Canonical Gravity

Lectures presented at the 16th Saalburg Summer School "Fundamentals and New Methods in Theoretical Physics", August 30 - September 10, 2010, Wolfersdorf, Germany

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Abstract

We review the decomposition of Einstein's equations into a set of constraints and a set of evolution equations. We then discuss how the constraints can be implemented in a Hamiltonian formalism of General Relativity (GR). Finally, assuming time-symmetry, we present Brill waves and (Multi) Schwarzschild black hole space-times as simple, non-trivial solutions to the vacuum constraint equations.

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1. Introduction and Motivation

General Relativity is usually formulated terms of equations for the space-time metric g (Einstein's equations). As such they are not in the form of evolution equations (spacetime does not evolve, spacetime simply is). Canonical gravity is a reformulation that puts Einstein's equations into the form of a constrained Hamiltonian system. These equations then allow for a well-defined initial value problem. The purpose of this approach is twofold: First, it is efficient and natural to treat physical problems as initial value problems, for example in the evaluation of gravitational radiation in scattering processes of black holes. Second, the canonical formulation allows to apply the canonical quantization rules to gravity as advocated by Dirac.

2. Some Elements of General Relativity

For what follows, we shall use the following notion of space-time:

Definition 2.1. *A space-time (or simply spacetime) is a pair* (M, g) *consisting of*

- *a smooth (i.e.* C^∞*) 4-dimensional manifold* M*,*
- *an at least piecewise* C^2 *Lorentzian metric* g *of signature* $(-,+,+,+)$*.*

Sometimes the use of $(+,-,-,-)$ *is more suitable, e.g. when introducing spinors. We note that the requirement of existence of a global Lorentzian metric puts a topological restriction on* M*: a vanishing Euler characteristic* $\chi(M) = 0$*. This last condition is equivalent to the existence of a global nowhere vanishing vector field up to sign (i.e. a "line-field"). However we require much stronger topological conditions on* M*:*

- M *is orientable, or equivalently, allows for a globally defined volume form,*
- M *is time orientable (there is a time direction),*
- M *has no closed timelike curves, which implies that* M *can not be compact because any Lorentzian compact manifold must have closed timelike curves [1],*
- M *is connected,*
- M *is globally hyperbolic.*

This last condition is equivalent to any of the following conditions:

- *the existence of a Cauchy surface,*
- *the existence of a* $SL(2, \mathbb{C})$ *spin-structure,*
- $M \cong \mathbb{R} \times \Sigma$*, where* Σ *is a 3-dimensional oriented manifold [1].*

We shall often (but not always) make use of index-notations. The conventions are

- *Greek indices* $\in \{0, 1, 2, 3\}$*,*
- *Latin indices* $\in \{1, 2, 3\}$*.*

The indices from the beginning of the alphabet, like $\alpha, \beta, \gamma, \ldots$ *and* $a, b, c, \ldots$ *refer to orthonormal frames, those from the middle of the alphabet, like* $\lambda, \mu, \nu, \ldots$ *and* $l, m, n, \ldots$ *to coordinate frames. The relation* $:=$ *(*$=:$*) defines the left (right) hand side.*

Newton	Einstein
equations of motion	equation of geodesic
$\ddot{\vec{x}}(t) = -\vec{\nabla}\phi(\vec{x}(t))$	$\ddot{x}^{\mu}(\tau) + \Gamma^{\mu}_{\lambda\sigma}(x(\tau))\dot{x}^{\lambda}(\tau)\dot{x}^{\sigma}(\tau)$=0
Poisson's Equation	Einstein's Equations
$\Delta\phi = 4\pi G\rho$	$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$
$G = 6,67.10^{-11}m^3kg^{-1}s^{-2}$	$\kappa = 8\pi G/c^4$
ρ = mass density	$T_{\mu\nu}$=energy-momentum tensor
single elliptic equation	system of 10 coupled equations, 4 (underdetermined) elliptic and 6 (underdetermined) hyperbolic ones
boundary data required	initial and boundary data required

Table 1: Equations of motion and field equations in Newton's theory compared with those of Einstein's gravity.

