# BRST Quantization: a Short Review

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### Abstract

Most of the known models describing the fundamental interactions have a gauge freedom. In the standard path integral, it is necessary to "fix the gauge" in order to avoid integrating over unphysical degrees of freedom. Gauge independence might then become a tricky issue, especially when the structure of the gauge symmetries is intricate. In the modern approach to this question, it is BRST invariance that effectively implements gauge invariance. This set of lectures briefly reviews some key ideas underlying the BRST-antifield formalism, which yields a systematic procedure to path-integrate any type of gauge system, while (usually) manifestly preserving spacetime covariance. The quantized theory possesses a global invariance under the so-called BRST transformation, which is nilpotent of order two. The cohomology of the BRST differential is the central element that controls the physics. Its relationship with the observables is sketched and explained. How anomalies appear in the "quantum master equation" of the antifield formalism is also discussed. These notes are based on lectures given by MH at the 10th Saalburg Summer School on Modern Theoretical Methods from the 30th of August to the 10th of September, 2004 in Wolfersdorf, Germany and were prepared by AF and AM. The exercises which were discussed at the school are also included.

## 1  Introduction

Gauge symmetries are omnipresent in theoretical physics, especially in particle physics. Well-known examples of gauge theories are QED and QCD. A common feature of gauge theories is the appearance of unphysical degrees of freedom in

the Lagrangian. Because of this, the naive path integral for gauge theories is meaningless since integrating over gauge directions in the measure would make it infinite-valued:

$$\int DA_\mu \, e^{iS} = \infty \tag{1.1}$$

The redundant gauge variables must be removed from the theory by considering gauge-fixing conditions. When this is done, gauge invariance is of course lost and it is not clear how to control the physics. The modern approach to cope with these problems in the case of general gauge theories was developed by Batalin and Vilkovisky [1, 2, 3], building on earlier work by Zinn-Justin [4], Kallosh [5] and de Wit and van Holten [6] . It goes under the name of BV or antifield formalism and is the method explained in these lectures. The BRST symmetry is central to it [7, 8, 9, 10]. A complete coverage of the topic and further references can be found in [11, 12, 13].

# 2 Structure of gauge symmetries

Let us give a brief overview of the different types of gauge theories that one may encounter.

## 2.1 Yang-Mills type

We are all familiar with Yang-Mills gauge theories. In the absence of matter, the action is given by

$$\begin{align}
S_0[A_\mu^a] &= -\frac{1}{4} \int F_{\mu\nu}^a F_a^{\mu\nu} d^n x \tag{2.1}\\
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^a{}_{bc} A_\mu^b A_\nu^c. \tag{2.2}
\end{align}$$

$F_{\mu\nu}^a$ is the field strength tensor, $A_\mu^a$ is the gauge field and $f^a{}_{bc}$ are the structure constants of the associated gauge group. The dimensionality of space-time is $n$. The gauge transformation takes infinitesimally the form

$$\begin{align}
A_\mu^a(x)' &= A_\mu^a(x) + \delta_\epsilon A_\mu^a \tag{2.3}\\
\delta_\epsilon A_\mu^a(x) &= D_\mu \epsilon^a(x) = \partial_\mu \epsilon^a + f^a{}_{bc} A_\mu^b \epsilon^c(x), \tag{2.4}
\end{align}$$

where $\epsilon^c(x)$ is a set of arbitrary functions, the gauge parameters. In these theories, the commutator of infinitesimal gauge transformations reads

$$[\delta_\varepsilon, \delta_\eta] X = \delta_\xi X, \quad \xi^a = f^a{}_{bc} \varepsilon^b \eta^c \tag{2.5}$$

with $\varepsilon$, $\eta$, $\xi$ gauge parameters and where $X$ can be any field. It is clear from this formula that the algebra of the gauge transformations closes off-shell as the commutator of the gauge transformations is again a gauge transformation of the same type, without using the equations of motion.

## 2.2 Closure only on-shell

Off-shell closure holds for Yang-Mills gauge theories but is not a general feature of gauge systems. The gauge transformations might close only when the equations of motion hold. Notable examples where this is the case are extended supergravity theories. Rather than discussing the gauge structure of supergravities, which is rather intricate, we shall illustrate "closure only on-shell" in the case of a much simpler (but of no direct physical interest) system. Consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\dot{q}^1 - \dot{q}^2 - \dot{q}^3)^2 = \frac{1}{2}\dot{y}^2, \quad y = q^1 - q^2 - q^3 \tag{2.6}$$

for a model with three coordinates $q^1$, $q^2$ and $q^3$. $\mathcal{L}$ is invariant under two different sets of gauge transformations, which can be taken to be:

$$\delta_\varepsilon q^1 = \varepsilon + \varepsilon q^2 \ddot{y}, \ \ \delta_\varepsilon q^2 = \varepsilon, \ \ \delta_\varepsilon q^3 = \varepsilon q^2 \ddot{y} \tag{2.7}$$

and

$$\delta_\eta q^1 = 0, \ \ \delta_\eta q^2 = \eta, \ \ \delta_\eta q^3 = -\eta \tag{2.8}$$

We can easily calculate the commutators of the gauge transformations on the fields:

$$[\delta_\varepsilon, \delta_\eta]\, q^1 = \varepsilon\eta\ddot{y}, \quad [\delta_\varepsilon, \delta_\eta]\, q^2 = 0, \quad [\delta_\varepsilon, \delta_\eta]\, q^3 = \varepsilon\eta\ddot{y}. \tag{2.9}$$

From (2.6), the equation of motion (eom) for $y$ is $\ddot{y} = 0$. We see that the algebra of the gauge transformations (2.7) and (2.2) is closed (in fact, abelian) only up to equations of motion, i.e., only on-shell.

## 2.3 Reducible gauge theories

The gauge transformations might also be "reducible", i.e., dependent. Consider the theory of an abelian 2-form $B_{\mu\nu} = -B_{\nu\mu}$. The field strength is given by $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. The Lagrangian reads:

$$\mathcal{L} = -\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} \tag{2.10}$$

and is invariant under gauge transformations

$$\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{2.11}$$

where $\Lambda$ is the gauge parameter. These transformations vanish for a class of parameters $\Lambda_\mu = \partial_\mu \epsilon$, meaning that the gauge parameters are not all independent. Such gauge transformations are called reducible, and the corresponding gauge theory is said to be reducible. Two-forms define a natural generalization of electromagnetism, $A_\mu \to B_{\mu\nu}$ and occur in many models of unification; the main difference with electromagnetism being the irreducibility of the latter.

## 2.4  Reducibility on-shell

The last feature that we want to illustrate is the possibility that the reducibility of the gauge transformations holds only on-shell. One can reformulate the previous free 2-form model by introducing an auxiliary field $A_\mu$. The Lagrangian is then given by:

$$\mathcal{L} = \frac{1}{12} A_\mu \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} - \frac{1}{8} A_\mu A^\mu. \tag{2.12}$$

This Lagrangian reduces to (2.10) by inserting the equation of motion for $A_\mu$:

$$A_\mu = \frac{1}{3} \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}. \tag{2.13}$$

The gauge transformations are of the form:

$$\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{2.14}$$
$$\delta_\Lambda A_\mu = 0. \tag{2.15}$$

We can introduce interactions by considering Lie-algebra-valued fields $A_\mu = A_\mu^a T_a$, $B_{\mu\nu} = B_{\mu\nu}^a T_a$ ($T_a$: generators of the gauge group) and covariant derivatives instead of partial derivatives, $\partial_\mu \to D_\mu$. This is the so-called Freedman-Townsend model [14].

