, \qquad \alpha = +, -, 3; \tag{3.26}$$

$$\mathcal{W}_{em}^B = -i \int d^4x \left(B_+ \frac{\delta}{\delta B_+} - B_- \frac{\delta}{\delta B_-} \right). \tag{3.27}$$

The matrix $O_{\alpha a}(\theta_W)$ is defined in (2.96) and $\epsilon_{\alpha\beta\gamma}$ as in (2.39). The abelian (hypercharge) operator is defined according to (2.47) by $\mathbf{w}_4^Q = \mathbf{w}_{em} - \mathbf{w}_3$. It is seen that the rigid as well as the gauge symmetries are broken by the gauge fixing. To be precise, the rigid $SU(2)$ symmetries are broken,

$$\mathcal{W}_\alpha \Gamma_{g.f.}^{(B,\xi)} = \Delta_\alpha^{g.f.}, \tag{3.28}$$

but electric charge is conserved,

$$\mathcal{W}_{em} \Gamma_{g.f.}^{(B,\xi)} = 0. \tag{3.29}$$

However, electromagnetic gauge symmetry is broken by a non-linear expression,

$$\left(e\mathbf{w}_{em} - \partial \frac{\delta}{\delta A} \right) \Gamma_{g.f.}^{(B,\xi)} = \Box B_A - i\partial_\mu (B_+ W_-^\mu - B_- W_+^\mu), \tag{3.30}$$

which reads in the R_ξ-gauges (3.24)

$$\Big(e\mathbf{w}_{em} - \partial\frac{\delta}{\delta A}\Big)\Gamma^{\xi}_{g.f.} = -\frac{1}{\xi_A}\Box\partial A + \frac{i}{\xi_W}\partial_\mu\Big((\partial W_+ - iM_W\zeta_W\phi^+)W^\mu_- - \text{h.c.}\Big). \tag{3.31}$$

From the last expression it is immediately clear that the situation is dramatically changed compared to pure QED (cf. (3.16)). In the case of QED we have derived from the QED Ward identity that ∂A is a free field in the classical theory (cf. the derivation of eq. (3.20)). The same arguments applied to eq. (3.31) show, that ∂A interacts with W_+ and W_- and will therefore indeed contribute to physical scattering processes. To cancel these contributions in the physical scattering matrix additional fields, the Faddeev-Popov ghost fields [32], have to be introduced into the theory and gauge symmetry has to be replaced by BRS symmetry [12, 14]. This is the topic of the following subsection 3.3.

Another complication of the gauge fixing in spontaneously broken theories and in particular in the Standard Model is that it does not even maintain rigid $SU(2)\times U(1)$ symmetry. Instead the gauge-fixing action and the gauge parameters (3.22) have been chosen as though they have been built around several $U(1)$ factors. In order not to spoil the group structure of global $SU(2)\times U(1)$ symmetry, the following choices for the gauge-fixing parameters are made:

$$\xi_A = \xi_W = \xi_Z = \xi \quad \text{and} \quad \zeta_W = \zeta_Z = \zeta. \tag{3.32}$$

Then the 4-dimensional terms of the gauge fixing are invariants

$$\mathcal{W}_\alpha \int d^4x \left(\frac{\xi}{2}B_a\tilde{I}_{ab}B_b + B_a\tilde{I}_{ab}\partial V_b\right) = 0, \tag{3.33}$$

whereas the 3-dimensional ones are seen to transform in the same covariant way as the fermion mass terms under $SU(2)\times U(1)$ transformations. The mass breakings of the gauge fixing cannot be coupled to the scalar doublet, since the corresponding expression vanishes identically, but we are able to couple it to an *external* scalar doublet $\hat{\Phi}$ and its hermitian conjugate:

$$\hat{\Phi} = \begin{pmatrix} \hat{\phi}^+ \\ \frac{1}{\sqrt{2}}(\hat{H} + i\hat{\chi}) \end{pmatrix}, \qquad \hat{\Phi}^\dagger = \begin{pmatrix} \hat{\phi}^- \\ \frac{1}{\sqrt{2}}(\hat{H} - i\hat{\chi}) \end{pmatrix}. \tag{3.34}$$

It is transformed in the same way as the scalar doublet Φ under rigid $SU(2)\times U(1)$ (see (2.87)), only the shift can be chosen differently in including the gauge parameter ζ of 3-dimensional breakings:

$$\mathcal{W}^{\hat{\Phi}}_\alpha = \int d^4x \Big(\big(\hat{\Phi} + \zeta \mathrm{v}\big)^\dagger \frac{i\tau^T_\alpha}{2}\frac{\delta}{\delta\hat{\Phi}^\dagger} - \frac{\delta}{\delta\hat{\Phi}}\frac{i\tau^T_\alpha}{2}\big(\hat{\Phi} + \zeta\mathrm{v}\big)\Big). \tag{3.35}$$

algebraically this construction is similar to the one that was applied when we did introduce the scalar doublet and spontaneous breaking of the symmetric gauge theory in section 2.2.5. However, since the construction here is done for a non-propagating scalar doublet, $\hat{\Phi}$ does not have a physical interpretation.

The gauge-fixing functions including the external fields read now:

$$F_a \to \mathcal{F}_a = \partial V_a - i\frac{e}{\sin\theta_W}\Big((\hat{\Phi}+\zeta\mathrm{v})^\dagger\frac{\tau_a^T(\hat{G})}{2}(\Phi+\mathrm{v}) - (\Phi+\mathrm{v})^\dagger\frac{\tau_a^T(\hat{G})}{2}(\hat{\Phi}+\zeta\mathrm{v})\Big).$$

Here we have introduced the following notations:

$$\begin{aligned} \tau_Z(G) &= \cos\theta_W\tau_3 + G\sin\theta_W\mathbf{1}, \\ \tau_A(G) &= -\sin\theta_W\tau_3 + G\cos\theta_W\mathbf{1}. \end{aligned} \tag{3.36}$$

When we choose the parameter $\hat{G}$

$$\hat{G} = -\frac{\sin\theta_W}{\cos\theta_W}, \tag{3.37}$$

we recover the gauge-fixing functions (3.23) with $\zeta_W = \zeta_Z = \zeta$. Explicitly, with this choice the gauge-fixing action at $\hat{\Phi} = 0$ reads:

$$\begin{aligned} &\int d^4x\Big(\frac{1}{2}\xi B_a\tilde{I}_{ab}B_b + B_a\tilde{I}_{ab}\mathcal{F}_b\Big)\Big|_{\substack{\hat{\Phi}=0\\ \hat{G}=-\tan\theta_W}} \qquad (3.38)\\ = &\int d^4x\Big(\frac{1}{2}\xi B_a\tilde{I}_{ab}B_b + B_a\tilde{I}_{ab}\partial V_b\ + i\zeta M_W(B_+\phi_- - B_-\phi_+) - \zeta M_Z B_A\chi\Big). \end{aligned}$$

The gauge fixing (3.38) is indeed a special gauge choice and has to be replaced by the most general one, compatible with rigid symmetry, in higher orders of perturbation theory:

$$\Gamma_{g.f.} = \int d^4x\left(\frac{1}{2}\xi B_a\tilde{I}_{ab}B_b + B_a\tilde{I}_{ab}\mathcal{F}_b(\hat{G},\zeta) + \frac{1}{2}\hat{\xi}(\sin\theta_W B_Z + \cos\theta_W B_A)^2\right). \tag{3.39}$$

Here the four parameters $\xi, \hat{\xi}, G$ and ζ are independent parameters of the gauge fixing. $\Gamma_{g.f.}$ (3.39) is characterized by being linear in the propagating fields, by CP invariance and by rigid invariance,

$$\mathcal{W}_\alpha\Gamma_{g.f.} = 0. \tag{3.40}$$

The operator of rigid $SU(2)$ transformations is now given by the sum of all field transformations introduced by now (cf. (2.56), (2.41), (2.87), (3.26) and (3.35))

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha^{fermion} + \mathcal{W}_\alpha^{scalar} + \mathcal{W}_\alpha^{vector} + \mathcal{W}_\alpha^{B} + \mathcal{W}_\alpha^{\hat{\Phi}} \tag{3.41}$$

3.3. BRS symmetry and Faddeev-Popov ghosts

In the previous subsection we have shown that the gauge fixing breaks local gauge symmetry non-linearly and we have argued that as a consequence of the broken gauge Ward identity the unphysical part of the vector bosons interacts and contributes to the physical scattering matrix in the tree approximation (cf. (3.31)). In order to cancel these interactions in the scattering matrix further fields, the Faddeev-Popov ghosts, are needed. The conventional way for introducing Faddeev-Popov fields into gauge theories does not start from unitarity arguments but from the path integral formulation of quantum field theory: To implement the gauge fixing program in path integrals one needs a compensating determinant. This determinant can be rewritten in the form a path integral over a set of anti-commuting scalar fields. Since these scalar fields have the wrong statistics (they should have been bosons instead of fermions) they are not physical and therefore called ghosts [32].

A third way of introducing Faddeev-Popov ghosts in the theory is provided by the algebraic method of BRS quantization. Since this method is close to the algebraic characterization of renormalized perturbation theory, we want to discuss it in the following: In a first step one considers BRS transformations as an alternative way to characterize the Lie algebra of the gauge group and replaces the infinitesimal parameters of gauge transformation $\epsilon_\alpha(x)$ by anti-commuting scalars $c_\alpha(x)$. With this substitution the infinitesimal transformations on the fermion, scalars and vectors are become BRS transformations denoted by $\mathrm{s}\varphi$:

$$\begin{aligned}
\mathrm{s}V_{\mu a} &= \partial_\mu c_a + \frac{e}{\sin\theta_W}\tilde{I}_{aa'}f_{a'bc}V_{\mu b}c_c, \\
\mathrm{s}\Phi &= i\frac{e}{\sin\theta_W}\frac{\tau_a(G_s)}{2}(\Phi + \mathrm{v})c_a, \\
\mathrm{s}F^L_\delta &= i\frac{e}{\sin\theta_W}\frac{\tau_a(G_\delta)}{2}F^L_{\delta_i}c_a \quad \text{with} \quad \delta = l, q \\
\mathrm{s}f^R_i &= -ieQ_f\frac{\sin\theta_W}{\cos\theta_W}f^R_i c_Z - ieQ_f f^R_i c_A
\end{aligned} \tag{3.42}$$

Here we have transformed the fields $c_\alpha, \alpha = +, -, 3, 4$ into physical fields by the orthogonal transformation matrix $O_{\alpha a}(\theta_W)$ (2.96),

$$c_\alpha = O_{\alpha a}(\theta_W)c_a, \tag{3.43}$$

and have given the transformations in the physical on-shell fields and in the QED-like parameterization (2.106). The structure constants are defined by

$$f_{abc} = \epsilon_{\alpha\beta\gamma}O_{\alpha a}(\theta_W)O_{\beta b}(\theta_W)O_{\gamma c}(\theta_W). \tag{3.44}$$

The matrices $\tau_a(G)$ are given in (3.36) and satisfy the algebra:

$$\big[\tau_a(G), \tau_b(G)\big] = f_{abc}\tilde{I}_{cc'}\tau_{c'}(G). \tag{3.45}$$

The parameters G_k are related to the weak hypercharge of the respective $SU(2)$-doublets.

$$G_k = -\frac{\sin\theta_W}{\cos\theta_W}Y_W^k \qquad \text{with} \qquad Y_W^k = \begin{cases} 1 & \text{for the scalar doublet} \quad (k=s), \\ -1 & \text{for the lepton doublet} \quad (k=l), \\ \frac{1}{3} & \text{for the quark doublet} \quad (k=q). \end{cases} \tag{3.46}$$

Since the action Γ_{GSW} is gauge invariant, it is is invariant under BRS transformations by construction

$$\mathrm{s}\Gamma_{GSW} = 0. \tag{3.47}$$

The Lie algebra of functional generators is translated into the nilpotency of the BRS operator s

$$\mathrm{s}^2 = 0. \tag{3.48}$$

It includes the commutation relations as well as the Jacobi identities. For illustration we calculate the BRS transformations of ghosts by requiring nilpotency of the BRS operator when acting on the fermion doublets:

$$\begin{aligned} 0 = \mathrm{s}^2 F^L &= \mathrm{s}(c_a\delta_a F^L) \\ &= \mathrm{s}c_a\delta_a F^L + c_b c_a \delta_b\delta_a F^L \\ &= (\mathrm{s}c_a + \frac{1}{2}f_{bda'}c_b c_d\tilde{I}_{aa'})\delta_a F^L \end{aligned} \tag{3.49}$$

From the last line one derives:

$$\mathrm{s}c_a = -\frac{1}{2}\tilde{I}_{aa'}f_{a'bc}c_b c_c. \tag{3.50}$$

In (3.49) δ_a denote the infinitesimal $SU(2)\times U(1)$ transformations transformed to physical field indices by applying the orthogonal rotation matrix $O_{\alpha a}(\theta_W)$. The obey the algebra

$$[\delta_a, \delta_b] = f_{abc}\tilde{I}_{cc'}\delta_{c'} \tag{3.51}$$

The second and crucial step of the construction is the observation that one is able to complete the gauge-fixing action in such a way that it is BRS invariant. We have shown in the previous subsection that on the gauge-fixing action gauge invariance is broken by non-linear field polynomials (3.31). But introducing the anti-ghosts $\bar{c}_a$ and their BRS transformations

$$\mathrm{s}\bar{c}_a = B_a \qquad \text{and} \qquad \mathrm{s}B_a = 0 \quad \text{with} \quad \mathrm{s}^2 = 0, \tag{3.52}$$

it is possible to enlarge the gauge fixing action by a ghost action in such a way that the sum is invariant under BRS transformations:

$$\mathrm{s}\Big(\Gamma_{g.f.} + \Gamma_{ghost}\Big) = 0. \tag{3.53}$$

If one writes for the ghost action $\Gamma_{ghost} = \int d^4x \bar{c}_a X_a$ one finds according to (3.53):

$$0 = \int d^4x \Big(B_a \tilde{I}_{aa'} \mathrm{s}\mathcal{F}_{a'} + B_a X_a - \bar{c}_a \mathrm{s} X_a\Big). \tag{3.54}$$

By identifying $X_a = -\tilde{I}_{aa'}\mathrm{s}\mathcal{F}_{a'}$ and noting that then $\mathrm{s}X_a = 0$ because of nilpotency, the ghost action is uniquely determined to:

$$\Gamma_{ghost} = -\int d^4x \bar{c}_a \tilde{I}_{aa'} \mathrm{s}\mathcal{F}_{a'}. \tag{3.55}$$

One has to note that in the B_a-gauges the gauge-fixing and ghost action is a BRS variation,

$$\Gamma_{g.f.} + \Gamma_{ghost} = \mathrm{s}\int d^4x \bar{c}_a (\frac{\xi}{2} B_a + \mathcal{F}_a), \tag{3.56}$$

and BRS invariant because of nilpotency of the BRS operator s. Finally we have to assign a BRS transformation to the external scalar doublet $\hat{\Phi}$. Since it couples to a BRS variation it is possible to transform the external scalar doublet into a scalar external ghost doublet $\mathbf{q}$ with Faddeev-Popov charge 1:

$$\mathrm{s}\hat{\Phi} = \mathbf{q} \qquad \mathrm{s}\mathbf{q} = 0. \tag{3.57}$$

The ghost action contains kinetic and mass terms for the fields. With the gauge choice (3.37) they read

$$\Gamma^{bil.}_{ghost} = -\int d^4x \left(\bar{c}_a \Box \tilde{I}_{ab} c_b + \zeta M_W^2 (\bar{c}_+ c_- + \bar{c}_- c_+) + \zeta M_Z^2 \bar{c}_Z c_Z\right). \tag{3.58}$$

For this reason they have to be considered as dynamical fields with the following free field propagators:

$$\begin{aligned}
\langle T c_+(x) \bar{c}_-(0)\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - \zeta M_W^2}, \\
\langle T c_Z(x) \bar{c}_Z(0)\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - \zeta M_Z^2}, \\
\langle T \bar{c}_A(x) \bar{c}_A(0)\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2}.
\end{aligned} \tag{3.59}$$

We want to note, that for the general gauge fixing there appear non-diagonal ghost propagators in the bilinear action. To diagonalize the ghost mass matrix one has to introduce in the BRS transformation of ghosts an additional ghost matrix which allows the diagonalization of the ghost mass matrix (for details see [16, 31] and Appendix A):

$$\mathrm{s}\bar{c}_a = B_b \hat{g}_{ba}. \tag{3.60}$$

For higher order loop calculations this observation is crucial for obtaining infrared finite results for off-shell Green's functions.