The Christoffel symbols are the components of the Levi-Civita connection (torsion-free connection on a pseudo-Riemannian manifold, i.e. $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$ for any vector fields X and Y) which are uniquely determined by the metric condition for the covariant derivative $\nabla g = 0$. They are given by

$$\Gamma^{\mu}_{\lambda\sigma} = \frac{1}{2}g^{\mu\nu}(-g_{\lambda\sigma,\nu} + g_{\nu\lambda,\sigma} + g_{\nu\sigma,\lambda}), \tag{1}$$

where $_{,\nu} = \partial_\nu = \frac{\partial}{\partial x^\nu}$. If ∇ is the covariant derivative, the Riemann curvature tensor is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2}$$

The map $(X,Y) \mapsto R(X,Y)$ is an endomorphism valued (values in the Lie-algebra $gl(4,\mathbb{R})$ or $so(1,3)$ for metric connections) 2-form. The components of the Riemann tensor in coordinate basis are obtained as follows

$$g(W, R(X,Y)Z) = W^{\mu}R_{\mu\lambda\nu\sigma}Z^{\lambda}X^{\nu}Y^{\sigma}. \tag{3}$$

Because R is a Lie-algebra valued 2-form, the first two indices of $R^{\mu}_{\lambda\nu\sigma}$ are Lie-algebra indices (for the gauge group $SO(1,3)$) while the last two indices are form indices. The components of the Ricci tensor are $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ and the Ricci scalar or curvature scalar is the trace of the Ricci tensor $R = g^{\mu\nu}R_{\mu\nu}$. If $\left\{\frac{\partial}{\partial x^\mu}\right\}$

is a coordinate basis for the tangent space then the Christoffel connection takes the form

$$\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = \Gamma^\lambda_{\mu\nu} \frac{\partial}{\partial x^\lambda}. \tag{4}$$

With the help of the last identity the Riemann tensor can be expressed as

$$R^\mu_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\lambda\kappa}\Gamma^\kappa_{\sigma\nu} - \Gamma^\mu_{\sigma\kappa}\Gamma^\kappa_{\lambda\nu}, \tag{5}$$

where the first two terms contain second order partial derivatives in the metric linearly and the last two terms contain quadratic products of first order partial derivatives. Similarly the Ricci tensor reads

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\kappa}\Gamma^\kappa_{\nu\mu} - \Gamma^\lambda_{\nu\kappa}\Gamma^\kappa_{\mu\lambda}. \tag{6}$$

Einstein's equations take the following form

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{7}$$

They are quasi-linear partial differential equations of second order. Schematically they take the form $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \partial^2 g - (\partial g)^2$ while the Christoffel connection has the structure $\Gamma = g^{-1}(-\partial g + \partial g + \partial g)$. On the other hand the energy-momentum tensor is

$$T^{\mu\nu} = \begin{pmatrix} W & \frac{1}{c} S^m \\ c g^m & t^{mn} \end{pmatrix}, \tag{8}$$

where W is the energy density, $\vec{S}$ is the energy current density, $\vec{g}$ is the momentum density and $\underline{t}$ is the momentum current density. Since the energy momentum tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, we have $\vec{S} = c^2 \vec{g}$ and $t^{mn} = t^{nm}$. The number of independent components of $T^{\mu\nu}$ is $1 + 3 + 6 = 10$. We note the physical dimensions (denoted by square brackets) of the quantities in Einstein's equations: $[T^{\mu\nu}] = kg \cdot m^{-1} \cdot s^{-2}$, $[\kappa] = s^2 \cdot kg^{-1} \cdot m^{-1}$ and $[R_{\mu\nu}] = m^{-2}$. The symmetries of the curvature tensor are such that it is antisymmetric under exchange of indices in the first pair, the second pair, and symmetric under slotwise exchange of the first pair with the second pair:

$$R_{\lambda\sigma\mu\nu} = -R_{\lambda\sigma\nu\mu} = -R_{\sigma\lambda\mu\nu} = R_{\mu\nu\lambda\sigma}\,. \tag{9}$$

Antisymmetry in the last pair stems from the very definition of the curvature tensor as endomorphism-valued two form. Antisymmetry in the first pair is a result of the requirement that the connection be metric preserving. This implies that the endomorphisms in which the curvature two-form takes its values must be contained in $SO(1,3) \subset gl(4,\mathbb{R})$. The last symmetry is a consequence of these two and the first Bianchi identity, to be discussed next.