For the Freedman-Townsend model, the gauge transformations vanish for parameters $\Lambda_\mu = D_\mu \epsilon$. But now, this vanishing occurs only if the equations of motion are satisfied. This is due to $[D_\mu, D_\nu] \propto F_{\mu\nu}$, and $F_{\mu\nu} = 0$ is the eom for $B$. The theory is said in that case to be reducible on-shell.

The antifield-BRST formalism is capable of handling all the gauge structures described here, while the original methods were devised only for off-shell closed, irreducible gauge algebras. This wide range of application of the antifield formalism is one of its main virtues.

Remark: recent considerations on the structure of gauge symmetries, including reducible ones, may be found in [15, 16].

# 3  Algebraic tools

BRST theory uses crucially cohomological ideas and tools. In the following, some definitions are collected and a useful technique for the computation of cohomologies is illustrated.

## 3.1  Cohomology

Let us consider a nilpotent linear operator $D$ of order 2: $D^2 = 0$. Because of this property, the image of $D$ is contained in the kernel of $D$, Im $D \subseteq$ Ker $D$. The cohomology of the operator $D$ is defined as the following quotient space:

$$H(D) \equiv \frac{\mathrm{Ker} D}{\mathrm{Im} D}. \tag{3.1}$$

## 3.2 De Rham $d$

As a familiar example, let us discuss the de Rham $d$-operator. This will enable us to introduce further tools.

In a coordinate patch, a $p$-form is an object of the form

$$w = \frac{1}{p!} w_{i_1 \ldots i_p} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{3.2}$$

where the coefficients $w_{i_1 \ldots i_p}$ are totally antisymmetric functions of the coordinates and $\wedge$ refers to the exterior product. [Appropriate transition conditions should hold in the overlap of two patches, but these will not be discussed here.] The vector space of $p$-forms on $M$ is denoted by $\Omega^p(M)$. The direct sum of $\Omega^p(M)$, $p = 0, \ldots, m \equiv \dim M$ defines the space of all forms on $M$:

$$\Omega^*(M) \equiv \Omega^0(M) \oplus \ldots \oplus \Omega^m(M) \tag{3.3}$$

The exterior derivative $d_p$ is a map $\Omega^p(M) \to \Omega^{p+1}(M)$ whose action on a $p$-form $w$ is defined by[1]

$$d_p w = \frac{1}{p!} \frac{\partial w_{i_1 \ldots i_p}}{\partial x^j} \, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{3.4}$$

Important properties of $d$ are its nilpotency of order two

$$d^2 = 0 \quad (\text{i.e. } d_{p+1} d_p = 0) \tag{3.5}$$

and the fact that it is an odd derivative:

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^p w \wedge d\eta. \tag{3.6}$$

The nilpotency can easily be proved by direct computation:

$$d^2 w = \frac{1}{p!} \frac{\partial^2 w_{i_1 \ldots i_p}}{\partial x^k \partial x^j} \, dx^k \wedge dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{3.7}$$

This expression clearly vanishes since the coefficients are symmetric in $k, j$ while $dx^k \wedge dx^j$ is antisymmetric. Therefore, $\text{Im } d_p \subset \text{Ker } d_{p+1}$.

An element of $\text{Ker } d_p$ is said to be "closed" or "a cocycle" (of $d$), $d\alpha = 0$. An "exact form" or "coboundary" (of $d$) lives in $\text{Im } d_{p-1}$; it is thus such that $\alpha = d\beta$, for some $(p-1)$-form $\beta$. The $p$th de Rham cohomology group is defined as:

$$H^p(d) = \frac{\text{Ker } d_p}{\text{Im } d_{p-1}}. \tag{3.8}$$

Another important operation is the interior product $i_X \colon \Omega^p(M) \to \Omega^{p-1}(M)$, where $X = X^j \partial/\partial x^j$. The action of $i_X$ on a $p$-form $w$ reads:

$$i_X w = \frac{1}{(p-1)!} X^j w_{j i_2 \ldots i_p} \, dx^{i_2} \wedge \ldots \wedge dx^{i_p}. \tag{3.9}$$

The interior product also satisfy (3.5) and (3.6) as well.

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[1]The subscript $p$ is often not written; the exterior derivative is then referred as $d$.

## 3.3 Poincaré lemma

A useful tool for computing cohomologies is given by contracting homotopies. We illustrate the techniques by computing the cohomology of $d$ in a special case.

Let $M$ be $\mathbb{R}^n$, and $w$ a closed $p$-form on M, $p > 0$. The Poincaré lemma states that all closed forms in degree $> 0$ are exact[2]. More precisely:

$$
\begin{aligned}
H^p(d) &= 0 & p > 0 \\
H^p(d) &\simeq \mathbb{R} & p = 0.
\end{aligned}
\tag{3.10}
$$

**Proof**:

Assume the coefficients $w_{i_1 \ldots i_p}$ to be polynomial in $x^i$ (this restriction is not necessary and is made only to simplify the discussion). Let:

$$
d = dx^i \frac{\partial}{\partial x^i}, \quad i_x = x^i \frac{\partial^L}{\partial (dx^i)} \ (\equiv i_X, \ X^j = x^j).
\tag{3.11}
$$

Define the counting operator $N$ as the combination[3] $N = d i_X + i_X d$. It is such that $N x^i = x^i$, $N dx^i = dx^i$. For a general form $\alpha$ we have $N\alpha = k\alpha$ with $k$ the total polynomial degree (in $x^i$ and $dx^i$). It then follows that for:

- $k \neq 0$:

$$
\alpha = \frac{k}{k}\alpha = N\left(\frac{1}{k}\alpha\right) = (d i_x + i_x d)\left(\frac{1}{k}\alpha\right).
$$

  Thus, if $d\alpha = 0$ then $\alpha = d(i_x\left(\frac{1}{k}\alpha\right))$.

- $k = 0$: Im $d_{-1}$ has no meaning, there is no such a thing as a $(-1)$-form. We can say that $\Omega^{-1}(M)$ is empty and $H^0(d) = \text{Ker } d_0$. So constants are the only members of the cohomology.

This proves the Poincaré lemma.

## 3.4 Local functions

A local function $f$ is a smooth function of the spacetime coordinates, the field variables and their respective derivatives up to a finite order, $f = f(x, [\varphi]) = f(x^\mu, \varphi^i, \partial_\mu \varphi^i, \cdots, \partial_{\mu_1 \cdots \mu_k} \varphi^i)$. In field theory local functions are usually polynomial in the derivatives. The following discussion is however more general than that, and it remains valid in the case of arbitrary smooth local functions.

The Euler-Lagrange derivative $\frac{\delta}{\delta \varphi^i}$ of a local function $f$ is defined by

$$
\frac{\delta f}{\delta \varphi^i} = \sum_{k \geq 0} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \frac{\partial f}{\partial (\partial_{\mu_1} \ldots \partial_{\mu_k} \varphi^i)}
\tag{3.12}
$$

---

[2]For $M \neq \mathbb{R}^n$, closed forms might not be globally exact (although they are locally so). The Poincaré lemma fails in such a case.

[3]It is the Lie derivative of a form along $X$, $Nw = \mathcal{L}_X w$.

(with $\partial_{\mu_1} \ldots \partial_{\mu_k} \varphi^i$) the last derivative of $\varphi$ occurring in $f$).

**Theorem :** A local function is a total derivative iff it has vanishing Euler-Lagrange derivatives with respect to all fields:

$$f = \partial_\mu j^\mu \quad \Leftrightarrow \quad \frac{\delta f}{\delta \varphi^i} = 0 \quad \forall \varphi^i. \tag{3.13}$$

A proof of this theorem will not be given here but can be found for instance in [17, 18, 13] (see also references given in [13]).