3.4. The defining symmetry transformations

The classical action of the electroweak Standard Model is given by:

$$\Gamma_{cl} = \Gamma_{GSW} + \Gamma_{g.f.} + \Gamma_{ghost}. \tag{3.61}$$

It is the starting point for the perturbative calculation of Green's functions and determines via the Gell-Mann–Low formula and the free field propagators the tree approximation completely. Higher orders are, however, subject of renormalization and have to be properly defined. For this reason we want to summarize the symmetry properties of the classical action. In the course of renormalization we have to show that these symmetry transformations determine the classical action uniquely, if one poses appropriate normalization conditions.

Due to the fact that gauge invariance is non-linearly broken by the gauge fixing we have to replace gauge invariance by invariance with respect to the nilpotent BRS transformations:

$$\mathrm{s}\Gamma_{cl} = 0, \qquad \mathrm{s}^2 = 0. \tag{3.62}$$

If we want to write BRS transformations in functional form we face the problem of non-linear symmetry transformations. These symmetry transformation become insertions[1], the classical action as the lowest order of the generating functional of 1PI Green's functions. To make them well-defined for ordinary as well as connected Green's functions, non-linear symmetry transformations have to be coupled to external fields $\rho_\alpha, \sigma_\alpha, Y, \Psi_l^R, \Psi_q^R$

[1]For an introduction to insertions and normal products see the reviews and books on renormalization [R2] – [R7]; see also section 4.2 of these lectures.

and ψ_f^L (see Appendix A for notations):

$$\begin{aligned}\Gamma_{ext.f.} = \int d^4x \Big(& \rho_3 O_{3a}(\theta_W) \mathrm{s} W_a + \sigma_3 O_{3a}(\theta_W) \mathrm{s} c_a + \rho_+ \mathrm{s} W_- + \rho_- \mathrm{s} W_+ \\ & + \sigma_+ \mathrm{s} c_- + \sigma_- \mathrm{s} c_+ + Y^\dagger \mathrm{s}\Phi + (\mathrm{s}\Phi)^\dagger Y \\ & + \big(\overline{\Psi_l^R} \mathrm{s} F_l^L + \overline{\Psi_q^R} \mathrm{s} F_q^L + \sum_f \overline{\psi_f^L} \mathrm{s} f_i^R + \mathrm{h.c.} \big) \Big).\end{aligned} \tag{3.63}$$

Adding the external field action to the classical action

$$\Gamma_{cl} \longrightarrow \Gamma_{cl} + \Gamma_{ext.f.}, \tag{3.64}$$

BRS invariance is rewritten into the Slavnov-Taylor identity:

$$\begin{aligned}\mathcal{S}(\Gamma_{cl}) \;=\; & \int d^4x \bigg(\big(\sin\theta_W \partial_\mu c_Z + \cos\theta_W \partial_\mu c_A \big) \Big(\sin\theta_W \frac{\delta\Gamma_{cl}}{\delta Z_\mu} + \cos\theta_W \frac{\delta\Gamma_{cl}}{\delta A_\mu} \Big) \\ & + \frac{\delta\Gamma_{cl}}{\delta\rho_3^\mu} \Big(\cos\theta_W \frac{\delta\Gamma_{cl}}{\delta Z_\mu} - \sin\theta_W \frac{\delta\Gamma_{cl}}{\delta A_\mu} \Big) + \frac{\delta\Gamma_{cl}}{\delta\sigma_3} \Big(\cos\theta_W \frac{\delta\Gamma_{cl}}{\delta c_Z} - \sin\theta_W \frac{\delta\Gamma_{cl}}{\delta c_A} \Big) \\ & + \frac{\delta\Gamma_{cl}}{\delta\rho_+^\mu} \frac{\delta\Gamma_{cl}}{\delta W_{\mu,-}} + \frac{\delta\Gamma_{cl}}{\delta\rho_-^\mu} \frac{\delta}{\delta W_{\mu,+}} + \frac{\delta\Gamma_{cl}}{\delta\sigma_+} \frac{\delta\Gamma_{cl}}{\delta c_-} + \frac{\delta\Gamma_{cl}}{\delta\sigma_-} \frac{\delta\Gamma_{cl}}{\delta c_+} + \frac{\delta\Gamma_{cl}}{\delta Y^\dagger} \frac{\delta\Gamma_{cl}}{\delta\Phi} + \frac{\delta\Gamma_{cl}}{\delta\Phi^\dagger} \frac{\delta\Gamma_{cl}}{\delta Y} \\ & + \sum_{i=1}^{N_F} \Big(\frac{\delta\Gamma_{cl}}{\delta\overline{\psi_{f_i}^L}} \frac{\Gamma_{cl}\delta}{\delta f_i^R} + \frac{\delta\Gamma_{cl}}{\delta\overline{\Psi_{\delta_i}^R}} \frac{\Gamma_{cl}\delta}{\delta F_{\delta_i}^L} + \mathrm{h.c.} \Big) \\ & + B_a \frac{\delta\Gamma_{cl}}{\delta\bar{c}_a} + \hat{\mathbf{q}} \frac{\delta\Gamma_{cl}}{\delta\hat{\Phi}} + \frac{\delta\Gamma_{cl}}{\delta\hat{\Phi}^\dagger} \hat{\mathbf{q}}^\dagger \bigg) = 0.\end{aligned} \tag{3.65}$$

The unitarity of the physical S-matrix, i.e. cancellation of unphysical particles in physical scattering processes, can be derived from the Slavnov-Taylor identity. To ensure that the physical interpretation also holds to higher orders the Slavnov-Taylor identity has to be established to higher orders of perturbation theory as defining symmetry identity of non-abelian and spontaneously broken gauge theories [12, 13, 33]. (For an introduction to unitarity proofs in gauge theories see [R1] and [Q4].)

In the Standard Model, due to the abelian factor group the Slavnov-Taylor identity does not completely characterize the theory. As we have already mentioned we have to require an abelian Ward identity for fixing electromagnetic current coupling and also $SU(2) \times U(1)$ rigid symmetry for being able to single out the abelian operator. Assigning to the external fields and to the Faddeev-Popov fields definite transformation properties with respect to rigid $SU(2) \times U(1)$ transformations, we have for the complete classical action

$$\mathcal{W}_\alpha \Gamma_{cl} = 0 \qquad \text{and} \qquad \mathcal{W}_{em} \Gamma_{cl} = 0, \tag{3.66}$$

The $\mathcal{W}_\alpha, \alpha = +, -, 3$ satisfy $SU(2)$ algebra:

$$[\mathcal{W}_\alpha, \mathcal{W}_\beta] = \epsilon_{\alpha\beta\gamma} \tilde{I}_{\gamma\gamma'} W_{\gamma'}. \tag{3.67}$$

The operators $\mathcal{W}_\alpha$ are the sum of all field operators (cf. (3.41))

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha^{fermion} + \mathcal{W}_\alpha^{scalar} + \mathcal{W}_\alpha^{vector} + \mathcal{W}_\alpha^{B} + \mathcal{W}_\alpha^{\hat{\Phi}} + + \mathcal{W}_\alpha^{ghosts} + \mathcal{W}_\alpha^{ext.f}. \tag{3.68}$$

The complete operators are given in Appendix A (A.46). Furthermore we find that the local abelian symmetry defined by the operator

$$\mathbf{w}_4^Q \equiv \mathbf{w}_{em} - \mathbf{w}_3 \qquad \left[\mathbf{w}_4^Q, \mathcal{W}_\alpha\right] = 0 \tag{3.69}$$

is broken linearly and can be interpreted as an abelian Ward identity for the generating functional of 1PI Green's functions:

$$\Big(\frac{e}{\cos\theta_W}\mathbf{w}_4^Q - \cos\theta_W \partial \frac{\delta}{\delta Z} - \sin\theta_W \partial \frac{\delta}{\delta A}\Big)\Gamma_{cl} = \Box(\sin\theta_W B_Z + \cos\theta_W B_a) \tag{3.70}$$

It allows to distinguish electromagnetic current coupling from coupling of lepton and baryon number conserving currents in the construction of the electroweak Standard Model.

As long as we do not consider family mixing in the fermion sector, CP invariance is a discrete symmetry of the Standard Model and conservation of lepton and baryon family are global abelian symmetries (2.109):

$$\mathcal{W}_l \Gamma_{cl} = 0 \qquad \text{and} \qquad \mathcal{W}_q \Gamma_{cl} = 0 \tag{3.71}$$

In the proof of renormalizability to all orders, it has to be shown that the Slavnov-Taylor identity (3.65), Ward identities of rigid symmetry (3.66) and the local abelian Ward identity (3.70) can be established to all orders of perturbation theory. Furthermore – and as important as the first part – it has to be shown that these symmetry transformations together with CP invariance and the global symmetries (3.71) uniquely determine all free parameters order to order in perturbation theory, if appropriate normalization conditions are imposed.

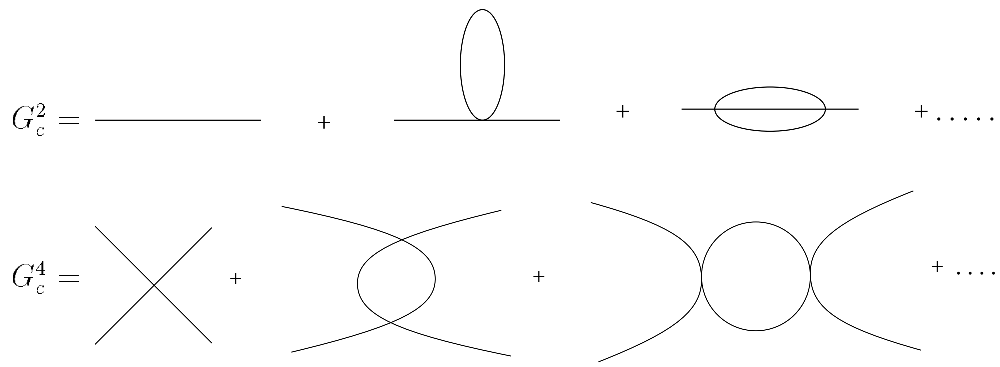
4. Proof of renormalizability to all orders

4.1. Scheme dependence of counterterms

For the purpose of illustrating general properties of renormalized perturbation theory we consider a simple quantum field theoretic model, the φ^4-theory, with the classical action

$$\Gamma_{cl} = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4\right). \tag{4.1}$$

As discussed in section 3.1 the perturbative expansion is formally governed by the Gell-Mann–Low (GML) formula (3.1) and can be diagrammatically expressed in the Feynman diagrams. There one assigns to propagators and vertices certain diagrammatic expressions and writes all topological distinct diagrams. If one assigns furthermore to the diagrams symmetry factors, diagrams are immediately translated into the mathematical expressions of Green's functions. The correspondence between diagrams and Green's functions is summarized in the Feynman rules. For example the connected 2- and 4- point Green's functions of the φ^4-theory are expanded diagrammatically as follows



When writing down the corresponding expressions to the loop diagrams, one sees that these integrals are not finite and therefore not well defined as they are given by the GML formula: the integral over the internal loop momenta is unbounded. To analyze these divergences it is useful to consider the one-particle-irreducible (1PI) Green's functions. In the above example they are obtained from the connected ones by amputating the external legs. For 1PI Green's functions the superficial degree of the ultraviolet divergence d_Γ of a specific loop diagram Γ is given to all orders by the following formula:

$$d_\Gamma = 4 - N_B - \frac{3}{2}N_F + \sum_V (d_V - 4). \tag{4.2}$$

Here N_B and N_F denote the number of external (amputated) boson (B) and fermion (F) legs. The sum is taken over all vertices V appearing in the respective 1PI diagram and d_V denotes the dimension of the vertex V. For the 2 and 4 point Green's functions in the φ^4-theory, we find $d_{\Gamma_2} = 2$ and $d_{\Gamma_4} = 0$ respectively, whereas all the 1PI Green's functions with more than four external (amputated) legs are finite. Since all the propagators behave not worse than $\frac{1}{p^2}$ when $p^2 \to \infty$ (cf. section 3.1 and 3.2) the formula (4.2) is also valid in the electroweak Standard Model. Furthermore, since all the interaction vertices of the classical action have dimension less than or equal 4 ($d_V \leq 4$), the divergencies of the Standard Model are restricted to 2-, 3-, and 4-point 1PI Green's functions and are quadratic, linear and logarithmic depending on the number of external fermion and boson lines. This property is called naive renormalizability by power counting (see e.g. [R2] for an introduction to renormalization).

Next we consider the explicit expressions of the divergent 1-loop 1PI diagrams in the φ^4-model[2]:

$$\begin{aligned} \Gamma_2(p^2) &= p^2 - m^2 - i\lambda\mathcal{R}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2}, \\ \Gamma_4(p_1,p_2,p_3,p_4) &= -\lambda - i\frac{\lambda^2}{2}\mathcal{R}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big(\frac{1}{k^2-m^2}\frac{1}{(k+p_1+p_2)^2-m^2} \\ &\qquad + (p_2 \leftrightarrow p_3) + (p_2 \leftrightarrow p_4)\Big). \end{aligned} \tag{4.3}$$

Here $\mathcal{R}$ denotes that the integral has to be made meaningful in the course of renormalization. There are several schemes which allow to define Green's functions to all orders consistently. Here we want to mention two of them: For practical calculations the most commonly used scheme is the scheme of dimensional regularization together with a subtraction prescription for removing the poles in the limit of 4 dimensions [10, 34]. In the abstract approach one refers to the momentum subtraction scheme in the version of BPHZ and, if one includes massless particles, to its generalization, the BPHZL scheme. (The scheme is called according to Bogoliubov, Parasiuk [35], Hepp [36] and Zimmermann [37] and in its massless version in addition to Lowenstein [38].)

[2]The definitions of 1PI Green's functions differs by a factor i from other conventions (cf. (4.22)), since we want to identify the lowest order contributions to 1PI Green's functions with the classical action (cf. 4.23).