2.1. First Bianchi Identity

For torsion-free connections, the first Bianchi identity reads

$$R_{\lambda[\sigma\mu\nu]} = 0\,. \tag{10}$$

Given the antisymmetry under exchange in the first and second index pair, it is easy to check that this equation is identically satisfied whenever at least two indices coincide. Hence it provides $\binom{n}{4}$ independent conditions (in $n \geq 4$ dimensions, and zero in less than 4 dimensions), namely one for each combination where no two indices $\lambda, \sigma, \mu, \nu$ coincide. We can now calculate the number of independent components. For this we regard $R_{\lambda\sigma\;\mu\nu}$ as a symmetric (in $(\lambda\sigma) \leftrightarrow (\mu\nu)$) bilinear form on the space of 2-forms. The linear space of bilinear forms on a vector space of dimension N is $\frac{1}{2}N(N+1)$ dimensional, where here $N = \frac{1}{2}n(n-1)$ is the dimensionality of the vector space of two forms on a tangent space of dimension n. From that the $\binom{n}{4}$ conditions from the non-trivial Bianchi identities are to be substracted if $n \geq 4$. The total number of independent components is then $\frac{1}{12}n^2(n^2-1)$ for $n \geq 4$, which is 20 in 4 spacetime dimensions. In 3 or 2 spacetime dimensions the number of independent components is 6 and 1 respectively. The Ricci tensor $R_{\mu\nu}$ has 10 independent components. Hence 10 components of the Riemann curvature tensor are determined by the matter content while the other 10 components remain free.

The Riemann tensor can be decomposed as follows:

$$\begin{aligned} R_{\lambda\sigma\mu\nu} = C_{\lambda\sigma\mu\nu} &+ \frac{1}{n-2}(g_{\lambda\mu}R_{\sigma\nu} + g_{\sigma\nu}R_{\lambda\mu} - g_{\lambda\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\lambda\nu}) \\ &- \frac{R}{(n-1)(n-2)}(g_{\lambda\mu}g_{\sigma\nu} - g_{\lambda\nu}g_{\sigma\mu}), \end{aligned}$$

where C is the totally trace-free part part of the Riemann tensor, known as the Weyl tensor. Since there are 10 independent traces of the Riemann tensor (as many as there are symmetric two-tensors), the Weyl tensor has 10 independent components in 4 spacetime dimensions. In 3 dimensions it vanishes identically, so that the Riemann tensor is determined by the Ricci tensor. This implies that in 3 spacetime dimensions, and for vanishing cosmological constant, spacetime must be flat on the complement of the support of the energy momentum tensor. This means that there are no propagating degrees of freedom (gravitational waves) for gravity in 3 spacetime dimensions. In $n \geq 4$ dimensions the vanishing of the Weyl tensor is equivalent to the condition that the manifold be conformally flat.

2.2. Second Bianchi Identity

The second Bianchi identity reads

$$R_{\lambda\kappa[\sigma\mu;\nu]} = 0, \tag{11}$$

where $; = \nabla$. Contracting this equation over λ and σ, and then taking again the contraction with $g^{\kappa\mu}$ gives

$$\nabla^\lambda G_{\lambda\nu} = 0, \tag{12}$$

which is therefore called the "twice contracted second Bianchi identity". When it is applied to the Einstein's equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{13}$$

it provides an integrability condition

$$\nabla^{\lambda} T_{\lambda\nu} = 0. \tag{14}$$

Let $\{x^{\mu}\}$ be a coordinate system such that x^0 is a time coordinate and x^k for $k = 1, 2, 3$ are space coordinates. Then

$$g_{00} := g(\partial_0, \partial_0) < 0, \tag{15a}$$
$$g_{kk} := g(\partial_k, \partial_k) > 0, \tag{15b}$$

(here there is no sum over the repeated index k). Then $\nabla_{\lambda} G^{\lambda\nu} = 0$ takes the explicit form

$$\partial_0 G^{0\nu} + \partial_k G^{k\nu} + \Gamma^{\mu}_{\mu k} G^{k\nu} + \Gamma^{\nu}_{\mu\lambda} G^{\mu\lambda} = 0. \tag{16}$$

This formula (16) is an identity in $g_{\mu\nu}$. Since the last 3 terms contain at most second time derivatives in $g_{\mu\nu}$, $G^{0\nu}$ contains at most first time derivatives. Hence the four Einstein's equations comprised by the $(0, 0)$ and $(0, k)$ components do not contain second time derivatives of $g_{\mu\nu}$ with respect to x^0. They are equations that the initial data have to satisfy. In other words: they are *constraints*. The six remaining equations comprised by the (k, j) components are evolution equations. Note that if the constraints are satisfied initially, then they remain satisfied under evolution given by the spatial components. This is easily seen (at least in the case of analytic evolutions) from

$$\partial_0 G^{0\nu} = -\partial_k G^{k\nu} - \Gamma^{\mu}_{\mu k} G^{k\nu} - \Gamma^{\mu}_{\nu\lambda} G^{\mu\lambda}, \tag{17}$$

which, upon taking further time derivatives, implies that all time derivatives of the constraints vanish initially.