## 3.5 Local differential forms

Local $p$-forms are differential forms whose coefficients are local functions:

$$w = \frac{1}{p!} w_{i_1 \ldots i_p}(x, [\varphi]) \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{3.14}$$

Consider a local $n$-form $w = f d^n x$ in $\mathbb{R}^n$; $w$ is trivially closed. It is further exact iff the function $f$ is a total derivative:

$$w = \mathrm{d}\alpha \quad \Leftrightarrow \quad f = \partial_\mu j^\mu \quad \Leftrightarrow \quad \frac{\delta f}{\delta \varphi^i} = 0 \quad \forall \varphi^i \tag{3.15}$$

## 3.6 Algebraic Poincaré lemma

The algebraic Poincaré lemma gives the cohomology of d in the algebra of local forms:

$$
\begin{aligned}
p = n : & \quad H^p(\mathrm{d}) \ \neq \ 0 \quad \text{and characterized above} \\
0 < p < n : & \quad H^p(\mathrm{d}) \ = \ 0 \\
p = 0 : & \quad H^p(\mathrm{d}) \ \simeq \ \mathbb{R}.
\end{aligned}
\tag{3.16}
$$

The distinguishing feature compared with the Poincaré lemma for ordinary exterior forms not depending on local fields is the appearance of a non-vanishing cohomology at $p = n$; non trivial local $n$-forms $w = f d^n x$ are such that at least one of the derivatives $\delta f/\delta \varphi^i$ is not identically zero. A proof of the algebraic Poincaré lemma in the case of polynomial dependence on derivatives can be found in [13]; for a more general proof see [17, 18].

**Example**. Consider local 1-forms in $\mathbb{R}^1$, $w = \mathcal{L}(t, q, \dot{q}, \ddot{q}, \ldots) \, dt$. For example:

$$w_1 = \mathcal{L}_1 \, dt = \frac{1}{2} \dot{q}^2 dt, \quad w_2 = \mathcal{L}_2 \, dt = \dot{q} q dt. \tag{3.17}$$

These are obviously closed, but are they exact? We work out the Euler-Lagrange derivatives of $\mathcal{L}_{1,2}$:

$$\frac{\delta \mathcal{L}_1}{\delta q} = -\ddot{q}, \quad \frac{\delta \mathcal{L}_2}{\delta q} = 0. \tag{3.18}$$

Therefore, from (3.15), $w_2$ is an exact form (and indeed, $w_2 = \mathrm{d}\left(\frac{1}{2}q^2\right)$), while $w_1$ cannot be written as the exterior derivative of a local function. $H^1(\mathrm{d})$ is clearly non trivial.

# 4  BRST construction

We stated in the introduction that gauge invariance is lost after the necessary gauge fixing. The central idea of the BRST construction [9, 10] is to replace the original gauge symmetry by a rigid symmetry, the BRST symmetry $s$, which is still present even after one has fixed the gauge. This is achieved by introducing extra fields in the theory: the ghost fields and the conjugate antifields. The operator $s$ acts on the enlarged space of fields, ghosts and antifields. An extended action involving all these variables can be constructed in such a way that it is BRST invariant. The operator $s$ is called BRST differential and it is nilpotent: $s^2 = 0$. Therefore, cohomological groups $H^k(s)$ can be constructed. The BRST differential fulfills:

$$H^0(s) = \text{Gauge invariant functions (``Observables'')} \tag{4.1}$$

In this way we recover the gauge symmetry. This is BRST theory in a nutshell. These important statements will now be discussed in more detail.

## 4.1  Master equation

Consider a gauge theory of fields $\varphi^i$ described by a classical action $S_0(\varphi^i)$ on a manifold $M$. The equations of motion constrain the fields to a submanifold, denoted $\Sigma$. The action is invariant under gauge transformations[4]

$$\delta_\varepsilon \varphi^i = R^i_\alpha \varepsilon^\alpha. \tag{4.2}$$

Assume the theory to be (on-shell) reducible, with no reducibility on the reducibility functions. In such a case there are relations among the gauge parameters but no relations among the relations. The relations among the gauge parameters can be written as:

$$Z^\alpha_\Delta R^i_\alpha = C^{ij}_\Delta \frac{\delta S_0}{\delta \varphi^j}. \tag{4.3}$$

We proceed as follows. For each commuting (anticommuting) gauge parameter $\varepsilon^\alpha$ one introduces a fermionic (bosonic) ghost variable $c^\alpha$. We also introduce ghosts of ghosts $c^\Delta$, one for each (independent) reducibility identity of the theory. The set of original fields, ghosts and ghosts of ghosts are collectively denoted as $\Phi^A$. We double now the configuration space by considering conjugate fields w.r.t. each of the $\Phi$' s: the anti-fields $\Phi^*_A$. They are postulated to have opposite (Grassmann) parity. Gradings are assigned to the various fields as displayed in table 1.

We define the antibracket[5] of two functionals $F(\Phi^A, \Phi^*_A)$, $G(\Phi^A, \Phi^*_A)$ by:

$$(F, G) = \frac{\delta^R F}{\delta \Phi^A} \frac{\delta^L G}{\delta \Phi^*_A} - \frac{\delta^R F}{\delta \Phi^*_A} \frac{\delta^L G}{\delta \Phi^A}. \tag{4.4}$$

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[4]We use De Witt's condensed notation; see Appendix I.

[5]Properties of anti-brackets are listed in Appendix II.

|  | puregh | antifd | gh |
|---|---|---|---|
| $\varphi^i$ | 0 | 0 | 0 |
| $c^\alpha$ | 1 | 0 | 1 |
| $c^\Delta$ | 2 | 0 | 2 |
| $\varphi^*_i$ | 0 | 1 | $-1$ |
| $c^*_\alpha$ | 0 | 2 | $-2$ |
| $c^*_\Delta$ | 0 | 3 | $-3$ |

**Table 1:** Pure ghost number, antifield number and gh $\equiv$ puregh $-$ antifd ("total ghost number"), for the different field types.

Here, $L$ (respectively $R$) refers to the standard left (respectively right) derivative. These are related as

$$\frac{\delta^R F}{\delta(\text{field})} \equiv (-1)^{\varepsilon_{\text{field}}(\varepsilon_F+1)} \frac{\delta^L F}{\delta(\text{field})}, \tag{4.5}$$

where $\varepsilon$ denotes the Grassmann-parity. The BRST transformation of any functional $F(\Phi^A, \Phi^*_A)$ can be written in terms of antibrackets:

$$sF = (S, F). \tag{4.6}$$

The generating function $S(\Phi^A, \Phi^*_A)$ of the BRST transformation is sometimes called the generalized action. The BRST transformation is nilpotent of order two; this is reflected in the (classical) master equation:

$$\boxed{(S,S)=0} \tag{4.7}$$

The solution to the master equation is unique up to canonical transformations. It can be constructed in a sequential form[6]:

$$S = S_0 + S_1 + S_2 + \dots \tag{4.8}$$

$$S_0 \equiv \text{classical action}, \quad S_1 = \varphi^*_i R^i_\alpha c^\alpha, \quad S_2 = c^*_\alpha Z^\alpha_\Delta c^\Delta + \dots$$

The proof of this statement (including the reducible case) can be found in [19, 20, 11, 12] and references therein. Locality of $S$ under general conditions is established in [21].

The solution $S$ of the master equation is key to the BRST-antifield formalism. It can be written down explicitly for the Yang-Mills theory and the abelian 2-form model introduced earlier:

**Yang-Mills.**

$$S = -\frac{1}{4} \int d^n x \ F^a_{\mu\nu} F^{\mu\nu}_a + \int d^n x \ A^{*\mu}_a D_\mu c^a + \frac{1}{2} \int d^n x \ c^*_a f^a{}_{bc} c^b c^c \tag{4.9}$$

---

[6]There is however no guarantee for this sequence to be finite!