1. *Dimensional regularization*
   In this scheme the dimension of space-time is analytically continued to D-dimensions in the complex plane. The integrals (4.3) become in D-dimensions:

$$\Gamma_2^{DIM} = p^2 - m^2 - i\lambda\mu^{4-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2-m^2}, \tag{4.4}$$
$$\stackrel{\epsilon\to 0}{=} p^2 - m^2 + m^2\lambda\frac{1}{(4\pi)^2}\Big(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi + 1 + \ln\frac{\mu^2}{m^2}\Big)$$
$$\Gamma_4^{DIM} = -\lambda - i\frac{\lambda^2}{2}\mu^{4-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2-m^2}\Big(\frac{1}{(k+p_1+p_2)^2-m^2} + \ 2\ \mathrm{Perm.}\Big) \tag{4.5}$$
$$\stackrel{\epsilon\to O}{=} -\lambda + \frac{1}{(4\pi)^2}\frac{3\lambda^2}{2}\Big(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi\Big)$$
$$- \frac{1}{(4\pi)^2}\frac{\lambda^2}{2}\Big(\int_0^1 \mathrm{d}z \ln\frac{m^2 - z(1-z)p^2 - i\varepsilon}{\mu^2} + 2\ \mathrm{Perm.}\Big),$$

   where $\epsilon = D - 4$. The auxiliary mass μ is introduced for having dimensionless couplings also in D dimensions. Γ^{DIM} denotes the dimensionally regularized integral. From there the finite renormalized Green's functions in 4 dimensions are defined by an additional subtraction prescription for removing the poles in the limit of 4 dimensions, i.e. $\epsilon \to 0$. This procedure is well-defined only up to constants: In the minimal subtraction scheme (MS) [39] only the poles $\frac{2}{\epsilon}$ are subtracted, whereas in the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$) [40] the poles $\frac{2}{\epsilon}$ and the constants $-\gamma_E + \ln 4\pi$ are removed from the D-dimensional expression. In the $\overline{\mathrm{MS}}$ scheme we find for the renormalized integrals:

$$\Gamma_2^{\overline{MS}} = p^2 - m^2 + m^2\lambda\Big(1 + \ln\frac{\mu^2}{m^2}\Big)$$
$$\Gamma_4^{\overline{MS}} = -\lambda + \frac{1}{(4\pi)^2}\frac{\lambda^2}{2}\Big(\int_0^1 \mathrm{d}z \ln\frac{m^2 - z(1-z)(p_1+p_2)^2 - i\varepsilon}{\mu^2} + 2\ \mathrm{Perm.}\Big). \tag{4.6}$$

2. *Momentum subtraction scheme of BPHZ*
   The renormalized Green's functions in the BPHZ scheme are defined without a regularization procedure. The finite Green's functions in 1-loop order are readily obtained by subtracting the first powers of the Taylor expansion in the external momenta p_i from the integrand at $p_i = 0$. The subtraction operator is denoted by $t^d_{p_1\ldots p_n}$. (Divergent subdiagrams of higher orders are subtracted according to the forest formula.) The order d of Taylor subtractions is called the subtraction degree

and coincides in the case considered here with the degree of divergency:

$$\Gamma_2^{BPHZ} = p^2 - m^2 + i\lambda(1-t_p^2)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2} = p^2 - m^2; \tag{4.7}$$

$$\begin{aligned}
\Gamma_4^{BPHZ} &= -\lambda - i\frac{\lambda^2}{2}(1-t_p^0)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\Big(\frac{1}{k^2-m^2}\frac{1}{(k+p_1+p_2)^2-m^2} + \ 2 \text{ Perm.}\Big) \\
&= -\lambda - i\left(\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big(\frac{1}{k^2-m^2}\frac{1}{(k+p_1+p_2)^2-m^2} - \frac{1}{(k^2-m^2)^2}\Big) + 2\text{ P.}\right) \\
&= -\lambda + \frac{1}{(4\pi)^2}\frac{\lambda^2}{2}\int_0^1 \mathrm{d}z\Big(\ln\frac{m^2 - z(1-z)(p_1+p_2)^2}{m^2} + 2\text{ Perm.}\Big).
\end{aligned} \tag{4.8}$$

Comparing the finite Green's functions of the MS and $\overline{\text{MS}}$ scheme with the ones of the BPHZL scheme it is seen, that the renormalized expressions differ by constants, but that the non-local (logarithmic) contributions coincide as they stand and illustrate the equivalence of different schemes. In particular we have

$$\begin{aligned}
\Gamma_2^{BPHZ} &= \Gamma_2^{\overline{\text{MS}}} - m^2\lambda(1+\ln\frac{\mu^2}{m^2}) \\
\Gamma_4^{BPHZ} &= \Gamma_4^{\overline{\text{MS}}} + \frac{3\lambda^2}{32\pi^2}\ln\frac{\mu^2}{m^2}.
\end{aligned} \tag{4.9}$$

These constants can be related to counterterms, which are added order by order to the classical action and appear in the GML formula in higher orders of perturbation theory. Of course these counterterms are restricted to have dimension less than or equal to 4 in order not to violate the properties of naive renormalizability. In the φ^4-theory the most general counterterms are given by

$$\Gamma_{ct} = \int d^4x \sum_{n=1}^{\infty}\big(a^{(n)}\lambda^n\varphi\Box\varphi + b^{(n)}\lambda^n m^2\varphi^2 + c^{(n)}\lambda^{n+1}\varphi^4 + \sqrt{\lambda}\lambda^n f^{(n)}\varphi^3\big). \tag{4.10}$$

In fact the above calculation has demonstrated, that these counterterms are fixed arbitrarily in different schemes and have to be defined uniquely by normalization conditions and symmetries. Let us consider first the φ^3-interaction, which can be added from pure power counting arguments. Since the classical action is invariant under the discrete transformation

$$\varphi \to -\varphi \qquad \Gamma_{cl}(\varphi) \to \Gamma_{cl}(-\varphi) = \Gamma_{cl}(\varphi) \tag{4.11}$$

and since the discrete symmetries are not violated in the course of renormalization, this term can be omitted from the counterterm action. All the other terms have to be fixed by normalization conditions and are interpreted as the wave function renormalization, coupling and mass renormalization. The classical action and the counterterms are summarized in a Γ_{eff}:

$$\begin{aligned}
\Gamma_{eff} &= \int d^4x \big(\frac{1}{2} z_\varphi^2 \partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(m+\delta m)^2 z_\varphi^2 \varphi^2 - \frac{z_\lambda \lambda}{4!} z_\varphi^4 \varphi^4\big) \\
&\equiv \int d^4x \big(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2\big) + \Gamma_{int}
\end{aligned} \tag{4.12}$$

with

$$z_\varphi = 1 + O(\hbar), \quad z_\lambda = 1 + O(\hbar), \quad \delta m = O(\hbar). \tag{4.13}$$

These coefficients are uniquely related to the coefficients a, b and c (4.10) order by order in perturbation theory. The arbitrary coefficients of counterterms have to be fixed by three normalization conditions, namely z_λ on the interaction vertex

$$\Gamma_4(p_1,p_2,p_3,p_4)\Big|_{\substack{p_i^2=\kappa^2 \\ p_i\cdot p_j = -\frac{1}{3}\kappa^2}} = -\lambda, \tag{4.14}$$

and z_φ and δm on the 2-point function:

$$\Gamma_2\Big|_{p^2=m^2} = 0 \qquad \text{and} \qquad \partial_{p^2}\Gamma_2\Big|_{p^2=\kappa^2} = 1. \tag{4.15}$$

The condition on the 4-point function fixes the coupling at the symmetric Euclidean momentum $p^2 = \kappa^2$ to its tree value. The first condition on the 2-point function means that the mass parameter appearing in the Green's functions is the physical mass since it is the pole of the propagator, the second condition fixes the residue of the pole to unity at the normalization momentum $p^2 = \kappa^2$. Applying the normalization conditions to the renormalized Green's functions, scheme dependence of counterterms is removed and the result is unique and does not depend on the scheme, which one has used for making the Green's functions finite:

$$\begin{aligned}
\Gamma_2(p^2) &= p^2 - m^2 \\
\Gamma_4(p^2) &= -\lambda + \frac{1}{(4\pi)^2}\frac{\lambda^2}{2}\int_0^1 \mathrm{d}z \Big(\ln\frac{m^2 - z(1-z)(p_1+p_2)^2 - i\varepsilon}{m^2 - z(1-z)\frac{4}{3}\kappa^2} + 2 \text{ Perm.}\Big)
\end{aligned} \tag{4.16}$$

The Γ_{eff} (4.12), however, which governs the evaluation of Green's functions in the GML formula, depends on the scheme which one has used for making finite the infinite integrals. Therefore, in a scheme independent proof of renormalizability one never refers to the properties of a Γ_{eff}, but only to properties of the finite renormalized Green's functions.

In the Standard Model there are a lot of counterterms, which can be added from pure power counting arguments to the action. As it was in the φ^4-model (cf. (4.11)), discrete and global symmetries as electric charge conservation and lepton and baryon number conservation are conserved in the procedure of renormalization and we are able to restrict the counterterms according to these symmetries. For example the general renormalizable action with bilinear terms in the vector-scalar fields is given by

$$\Gamma^{bil}_{gen} = \int d^4x \Big(-\frac{1}{4} Z^V_{ab} (\partial^\mu V^\nu_a - \partial^\nu V^\mu_a)(\partial_\mu V_{b\nu} - \partial_\nu V_{b\mu}) - \frac{1}{2} \tilde{Z}^V_{ab} (\partial^\mu V_{a\mu})^2 + \frac{1}{2} \mathcal{M}^V_{ab} V^\mu_a V_{\mu b}$$
$$+ \frac{1}{2} Z^S_a \tilde{I}_{ab} \partial^\mu \phi_a \partial_\mu \phi_b - \frac{1}{2} m^S_a \phi_a \tilde{I}_{ab} \phi_b + D_{a,b} V^\mu_a \partial_\mu \varphi_b \Big). \tag{4.17}$$

Applying CP invariance and charge neutrality it is seen that Z^S as well as m^S are diagonal in the neutral sector and off-diagonal and real in the charged sector, whereas Z^V and M^V are real but non-diagonal in the neutral sector. Of course some of these constants are fixed by normalization conditions, as it is for the mass matrix of vectors and the mass of the Higgs. Other counterterms in (4.17) as $D_{a,b}$ are determined by the symmetries as it is seen from the classical action.

In the (complete) on-shell scheme the mass matrix of vectors is fixed by the following normalization conditions [24, 25, 41]:

$$\mathrm{Re}\, \Gamma_{W^+W^-}\Big|_{p^2=M_W^2} = 0, \qquad \mathrm{Re}\, \Gamma_{ZZ}\Big|_{p^2=M_Z^2} = 0, \qquad \Gamma_{AA}\Big|_{p^2=0} = 0,$$
$$\mathrm{Re}\, \Gamma_{ZA}\Big|_{p^2=M_Z^2} = 0, \qquad \Gamma_{ZA}\Big|_{p^2=0} = 0. \tag{4.18}$$

With these conditions the mass matrix of vectors is diagonalized on-shell. Since Z and $W^\pm$ are unstable particles, their self energies are not real. The on-shell conditions do not seem to be the appropriate conditions for describing unstable particle in higher orders, but we want to indicate here that there are free counterterms available for fixing the masses of particles and for diagonalizing the mass matrix. Then on-shell conditions can be replaced immediately by the appropriate normalization conditions as for example pole conditions in higher orders.

In the course of algebraic renormalization counterterms have to be characterized algebraically by the symmetries of the model. In particular one has to distinguish the invariant counterterms that are fixed by normalization conditions from non-invariant counterterms which are fixed by the symmetries. This classification is carried out when one solves the defining symmetries, the Ward identities and the ST identity, for the most general local field polynomial compatible with power counting renormalizability. The proof of renormalizability is finished by proving that the defining symmetries of the model can be established in higher orders by adjusting non-invariant counterterms appropriately. (For an introduction to algebraic renormalization see [R5,R6].) The basis for this proof is the quantum action principle for off-shell Green's functions, whose content and consequences for renormalization we outline in the following subsection.

4.2. The quantum action principle

The classical action of the Standard Model satisfies the ST identity (3.65)

$$\mathcal{S}(\Gamma_{cl}) = 0, \tag{4.19}$$

Ward identities of rigid $SU(2)$ symmetry and global electromagnetic charge conservation (3.66)

$$\mathcal{W}_\alpha \Gamma_{cl} = 0, \qquad \mathcal{W}_{em} \Gamma_{cl} = 0. \tag{4.20}$$

Starting from the classical action one can immediately calculate the Green's functions of 1-loop order, by using the Gell-Mann–Low formula and Wick's theorem, or equivalently using Feynman diagrams and Feynman rules as described in the last subsection. The divergent Green's functions are renormalized by a well-defined subtraction scheme as we have presented in the example of the φ^4-theory. (Feynman rules of the Standard Model and standard 1-loop diagrams evaluated in dimensional regularization are given in several publications. See e.g. [25, 28, 41]).

The finite 1PI Green's functions are summarized in the generating functional of 1PI Green's functions.

$$\Gamma[\varphi_k] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 x_1 \dots \mathrm{d}^4 x_n \sum_{i_1, \dots i_n} \varphi_{i_1}(x_1) \varphi_{i_2}(x_2) \cdots \varphi_{i_n}(x_n) \Gamma_{\varphi_{i_1} \cdots \varphi_{i_n}}(x_1, \dots x_n) \tag{4.21}$$

Here φ_k denotes the different fields of the Standard Model and $\Gamma_{\varphi_{i_1} \cdots \varphi_{i_n}}$ the 1PI Green's functions with external amputated legs $\varphi_{i_1}, \dots \varphi_{i_n}$

$$\Gamma_{\varphi_{i_1} \cdots \varphi_{i_n}}(x_1, \dots x_n) = \frac{1}{i} < T \varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n) > \Big|_{\substack{\text{1PI diagrams} \\ \text{amputated legs}}} \tag{4.22}$$

In perturbation theory the generating functional of 1PI Green's functions is expanded in orders of $\hbar$, which agrees with the loop order and the expansion in the coupling constant. The lowest order is the classical action:

$$\Gamma = \sum_{k=0}^{\infty} \Gamma^{(k)} \qquad \Gamma^{(0)} = \Gamma_{cl} \tag{4.23}$$

The proof of renormalizability is an induction proof; therefore we have in a first step to prove that the symmetries of the tree approximation can be established also in 1-loop order

$$\mathcal{S}(\Gamma_{cl}) = 0 \implies \Big(\mathcal{S}(\Gamma)\Big)^{(\leq 1)} = 0 \qquad (4.24)$$
$$\mathcal{W}_\alpha \Gamma_{cl} = 0 \implies \Big(\mathcal{W}_\alpha \Gamma\Big)^{(\leq 1)} = 0$$

Finally we have also to establish the local Ward identity (3.70) in 1-loop order. The global symmetries as electric charge conservation, lepton and baryon number conservation as well as discrete CP symmetry are trivially established. Having carried out the step from the classical approximation to 1-loop order the step from order n to $n+1$ can be done in analogy if none of the initial conditions as power counting renormalizability and infrared existence have changed.