3. The 3+1 - Split

In this section we review the $3+1$ - split which allows us to formulate Einstein's equations as an initial value problem. As explained in the last section, the ten Einstein equations can be classified into four constraints and six second order evolution equations for the space-time metric $g_{\mu\nu}$. In the $3+1$ - decomposition we parameterize the $g_{\mu\nu}$ in such a way that it decomposes into four functions – the *lapse* function and the three components of the *shift* vector-field – whose evolution is not determined by the equations of motion and can be chosen at will (they are "gauge functions") and the remaining six components which are subject to evolution equations.

To start, we pretend we were already given a 4-dim differentiable manifold M with Lorentzian metric g. We can then derive the evolution equations for the intrinsic and extrinsic geometry of a 3-dim spacelike manifold Σ as we "move" it through M. If M satisfies Einstein's equations, this motion of Σ through M is equivalent to a constrained dynamical system whose configuration variable is identified with the Riemannian 3-metric h on Σ. Now, in the initial value formulation of GR we start off from this Hamiltonian system without (M, g) being given to us. Rather, we use the evolution equations for Σ's geometry to construct the space-time (M, g) from the family $\{(\Sigma, h_t) \,|\, t \in \mathbb{R}\}$. Our equations will then guarantee that the spacetime (M, g) so constructed will indeed satisfy Einstein's equations.

In the rest of this chapter we will see how this works technically in terms of decomposing spacetime into space and time (3+1 decomposition). The Hamiltonian formulation of GR will then be discussed in detail in section 4. Here we will mainly follow the ideas outlined in [2] and [3].

3.1. The $(3+1)$ *- Decomposition of 4-dim Geometry*

The constraint that M be globally hyperbolic (see Definition 2.1) implies that space-time is topologically (i.e. is homeomorphic to) the product $M \cong \mathbb{R} \times \Sigma$, where Σ denotes an arbitrary spacelike 3-manifold. (It can be shown that there exist *some* initial data for *any* 3-manifold Σ. Hence Einstein's equations do not pose a topological constraint on Σ.) We foliate M by a one-parameter family of embeddings

$$\begin{aligned} e_t : \Sigma \hookrightarrow M,\, t \in \mathbb{R}, \\ \Sigma_t := e_t(\Sigma) \subset M. \end{aligned} \tag{18}$$

Here Σ_t is the image of e_t in M for fixed topological "time" t; it is the t'th leaf of the foliation, see Fig. 1. We assume all leaves Σ_t to be spacelike with respect to g in M. Therefore, there exists a timelike field of normals n to leaves Σ_t in M. With n fixed to one of its two possible orientations, the notions of future and past are specified: A timelike vector X is called "future pointing" iff $g(X, n) < 0$. We can split the tangent-bundle $T(M)$ into the orthogonal sum of the subbundle of spacelike vectors $T_{\|}(M)$ and the normal bundle $T_{\perp}(M)$.

The associated projection maps $P_{\|}$ ($\|$ to leaves Σ_t) and $P_{\perp}$ ($\perp$ to leaves Σ_t)

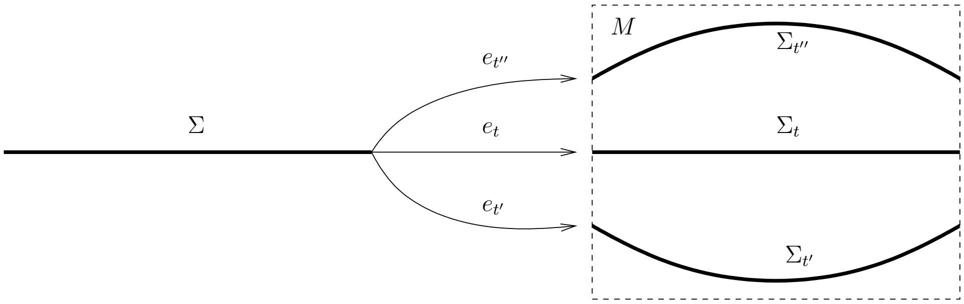

Figure 1: Foliation of space-time M by a one-parameter family of embeddings of the 3-manifold Σ into M. Σ_t is the image in M of Σ under e_t. Here the leaf $\Sigma_{t'}$ lies to the past and $\Sigma_{t''}$ lies to the future of Σ_t [3].