**Abelian 2-form.**

$$S = -\frac{1}{12} \int d^n x \; H_{\mu\nu\rho} H^{\mu\nu\rho} + \int d^n x \; B^{*\mu\nu}(\partial_\mu c_\nu - \partial_\nu c_\mu) + \int d^n x \; c^{*\mu} \partial_\mu c \quad (4.10)$$

In those cases, the solution $S$ of the master equation is linear in the antifields. For gauge systems with an "open algebra" (i.e., for which the gauge transformations close only on-shell), or for on-shell reducible gauge theories, the solution of the master equation is more complicated. It contains indeed terms that are non linear in the antifields. These terms are essential for getting the correct gauge fixed action below. Without them, one would not derive the correct Feynman rules leading to gauge-independent amplitudes.

**Exercises.**

1.) Consider a nilpotent operator $\Omega$ of order $N$ (i.e. $\Omega^N \equiv 0$).

   (i) prove that the only eigenvalue of $\Omega$ is zero.
   (ii) for $N = 2$, analyze the cohomology of $\Omega$ in terms of its Jordan decomposition.
   (iii) for $N = 3$, Im $\Omega^2 \subset$ Ker $\Omega$. The corresponding cohomologies are defined as

$$H_{(1)}(\Omega) \equiv \frac{\text{Ker}\Omega}{\text{Im}\Omega^2}, \quad H_{(2)}(\Omega) \equiv \frac{\text{Ker}\Omega^2}{\text{Im}\Omega}$$

Calculate these.
Hint: The Jordan decomposition of a matrix is a block-diagonal form. Each such block, called Jordan block, has on its diagonal always the same eigenvalue and 1 in the upper secondary diagonal.

2.) Prove that $P(\varphi^i, \partial_\mu \varphi^i, \ldots, \partial_{\mu_1 \ldots \mu_k} \varphi^i) \, d^n x$ is exact iff $\frac{\delta P}{\delta \varphi^i} = 0$, where $\frac{\delta}{\delta \varphi^i}$ is the Euler-Lagrange derivative.

Hint: Relate $N = \varphi^i \frac{\partial}{\partial \varphi^i} + (\partial_\mu \varphi^i) \frac{\partial}{\partial(\partial_\mu \varphi^i)} + \ldots$   to $\frac{\delta P}{\delta \varphi^i}$

3.) Write explicitly $R^i_\alpha$, $Z^\alpha_\Delta$, $C^{ij}_\Delta$ for the Freedman-Townsend model.

4.) Write the Noether's identities (see Appendix I) for Yang-Mills, gravity and an abelian 2-form gauge theory.

5.) Check the properties of anti-brackets given in Appendix II.

6.) Consider an irreducible gauge theory with gauge transformations closing off-shell and forming a group. The solution of the master equation is given by

$$S = S_0 + \varphi^*_i R^i_\alpha c^\alpha + \frac{1}{2} c^*_\alpha f^\alpha_{\beta\gamma} c^\beta c^\gamma \quad\quad\quad (4.11)$$

where $f^\alpha_{\ \beta\gamma}$ are the structure constants of the gauge group. Verify that $S$ satisfies the master equation $(S, S) = 0$.

7.) Define the operator $\Delta$ as

$$\Delta F = (-1)^{\varepsilon_A} \frac{\delta^L}{\delta\phi^A} \frac{\delta^L F}{\delta\phi^*_A}. \tag{4.12}$$

Prove the following statements:

   (i) $\varepsilon(\Delta) = 1$
   (ii) $\Delta^2 = 0$
   (iii) $\mathrm{gh}\,\Delta = 1$
   (iv) $\Delta(\alpha, \beta) = (\Delta\alpha, \beta) - (\alpha, \Delta\beta)(-1)^{\varepsilon_\alpha}$
   (v) $\Delta(\alpha\beta) = (\Delta\alpha)\beta + (-1)^{\varepsilon_\alpha}\alpha(\Delta\beta) + (-1)^{\varepsilon_\alpha}(\alpha, \beta)$

Verify that the superjacobian for the change of variables $\phi^A \to \phi'^A + (\mu S, \phi^A)$ is $1 - (\Delta S)\mu$, where $\mu$ is a fermionic constant.

# 5   Observables

It is now time to substantiate the claim

$$H^0(s) \simeq \text{Observables} \tag{5.1}$$

where, as we have just seen, the BRST differential is given by $sF = (S, F)$. In particular,

$$s\Phi^A \;=\; (S, \Phi^A) = -\frac{\delta^R S}{\delta\Phi^*_A} = -\sum_k \frac{\delta^R S_k}{\delta\Phi^*_A} \tag{5.2}$$

$$s\Phi^*_A \;=\; (S, \Phi^*_A) = \frac{\delta^R S}{\delta\Phi^A} = \sum_k \frac{\delta^R S_k}{\delta\Phi^A}. \tag{5.3}$$

Note that the ghost number of $S$ is 0, the ghost number of the BRST transformation is 1 as well as the one of the antibracket, so that the gradings of both sides of the equation $sF = (S, F)$ match. We shall actually not provide the detailed proof of (5.1) here, but instead, we give only the key ingredients that underlie it, referring again to [20, 11, 12] for more information.

To that end, we expand the BRST transformations of all the variables according to the antifield number, as in [22, 20, 11, 12]. So one has,

$$S \;=\; \sum_{k\geq 0} S_k \tag{5.4}$$

$$s \;=\; \delta + \gamma + \sum_{i>0} s_i. \tag{5.5}$$

The first term $\delta$ has antifield number -1 and is called the Koszul-Tate differential, the second term $\gamma$ has antifield number 0 and is called the longitudinal differential, and the next terms $s_i$ have antifield number $i$. Although the expansion stops at $\delta + \gamma$ for Yang-Mills (as it follows from the solution of the master equation given above), higher order terms are present for gauge theories with an open algebra, or on-shell reducible theories[7].

Explicitly, one finds for the Koszul-Tate differential $\delta\varphi^i = 0$, as there is no field operator of anti-field number -1 and $\delta\varphi_i^* = \delta S_0/\delta\varphi^i$ . For the longitudinal differential it follows in the same way that

$$\gamma\varphi^i = R_\alpha^i c^\alpha. \tag{5.6}$$

Observe also that

$$0 = s^2 = \delta^2 + \{\delta, \gamma\} + (\gamma^2 + \{\delta, s_1\}) + ... \tag{5.7}$$

In this equation, each term has to vanish separately, as each term is of different antifield number.

Let $A$ be a BRST-closed function(al), $sA = 0$. We must compute the equivalence class

$$A \sim A + sB. \tag{5.8}$$

Since we are dealing with observables, the only relevant operators are of of ghost number 0, thus $\mathrm{gh}A = 0$ and $\mathrm{gh}B = -1$. The latter can only be satisfied, if $B$ contains at least one anti-field. Expanding $A$ and $B$ in antifield number yields

$$A = \sum_{k \geq 0} A_k = A_0 + A_1 + A_2 + ... \tag{5.9}$$

$$B = \sum_{k \geq 1} B_k = B_1 + B_2 + B_3 + ... \tag{5.10}$$

Acting with $s$ on $A$ using the expansion (5.5) then gives

$$(\delta + \gamma + ...)(A_0 + A_1 + ...) = (\gamma A_0 + \delta A_1) + ..., \tag{5.11}$$

where the term in parentheses collects all antifield number zero contributions. The condition $sA = 0$ implies that this term must vanish on its own and thus $\gamma A_0 = -\delta A_1$. Furthermore, one finds

$$A + sB = A + (\delta + \gamma + ...)B = (A_0 + \delta B_1) + ... \tag{5.12}$$

where the last term in parentheses is again the antifield zero contribution. Using (5.3),

$$\delta B_1 = \frac{\delta B_1}{\delta \Phi^*} \frac{\delta S_0}{\delta \Phi^i}, \tag{5.13}$$

we see that the second term of the antifield zero contribution in $A + sB$ vanishes when the equations of motions are fulfilled, i.e. on-shell. A similar property holds

---

[7]The expansion of $s$ is connected to spectral sequences, which will not be discussed in detail here [12].

for $\delta A_1$. There is therefore a clear connection of $\delta$ to the dynamics and the equations of motions. Note that the "on-shell functions" can be viewed as the equivalence classes of functions on $M$ identified when they coincide on $\Sigma$, i.e., $C^\infty(\Sigma) = C^\infty(M)/\mathcal{N}$, where the ideal $\mathcal{N}$ contains all the functions that vanish on-shell.