In the following we denote the finite scheme-dependent renormalized 1-loop Green's functions by $\Gamma^{(1)}_{ren}$. As in the example of the φ^4-theory we are able to add arbitrary counterterms in 1-loop order. The Green's functions of the Standard Model are finally determined as a sum of the renormalized scheme-dependent contributions and local counterterms (see (4.17)):

$$\Gamma^{(\leq 1)} = \Gamma_{cl} + (\Gamma^{(1)}_{ren} + \Gamma^{(1)}_{ct}) \qquad (4.25)$$

Applying the ST operator and the Ward operators of the tree approximation (3.65) and (3.66) to this expression we obtain:

$$\begin{aligned} \mathcal{S}(\Gamma^{(\leq 1)}) &= \mathcal{S}\big(\Gamma_{cl} + (\Gamma^{(1)}_{ren} + \Gamma^{(1)}_{ct})\big) \qquad (4.26) \\ &= \mathrm{s}_{\Gamma_{cl}}\Gamma^{(1)}_{ren} + \mathrm{s}_{\Gamma_{cl}}\Gamma^{(1)}_{ct} + O(\hbar^2) \\ \mathcal{W}_\alpha \Gamma^{(\leq 1)} &= \mathcal{W}_\alpha\big(\Gamma_{cl} + (\Gamma^{(1)}_{ren} + \Gamma^{(1)}_{ct})\big) \\ &= \mathcal{W}_\alpha \Gamma^{(1)}_{ren} + \mathcal{W}_\alpha \Gamma^{(1)}_{ct} \end{aligned}$$

The operator s_Γ is the linearized version of the ST operator:

$$\begin{aligned} \mathrm{s}_\Gamma = \int d^4x \Big(&\big(\sin\theta_W \partial_\mu c_Z + \cos\theta_W \partial_\mu c_A\big)\Big(\sin\theta_W \frac{\delta}{\delta Z_\mu} + \cos\theta_W \frac{\delta}{\delta A_\mu}\Big) \qquad (4.27) \\ &+ B_a \frac{\delta}{\delta \bar{c}_a} + \hat{\mathbf{q}} \frac{\delta}{\delta \hat{\Phi}} + \frac{\delta}{\delta \hat{\Phi}^\dagger}\hat{\mathbf{q}}^\dagger \quad + \sum_{\varphi_k, \Upsilon_k} u_k \Big(\frac{\delta \Gamma}{\delta \Upsilon_k}\frac{\delta}{\delta \varphi_k} + \frac{\delta \Gamma}{\delta \varphi_k}\frac{\delta}{\delta \Upsilon_k}\Big)\Big) \end{aligned}$$

We have generically written φ_k for the fields and Υ_k for the corresponding external sources in the theory.

If we want to prove that symmetries can be established in 1-loop order, we have to show that all scheme dependent breakings in 1-loop order can be cancelled by adding appropriate counterterms, i.e.:

$$\mathcal{W}_\alpha \Gamma^{(1)}_{ren} \stackrel{!}{=} -\mathcal{W}_\alpha \Gamma^{(1)}_{ct} \qquad \mathrm{s}_{\Gamma_{cl}} \Gamma^{(1)}_{ren} \stackrel{!}{=} -\mathrm{s}_{\Gamma_{cl}} \Gamma^{(1)}_{ct} \tag{4.28}$$

(Here $\stackrel{!}{=}$ denotes that the equality of both sides of these equations has to be proven.) For proving these equalities the most important input comes from the quantum action principle [42, 43]. It relates differentiations with respect to parameters and with respect to fields to insertions (see [R3] – [R6] and below, in particular (4.38)). In its most general form it has been formulated even independent of a specific renormalization scheme [43]. Applying the quantum action principle to the symmetry operators involved here we find that the symmetries of the tree approximation can be at most broken by *local* field polynomials with UV-dimension less or equal than 4 in 1-loop order:

$$\begin{aligned} \mathcal{S}(\Gamma_{cl} + \Gamma^{(1)}_{ren}) &= \Delta^{(1)}_{brs} + O(\hbar^2), & (4.29)\\ \mathcal{W}_\alpha(\Gamma_{cl} + \Gamma^{(1)}_{ren}) &= \Delta^{(1)}_{\alpha} + O(\hbar^2) & (4.30) \end{aligned}$$

with

$$\dim^{UV} \Delta^{(1)}_{brs} \leq 4 \qquad \text{and} \qquad \dim^{UV} \Delta^{(1)}_{\alpha} \leq 4. \tag{4.31}$$

In particular the proof of renormalizability is completely traced back to an (algebraic) analysis of local field polynomials and (4.28) is simplified to the expressions

$$\begin{aligned} \Delta^{(1)}_{brs} + \mathrm{s}_{\Gamma_{cl}} \Gamma^{(1)}_{ct} &\stackrel{!}{=} 0 \\ \Delta^{(1)}_{\alpha} + \mathcal{W}_\alpha \Gamma^{(1)}_{ct} &\stackrel{!}{=} 0. \end{aligned} \tag{4.32}$$

A characterization of all possible breakings is obtained by the algebraic method, which will be presented in the following section. Before we turn to the algebraic method we want to make a few remarks on the quantum action principle.

In its general form the action principle relates field and parameter differentiations acting on the generating functional of Green's functions to insertions into the respective Green's functions. According to the dimension of fields appearing in the differential operators the field polynomials of the insertions have a definite upper UV dimension in all power counting renormalizable theories. In the BPHZL scheme the quantum action principle takes a simple form and relates the differential operators to Zimmermann's normal products [42, 44]. Furthermore the insertions can be expressed in terms of the (scheme-dependent) Γ_{eff}. Here we will restrict ourselves to the most important properties of insertions. First we want to give the definition of an insertion. Green's functions with insertions are quite analogously determined as ordinary Green's functions: Factors and Feynman rules are given by the formal expansion of the generalized Gell-Mann–Low formula, which defines Green's functions with insertions in a formal way:

$$\langle TO(x)\varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle = \mathcal{R}\langle T : O^{(0)}(x) : \varphi^{(0)}_{i_1}(x_1)\cdots\varphi^{(0)}_{i_n}(x_n)e^{i\Gamma_{int}(\varphi^{(0)}_k)}\rangle. \tag{4.33}$$

With $O(x)$ we denote an arbitrary field polynomial composed of propagating fields of the model. Examples for such polynomials are $\varphi^2(x)$ and $\varphi^4(x)$ in the φ^4-theory. Integrated insertions usually denoted by Δ are defined by carrying out the x integration in the above formula $\Delta = \int \mathrm{d}^4 x O(x)$. The Green's functions with a certain insertion are again summarized in the generating functional of Green's functions with insertion. From here one is able to define connected and finally 1PI Green's functions with insertion by Legendre transformation. The generating functional of 1PI Green's functions with the non-integrated insertion $O(x)$ and the integrated insertion Δ are denoted by

$$[O(x)]\cdot\Gamma \qquad \text{and} \qquad [\Delta]\cdot\Gamma \tag{4.34}$$

and

$$\frac{\delta^n}{\delta\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n)}[O(x)]\cdot\Gamma = \mathcal{R} < T : O(x) : \varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n) > \Big|_{\substack{\text{1PI diagrams}\\ \text{amputated legs}}}$$

1PI Green's functions have the same obvious diagrammatic interpretation as ordinary Green's functions. It is important to note that the lowest order in the perturbative expansion is a local expression and given by the field polynomial $O(x)$:

$$[O(x)]\cdot\Gamma = O(x) + O(\hbar). \tag{4.35}$$

(This is analogous to the observation, that Γ_{cl} is the lowest order of the generating functional of 1PI Green's functions (see (4.23).)

Of course insertions of field polynomials into loop diagrams are in general divergent and have also to be made meaningful by renormalization. Similarly as for ordinary 1PI Green's functions we find the following degree of divergency d_{Γ_O} of a 1PI Green's functions with one insertion $O(x)$:

$$d_{\Gamma_O} = 4 - N_B - \frac{3}{2}N_F + \sum_V (d_V - 4) + (d_O - 4). \tag{4.36}$$

Here the notation is the same as in (4.2), and d_O denotes the dimension of the field polynomial $O(x)$. (For example in the φ^4-theory we have $d_{\varphi^4} = 4$ and $d_{\varphi^2} = 2$.) In the BPHZ scheme the renormalized Green's functions with insertions are defined by Taylor subtraction. The number of Taylor subtractions are given by the subtraction degree, which is in a renormalizable theory given by

$$\delta_{\Gamma_O} = 4 - N_B - \frac{3}{2}N_F + (\delta_O - 4), \tag{4.37}$$

and $\delta_O \geq d_O$ defines the subtraction degree. (For example in the φ^4-theory one often has to consider φ^2-insertions with $\delta_{\varphi^2} = 4$.) In the BPHZ scheme the Green's functions with insertions are therefore given together with their subtraction degree δ: With the notation

$$[O(x)]_\delta \cdot \Gamma$$

the Green's functions with insertion are completely defined.

The quantum action principle relates field differentiations to insertions with a well defined UV-degree δ. For the purpose of the present lectures we need the following forms of the action principle: variations of propagating fields as they appear in the Ward operators of rigid symmetry (3.66) and products of field variations with respect to a propagating and an external fields as they appear in the ST operator (3.65):

$$\begin{aligned}
\varphi_k(x)\frac{\delta\Gamma}{\delta\varphi_l(x)} &= [O(x)]_{\delta_O}\cdot\Gamma \quad \text{with} \quad \delta_O = 4 - \dim^{UV}\varphi_l + \dim^{UV}\varphi_k \quad (4.38)\\
\int \mathrm{d}^4x \frac{\delta\Gamma}{\delta\Upsilon_k(x)}\frac{\delta\Gamma}{\delta\varphi_l(x)} &= [\Delta]_{\delta_\Delta}\cdot\Gamma \quad \text{with} \quad \delta_\Delta = 4 - \dim^{UV}\varphi_l + 4 - \dim^{UV}\Upsilon_k
\end{aligned}$$

The lowest order of Δ and $O(x)$ is given in expressions of the classical action:

$$\begin{aligned}
O(x) &= \varphi_k\frac{\delta\Gamma_{cl}}{\delta\varphi_l} + O(\hbar) \qquad (4.39)\\
\Delta &= \int \mathrm{d}^4x \frac{\delta\Gamma_{cl}}{\delta\Upsilon_k(x)}\frac{\delta\Gamma_{cl}}{\delta\varphi_l(x)} + O(\hbar).
\end{aligned}$$

Field polynomials appearing in higher orders are scheme dependent but restricted by the UV-degree δ_O and δ_Δ:

$$\dim^{UV} O(x) \leq \delta_O \qquad\qquad \dim^{UV}\Delta \leq \delta_\Delta. \qquad (4.40)$$

Applying the quantum action principle as given in the above formula to the Standard Model we find

$$\begin{aligned}
\mathcal{S}(\Gamma) &= [\Delta_{brs}]_4\cdot\Gamma \quad \text{with} \quad \Delta_{brs} = \mathcal{S}(\Gamma_{cl}) + O(\hbar) = O(\hbar), \qquad (4.41)\\
\mathcal{W}_\alpha\Gamma &= [\Delta_\alpha]_4\cdot\Gamma \quad \text{with} \quad \Delta_\alpha = \mathcal{W}_\alpha\Gamma_{cl} + O(\hbar) = O(\hbar). \qquad (4.42)
\end{aligned}$$

Using that the lowest order of the insertion is a local field polynomial we arrive immediately at (4.29) and (4.30), where the upper UV dimension of field polynomials is given by the subtraction degree of the insertion (4.31).

In the Standard Model and quite generally in gauge theories with unbroken gauge groups there are massless particles. For this reason, one has to assign to every field also an infrared (IR) dimension [38]. Insertions are defined by giving an subtraction degree not only with respect to their UV but also with respect to their IR dimension [44]. Then the local field polynomials Δ_{brs} and Δ_α are in addition restricted with respect to their infrared degree [38, 44]. In the Standard Model we obtain from the pure power counting analysis

$$\dim^{IR} \Delta_{brs} \geq 3, \qquad \dim^{IR} \Delta_\alpha \geq 2. \tag{4.43}$$

A complete list of the UV- and IR-dimension of the fields appearing the Standard Model is given in ref. [16].

4.3. The algebraic method

With the algebraic method one has to characterize the counterterms and the breakings by the defining symmetries of the model. In the algebraic characterization of counterterms the free parameters of the model are determined and the normalization conditions and symmetries are identified. Then the Green's functions can be uniquely defined independently of a specific (invariant) scheme. In the second step the possible breakings of the symmetry operators are restricted by algebraic consistency, and in this way it is possible to find out, if eq. (4.32) can be solved by adjusting appropriate counterterms.

The first step is called in the literature the general classical solution, since one solves the defining symmetry identities for all integrated field polynomials allowed by the power counting renormalizability. Neglecting in a first step the local Ward identity (3.70), the defining symmetries are the ST identity (3.65) and the Ward identities of rigid symmetry (3.66):

$$\mathcal{S}(\Gamma) = 0, \qquad \mathcal{W}_\alpha \Gamma = 0 \qquad \text{and} \qquad \mathcal{W}_{em} \Gamma = 0. \tag{4.44}$$

In usual gauge theories with simple gauge groups these symmetry operators are defined by their tree approximation. Since the gauge group of the Standard Model is non-semisimple and since the unbroken gauge group does not correspond to the $U(1)$-group, such a procedure is not satisfactory for renormalizing the Standard Model. In particular, when we try to proceed as usually, it is seen that there are not available enough free parameters to establish the normalization conditions of the on-shell scheme for the vector and ghost fields (cf. (4.18)). Due to the presence of the massless photon such normalization conditions are crucial for obtaining off-shell finite Green's functions in higher orders. Therefore the symmetry operators have to be themselves subject of renormalization, especially the weak mixing angle expressed in the on-shell scheme by the mass ratio of vector bosons, does get higher order corrections and cannot be fixed to its tree value in the symmetry operators.

For this reason we have to generalized the notion of invariant counterterms: Instead of taking the ST identity and Ward identities of the tree approximation, we take the most general operators compatible with the algebra (4.45) – (4.47) and call counterterms invariant if they satisfy these generalized identities (4.48). For the ST operator we require the following properties of nilpotency:

$$\begin{aligned} \mathrm{s}_\Gamma \, \mathcal{S}(\Gamma) &= 0 \quad \text{for any functional}\, \Gamma, \\ \mathrm{s}_\Gamma \, \mathrm{s}_\Gamma &= 0 \quad \text{if} \quad \mathcal{S}(\Gamma) = 0. \end{aligned} \tag{4.45}$$

The Ward operators $\mathcal{W}_\alpha$ are required to fulfil the $SU(2)$ algebra

$$[\mathcal{W}_\alpha, \mathcal{W}_\beta] = \epsilon_{\alpha\beta\gamma} \tilde{I}_{\gamma\gamma'} \mathcal{W}_{\gamma'}. \tag{4.46}$$

Finally ST operator and the Ward operators have to fulfil the consistency equation:

$$\mathrm{s}_\Gamma \mathcal{W}_\alpha \Gamma - \mathcal{W}_\alpha \mathcal{S}(\Gamma) = 0. \tag{4.47}$$

These properties are valid for the operators of the tree approximation (3.65) and (4.27) and (3.66).