are given by

$$P_{\|} : T(M) \to T_{\|}(M), \qquad X \mapsto X + ng(n, X), \tag{19}$$
$$P_{\perp} : T(M) \to T_{\perp}(M), \qquad X \mapsto -ng(n, X)\,. \tag{20}$$

We endow Σ_t with a Riemannian metric h by restricting the Lorentzian metric g of M to the tangent vectors of Σ_t

$$h := P_{\|} g = g + \underline{n} \otimes \underline{n}\,, \tag{21}$$

where $\underline{n} := g(n, \cdot\,)$ (1-form).
Let X, Y be any "spatial" vector fields, i.e. with values in $T_{\|}(M)$; then we have

$$\nabla_X Y = P_{\|}(\nabla_X Y) + P_{\perp}(\nabla_X Y) =: D_X Y + nK(X, Y), \tag{22}$$

where we defined the spatial covariant derivative D and the *extrinsic curvature* K of the embedded leaves $\Sigma_t \subset M$ by

$$D_X := P_{\|} \circ \nabla_X, \tag{23}$$
$$K(X, Y) := -g(\nabla_X Y, n). \tag{24}$$

Proposition 3.1. (a) *K is a symmetric section in $T^*_{\|}(M) \otimes T^*_{\|}(M)$.*
(b) *D is the Levi-Civita Connection on $T_{\|}(M)$ with respect to h (defined in (21)).*

Proof of Proposition 3.1. (a) To prove that K is a tensor we need to show that $K(fX, Y) = K(X, fY) = fK(X, Y)$, $\forall f \in C^\infty(M)$. Now, for X, Y parallel to the leaves Σ_t, $P_{\perp}(\nabla_{fX} Y) = fP_{\perp}(\nabla_X Y)$ is trivially fulfilled and $P_{\perp}(\nabla_X fY) = P_{\perp}(X(f)Y + f\nabla_X Y) = fP_{\perp}(\nabla_X Y)$, since $P_{\perp}(Y) = 0$. To see the symmetry of the extrinsic curvature tensor K we write

$$\begin{aligned} K(X, Y) &= -g(n, \nabla_X Y) \\ &= -g(n, \nabla_Y X + [X, Y] + T(X, Y)) \\ &= -g(n, \nabla_Y X), \end{aligned} \tag{25}$$

where we have used the definition of the torsion tensor

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]. \tag{26}$$

The last equality (25) follows from the vanishing torsion of ∇ ($T = 0$) and the fact that $[X,Y]$ is spatial, i.e. $[X,Y] \in T_{\|}(M)$ for $X,Y \in$ sections in $T_{\|}(M)$. ($T_{\|}(M)$ is clearly an integrable subbundle of $T(M)$, since it is tangent to the leaves of a foliation by definition).

(b) We must first prove that D is a connection/covariant derivative (in the sense of Kozul):
1.) $D_{X_1+X_2}Y = D_{X_1}Y + D_{X_2}Y$
2.) $D_X(Y_1+Y_2) = D_X Y_1 + D_X Y_2$
3.) $D_{fX} = fD_X Y$
4.) $D_X(fY) = fD_X Y + X(f)Y$
$\forall\, X_i, Y_i \in \text{Sec}\; T_{\|}(M)$ and $f \in C^\infty(M)$.
Since these properties are satisfied by ∇, the relations 1.)- 3.) follow immediately. To see 4.), we write

$$\begin{aligned} D_X(fY) &= P_{\|}(\nabla_X fY) = P_{\|}(X(f)Y + f\nabla_X Y) \\ &= X(f)Y + fD_X Y, \end{aligned}$$

where we used the definition of the spatial covariant derivative D (23) and $P_{\|}(Y) = Y$.
Next we must prove that D is preserving the metric h and is torsion free. The metricity follows from

$$D_X h = P_{\|} \nabla_X (g + \underline{n} \otimes \underline{n}) = 0,$$

since $\nabla_X g = 0$ and $P_{\|}\, \underline{n} = 0$.
Vanishing torsion is also immediate:

$$\overset{D}{T}(X,Y) = D_X Y - D_Y X - [X,Y] = P_{\|}(\nabla_X Y - \nabla_Y X - [X,Y]) = 0.$$

Here we used the vanishing torsion of ∇ ($\overset{\nabla}{T} = 0$) and $P_{\|}[X,Y] = [X,Y]$.