¿From (5.6) and the fact that $\gamma$ is a derivation, one gets for[8] $\gamma A_0$

$$\gamma A_0 = \frac{\delta A_0}{\delta \varphi^i} R^i_\alpha c^\alpha. \tag{5.14}$$

As this is a gauge transformation (with gauge parameters replaced by the ghosts), $\gamma A_0$ vanishes if $A_0$ is gauge-invariant. The longitudinal differential is associated with gauge transformations. We thus see that a necessary condition for $A$ to be BRST-closed is that its first term $A_0$ be gauge-invariant on-shell. And furthermore, two such $A_0$'s are equivalent when they coincide on-shell.

It turns out that the condition on $A_0$ is also sufficient for $A$ to be BRST-closed, in the sense that given an $A_0$ that is gauge-invariant on-shell, one can complete it by terms $A_1$, $A_2$ ... of higher antifield number so that $sA = 0$.

To summarize: the term that determines the cohomological class of a BRST cocycle is the first term $A_0$. This term must be an observable, in that it must be gauge invariant on-shell. We can therefore conclude that $H^0(s)$ captures indeed the concept of observables. The differential $\delta$ reduces from the manifold $M$ to the on-shell manifold $\Sigma$ and $\gamma$ further to $\Sigma/G$, the set of all gauge-invariant functions, where $G$ is the set of all gauge orbits.

# 6 Path Integral and Gauge-fixing

We first consider the Yang-Mills case. To perform actual path-integral calculations, it is necessary to gauge-fix the theory. To perform this task, it is convenient to add additional fields, the anti-ghost $\bar{c}_a$ and auxiliary fields, the Nakanishi-Lautrup fields $b_a$. They transform as $s\bar{c}_a \sim b_a$ and $sb_a = 0$. We take $\bar{c}_a$ and $b_a$ to have ghost number -1 and 0, respectively. The corresponding antifields $\bar{c}^{a*}$ and $b^{a*}$ have thus ghost number 0 and -1, respectively. Furthermore a contracting homotopy argument similar to the one given above for the Poincaré lemma shows that the counting operator of $\bar{c}_a$, $b_a$ and their conjugate antifields is BRST exact. Hence the cohomology is not altered by the introduction of these new variables. In particular, the set of observables is not affected. The solution of the master equation with the new variables included reads, for Yang-Mills theory

$$
\begin{aligned}
S &= -\frac{1}{4} \int d^n x F^a_{\mu\nu} F^{\mu\nu}_a + \int d^n x A^{\mu*}_a D_\mu c^a \\
&\quad + \frac{1}{2} \int d^n x c^*_a f^a_{bc} c^b c^c - i \int d^n x \bar{c}^{*a} b_a.
\end{aligned} \tag{6.1}
$$

The last term is called the non-minimal part.

---

[8] $A_0$ can only depend on $\phi_i$, as it has pure ghost and antifield number 0.

**Theorem**
*The generating functional*

$$\mathcal{Z} = \int \mathcal{D}\Phi^A \exp\left(\frac{i}{\hbar}S_\psi[\Phi^A]\right), \qquad (6.2)$$

*does not depend on the choice of $\psi$. Here, $\psi$ is called the gauge-fixing fermion, and has Grassmann-parity 1 (hence its name) and ghost number -1. In (6.2), the notation*

$$\Phi^a = (A_\mu^a, c^a, \bar{c}^a, b_a) \qquad (6.3)$$

$$\Phi_a^* = (A_\mu^{a*}, c^{a*}, \bar{c}^{a*}, b^{a*}), \qquad (6.4)$$

*has been used and the "gauge-fixed action" $S_\psi[\Phi^A]$ is given by*

$$S_\psi[\Phi^A] = S\left[\Phi^A, \Phi_A^* = \frac{\delta\psi}{\delta\Phi^A}\right]. \qquad (6.5)$$

This theorem is proved in section 8 below.

Before turning to the proof, we want to illustrate formula (6.2) by showing how one can choose the gauge-fixing fermion $\psi$ to reproduce familiar expressions for the path integral of the Yang-Mills field. A possible choice, which leads to non-degenerate propagators for all fields and ghosts, is given by

$$\psi = i\int d^n x\, \bar{c}^a\left(\mathcal{F}^a + \frac{\alpha}{2}b^a\right), \qquad (6.6)$$

where $\mathcal{F}^a$ is the gauge condition, e.g. $\mathcal{F}^a = \partial^\mu A_\mu^a$ for covariant gauges and $\alpha$ is the gauge parameter. This leads to

$$\bar{c}^{a*} = \frac{\delta\psi}{\delta\bar{c}_a} = i\left(\mathcal{F}^a + \frac{\alpha}{2}b^a\right) \qquad (6.7)$$

$$A_{a\mu}^* = \frac{\delta\psi}{\delta A_\mu^a} = -i\partial_\mu\bar{c}_a \qquad (6.8)$$

$$b^{a*} = \frac{\delta\psi}{\delta b_a} = i\frac{\alpha}{2}\bar{c}^a \qquad (6.9)$$

$$c_a^* = \frac{\delta\psi}{\delta c^a} = 0. \qquad (6.10)$$

One then gets the familiar gauge-fixed Yang-Mills action

$$S_\psi[A_\mu^a, c^a, \bar{c}_a, b_a]$$
$$= \int d^n x\left(-\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - i\partial^\mu\bar{c}_a D_\mu c^a + \left(\mathcal{F}^a + \frac{\alpha}{2}b^a\right)b_a\right), \qquad (6.11)$$

which is usually obtained by the Fadeev-Popov procedure [23]. The conventional Landau gauge is recovered by setting $\alpha = 0$. As the resulting path integral can be written as

$$\int \mathcal{D}[A_\mu^a \bar{c}^a c^a]\delta(\partial^\mu A_\mu^a)e^{\left(\frac{i}{\hbar}S_{gf}[A_\mu^a, \bar{c}^a, c^a]\right)}, \qquad (6.12)$$

the transversality of the gauge boson is directly implemented. Landau gauge is thus called a strict gauge. An example of a non-strict gauge is Feynman gauge, $\alpha = 1$, in which case the equations of motion does not imply $\mathcal{F}^a = 0$ but yield instead $b^a \sim \mathcal{F}^a$.

The choice of the gauge-fixing fermion is not unique. One can add to $\psi$ the term $\psi \sim \bar{c}_a \bar{c}_b c^c$, which yields quartic ghost couplings. Quartic ghost renormalizations may even be needed without such explicit terms, e.g. when using $\mathcal{F}^a = \partial^\mu A_\mu^a + d^a{}_{bc} A_\mu^b A^{\mu c}$, where $d^a{}_{bc}$ is a symmetric tensor in color space (see [24, 25]).