For determining the general classical solution of general symmetry operators, i.e. the invariant counterterms, one has to solve the algebra as well as the defining symmetry identities for the most general power counting renormalizable action:

$$\mathcal{S}^{gen}(\Gamma^{gen}_{cl}) = 0 \qquad \mathcal{W}^{gen}_\alpha(\Gamma^{gen}_{cl}) = 0 \tag{4.48}$$

and $\mathcal{W}^{gen}_\alpha$ and $\mathcal{S}^{gen}$ fulfil equations (4.45), (4.46) and (4.47) and

$$\dim^{UV} \Gamma^{gen}_{cl} \leq 4 \tag{4.49}$$

Γ^{gen}_{cl} as well as the symmetry operators are restricted according to the global and discrete symmetries (CP invariance !) of the model (cf. the discussion after eq. (4.17)). An outline of the main steps of the solution can be found in [16]. Here we give the most important results:

The most general solution is gained from the special solution of the classical approximation Γ_{cl} by redefining all fields with the most general matrix allowed by discrete and global symmetries. Of course these field redefinitions have to be carried out in the ST operator and in the Ward operators of rigid symmetry. It is seen that such field redefinitions renormalize the operators in accordance with the algebra. For the vectors one is able to introduce a non-diagonal wave function redefinition matrix z^V_{ab} in the neutral sector, whereas the redefinition matrix of scalars is diagonal due to CP invariance.

$$z^V_{ab} = \begin{pmatrix} \hat{z}_W & 0 & 0 & \\ 0 & z_W & 0 & 0 \\ 0 & 0 & z_Z \cos\theta_Z & -z_A \sin\theta_A \\ 0 & 0 & z_Z \sin\theta_Z & z_A \cos\theta_A \end{pmatrix} \qquad z^S_{ab} = \begin{pmatrix} z_+ & 0 & 0 & 0 \\ 0 & z_+ & 0 & 0 \\ 0 & 0 & z_H & 0 \\ 0 & 0 & 0 & z_\chi \end{pmatrix} \tag{4.50}$$

Similar general field redefinitions can be carried out for the Faddeev-Popov ghosts and the fermions. Further free parameters are the parameters listed in (2.102) and the gauge parameters $\xi, \hat{\xi}, \zeta$ and G of the general gauge fixing (3.39). In this way one is able to carry out mass diagonalization on-shell and to give normalization conditions for all the residua in accordance with the symmetry operators. Due to the fact, that the general field redefinitions enter the Ward operators, invariant counterterms in 1-loop order are characterized by the equations

$$\mathrm{s}_{\Gamma_{cl}} \Gamma^{(1)}_{inv} + \delta \mathcal{S}^{(1)} \Gamma_{cl} = 0, \qquad \mathcal{W}_\alpha \Gamma^{(1)}_{inv} + \delta \mathcal{W}^{(1)}_\alpha \Gamma_{cl} = 0. \tag{4.51}$$

Consequently non-invariant counterterms are called such counterterms which cannot be arranged to fulfil equations (4.51) by an adjustment of parameters in the 1-loop operators. By solving the general classical approximation we have now splitted uniquely the counterterms into invariant and non-invariant counterterms and have specified at the same time all the possible normalization conditions. In the fermion sector of course not all the abelian couplings are specified by the solution the ST identity and Ward identities of rigid symmetry, but we find the couplings of the abelian field combination to lepton and baryon number conserving currents order to order as free parameters of the model. For this reason one has finally to establish the Ward identity of local abelian gauge symmetry (3.70) also in higher orders.

According to eq. (4.32) we have finally to prove that all breakings can be written as variations of the counterterm action. Again scheme invariance of global and discrete symmetries immediately restricts breakings according to their electric and Faddeev-Popov charge and according to their behaviour under CP transformations. Then we apply the classical ST operator $\mathrm{s}_{\Gamma_{cl}}$ and Ward operators $\mathcal{W}_\alpha$ on eqs. (4.29) and (4.30). Using the algebraic properties of the operators (4.45) – (4.47) we get:

$$\begin{aligned} \mathrm{s}_{\Gamma_{cl}} \Delta^{(1)}_{brs} &= 0, \\ \mathrm{s}_{\Gamma_{cl}} \Delta^{(1)}_\alpha - \mathcal{W}_\alpha \Delta^{(1)}_{brs} &= 0, \\ \mathcal{W}_\alpha \Delta^{(1)}_\beta - \mathcal{W}_\beta \Delta^{(1)}_\alpha &= 0. \end{aligned} \tag{4.52}$$

These equations restrict strongly the possible breakings. It is seen immediately that all breakings, which are variations

$$\Delta^{var}_{brs} = \mathrm{s}_{\Gamma_{cl}} P_{ct} \qquad \Delta^{var}_{\alpha} = \mathcal{W}_\alpha P_{ct} \tag{4.53}$$

and

$$\dim^{UV} P_{ct} \leq 4 \tag{4.54}$$

satisfy the above consistency equations. Further solutions of the equations, which cannot be written in the form of a variation are the Adler-Bardeen anomalies [45, 46, 47]. For their explicit form in the Standard Model we refer to [16]. They are seen to cancel in 1-loop order according to the appearance of lepton and quark pairs and vanish to all orders according to the non-renormalization theorems proven in [48]. Therefore all breakings can be written as variations of 4-dimensional field polynomials.

Finally we have to show, that we are able to add the field polynomials P_{ct} to the counterterm action without being in conflict with infrared existence and on-shell normalization conditions conditions. Indeed it turns out that on-shell schemes and a complete normalization of residua fix uniquely all field polynomials appearing in Γ_{inv}. Establishing the normalization conditions by adding such counterterms we find

$$\begin{aligned} \mathcal{S}(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv}) &= \mathrm{s}_{\Gamma_{cl}} P^{(1)}_{ct} + \mathrm{s}_{\Gamma_{cl}} \Gamma^{(1)}_{inv} + O(\hbar^2) && (4.55)\\ \mathcal{W}_\alpha(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv}) &= \mathcal{W}_\alpha P^{(1)}_{ct} + \mathcal{W}_\alpha \Gamma^{(1)}_{inv} + O(\hbar^2) && (4.56) \end{aligned}$$

From the definition of invariant counterterms (4.51) it is obvious that some invariants are naive invariants of the tree operators and other invariants break the symmetry of the tree operators:

$$\begin{aligned} &\Gamma_{inv} = \Gamma^o_{inv} + \Gamma^o_{break} \qquad \text{with} \\ &\mathrm{s}_{\Gamma_{cl}} \Gamma^o_{break} = -\delta \mathcal{S}^{(1)} \Gamma_{cl} \quad \text{and} \quad \mathrm{s}_{\Gamma_{cl}} \Gamma^o_{inv} = 0 \end{aligned} \tag{4.57}$$

(The superscript o indicates that we have splitted the generalized invariants (4.51) into invariants and breakings of tree operators.) In the same way P_{ct} can be splitted into non-invariant counterterms and such counterterms which are invariant in the generalized sense of (4.51) but break the symmetry of the tree operators. Having already disposed of invariant counterterms for establishing the normalization conditions we are not able to dispose of the invariant counterterms for establishing the symmetry. But according to their definitions these breakings can just be absorbed into a redefinition of the ST operator and Ward operators. These redefinitions become unique if we take into account the algebraic properties of the symmetry operators. Finally we obtain the following equations:

$$\begin{aligned}
\mathcal{S}(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv}) &= \Delta^{(1)}_{brs} + \mathrm{s}_{\Gamma_{cl}}\big(\Gamma^{o}_{break}\big)^{(1)} + O(\hbar^2) && (4.58)\\
&= \mathrm{s}_{\Gamma_{cl}}(P^{o}_{break} + \Gamma^{o}_{break})^{(1)} + \mathrm{s}_{\Gamma_{cl}} P^{(1)}_{noninv} + O(\hbar^2)\\
&= -\delta\mathcal{S}^{(1)}\Gamma_{cl} - \mathrm{s}_{\Gamma_{cl}}\Gamma^{(1)}_{noninv} + O(\hbar^2)\\
\mathcal{W}_\alpha(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv}) &= \Delta^{(1)}_{\alpha} + \mathcal{W}_\alpha\big(\Gamma^{o}_{break}\big)^{(1)} + O(\hbar^2) && (4.59)\\
&= \mathcal{W}_\alpha(P^{o}_{break} + \Gamma^{o}_{break})^{(1)} + \mathcal{W}_\alpha P^{(1)}_{noninv} + O(\hbar^2)\\
&= -\delta\mathcal{W}^{(1)}_\alpha\Gamma_{cl} - \mathcal{W}_\alpha\Gamma^{(1)}_{noninv} + O(\hbar^2)
\end{aligned}$$

Therefore we are able to establish all normalization conditions and to remove all the breakings by adjusting non-invariant counterterms and symmetry operators:

$$\begin{aligned}
\mathcal{S}^{gen}(\Gamma) \equiv \mathcal{S}(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv} + \Gamma^{(1)}_{noninv}) + \delta\mathcal{S}^{(1)}\Gamma_{cl} + O(\hbar^2) &= O(\hbar^2)\\
\mathcal{W}^{gen}_\alpha\Gamma \equiv \mathcal{W}_\alpha(\Gamma_{cl} + \Gamma^{(1)}_{ren} + \Gamma^{(1)}_{inv} + \Gamma^{(1)}_{noninv}) + \delta\mathcal{W}^{(1)}_a\Gamma_{cl} + O(\hbar^2) &= O(\hbar^2). && (4.60)
\end{aligned}$$

The proof to all orders can be immediately finished by induction, i.e. one has to go through all the steps above from order n to order $n+1$ and one has to realize that none of the initial conditions as power counting renormalizability, infrared existence and global symmetries have changed by the adjustment of 1-loop counterterms. Then the quantum action principle can be applied in the same way as in 1-loop order. The point where one has to be careful in proving renormalizability of the Standard Model is infrared existence of Green's functions. Due to the fact, that the mass matrix of vector bosons (and the one of Faddeev Popov ghosts) can be diagonalized in accordance with the symmetries on-shell, we are indeed able to proceed to higher orders as it was from the classical approximation to the 1-loop order and the renormalizability as well as infrared existence is proven to all orders.

5. Summary

In these lecture notes we have discussed the renormalization of the electroweak Standard Model by using the method of algebraic renormalization. According to the fact, that the renormalization of the electroweak Standard Model cannot be based on an invariant scheme, we have to characterize the model completely by its symmetries. Due to the non-semisimple gauge group and the specific form of the spontaneous symmetry breaking the characterization by symmetries requires quite a few generalizations compared to theories with simple groups. For clarity we review the main steps of our lectures here briefly again.

We started from the free massless Dirac action of fermions and constructed the symmetry operators which produce the currents of weak and electromagnetic interactions. In this way we found quite naturally to the $SU(2) \times U(1)$ gauge structure of electroweak interactions. When we coupled the currents to vector fields, we required a local gauge symmetry to hold for the enlarged theory. Then the interactions as well as the transformation of vectors are fixed.

So far we have worked with the massless gauge theory. Mass terms for fermions were not allowed since they break $SU(2) \times U(1)$ symmetry of the theory. We noted however that the mass terms transform covariantly under $SU(2) \times U(1)$. Therefore we are able to couple them to scalars and require again that the transformations satisfy the $SU(2) \times U(1)$ algebra. Then the transformation of scalars is fixed. The action of the Glashow-Salam-Weinberg model is then constructed by giving the most general 4-dimensional action invariant under the spontaneously broken symmetry transformations. Apart from the $SU(2) \times U(1)$ gauge symmetry with the unbroken electromagnetic gauge symmetry, we identified two further global symmetries: the conservation of lepton and baryon family number. In these lectures we did not consider mixing of different fermion families, especially we have been able to require CP invariance in the construction of higher orders.

In order to have renormalizability by power counting we added to the Glashow-Salam-Weinberg action the gauge-fixing functions in the so called R_ξ-gauges. For having nilpotency of the BRS transformations the gauge fixing functions have been coupled to the auxiliary fields B_a. Furthermore it was noticed that the R_ξ-gauges break not only local but also rigid symmetry. For maintaining rigid $SU(2) \times U(1)$ invariance external scalars have been introduced. In this way one is able to construct even a local abelian Ward identity in the tree approximation. This Ward identity proven to all orders ensures electromagnetic current coupling in the model and is the functional form of the Gell-Mann–Nishijima relation. The gauge fixing breaks the gauge symmetry non-linearly. Therefore one had to replace gauge invariance by BRS invariance, introducing the Faddeev-Popov ghost fields.

BRS transformations act on the matter fields and vectors as gauge transformations, but allow to complete the gauge fixing to a BRS symmetric action by adding the ghost action. The algebra of $SU(2) \times U(1)$ transformations is then translated to nilpotency of the BRS transformations. Having determined the gauge fixing and ghost part, the construction of the classical action has been finished by giving all the symmetry transformations in their functional form. BRS invariance is replaced by the Slavnov-Taylor identity and invariance under rigid and local gauge transformations by the Ward identities. In the proof of renormalizability it has to be proven, that these symmetries can be established to all orders of perturbation theory and define the Green's functions of the Standard Model uniquely to all orders.

In the last section we first illustrated in the φ^4-theory some special properties of renormalized perturbation theory. By comparing two renormalization prescriptions, dimensional regularization with (modified) minimal subtraction and the BPHZ momentum subtraction scheme, we have shown, that in the procedure of renormalization Green's functions are only defined up to local counterterms. To remove this scheme dependence one has to introduce normalization conditions for the free parameters of the model. For the Standard Model we have chosen a normalization scheme, which allows to fix all mass parameters of the theory and all the residua independently. In particular we required the photon and Z boson mass matrix to be diagonal at the Z-mass and at $p^2 = 0$. The latter normalization condition is crucial for ensuring infrared existence for off-shell Green's functions.

Finally the most important ingredient for the algebraic proof of renormalizability, the quantum action principle, has been given. In particular we have discussed consequences of the quantum action principle for the symmetries of the Standard Model to higher orders. The notes ended with an outline of the algebraic method. We have shown, that by the algebraic characterization of all possible counterterms and all possible breakings renormalizability can be proven in a scheme independent way. Indeed the symmetries, the Slavnov-Taylor identity, the rigid $SU(2)$ and the local abelian Ward identity, which we have derived in the classical approximation, completely characterize the model and can be established to all orders of perturbation theory since the anomalies are cancelled by the lepton and quark loops.

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Appendix A: List of important formulae

In this appendix we summarize the important formulae of the electroweak Standard Model, the action and the defining symmetry operators, in the tree approximation. All expressions are given in the QED-like on-shell parameterization (2.106), in particular we use the on-shell definition of the weak mixing angle throughout (2.103): $\cos\theta_W \equiv \frac{M_W}{M_Z}$.