Let $\{e_\alpha\} = \{e_0, e_a\}$ be an orthonormal frame of $T(M)$ with respect to g adapted to the foliation, i.e. $e_0 = n$, $e_a = P_{\|}(e_a)$ and let $\{e^\alpha\} = \{e^0, e^a\}$ be its dual:

$$e^\alpha(e_\beta) = \delta^\alpha_{\ \beta}. \tag{27}$$

Then from (21) with $\underline{n} = e^0$ we have

$$g = -e^0 \otimes e^0 + h = -e^0 \otimes e^0 + \sum_{a=1}^{3} e^a \otimes e^a. \tag{28}$$

The one-parameter family of embeddings $t \mapsto e_t : \Sigma \hookrightarrow M$ defines a vector field $\partial_t := \partial/\partial t$. This vector field acts on any smooth function f at $e_t(p)$, $p \in \Sigma_t$ in the following way:

$$\partial_t|_{e_t(p)} f := \frac{d}{dt} f \circ e_t(p), \tag{29}$$

(considered as usual as the derivation of an algebra of (germs of) C^∞ functions).

We decompose the vector field ∂_t in its normal and tangential components to the leaves Σ_t

$$\partial_t = \alpha n + \beta = \alpha e_0 + \beta^a e_a, \tag{30}$$

where n is the normal to Σ_t. Here, the scalar field α is called *lapse* (function) and β the *shift* (vector field) with values in $P_\parallel T(M)$, respectively. The normal component α advances one leaf Σ_t to the next one $\Sigma_{t+\mathrm{dt}}$, whereas β generates intrinsic diffeomorphisms on each Σ_t; see Fig. 2. The lapse and shift were first introduced in [4].

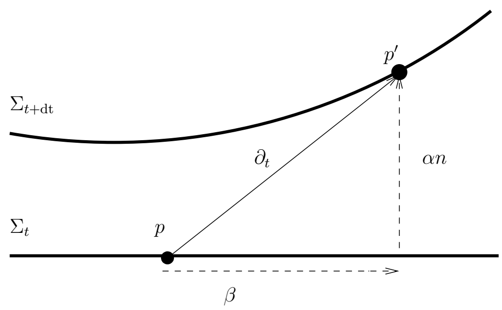

Figure 2: Infinitesimally nearby leaves Σ_t and $\Sigma_{t+\mathrm{dt}}$. For some point $q \in \Sigma$, the image points $p = e_t(q)$ and $p' = e_{t+\mathrm{dt}}(q)$ are connected by the vector $\partial_t|_p$, whose components tangential and normal to Σ_t are β and αn, respectively. Here n is the normal to Σ_t in M, β is called the "shift vector field" and α the "lapse function" on Σ_t [3].

(α, β) are interpreted as normal and tangential components (with respect to a given foliation $t \to e_t$) of four-velocities, measured in units of t, of the flow lines of Σ's points. If we fix a coordinate system $\{x^\kappa\}$ on Σ, then ∂_t is the four-velocity, measured in units of t, of points with fixed spatial coordinates x^k.

If $U \subset \Sigma$ is a coordinate in neighborhood with coordinate $y^k : U \to \mathbb{R}$, we define a coordinate system $\{x^\mu, V\}$ on M via

$$V = \bigcup_{t \in \mathbb{R}} e_t(U), \tag{31a}$$

$$x^0(p) = t \qquad \text{if } p \in \Sigma_t, \tag{31b}$$

$$x^k(p) = y^k \circ e_t^{-1}(p) \qquad \text{if } p \in \Sigma_t. \tag{31c}$$

Let ∂_t, ∂_k be the corresponding vector fields in V, then we can use the following obvious matrix notation

$$\begin{pmatrix} \partial_t \\ \partial_k \end{pmatrix} = \begin{pmatrix} \alpha & \beta^a \\ 0 & A^a_k \end{pmatrix} \begin{pmatrix} e_0 \\ e_a \end{pmatrix}, \tag{32a}$$

$$\Rightarrow \begin{pmatrix} e^0 \\ e^a \end{pmatrix} = (dt, dx^k) \begin{pmatrix} \alpha & \beta^a \\ 0 & A^a_k \end{pmatrix}. \tag{32b}$$

Replacing (e^0, e^a) in (28) with (dt, dx^k) via (32b) gives the $3+1$ - split form of the metric g

$$g = -\alpha^2 dt \otimes dt + h_{ik} \left(dx^i + \beta^i dt\right) \otimes \left(dx^k + \beta^k dt\right), \tag{33}$$

where

$$h_{ik} := h\left(\partial_i, \partial_k\right) = \sum_{a=1}^{3} A^a_i A^a_k, \tag{34}$$

and

$$\beta^a = A^a_k \beta^k. \tag{35}$$

The volume 4-form $d\mu$ is given by

$$d\mu = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \sqrt{\mathrm{det} g_{\mu\nu}}\, \mathrm{d^4x} = \alpha \sqrt{\mathrm{deth_{ij}}}\, \mathrm{dt} \wedge \mathrm{d^3x}. \tag{36}$$