By appropriately choosing the gauge fixing fermion, one can reduce the path integral to an expression that involves only the physical (transverse) degrees of freedom and which is manifestly unitary in the physical subspace (equal to the reduced phase space path integral). Independence on the choice of $\psi$ (still to be proved) guarantees then that the expression (6.2) is correct. We shall not demonstrate here the equivalence of (6.2) with the reduced phase space path integral. The reader may find a discussion of that point in [12].

As a final point, we note that in order for (6.2) to be indeed independent on the choice of $\psi$, it is necessary that the measure be BRST invariant. This can be investigated using the operator $\Delta$, already defined in the exercises in (4.12) as

$$\Delta = (-1)^{\epsilon_A} \frac{\delta^L}{\delta \Phi^A} \frac{\delta^L}{\delta \Phi_A^*}. \tag{6.13}$$

The BRST transformation can be written as

$$\Phi^A \to \Phi^{A\prime} = \Phi^A + (\mu S, \Phi^A) = \Phi^A - (\Phi^A, S)\mu = \Phi^A - \frac{\delta^L S}{\delta \Phi_A^*}\mu, \tag{6.14}$$

where $\mu$ is a constant, anti-commuting parameter. The Jacobian of this transformation is given by

$$J_{AB} = \frac{\delta^L \Phi^{A\prime}}{\delta \Phi^B} = \delta^{AB} - \frac{\delta^L}{\delta \Phi^B} \frac{\delta^L S}{\delta \Phi_A^*}\mu. \tag{6.15}$$

[As the Jacobian involves commuting and anti-commuting fields, the Jacobian "determinant" is actually a super-determinant.] Therefore the measure transforms as

$$\mathcal{D}\Phi^A \to \mathrm{sdet} J \, \mathcal{D}\Phi^{A\prime}. \tag{6.16}$$

For an infinitesimal transformation, the super-determinant can be approximated by the super-trace

$$\mathrm{sdet} J \approx 1 + \mathrm{str}\left(-\frac{\delta^L \delta^L S}{\delta \Phi_B \delta \Phi^{A*}}\mu\right) = 1 - (-1)^{\epsilon_A} \frac{\delta^L \delta^L S}{\delta \Phi_A \delta \Phi^{A*}}\mu = 1 + \Delta S. \tag{6.17}$$

It follows that the measure is BRST-invariant iff $\Delta S = 0$. The property $\Delta S = 0$ can be shown by explicit calculation for pure Yang-Mills theory [26]. The more general case will be treated in the last section. Further interesting properties of $\Delta$ and of the formalism are developed in [27].

**Exercises.**

8.) For Yang-Mills theory in 4 dimensions, compute the dimensionality of all fields. Is there any freedom? What is the most general gauge fixing fermion $\psi$ of mass dimension 3? What are the restrictions on the gauge-fixing fermion, if the action is required to be invariant under the transformation $\bar{c}^a \rightarrow \bar{c}^a + \epsilon^a$?

9.) (a) Show that for a functional $W$ with $\mathrm{gh}W = 0$ and Grassmann-parity 0

$$\Delta e^{\frac{i}{\hbar}W} = 0 \Leftrightarrow \frac{1}{2}(W,W) - i\hbar\Delta W = 0, \tag{6.18}$$

where $\Delta$ is the operator defined by (6.13).

(b) Define the operator $\sigma$ as

$$\sigma\alpha \equiv (W,\alpha) - i\hbar\alpha, \tag{6.19}$$

where $W$ satisfies (6.18). Then show that

$$\sigma\alpha = 0 \Leftrightarrow \Delta(\alpha e^{\frac{i}{\hbar}W}) = 0. \tag{6.20}$$

(c) Show that $\sigma$ is nilpotent, $\sigma^2 = 0$.

(d) Show that if $\alpha = \sigma\beta$, then $\alpha\exp iW/\hbar = \Delta(\text{something})$.

10.) Write explicitly the action of $s$, $\delta$, and $\gamma$ on all fields and antifields for Yang-Mills theory.

11.) For Yang-Mills theory, consider $H(\gamma)$ in the space of polynomials in the ghost fields $c^a$, i.e. $a_0 + a_a c^a + a_{ab}c^a c^b + ....$ Compute $H^0(\gamma)$ and $H^1(\gamma)$. Show in particular that $H^2(\gamma)$ parameterizes the non-trivial central extensions $h_{ab}$, i.e., non trivial modifications of the algebra of the form $[X_a, X_b] = f^a{}_{bc}X_a + h_{ab}1$.

# 7  Beyond Yang-Mills

The results of the previous section generalize straightforwardly to gauge theories other than Yang-Mills. The solution of the master equation $S$ is of the form

$$S = S_0 + \phi_i^* R_\alpha^i c^\alpha + ..., \tag{7.1}$$

where $S_0$ is the classical action. The second term is uniquely determined by the gauge transformation (4.2), and all further terms depend on the specific theory. While the expansion of the solution of the master equation stops at antifield number one in the Yang-Mills case, one gets higher order terms in the case of open gauge systems. It is again often convenient to extend to the non-minimal sector by introducing $\bar{c}_\alpha$ and $b_\alpha$ and the corresponding antifields in a similar manner to what has been done in the case of Yang-Mills theory.

Assuming again $\Delta S = 0$, the quantized theory follows from a path integral

$$\mathcal{Z} = \int \mathcal{D}\Phi^A \exp\left(\frac{i}{\hbar}S_\psi[\Phi^A]\right). \tag{7.2}$$

$S_\psi$ is the solution $S$ of the master equation $(S,S) = 0$, in which the antifields have been eliminated by use of the gauge-fixing fermion $\psi$ as before,

$$S_\psi[\Phi^A] = S_\psi\left[\Phi^A, \Phi_A^* = \frac{\delta\psi}{\delta\Phi^A}\right].\tag{7.3}$$

The gauge-fixing fermion has again odd Grassmann-parity and ghost number -1. It is in general given by a local expression

$$\psi = \int d^n x \chi(\Phi^A, \partial_\mu\Phi^A, ..., \partial_{\mu_1}...\partial_{\mu_k}\Phi^A).\tag{7.4}$$

For theories with an open algebra, the terms quadratic in the antifields will lead to quartic (or higher) ghost-antighost vertices in the gauge-fixed action. While these terms are a gauge-dependent option in the Yang-Mills case, they have an unavoidable character (in relativistic gauges) for open gauge algebras. These terms, which follow directly from the general construction of the gauge-fixed action, cannot be obtained through the exponentiation of a determinant, since this procedure always produces an expression which is quadratic in the ghosts.

A useful concept is that of gauge-fixed BRST transformation, which is what becomes $s$ after gauge-fixing. It is denoted by $s_\psi$ and defined as

$$s_\psi\Phi^A = (s\Phi^A)|_{\Phi_A^* = \delta\psi/\delta\Phi^A}.\tag{7.5}$$

Note that $s_\psi^2 = 0$ is in general only valid on (gauge-fixed) shell, i.e. for field configurations satisfying $\delta S_\psi/\delta\Phi^A = 0$.

If $S$ is linear in the antifields, i.e. if the gauge algebra closes off-shell, $s_\psi\Phi^A = s\Phi^A$. This follows directly from

$$s\Phi^A = (S, \Phi^A) = -\frac{\delta^R S}{\delta\Phi_A^*},\tag{7.6}$$

which is independent of the antifields, if $S$ depends only linearly on the antifields. In that case $s_\psi^2 = 0$ even off-shell provided one keeps all the variables. This is the case in Yang-Mills theory. [In Yang-Mills theory, one often eliminates the auxiliary fields $b_a$ by means of their own equations of motion. One then looses off-shell nilpotency on the antighosts, for which $s\bar{c}_a \sim b_a$, even though $s_\psi^2 = 0$ is true off-shell beforehand.]