Fields of the Standard Model

Left-handed fermion doublets:

$$F^L_{l_i} = \begin{pmatrix} \nu^L_i \\ e^L_i \end{pmatrix}, \qquad F^L_{q_i} = \begin{pmatrix} u^L_i \\ d^L_i \end{pmatrix}, \qquad i = 1 \ldots N_F, \tag{A.1}$$

right-handed-fermion singlets:

$$f^R_i = e^R_i, u^R_i, d^R_i, \qquad i = 1 \ldots N_F. \tag{A.2}$$

With three generations of fermions ($N_F = 3$) one has explicitly:

$$\begin{array}{rcccc} \nu_{e_i} &=& \nu_e, & \nu_\mu, & \nu_\tau \\ e_i &=& e, & \mu, & \tau \\ u_i &=& u, & c, & t \\ d_i &=& d, & s, & b; \end{array} \tag{A.3}$$

quarks are colour vectors, $q = (q_r, q_b, q_g), q = u_i, d_i$.
Vector fields:

$$V^\mu_a = (W^\mu_+, W^\mu_-, Z^\mu, A^\mu), \tag{A.4}$$

auxiliary B-fields:

$$B_a = (B_+, B_-, B_Z, B_A), \tag{A.5}$$

Faddeev-Popov ghosts with Faddeev-Popov charge $1(c_a)$ and $-1(\bar{c}_a)$:

$$c_a = (c_+, c_-, c_Z, c_A) \qquad \bar{c}_a = (\bar{c}_+, \bar{c}_-, \bar{c}_Z, \bar{c}_A). \tag{A.6}$$

The scalar doublet and its hermitian conjugate:

$$\Phi \equiv \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(H(x) + i\chi(x)) \end{pmatrix} \qquad \tilde{\Phi} \equiv i\tau_2 \Phi^* = \begin{pmatrix} \frac{1}{\sqrt{2}}(H(x) - i\chi(x)) \\ -\phi^-(x) \end{pmatrix}, \tag{A.7}$$

the external scalar doublet and its hermitian conjugate:

$$\hat{\Phi} \equiv \begin{pmatrix} \hat{\phi}^+(x) \\ \frac{1}{\sqrt{2}}(\hat{H}(x) + i\hat{\chi}(x)) \end{pmatrix} \qquad \tilde{\hat{\Phi}} \equiv i\tau_2 \hat{\Phi}^* = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{H}(x) - i\hat{\chi}(x)) \\ -\hat{\phi}^-(x) \end{pmatrix}. \tag{A.8}$$

External fields with Faddeev-Popov charge $-1(\rho_\alpha^\mu)$ and $-2(\sigma_\alpha)$:

$$\rho_\alpha = (\rho_+, \rho_-, \rho_3) \qquad \sigma_\alpha = (\sigma_+, \sigma_-, \sigma_3), \tag{A.9}$$

scalar fields with Faddeev-Popov charge -1:

$$Y \equiv \begin{pmatrix} Y^+ \\ \frac{1}{\sqrt{2}}(Y_H + iY_\chi) \end{pmatrix} \qquad Y^* = \begin{pmatrix} Y^- \\ \frac{1}{\sqrt{2}}(Y_H - iY_\chi) \end{pmatrix}, \tag{A.10}$$

right-handed fermion doublets with Faddeev-Popov charge -1:

$$\Psi_{l_i}^R = \begin{pmatrix} \psi_{\nu_i}^R \\ \psi_{e_i}^L \end{pmatrix} \qquad \Psi_{q_i}^R = \begin{pmatrix} \psi_{u_i}^R \\ \psi_{d_i}^R \end{pmatrix}, \tag{A.11}$$

left-handed fermion singlets with Faddeev-Popov charge -1

$$\psi_{f_i}^L = \psi_{e_i}^L, \psi_{u_i}^L, \psi_{d_i}^L. \tag{A.12}$$

The BRS variation of the external scalar doublet $\hat{\Phi}$ with Faddeev-Popov charge 1:

$$\mathbf{q} \equiv \begin{pmatrix} q^+ \\ \frac{1}{\sqrt{2}}(q_H + iq_\chi) \end{pmatrix} \qquad \mathbf{q}^* = \begin{pmatrix} q^- \\ \frac{1}{\sqrt{2}}(q_H - iq_\chi) \end{pmatrix}. \tag{A.13}$$

The classical action

The classical action of the Standard Model can be decomposed in the gauge invariant Glashow-Salam-Weinberg action and the gauge-fixing and ghost action:

$$\Gamma_{cl} = \Gamma_{GSW} + \Gamma_{g.f.} + \Gamma_{ghost} \tag{A.14}$$

The Glashow-Salam-Weinberg action is given by

$$\Gamma_{GSW} = \Gamma_{YM} + \Gamma_{scalar} + \Gamma_{matter} + \Gamma_{Yuk}, \tag{A.15}$$

with

$$\Gamma_{YM} = -\frac{1}{4}\int d^4x G^{\mu\nu}_a \tilde{I}_{aa'} G_{\mu\nu a'} \tag{A.16}$$

$$\Gamma_{scalar} = \int d^4x \Big((D^\mu(\Phi + \mathrm{v}))^\dagger D_\mu(\Phi+\mathrm{v}) - \frac{1}{8}\frac{m_H^2}{M_W^2}\frac{e^2}{\sin^2\theta_W}(\Phi^\dagger\Phi + \mathrm{v}^\dagger\Phi + \Phi^\dagger \mathrm{v})^2\Big) \tag{A.17}$$

$$\Gamma_{matter} = \sum_{i=1}^{N_F}\int d^4x \Big(\overline{F^L_{l_i}} i\not{D} F^L_{l_i} + \overline{F^L_{q_i}} i\not{D} F^L_{q_i} + \overline{f^R_i} i\not{D} f^R_i\Big) \tag{A.18}$$

$$\Gamma_{Yuk} = \sum_{i=1}^{N_F}\int d^4x \frac{-e}{M_W\sqrt{2}\sin\theta_W}\Big(m_{e_i}\overline{F^L_{l_i}}(\Phi+\mathrm{v})e^R_i + m_{u_i}\overline{F^L_{q_i}}(\Phi+\mathrm{v})u^R_i$$

$$+ m_{d_i}\overline{F^L_{q_i}}(\tilde{\Phi}+\tilde{\mathrm{v}})d^R_i + \mathrm{h.c.}\Big), \tag{A.19}$$

The gauge-fixing in the most general linear gauge compatible with rigid symmetry is given by

$$\Gamma_{g.f.} = \int d^4x \Big(\frac{1}{2}\xi B_a \tilde{I}_{ab} B_b + \frac{1}{2}\hat{\xi}(\sin\theta_W B_Z + \cos\theta_W B_A)^2 + B_a \tilde{I}_{ab}\partial V_b \tag{A.20}$$

$$- \frac{ie}{\sin\theta_W}\big((\hat{\Phi}+\zeta \mathrm{v})^\dagger \frac{\tau_a(\hat{G})}{2} B_a(\Phi+\mathrm{v}) - (\Phi+\mathrm{v})^\dagger \frac{\tau_a(\hat{G})}{2} B_a(\hat{\Phi}+\zeta\mathrm{v})\big)\Big).$$

The Faddeev-Popov ghost action for arbitrary $\hat{G}$ is derived from the BRS transformations (A.40) by postulating $\mathrm{s}\Gamma_{ghost} + \mathrm{s}\Gamma_{g.f.} = 0$. (The matrix $\hat{g}$ depends on $\hat{G}$ and θ_W; it is defined in (A.36)) :

$$\begin{aligned}\Gamma_{ghost} \;=\; & \int d^4x \Big(-\bar{c}_a \Box \tilde{I}_{ab} c_b - \frac{e}{\sin\theta_W}\bar{c}_a \hat{g}^{-1}_{aa'} f_{a'bc'}\partial(V_b \hat{g}_{c'c} c_c) \\ & + i\frac{e}{2\sin\theta_W}\bar{c}_a\hat{g}^{-1}_{aa'}(\hat{\mathbf{q}}^\dagger \tau_{a'}(\hat{G})(\Phi+\mathrm{v}) - (\Phi+\mathrm{v})^\dagger\tau_{a'}(G_s)\hat{\mathbf{q}}) \\ & - \frac{e^2}{4\sin^2\theta_W}\Big(\bar{c}_a\hat{g}^{-1}_{aa'}(\hat{\Phi}+\zeta\mathrm{v})^\dagger\tau_{a'}(\hat{G})\tau_{b'}(G_s)(\Phi+\mathrm{v}) \\ & + (\Phi+\mathrm{v})^\dagger\tau_{b'}(G_s)\tau_{a'}(\hat{G})(\hat{\Phi}+\zeta\mathrm{v})\Big)\hat{g}_{b'b}c_b\Big)\end{aligned} \tag{A.21}$$

We want to note that the bilinear part of the ghost action is diagonal with arbitrary ghost masses $\zeta_W M_W^2$ and $\zeta_Z M_Z^2$

$$\Gamma^{bil}_{ghost} = \int d^4x\Big(-\bar{c}_a\Box\tilde{I}_{ab}c_b - \zeta_W M_W^2(\bar{c}_+ c_- + \bar{c}_- c_+) - \zeta_Z M_Z^2 \bar{c}_Z c_Z\Big) + \Gamma^{int}_{ghost} \tag{A.22}$$

with $\zeta_W \equiv \zeta$ and $\zeta_Z = \zeta\cos\theta_W(\cos\theta_W - \hat{G}\sin\theta_W)$.

In the above formulae we have used the following conventions and abbreviations: v denotes the shift of the scalar field doublet:

$$\mathrm{v} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v \end{pmatrix} \quad \text{with} \quad v = \frac{2}{e}M_Z\cos\theta_W\sin\theta_W\ . \tag{A.23}$$

The field strength tensor and the covariant derivatives have the form

$$
\begin{aligned}
G_a^{\mu\nu} &= \partial^\mu V_a^\nu - \partial^\nu V_a^\mu + \frac{e}{\sin\theta_W}\tilde{I}_{aa'} f_{a'bc} V_b^\mu V_c^\nu && \text{(A.24)} \\
D_\mu \Phi &= \partial_\mu \Phi - i\frac{e}{\sin\theta_W}\frac{\tau_a(G_s)}{2}\Phi V_{\mu a} && \text{(A.25)} \\
D_\mu F_{\delta_i}^L &= \partial_\mu F_{\delta_i}^L - i\frac{e}{\sin\theta_W}\frac{\tau_a(G_\delta)}{2} F_{\delta,i}^L V_{\mu a} \qquad \delta = l, q && \text{(A.26)} \\
D_\mu f_i^R &= \partial_\mu f_i^R + ieQ_f \frac{\sin\theta_W}{\cos\theta_W} f_i^R Z_\mu + ieQ_f f_i^R A_\mu \, . && \text{(A.27)}
\end{aligned}
$$

The structure constants are defined by the antisymmetric tensor

$$
f_{abc} = \begin{cases} f_{+-Z} = -i\cos\theta_W \\ f_{+-A} = i\sin\theta_W \end{cases} \tag{A.28}
$$

The matrices $\tau_a \quad (a = +, -, Z, A)$ form a representation of $SU(2) \times U(1)$:

$$
\left[\frac{\tau_a}{2}, \frac{\tau_b}{2}\right] = i f_{abc} \tilde{I}_{cc'} \frac{\tau_{c'}}{2} \, . \tag{A.29}
$$

They are explicitly given by

$$
\begin{aligned}
\tau_+ &= \tfrac{1}{\sqrt{2}}(\tau_1 + i\tau_2) \qquad & \tau_Z(G) &= \tau_3 \cos\theta_W + G\mathbf{1}\sin\theta_W \\
\tau_- &= \tfrac{1}{\sqrt{2}}(\tau_1 - i\tau_2) \qquad & \tau_A(G) &= -\tau_3 \sin\theta_W + G\mathbf{1}\cos\theta_W \, .
\end{aligned} \tag{A.30}
$$

τ_i, $i = 1, 2, 3$, are the Pauli matrices and

$$
\tau_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \quad \tau_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.31}
$$

The matrices τ_Z and τ_A depend on the abelian coupling G, which is related to the weak hypercharge Y_W of the different $SU(2)$-doublets:

$$
G_k = -Y_W^k \frac{\sin\theta_W}{\cos\theta_W} \qquad Y_W^k = \begin{cases} 1 & \text{for the scalar } (k = s) \\ \text{-1} & \text{for the lepton doublets } (k = l) \\ \frac{1}{3} & \text{for the quark doublets } (k = q) \, . \end{cases} \tag{A.32}
$$

The matrix $\tilde{I}_{aa'}$ guarantees the charge neutrality of the classical action

$$
\begin{aligned}
\tilde{I}_{+-} &= \tilde{I}_{-+} = \tilde{I}_{ZZ} = \tilde{I}_{AA} = 1 && \text{(A.33)} \\
\tilde{I}_{ab} &= 0 \text{ else.}
\end{aligned}
$$

The parameter $\hat{G}$ appearing in the gauge-fixing and ghost action is arbitrary and not restricted by the symmetries of the Standard Model. Defining

$$
\hat{G} = -\frac{\sin\theta_G}{\cos\theta_G} \tag{A.34}
$$

the matrix $\hat{g}_{ab}$ depends on $\hat{G}$ in the following way:

$$\begin{aligned} \hat{g}_{+-} &= 1 & \hat{g}_{-+} &= 1 \\ \hat{g}_{ZZ} &= \cos(\theta_W - \theta_G) & \hat{g}_{AZ} &= -\sin(\theta_W - \theta_G) \\ \hat{g}_{ZA} &= 0 & \hat{g}_{AA} &= 1. \end{aligned} \tag{A.35}$$

In matrix notation it reads:

$$\hat{g}_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta_W - \theta_G) & 0 \\ 0 & 0 & -\sin(\theta_W - \theta_G) & 1 \end{pmatrix}. \tag{A.36}$$

A natural choice in the tree approximation is given by

$$\hat{G} \equiv -\frac{\sin\theta_W}{\cos\theta_W}; \tag{A.37}$$

then one has

$$\hat{g}_{ab} = \delta_{ab} \qquad \text{and} \qquad \zeta_W = \zeta_Z. \tag{A.38}$$

It turns out, that this choice is not stable under renormalization.

BRS transformations

The classical action Γ_{cl}, explicitly Γ_{GSW} and $\Gamma_{g.f.} + \Gamma_{ghost}$, are invariant under BRS transformations:

$$\mathrm{s}\Gamma_{cl} = 0 \qquad \text{and} \qquad \mathrm{s}\Gamma_{GSW} = 0, \qquad \mathrm{s}\big(\Gamma_{g.f.} + \Gamma_{ghost}\big) = 0. \tag{A.39}$$

BRS transformations are given by:

$$\begin{aligned} \mathrm{s}V_{\mu a} &= \partial_\mu c_a + \frac{e}{\sin\theta_W}\tilde{I}_{aa'} f_{a'bc} V_{\mu b} \hat{g}_{cc'} c_{c'} \\ \mathrm{s}\Phi &= i\frac{e}{\sin\theta_W}\frac{\tau_a(G_s)}{2}(\Phi + \mathrm{v})\hat{g}_{aa'} c_{a'} \\ \mathrm{s}F^L_{\delta_i} &= i\frac{e}{\sin\theta_W}\frac{\tau_a(G_\delta)}{2}F^L_{\delta_i}\hat{g}_{aa'} c_{a'} \qquad \delta = l, q \\ \mathrm{s}f^R_i &= -ieQ_f\frac{\sin\theta_G}{\cos\theta_W}f^R_i c_Z - ieQ_f f^R_i c_A \\ \mathrm{s}c_a &= -\frac{e}{2\sin\theta_W}\tilde{I}_{aa'} f_{a'bc}\hat{g}_{bb'} c_{b'}\hat{g}_{cc'} c_{c'} \\ \mathrm{s}\bar{c}_a &= \hat{B}_{a'}\hat{g}_{a'a} \text{ (i.e. } \mathrm{s}\bar{c}_Z = \cos(\theta_W - \theta_G)B_Z - \sin(\theta_W - \theta_G)B_A,\ \mathrm{s}\bar{c}_A = B_A) \\ \mathrm{s}B_a &= 0 \\ \mathrm{s}\hat{\Phi} &= \hat{\mathbf{q}} \\ \mathrm{s}\hat{\mathbf{q}} &= 0 \end{aligned} \tag{A.40}$$

The BRS transformations are nilpotent on all fields:

$$\mathrm{s}^2\varphi_k = 0 \tag{A.41}$$

We have given the BRS transformations for arbitrary $\hat{G}$ (cf (A.34)).