Through the foliation $t \mapsto e_t : \Sigma \hookrightarrow M$ we introduce the notions of "simultaneity" (equivalence relation), "time", and " flow of time"; the latter through the Lie-derivative with respect to ∂_t. Therefore, in what follows, we need to take Lie derivatives with respect to ∂_t and, since $\partial_t = \alpha n + \beta$ (30), with respect to n and β. Concerning the former we have

Proposition 3.2. *Let T be any spatial, covariant tensor field, then for any $f \in C^\infty(M)$*

$$P_\| L_{fn} T = L_{fn} T = f L_n T. \tag{37}$$

Proof of Proposition 3.2. T is a sum of tensor products of spatial 1-forms. Hence, by Leibnitz' rule, it suffices to prove (37) for them. By the general rule $L_X = i_X \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i_X}$ for forms (here applied to 1-forms), we have

$$\begin{aligned} L_{fn} T &= i_{fn} \circ \mathrm{d}T + \mathrm{d} \circ i_{fn} T \\ &= f i_n \mathrm{dT} + \mathrm{d}\big(\mathrm{f}\, \underbrace{(i_n T)}_{0}\big) \\ &= f\left(i_n \circ \mathrm{d} + \mathrm{d} \circ i_n\right) T = f L_n T, \end{aligned} \tag{38}$$

which proves the 2^{nd} equality of (37).
The 1^{st} equality of (37), i.e. that $L_n T$ is spatial if T is, follows from the general formula (for forms)

$$L_X \circ i_Y = i_{[X,Y]} + i_Y \circ L_X. \tag{39}$$

Applied to $X = Y = n$ we have $[L_n, i_n] = 0$, hence $i_n L_n T = L_n (i_n T) = 0$.

Defining the "doting" by

$$\dot{h} := P_{\|} L_{\partial_t} h, \tag{40}$$

we get with Proposition 3.2 and equation (30)

$$\dot{h} = \alpha L_n h + P_{\|} L_\beta h. \tag{41}$$

We note that the projector $P_{\|}$ in front of $L_\beta h$ in equation (41) is really necessary, because we have for 1-forms:

$$i_n L_\beta T = -i_{[\beta,n]} T \neq 0 \quad \text{if } [\beta, n] \nparallel n. \tag{42}$$

The 1^{st} term in (41) can be rewritten in terms of the extrinsic curvature K. To see this, we show that $L_n h$ is just twice the extrinsic curvature.

Proposition 3.3.

$$L_n h = 2K. \tag{43}$$

Proof of Proposition 3.3. Let X, Y be any spatial vector fields. We compute

$$\begin{aligned}
(L_n h)(X, Y) &= n\,(h(X, Y)) - h\,([n, X], Y) - h\,(X, [n, Y]) \\
&= \nabla_n\,(h(X, Y)) - h\,([n, X], Y) - h\,(X, [n, Y]) \\
&= (\nabla_n h)(X, Y) + h(\nabla_n X - [n, X], Y) + h(X, \nabla_n Y - [n, Y]).
\end{aligned}$$

Since ∇ is metric $\nabla_n h = \nabla_n (g + \underline{n} \otimes \underline{n}) = (\nabla_n \underline{n}) \otimes \underline{n} + \underline{n} \otimes \nabla_n \underline{n}$, which vanishes if applied to spatial vectors X, Y. Further, since ∇ has vanishing torsion, we have $\nabla_n X - [n, X] = \nabla_X n$ and $\nabla_n Y - [n, Y] = \nabla_Y n$. Hence

$$(L_n h)(X, Y) = h\,(\nabla_X n, Y) + h\,(n, \nabla_Y n)\,.$$

Now recall definition (22) $\nabla_X Y = P_{\|}(\nabla_X Y) + nK(X, Y)$ and therefore

$$K(X, Y) = -g\,(n, \nabla_X Y) = g\,(\nabla_X n, y) = h\,(\nabla_X n, Y) = h\,(\nabla_Y n, X)\,,$$

which finally gives (43).