If $S$ is linear in the antifields, one may in fact write it as

$$S = S_0 - (s\Phi^A)\Phi_A^*,\tag{7.7}$$

by virtue of (7.6). The gauge fixed version is then

$$S_\psi = S_0 - (s_\psi\Phi^A)\frac{\delta\psi}{\delta\Phi^A} = S_0 - s_\psi\psi.\tag{7.8}$$

Therefore $s_\psi S_\psi = 0$, as the first term is BRST invariant, and the second term is annihilated by $s_\psi$ by virtue of $s_\psi^2 = 0$, which holds off-shell when $S$ is linear in the antifields as we have just pointed out. The property $s_\psi S_\psi = 0$ is actually quite

general and holds even when $S$ is not linear in the antifields. It can be proved directly as follows,

$$s_\psi S_\psi = (s_\psi \Phi^A) \frac{\delta^L S_\psi}{\delta \Phi^A} = -\frac{\delta^R S}{\delta \Phi_A^*} \frac{\delta^L S_\psi}{\delta \Phi^A}. \tag{7.9}$$

The left-derivative in this expression is a total derivative, as $S_\psi$ depends on $\Phi_A$ directly and though the gauge-fixing fermion. Using the chain rule, this yields by virtue of (7.3)

$$-\frac{\delta^R S}{\delta \Phi_A^*} \left( \frac{\delta S}{\delta \Phi^A} + \frac{\delta^2 \psi}{\delta \Phi^A \delta \Phi^B} \frac{\delta S}{\delta \Phi_B^*} \right) = 0. \tag{7.10}$$

The second term vanishes because the product of the functional derivatives of $S$ have a symmetry in $(A, B)$ opposite to that of the second functional derivative of $\psi$. The first term vanishes by the master equation, thus proving the claim.

Further information on the gauge-fixed action and the gauge-fixed cohomology can be found in [28, 29, 30].

# 8 Quantum Master Equation

In order to prove gauge independence of the expressions given above, it is necessary to discuss two important features of the path integral.

- Assume that a theory of fields $\chi^\alpha$ is given, governed by the action $S[\chi^\alpha]$, with no gauge invariance (this could be the gauge-fixed action). Expectation values are, after proper normalization, calculated as

$$< F > = \int \mathcal{D}\chi F \exp\left( \frac{i}{\hbar} S[\chi] \right). \tag{8.1}$$

The Dyson-Schwinger equations can be directly derived from the vanishing of the path integral of a total derivative,

$$\int \mathcal{D}\chi \frac{\delta}{\delta \chi^\alpha} \left( F e^{\frac{i}{\hbar} S} \right) = 0 \tag{8.2}$$

(which is itself a consequence of translation invariance of the measure). This leads to

$$\left\langle \frac{\delta F}{\delta \chi^\alpha} + \frac{i}{\hbar} F \frac{\delta S}{\delta \chi^\alpha} \right\rangle = 0, \tag{8.3}$$

which is equivalent to

$$\left\langle F \frac{\delta S}{\delta \chi^\alpha} \right\rangle = i\hbar \left\langle \frac{\delta F}{\delta \chi^\alpha} \right\rangle. \tag{8.4}$$

This expression contains in the l.h.s. the expectation values of the classical equations of motions. In the classical limit $\hbar \to 0$, the r.h.s. vanishes and the classical equations of motion hold.

- There is another aspect that we shall have to take into account. If $\chi^\alpha$ is changed under a transformation, $\chi^\alpha \to \chi^\alpha + \epsilon^\alpha$, where $\epsilon^\alpha$ depends on the fields, the expectation value (8.1) is in general not invariant. Furthermore, invariance of the classical action is not sufficient to guarantee that the path integral is invariant. One needs also invariance of the measure.

  We shall be concerned with BRST invariance of the path integral constructed above. We have seen that the gauge-fixed action is BRST invariant. But the measure might not be. If it is not, one may, in some cases, restore invariance by taking a different measure. [The measure is in fact dictated by unitarity and may indeed not be equal to the trivial measure $\mathcal{D}\Phi$.] Invariance is quite crucial in the case of BRST symmetry, since it is BRST symmetry that guarantees gauge-independence of the results. The non-trivial measure terms can be exponentiated in the action. Since there is an overall $(1/\hbar)$ in front of $S$, the measure terms appear as quantum corrections to $S$. So, one replaces the classical action by a "quantum action"

$$W = S + \hbar M_1 + \hbar^2 M_2 + ..., \tag{8.5}$$

  where the functionals $M_i$ stem from non-trivial measure factors. The theorem proved below states that quantum averages are gauge-independent if the master equation is replaced by the "quantum master equation"

$$\frac{1}{2}(W, W) = i\hbar \Delta W, \tag{8.6}$$

  where $\Delta$ is defined in (6.13). Note that if $\Delta S = 0$, the Jacobian is unity for the BRST transformation and $W$ might then be taken equal to $S$. The quantum master equation reduces to the classical master equation considered above, which is solved by $S$. While there is always a solution to the classical master equation, the solution to the quantum master equation might get obstructed. We shall investigate this question below. For the moment, we assume that there is no obstruction.

We can now state the correct, general rules, for computing expectation values of observables (including 1): these are the quantum averages, weighted by $\exp(\frac{i}{\hbar}W)$, of the BRST observables corrected by the addition of appropriate $\hbar$ (and possibly also higher) order terms. Namely, consider a classical observable $A_0$. Construct its (in fact, one of its) BRST-invariant extension $A = A_0 +$ ghost terms, so that $(S, A) = 0$. The BRST cocycle $A$ has to be augmented as

$$A \to \alpha = A_0 + \hbar B_1 + \hbar^2 B_2 + ... \tag{8.7}$$

where the terms of order $\hbar$ and higher must be such that $\sigma\alpha = 0$, where $\sigma$ was defined in the exercises as

$$\sigma\alpha \equiv (W, \alpha) - i\hbar\Delta\alpha, \tag{8.8}$$

with $W$ the solution of the quantum master equation. The operator $\sigma$ is the quantum generalization of $s$. The $\psi$-independent expectation value $< A_0 >$ of the

observable $A_0[\phi^i]$ is computed from $\alpha$ as

$$< A_0 >= \int \mathcal{D}\Phi^A \alpha \left( \Phi^A, \Phi_A^* = \frac{\delta\psi}{\delta\Phi^A} \right) \exp \left( \frac{i}{\hbar} W \left[ \Phi^A, \Phi_A^* = \frac{\delta\psi}{\delta\Phi^A} \right] \right). \quad (8.9)$$

The claim is that this expectation value does not depend on the choice of the gauge-fixing fermion

$$\psi' = \psi + \delta\psi \equiv \psi + \mu \quad (8.10)$$

$$< A_0 >_{\psi'} =< A_0 >_{\psi}, \quad (8.11)$$

where $\mu$ is an arbitrary modification of $\psi$.