Slavnov-Taylor identity

For renormalization the BRS transformations are encoded in the Slavnov-Taylor identity. For this reason one adds to the classical action the external field part:

$$\Gamma_{cl} \longrightarrow \Gamma_{cl} = \Gamma_{GSW} + \Gamma_{g.f.} + \Gamma_{ghost} + \Gamma_{ext.f.} \tag{A.42}$$

with

$$\begin{aligned}\Gamma_{ext.f.} \;=\; & \int d^4x \Big(\rho_+^\mu \mathrm{s}W_{\mu,-} + \rho_-^\mu \mathrm{s}W_{\mu,+} + \rho_3^\mu(\cos\theta_W \mathrm{s}Z_\mu - \sin\theta_W \mathrm{s}A_\mu) \\ & + \sigma_+ \mathrm{s}c_- + \sigma_- \mathrm{s}c_+ + \sigma_3(\cos\theta_G \mathrm{s}c_Z - \sin\theta_W \mathrm{s}c_A) \\ & + Y^\dagger \mathrm{s}\Phi + (\mathrm{s}\Phi)^\dagger Y + \sum_{i=1}^{N_F}\big(\overline{\Psi^R_{l_i}}\mathrm{s}F^L_{l_i} + \overline{\Psi^R_{q_i}}\mathrm{s}F^L_{q_i} + \sum_f \overline{\psi^L_{f_i}}\mathrm{s}f_i^R + \mathrm{h.c.}\big)\Big).\end{aligned} \tag{A.43}$$

The Slavnov-Taylor identity of the tree approximation reads (again for arbitrary $\hat{G}$ (A.34) – (A.36)):

$$\begin{aligned}\mathcal{S}(\Gamma_{cl}) = & \int d^4x \bigg(\Big(\sin\theta_G \partial_\mu c_Z + \cos\theta_W \partial_\mu c_A\Big)\Big(\sin\theta_W \frac{\delta\Gamma}{\delta Z_\mu} + \cos\theta_W \frac{\delta\Gamma}{\delta A_\mu}\Big) \\ & + \frac{\delta\Gamma}{\delta\rho_3^\mu}\Big(\cos\theta_W \frac{\delta\Gamma}{\delta Z_\mu} - \sin\theta_W \frac{\delta\Gamma}{\delta A_\mu}\Big) + \frac{\delta\Gamma}{\delta\sigma_3}\Big(\cos\theta_W \frac{\delta\Gamma}{\delta c_Z} - \sin\theta_G \frac{\delta\Gamma}{\delta c_A}\Big)\frac{1}{\cos(\theta_W - \theta_G)} \\ & + \Big(\cos(\theta_W - \theta_G) B_Z - \sin(\theta_W - \theta_G) B_A\Big)\frac{\delta\Gamma}{\delta\bar{c}_Z} + B_A \frac{\delta\Gamma}{\delta\bar{c}_A} \\ & + \frac{\delta\Gamma}{\delta\rho_+^\mu}\frac{\delta\Gamma}{\delta W_{\mu,-}} + \frac{\delta\Gamma}{\delta\rho_-^\mu}\frac{\delta\Gamma}{\delta W_{\mu,+}} + \frac{\delta\Gamma}{\delta\sigma_+}\frac{\delta\Gamma}{\delta c_-} + \frac{\delta\Gamma}{\delta\sigma_-}\frac{\delta\Gamma}{\delta c_+} + B_+ \frac{\delta\Gamma}{\delta\bar{c}_+} + B_- \frac{\delta\Gamma}{\delta\bar{c}_-} \\ & + \sum_{i=1}^{N_F}\Big(\sum_f \frac{\delta\Gamma}{\delta\overline{\psi^L_{f_i}}}\frac{\delta\Gamma}{\delta f_i^R} + \sum_{\delta=l,q}\frac{\delta\Gamma}{\delta\overline{\Psi^R_{\delta_i}}}\frac{\delta\Gamma}{\delta F^L_{\delta_i}} + \mathrm{h.c.}\Big) + \Big(\frac{\delta\Gamma}{\delta Y^\dagger}\frac{\delta\Gamma}{\delta\Phi} + \mathbf{q}\frac{\delta\Gamma}{\delta\hat{\Phi}} + \mathrm{h.c.}\Big)\bigg) \\ = & \; 0.\end{aligned} \tag{A.44}$$

Ward identities of rigid symmetry

The classical action including the gauge-fixing and ghost action and external field action (A.42) is constructed in a way, that it is invariant under rigid $SU(2) \times U(1)$ transformations. The Ward operators of rigid SU(2) transformations satisfy the algebra

$$[\mathcal{W}_\alpha, \mathcal{W}_\beta] \;=\; \varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathcal{W}_{\gamma'} \qquad \alpha = +,-,3 \tag{A.45}$$

The structure constants are defined by the completely antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$ with $\varepsilon_{+-3} = -i$. The Ward identities of rigid $SU(2)$ transformations are given in the tree approximation by

$$
\begin{aligned}
\mathcal{W}_\alpha \Gamma_{cl} = \tilde{I}_{\alpha\alpha'} \int d^4x \Bigg(& V_b^\mu \hat{\varepsilon}_{bc,\alpha'} \tilde{I}_{cc'} \frac{\delta}{\delta V_{c'}^\mu} + B_b \hat{\varepsilon}_{bc,\alpha'} \tilde{I}_{cc'} \frac{\delta}{\delta B_{c'}} \\
+ \; & c_b \hat{g}_{bb'}^T \hat{\varepsilon}_{b'c',\alpha'} \hat{g}_{c'c}^{-1T} \tilde{I}_{cd} \frac{\delta}{\delta c_d} + \bar{c}_b \hat{g}_{bb'}^{-1} \hat{\varepsilon}_{b'c',\alpha'} \hat{g}_{c'c} \tilde{I}_{cd} \frac{\delta}{\delta \bar{c}_d} \\
+ \; & \rho_\beta^\mu \varepsilon_{\beta\gamma\alpha'} \tilde{I}_{\gamma\gamma'} \frac{\delta}{\delta \rho_{\gamma'}^\mu} + \sigma_\beta \varepsilon_{\beta\gamma\alpha'} \tilde{I}_{\gamma\gamma'} \frac{\delta}{\delta \sigma_{\gamma'}} \\
+ \; & i(\Phi + \mathrm{v})^\dagger \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta \Phi^\dagger} - i \frac{\overleftarrow{\delta}}{\delta \Phi} \frac{\tau_{\alpha'}}{2} (\Phi + \mathrm{v}) \\
+ \; & i(\hat{\Phi} + \zeta \mathrm{v})^\dagger \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta \hat{\Phi}^\dagger} - i \frac{\overleftarrow{\delta}}{\delta \hat{\Phi}} \frac{\tau_{\alpha'}}{2} (\hat{\Phi} + \zeta \mathrm{v}) \\
+ \; & i Y^\dagger \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta Y^\dagger} - i \frac{\overleftarrow{\delta}}{\delta Y} \frac{\tau_{\alpha'}}{2} Y + i \mathbf{q}^\dagger \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta \mathbf{q}^\dagger} - i \frac{\overleftarrow{\delta}}{\mathbf{q}} \frac{\tau_{\alpha'}}{2} \mathbf{q} \\
+ \; & \sum_{i=1}^{N_F} \sum_{\delta = l,q} \Bigg(i \overline{F_{\delta_i}^L} \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta \overline{F_{\delta_i}^L}} - i \frac{\overleftarrow{\delta}}{\delta F_{\delta_i}^L} \frac{\tau_{\alpha'}}{2} F_{\delta_i}^L \\
& \qquad + i \overline{\Psi_{\delta_i}^R} \frac{\tau_{\alpha'}}{2} \frac{\overrightarrow{\delta}}{\delta \overline{\Psi_{\delta_i}^R}} - i \frac{\overleftarrow{\delta}}{\delta \Psi_{\delta_i}^R} \frac{\tau_{\alpha'}}{2} \Psi_{\delta_i}^R \Bigg) \Bigg) \Gamma_{cl} = 0
\end{aligned}
\tag{A.46}
$$

The matrix $\hat{g}_{ab}$ is defined in (A.36), the tensor $\hat{\varepsilon}_{bc,\alpha}$, $b, c = +, -, Z, A$ and $\alpha = +, -, 3$ is given by

$$
O_{b\beta}^T(\theta_W) \varepsilon_{\beta\gamma\alpha} O_{\gamma c}(\theta_W) \equiv \hat{\varepsilon}_{bc,\alpha} = \begin{cases} \hat{\varepsilon}_{Z+,-} & = \; -i \cos\theta_W \\ \hat{\varepsilon}_{A+,-} & = \; i \sin\theta_W \\ \hat{\varepsilon}_{+-,3} & = \; -i \end{cases}
\tag{A.47}
$$

The matrix $O_{\alpha a}(\theta_W)$ (2.96) transforms the $SU(2) \times U(1)$ fields $W_+^\mu, W_-^\mu, W_3^\mu, W_4^\mu$ to physical on-shell fields $W_+^\mu, W_-^\mu, Z^\mu, A^\mu$.

In the Standard Model there are three types of abelian rigid symmetries: the abelian operator connected with electromagnetic charge conservation $\mathcal{W}_4^Q = \mathcal{W}_{em} - \mathcal{W}_3$, and the abelian operators of lepton and baryon conservation $\mathcal{W}_{l_i}$ and $\mathcal{W}_{q_i}$:

$$
[\mathcal{W}_\alpha, \mathcal{W}_4^Q] = 0, \quad [\mathcal{W}_\alpha, \mathcal{W}_{l_i}] = 0, \quad [\mathcal{W}_\alpha, \mathcal{W}_{q_i}] = 0
\tag{A.48}
$$

These operators are given by

$$\mathcal{W}_{em} = -i\int d^4x \Big(W_+^\mu \frac{\delta}{\delta W_+^\mu} - W_-^\mu \frac{\delta}{\delta W_-^\mu} + B_+ \frac{\delta}{\delta B_+} - B_- \frac{\delta}{\delta B_-} \tag{A.49}$$
$$+ c_+ \frac{\delta}{\delta c_+} - c_- \frac{\delta}{\delta c_-} + \bar{c}_+ \frac{\delta}{\delta \bar{c}_+} - \bar{c}_- \frac{\delta}{\delta \bar{c}_-}$$
$$+ \rho_+ \frac{\delta}{\delta \rho_+} - \rho_- \frac{\delta}{\delta \rho_-} + \sigma_+ \frac{\delta}{\delta \sigma_+} - \sigma_- \frac{\delta}{\delta \sigma_-}$$
$$+ \phi^+ \frac{\delta}{\delta \phi^+} - \phi^- \frac{\delta}{\delta \phi^-} + Y^+ \frac{\delta}{\delta Y^+} - Y^- \frac{\delta}{\delta Y^-}$$
$$+ \hat{\phi}^+ \frac{\delta}{\delta \hat{\phi}^+} - \hat{\phi}^- \frac{\delta}{\delta \hat{\phi}^-} + q^+ \frac{\delta}{\delta q^+} - q^- \frac{\delta}{\delta q^-}$$
$$- \sum_{i=1}^{N_F} \Big(Q_e (\bar{e}_i \frac{\delta}{\delta \bar{e}_i} - \frac{\delta}{\delta e_i} e_i + \bar{\psi}_{e_i} \frac{\delta}{\delta \bar{\psi}_{e_i}} - \frac{\delta}{\delta \psi_{e_i}} \psi_{e_i})$$
$$+ Q_u (\bar{u}_i \frac{\delta}{\delta \bar{u}_i} - \frac{\delta}{\delta u_i} u_i + \bar{\psi}_{u_i} \frac{\delta}{\delta \bar{\psi}_{u_i}} - \frac{\delta}{\delta \psi_{u_i}} \psi_{u_i})$$
$$+ Q_d (\bar{d}_i \frac{\delta}{\delta \bar{d}_i} - \frac{\delta}{\delta d_i} d_i + \bar{\psi}_{d_i} \frac{\delta}{\delta \bar{\psi}_{d_i}} - \frac{\delta}{\delta \psi_{d_i}} \psi_{d_i}) \Big) \Big)$$

and

$$\mathcal{W}_{l_i} = i\int d^4x \Big(\bar{e}_i \frac{\delta}{\delta \bar{e}_i} - \frac{\delta}{\delta e_i} e_i + \bar{\psi}_{e_i} \frac{\delta}{\delta \bar{\psi}_{e_i}} - \frac{\delta}{\delta \psi_{e_i}} \psi_{e_i} \tag{A.50}$$
$$+ \overline{\nu_i^L} \frac{\delta}{\delta \overline{\nu_i^L}} - \frac{\delta}{\delta \nu_i^L} \nu_i^L + \overline{\psi_{\nu_i}^R} \frac{\delta}{\delta \overline{\psi_{\nu_i}^R}} - \frac{\delta}{\delta \psi_{\nu_i}^R} \psi_{\nu_i}^R \Big)$$

$$\mathcal{W}_{q_i} = i\int d^4x \Big(\bar{d}_i \frac{\delta}{\delta \bar{d}_i} - \frac{\delta}{\delta d_i} d_i + \bar{\psi}_{d_i} \frac{\delta}{\delta \bar{\psi}_{d_i}} - \frac{\delta}{\delta \psi_{d_i}} \psi_{d_i} \tag{A.51}$$
$$+ \bar{u}_i \frac{\delta}{\delta \bar{u}_i} - \frac{\delta}{\delta u_i} u_i + \bar{\psi}_{u_i} \frac{\delta}{\delta \bar{\psi}_{u_i}} - \frac{\delta}{\delta \psi_{u_i}} \psi_{u_i} \Big).$$

The classical action is invariant under these global symmetries:

$$\mathcal{W}_{em}\Gamma_{cl} = 0 \qquad \mathcal{W}_{l_i}\Gamma_{cl} = 0 \qquad \mathcal{W}_{q_i}\Gamma_{cl} = 0. \tag{A.52}$$

Since these global symmetries are not broken by renormalization the generating functional of 1PI Green's functions Γ is invariant by construction to all orders.