To prove the claim, we denote the argument of the integral by $V$ for convenience in the following. It has been already shown in the exercises that

$$\Delta V = 0 \Leftrightarrow \sigma\alpha = 0. \quad (8.12)$$

Now, perform the variation of the gauge fixing functional (8.10). The variation of the quantum average (8.9) is equal to

$$\int \mathcal{D}\Phi \frac{\delta^L \mu}{\delta\Phi^A} \frac{\delta^L V}{\delta\Phi_A^*}. \quad (8.13)$$

To evaluate this expression, note that

$$\frac{\delta^L}{\delta\Phi^A} \left( \mu \frac{\delta^L V}{\delta\Phi_A^*} \right) = \frac{\delta^L \mu}{\delta\Phi^A} \frac{\delta^L V}{\delta\Phi_A^*} + (-1)^{\epsilon_A} \frac{\delta^L \delta^L V}{\delta\Phi^A \delta\Phi_A^*}. \quad (8.14)$$

The derivatives are total ones. Denoting partial derivatives by a prime ', the last term can be rewritten as

$$\frac{\delta^L \delta^L V}{\delta\Phi^A \delta\Phi_A^*} = \frac{\delta'^L \delta^L V}{\delta'\Phi^A \delta\Phi_A^*} + \frac{\delta^2 \psi}{\delta\Phi^A \delta\Phi^B} \frac{\delta^L \delta^L V}{\delta\Phi_B^* \delta\Phi_A^*} = \Delta V = 0. \quad (8.15)$$

As in the case of equation (7.10), the second term vanishes by parity arguments. Thus the integral (8.13) can be rewritten as a total derivative in field space, which vanishes in view of translation invariance of the standard measure. Therefore the path integral does not get modified if one changes the gauge-fixing fermion, as claimed.

Given $A_0$, its BRST extension is determined up to a BRST exact term $sB$, see (5.8). This ambiguity can be extended to higher orders in $\hbar$ as

$$\alpha \to \alpha + \sigma\beta. \quad (8.16)$$

It has been shown in the exercises that

$$(\sigma\beta) \exp \left( \frac{i}{\hbar} W \right) \sim \Delta \left( \beta \exp \left( \frac{i}{\hbar} W \right) \right). \quad (8.17)$$

As the r.h.s is a total derivative by its definition, (6.13), the path-integral over the l.h.s. vanishes

$$\int \mathcal{D}\Phi \sigma\beta \exp \left( \frac{i}{\hbar} W \right) = 0. \quad (8.18)$$

Therefore adding any element in the image of $\sigma$ does not alter the quantum averages. The path integral associates a unique answer to a given cohomological class of $\sigma$, i.e., does not depend on the choice of representative.

Note that the ambiguity in $\alpha$, given $A_0$, is more than just adding a $\sigma$-trivial term to $\alpha$. At each order in $\hbar$ one may add a non trivial new observable since this does not modify the classical limit. This addition is relevant, in the sense that it changes the expectation value by terms of order $\hbar$ or higher. This is an unavoidable quantum ambiguity. A similar ambiguity exists for the quantum measure (i.e., the $M_1$, $M_2$ etc. terms in $W$). These terms do not spoil BRST invariance and must be determined by other criteria, e.g., by comparison with the Hamiltonian formalism.

# 9 Anomalies

We close this brief survey by analyzing the possible obstructions to the existence of a solution $W$ to the quantum master equation. This leads to the important concept of anomalies. The fact that anomalies in the Batalin-Vilkovisky formalism appear as an incurable violation of the BRST invariance of the measure was first investigated in [31].

Analyzing the obstructions to the existence of a solution to the quantum master equation can be done most easily from a direct $\hbar$ expansion. To order $\hbar^0$ one gets from the quantum master equation

$$\frac{1}{2}(S, S) = 0, \tag{9.1}$$

which is the classical master equation. This equation is certainly fulfilled, since there is no obstruction to the existence of $S$.

To the next order $\hbar$, the quantum master equation yields

$$sM_1 = (S, M_1) = i\Delta S. \tag{9.2}$$

Given the $\hbar^0$-term $S$, this equation has a solution for $M_1$ if $s\Delta S = (S, \Delta S) = 0$. This condition is necessary but in general not sufficient (see below). To prove that the condition $(S, \Delta S) = 0$ holds, we note that

$$\Delta(\alpha, \beta) = (\Delta\alpha, \beta) - (-1)^{\epsilon_\alpha}(\alpha, \Delta\beta), \tag{9.3}$$

as was proven in the exercises. This property uses $\Delta^2 = 0$ and the generalized Leibniz rule

$$\Delta(\alpha\beta) = (\Delta\alpha)\beta + (-1)^{\epsilon_\alpha}\alpha\Delta\beta + (-1)^{\epsilon_\alpha}(\alpha, \beta). \tag{9.4}$$

For $\alpha = \beta = S$, the l.h.s of (9.3) vanishes by $(S, S) = 0$. In view of the gradings of $S$, the r.h.s yields $2(\Delta S, S)$. Thus, $(\Delta S, S) = 0$, that is, $\Delta S$ is closed, as requested.

This does not imply that $\Delta S$ is exact, however, unless the cohomological group $H^1(s)$ vanishes at ghost number one (recall that $\Delta S$ has ghost number one). But (9.2) requires $\Delta S$ to be exact. If $\Delta S$ is not exact, there is no $M_1$ and therefore, no way to define a BRST invariant measure such that the quantum averages do not depend on the gauge fixing fermion. This presumably signals a serious pathology

of the theory. If $\Delta S$ is exact, $M_1$ exists and one can investigate the problem of existence of the next term $M_2$. One easily verifies that it is again $H^1(s)$ that measures the potential obstructions to the existence of this next term $M_2$, as well as the existence of the subsequent terms $M_3$ etc.

In particular, if one can show that $H^1(s)$ vanishes, one is guaranteed that a solution of the quantum master equation exists. If $H^1(s) \neq 0$, further work is required since one must check that one does not hit an obstruction. Note that $W$ should be a local functional (with possibly infinite coupling constants), so that the relevant space in which to compute the cohomology is that of local functionals.

The computation of the local cohomology of the BRST operator for Yang-Mills gauge theory has been carried out in [32, 33, 34], following earlier work without antifields [35, 36, 17, 37, 38, 39]. See [13] for a review.

# Acknowledgements

# A    Appendix I: De Witt's notation.

We review the De Witt's condensed notation. This notation makes it possible to write gauge transformations in a more compact form:

$$\delta_\varepsilon \varphi^i = R^i_\alpha \varepsilon^\alpha \leftrightarrow \delta_\varepsilon \varphi^i(x) = \int d^n y \, R^i_\alpha(y,x) \varepsilon^\alpha(y). \tag{A.1}$$

For example, the transformation of the Yang-Mills gauge field

$$\delta_\varepsilon A^a_\mu = D_\mu \varepsilon^a = \partial_\mu \varepsilon^a + f^a{}_{cb} A^b_\mu \varepsilon^c \tag{A.2}$$

can be written as

$$\delta_\varepsilon A^a_\mu = R^a_{\mu b} \varepsilon^b, \quad R^a_{\mu b}(x,y) = \partial_\mu \delta(x-y) \delta^a_b + f^a{}_{bc} A^b_\mu \delta(x-y). \tag{A.3}$$

The Noether's identities have the simple form:

$$\frac{\delta S_0}{\delta \varphi^i} R^i_\alpha = 0. \tag{A.4}$$

# B  Appendix II: Properties of anti-brackets.

(i) $(F,G) = -(-1)^{(\varepsilon_F+1)(\varepsilon_G+1)}(G,F)$, where $\varepsilon_F = 0$ (1) for $F$ bosonic (fermionic); the anti-bracket is symmetric if both $F$ and $G$ are bosonic, and antisymmetric otherwise.

(ii) Jacobi identitiy:
$$(-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}(F,(G,H)) + \text{cyclic permutations} = 0$$

(iii) $(FG,H) = F(G,H) + (F,H)G\,(-1)^{\varepsilon_G(\varepsilon_H+1)}$;
$(F,GH) = (F,G)H + G(F,H)\,(-1)^{\varepsilon_G(\varepsilon_F+1)}$

(iv) $\text{gh}\,((F,G)) = \text{gh}F + \text{gh}G + 1$

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