The local U(1) Ward identity

The local $U(1)$ Ward operator, which is defined by the relation

$$\mathcal{W}_4^Q = \int d^4x \, \mathbf{w}_4^Q, \tag{A.53}$$

is (to all orders of perturbation theory) given by the following expression:

$$\begin{aligned}\mathbf{w}_4^Q &= \frac{i}{2}(\Phi+\mathrm{v})^\dagger\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger}-\frac{i}{2}\frac{\overleftarrow{\delta}}{\delta\Phi}(\Phi+\mathrm{v})+\{Y,\hat{\Phi}+\zeta\mathrm{v},\mathbf{q}\} \\ &\quad+\sum_{i=1}^{N_F}\Big(\sum_{\delta=l,q}Y_W^\delta\Big(\frac{i}{2}\overline{F_{\delta_i}^L}\frac{\overrightarrow{\delta}}{\delta\overline{F_{\delta_i}^L}}-\frac{i}{2}\frac{\overleftarrow{\delta}}{\delta F_{\delta_i}^L}F_{\delta_i}^L+\{\Psi_{\delta_i}^R\}\Big) \\ &\quad+\sum_f Q_f\Big(i\overline{f_i^R}\frac{\overrightarrow{\delta}}{\delta\overline{f_i^R}}-i\frac{\overleftarrow{\delta}}{\delta f_i^R}f_i^R+\{\psi_{f_i}^L\}\Big)\Big).\end{aligned} \tag{A.54}$$

This operator is continued to a local $U(1)$ Ward identity which is the functional form of the Gell-Mann–Nishijima relation:

$$\Big(\frac{e}{\cos\theta_W}\mathbf{w}_4^Q-\Big(\sin\theta_W\partial\frac{\delta}{\delta Z}+\cos\theta_W\partial\frac{\delta}{\delta A}\Big)\Big)\Gamma_{cl}=(\sin\theta_W\Box B_Z+\cos\theta_W\Box B_A) \tag{A.55}$$

Appendix B: Exercises

1. Parity transformation P is defined by $(x^0,\vec{x})\xrightarrow{P}(x^0,-\vec{x})$.
   Show that under the parity transformation P

   (a) $\psi(x^\mu)\xrightarrow{P}\eta_P\gamma^0\psi(x^o,-\vec{x})$, where η_P is a phase factor,

   (b) $\bar\psi\gamma^\mu\psi\xrightarrow{P}(\bar\psi\gamma^0\psi,-\bar\psi\vec{\gamma}\psi)$,

   (c) $\bar\psi\gamma^\mu\gamma^5\psi\xrightarrow{P}(-\bar\psi\gamma^0\gamma^5\psi,-\bar\psi\vec{\gamma}\gamma^5\psi)$,

   (d) $\int\bar\psi\gamma_\mu\psi V^\mu\xrightarrow{P}\int\bar\psi\gamma_\mu\psi V^\mu$, if V^μ is a vector,

   (e) $\int\bar\psi\gamma_\mu\gamma^5\psi A^\mu\xrightarrow{P}\int\bar\psi\gamma_\mu\gamma^5\psi A^\mu$, if A^μ is an axial vector.

   (f) Show that one cannot assign a well-defined parity to $\int d^4xW_-^\mu\bar{e}\gamma^\mu\frac{1}{2}(\mathbb{1}-\gamma^5)\nu$.

2. Left- and right-handed projectors $P^L=\frac{1}{2}(\mathbb{1}-\gamma^5)$ and $P^R=\frac{1}{2}(\mathbb{1}+\gamma^5)$.

   (a) Verify the projector properties

   $$P^L+P^R=\mathbb{1},\qquad P^iP^i=P^i,\ i=L,R,\qquad P^LP^R=0.$$

   (b) Show that $\overline{f^L}=\bar{f}P^R$.

   (c) Express the action Γ_{Dirac}^{bil} (2.5) in terms of left- and right-handed fermion fields.

3. Fields are representations of the $SU(2) \times U(1)$ symmetry.

   (a) Transformations have been given in terms of left- and right-handed fields. Calculate on the Dirac spinors the transformations $\delta_+ f$, $\delta_- f$ and $\delta_3 f$ for $f = u, d, e, \nu$ explicitly. Use these results to find an expression for the local functional operators $\mathbf{w}_\alpha$ in expressions of Dirac spinors. Show that the transformations depend on γ_5.

   (b) Verify that δ_+ raises and δ_- lowers the electric charge.

   (c) Check the commutator relation of local $SU(2)$-operators

$$[\mathbf{w}_\alpha(x), \mathbf{w}_\beta(y)] = \varepsilon_{\alpha\beta\gamma} \mathbf{w}_\gamma^\dagger(x) \delta^4(x-y).$$

4. The $SU(2)$ invariance is broken by mass terms.

   (a) Calculate the currents $J_\pm^\mu, J_3^\mu$ and j_{em}^μ from the identities

$$\mathbf{w}_\alpha \Gamma_{Dirac}^{bil}\Big|_{m_f=0} = -\partial_\mu J_\alpha^\mu,$$
$$\mathbf{w}_{em} \Gamma_{Dirac}^{bil} = -\partial_\mu j_{em}^\mu.$$

   (b) Take a non-vanishing electron mass $m_e \neq 0$ and show that

$$\mathbf{w}_\alpha \Gamma_{leptons}^{Dirac}(m_e \neq 0) = \partial_\mu J^\mu_{\alpha\ leptons} + i Q_\alpha^{leptons}$$

   with

$$Q_+^{leptons} = -m_e \frac{1}{\sqrt{2}} \bar{e}^R \nu^L,$$
$$Q_-^{leptons} = m_e \frac{1}{\sqrt{2}} \bar{\nu}^L e^R,$$
$$Q_3^{leptons} = m_e \frac{1}{2} (\bar{e}^R e^L - \bar{e}^L e^R).$$

5. This exercise shows that $\phi^\pm$ and χ are not physical fields in the Standard Model.

   (a) Give the bilinear part of the Glashow-Salam-Weinberg action Γ_{GSW}.

   (b) Eliminate the fields $\phi^\pm$ and χ from the free field action by redefining the W and Z bosons:

$$W'^\mu_\pm = W_\pm^\mu \pm \frac{i}{M_W} \partial^\mu \phi_\pm,$$
$$Z'^\mu = Z^\mu + \frac{1}{M_Z} \partial^\mu \chi.$$

(c) Give the respective free field equations for these redefined fields.

6. Calculate Γ_{matter} in terms of the physical fields $W_+^\mu, W_-^\mu, Z^\mu, A^\mu$.

7. Lepton and quark numbers.

   (a) Show that the operators of lepton and quark number conservation commute with rigid $SU(2)$-operators (see (A.46) and (A.50)):

$$[\mathcal{W}_l, \mathcal{W}_\alpha] = 0, \qquad [\mathcal{W}_q, \mathcal{W}_\alpha] = 0.$$

   (b) Determine the corresponding currents j_μ^l and j_μ^q in the classical approximation.

   (c) Construct the $SU(2) \times U(1)$ gauge theory, in which apart from $J_\mu^\pm$ and J_μ^3 the currents $q_l j_\mu^l + q_q j_\mu^q$ are gauged, but not the electromagnetic current. Discuss the result!

8. Consider the renormalization of the φ^4-theory as discussed in the text (section 4.1). Two renormalization prescriptions were given: dimensional regularization with MS-subtraction and BPHZ renormalization. Take the one-loop expressions we have given in (4.4) and (4.5), (4.7) and (4.8). Compare the Γ_{eff}, i.e. the 1-loop counterterms, in different schemes. The vertex functions are normalized according to the conditions (4.14) and (4.15).

   (a) Calculate the counterterms to the Γ_{eff} in dimensional regularization with MS-subtraction.

   (b) Calculate the counterterms to the Γ_{eff} in BPHZL subtraction.

   (c) Discuss the result!

9. The 't Hooft gauges versus unitary gauge.

   (a) Calculate the propagators of the scalar and vector fields of the Standard Model in the 't Hooft gauges.

   (b) Compare these results with the unitary gauge.

10. BRS transformations.

   (a) Check the nilpotency of the BRS operator s on the vector and Higgs fields explicitly (see (A.40)).

   (b) Determine the bilinear part of the Faddeev-Popov ghost action (see (A.21) and (A.22)).

Books and reviews on quantum field theory and renormalization

Quantum field theory

[Q1] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley-Intersciences, NewYork, 1959.

[Q2] S. Gasiorowicz, *Elementary Particle Physics*, J. Wiley & Sons, Inc., New York, 1966.

[Q3] C. Itzykson, J.-B. Zuber, *Quantum field theory*, McGraw-Hill, New York, 1985.

[Q4] T. Kugo, *Eichtheorien*, Springer, Berlin, Heidelberg, 1997.

[Q5] S. Weinberg, *The Quantum Theory of Fields, Volume I Foundations*, Cambridge University Press, Cambridge, 1995.

Renormalization

[R1] C. Becchi, *Lectures on the Renormalization of Gauge Theories* , in 'Relativity, groups and topology II', Les Houches 1983 (eds. B.S. DeWitt and R. Stora) North Holland, Amsterdam, 1984.

[R2] J.C. Collins, *Renormalization*, Cambridge University Press, Cambridge, 1984.

[R3] J.H. Lowenstein, *Seminars on Renormalization Theory* vol.II, University of Maryland Techn. Rep. No 73-068 (1972).

[R4] J.H. Lowenstein, *BPHZ Renormalization*, in 'Renormalization Theory' (eds. G. Velo and A.S. Wightman) D. Reidel, Dordrecht, 1976.

[R5] O. Piguet, *Renormalisation en théorie quantique des champs* and *Renormalisation des théories de jauge*, Lectures of the 'Troisieme cycle de la physique en suisse romande', 1982-83.

[R6] O. Piguet and S. Sorella, *Algebraic Renormalization*, Lecture Notes in Physics 28, Springer, Berlin, Heidelberg, 1995.

[R7] W. Zimmermann, *Local Operator Products and Renormalization* in 'Lectures on Elementary Particles and Quantum Field Theory', Vol. I, 1970 Brandeis Lectures (eds. S. Deser, M. Grisaru, H. Pendleton) M.I.T. Press, Cambridge, 1970.

References

[1] S.L. Glashow, Nucl. Phys. **B22** (1961) 579.

[2] S. Weinberg, Phys. Rev. Lett. **19** (1967) 1264.

[3] A. Salam, in: *Proceedings of the 8th Nobel Symposium, p. 367*, ed. N. Svartholm, Almqvist and Wiksell, Stockholm 1968.

[4] LEP Collaboration ALEPH, DELPHI, L3, OPAL, the LEP Electroweak Working Group and the SLD Heavy Flavour Group, CERN-PPE/97-154.

[5] W. Hollik, in: *Precision tests of the standard elektroweak model*, ed. P.Langacker, World Scientific, 1995.

[6] F. Jegerlehner, *Proceedings of the 1990 Theoretical Advanced Study Institute in Elementary Particle Physics*, Boulder, Colorado, eds. M. Cvetic and P. Langacker, 1991, Singapore.

[7] W. Hollik, *Review: Status of the Standard Model*, Talk at the International Workshop on Quantum Effect in the MSSM, Barcelona, 9 -13 September 1997, Preprint Karlsruhe KA-TP-22-1197, hep-ph/9711492.

[8] *Reports of the Working Group on Precision Calculations for the Z-resonance*, CERN Yellow Report, CERN 95-03, eds. D. Bardin, W. Hollik and G. Passarino.

[9] W. Beenakker and F.A. Berends (conveners), *WW Cross-Sections and Distributions*,CERN Report 1996, hep-ph/9602351.

[10] G. 't Hooft and M. Veltman Nucl. Phys. **B44** (1972) 189.

[11] G. 't Hooft and M. Veltman Nucl. Phys. **B50** (1972) 318.

[12] C. Becchi, A. Rouet and R. Stora, Phys. Lett. **B52** (1974) 344; Comm. Math. Phys. **42** (1975) 127.

[13] C. Becchi, A. Rouet and R. Stora, Ann. Phys. (NY) **98** (1976) 287.

[14] I.V. Tyutin, "Gauge invariance in field theory and statistical mechanics", Lebedev preprint FIAN, no **39** (1975), unpublished.

[15] G. Bandelloni, C. Becchi, A. Blasi and R. Collina, Ann. Inst. Henri Poincaré **28** (1978) 225, 255.

[16] E. Kraus, Ann. Phys. (NY) **262** (1998) 155.

[17] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531,
M. Kobayashi and T. Maskawa, Prog. Theor. Phys **49** (1972) 652.

[18] P.W. Higgs Phys. Lett. 12 (1964) 132.
P.W. Higgs Phys. Rev. Lett. 13 (1964) 508.

[19] R. Brout, F. Englert Phys. Lett. 13 (1964) 321.

[20] Particle Data Group, Phys. Rev. **D54**(1996) 1.

[21] A. Sirlin, Phys. Rev. **D22** (1980) 971. W.J. Marciano, A. Sirlin, Phys. Rev. **22** (1980) 2695. A. Sirlin, W.J. Marciano, Nucl. Phys. **189** (1981) 442.

[22] G. Passarino and M. Veltman, Nucl. Phys. **B160** (1979) 151.

[23] M. Consoli, Nucl. Phys. **B160** (1979) 208.

[24] J. Fleischer and F. Jegerlehner, Phys. Rev. **D23** (1981) 2001.

[25] K.I. Aoki, Z. Hioki, R. Kawabe, M. Konuma, T. Muta, Suppl. Prog. Theor. Phys. **73** (1982) 1; Z. Hoki, Phys. Rev. Lett. **65** (1990) 683, E:1692; Z. Phys. **C49** (1991) 287.

[26] D.Yu. Bardin, P.Ch. Christova, O.M. Fedorenko, Nucl. Phys. **B175** (1980) 435; Nucl. Phys. **B197** (1982) 1.

[27] M. Consoli, S. LoPresti and L. Maiani, Nucl. Phys. **B223** (1983) 474.

[28] M. Böhm, W. Hollik and H. Spiesberger, Fortschr. Phys. **34** (1986) 687.

[29] W. Hollik, Fortschr. Phys. **38** (1990) 165.

[30] G. 't Hooft, Nucl. Phys. **B35** (1971) 167.

[31] E. Kraus and G. Weiglein, Preprint Bonn BN-TH-98-16 and Karlsruhe KA-TP-12-1998, hep-ph/9807548.

[32] L.D. Faddeev and V.N. Popov, Phys. Lett. **B25** (1967) 29.

[33] T. Kugo and I. Ojima, Phys. Lett. **B73** (1978) 458;
Prog. theor. Phys. suppl. **66** (1979) 1.

[34] P. Breitenlohner and D. Maison, Comm. Math. Phys. **52** (1977) 11,39,55.

[35] N.N. Bogoliubov, O. Parasiuk, Acta Math. **97** (1957) 227.

[36] K. Hepp, Comm. Math. Phys. **2** (1966) 301.

[37] W. Zimmermann, Comm. Math. Phys. **15** (1969) 208.

[38] J.H. Lowenstein, Comm. Math. Phys. **47** (1976) 53.

[39] G. 't Hooft, Nucl. Phys. **B61** (1973) 455.

[40] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. **D18** (1978) 3998.

[41] A. Denner, Fortschr. Phys. **41** (1993) 307.

[42] J.H. Lowenstein, Phys. Rev. **D4** (1971) 2281.
J.H. Lowenstein, Comm. Math. Phys. **24** (1971) 1.

[43] Y.M.P. Lam, Phys. Rev. **D6** (1972) 2145; Phys. Rev. **D7** (1973) 2943.

[44] T.E. Clark, J.H. Lowenstein, Nucl. Phys. **B113** (1976) 109.

[45] S.L. Adler, Phys. Rev. **117** (1969) 2426.

[46] J.S. Bell and R. Jackiw, Nuovo Cimento **60** (1969) 47.

[47] W.A. Bardeen, Phys. Rev. **184** (1969) 48.

[48] G. Bandelloni, C. Becchi, A. Blasi, R. Collina, Comm. Math. Phys. **72** (1980) 239.