Using (43) in (41) yields:

$$K = \frac{1}{2\alpha}\left(\dot{h} - P_{\|} L_\beta h\right). \tag{44}$$

Finally, we may rewrite the projected Lie-derivative in (44) in terms of the spatial covariant derivative. In components this reads

$$\left(L_\beta h\right)_{\mu\nu} = \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu \Rightarrow \left(P_\parallel L_\beta h\right)_{mn} = D_m \beta_n + D_n \beta_m. \tag{45}$$

Hence we arrive at

$$K_{mn} = \frac{1}{2\alpha}\left(\dot{h}_{mn} - D_m \beta_n - D_n \beta_m\right), \tag{46}$$

or alternatively

$$\dot{h}_{mn} = 2\alpha K_{mn} + D_m \beta_n + D_n \beta_m. \tag{47}$$

3.2. The $(3+1)$ *- Decomposition of Curvature Tensor*

The splitting formula (22) for the connection ∇ in terms of the spatial covariant derivative D and the extrinsic curvature K enables us to write down the Gauss-Codazzi and Codazzi-Mainardi evolution equations.

Proposition 3.4. *Let* $\{e_0, e_a\}$ *be an adapted orthonormal basis. In covariant components with respect to* $\{e_0, e_a\}$ *the Riemann tensors for* (M, g) *and* (Σ_t, h) *are defined by*

$$R_{\alpha\beta\gamma\delta} := g\left(e_\alpha, R\left(e_\gamma, e_\delta\right) e_\beta\right), \tag{48a}$$

$$R^{(3)}_{abcd} := h\left(e_a, R^{(3)}\left(e_c, e_d\right) e_b\right), \tag{48b}$$

with R *denoting the curvature of* ∇ *and* $R^{(3)}$ *the curvature of* D

$$R = \overset{\nabla}{R}, \quad R^{(3)} = \overset{D}{R}. \tag{49}$$

Then the "Gauß-Codazzi" and "Codazzi-Mainardi" equations read respectively:

$$R_{abcd} = R^{(3)}_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}, \tag{50}$$

$$R_{0abc} = D_c K_{ab} - D_b K_{ac}. \tag{51}$$

Proof of Proposition 3.4. Let X, Y and Z be any spatial vector fields and n the normal field. We make repeated use of the decomposition formula of covariant derivative (22) in the defining equation for the Riemann curvature tensor (2)

$$\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_X \left(D_Y Z + nK(Y,Z)\right) - \nabla_Y \left(D_X Z + nK(X,Z)\right) \\
&\quad - D_{[X,Y]} Z - nK\left([X,Y], Z\right) \\
&= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z + n[K(X, D_Y Z] + X(K(Y,Z)) \\
&\quad - K(Y, D_X Z) - Y(K(X,Z)) - K([X,Y], Z)] \\
&\quad + \left(\nabla_X n\right) K(Y,Z) - \left(\nabla_Y n\right) K(X,Z) \\
&= R^{(3)}(X,Y)Z + nK(\overset{D}{T}(X,Y), Z) + n[(D_X K)(Y,Z) \\
&\quad - (D_Y K)(X,Z)] + (\nabla_X n)K(Y,Z) - (\nabla_Y n)K(X,Z),
\end{aligned}$$

where the second term in the last equality vanishes because of Proposition 3.1 (b).

a) Taking the inner product with W spatial:

$$\begin{aligned} g(W, R(X,Y)Z) &= h(W, R^{(3)}(X,Y)Z) + \underbrace{g(W, \nabla_X n)}_{K(W,X)} K(Y,Z) \\ &\quad - \underbrace{g(W, \nabla_Y n)}_{K(W,Y)} K(X,Z) \\ &= h(W, R^{(3)}(X,Y)Z) + K(W,X)K(Z,Y) \\ &\quad - K(W,Y)K(Z,X). \end{aligned}$$

For $W = e_a$, $Z = e_b$, $X = e_c$, $Y = e_d$ this equals the Gauß-Codazzi equation (50).

b) Taking the inner product with the normal field n:
Since $g(n, \nabla_X n) = 0$ etc. only terms $\sim DK$ contribute:

$$g(n, R(X,Y)Z) = -(D_X K)(Y,Z) + (D_Y K)(X,Y). \tag{52}$$

Which for $X = e_b$, $Y = e_c$, $Z = e_a$ is just the Codazzi-Mainardi equation (51).

The Gauß-Codazzi equation (50) allows to express the 00 component of the Einstein tensor $G_{00} = G(n,n)$ in terms of the spatial components of ∇'s curvature:

Proposition 3.5. *$G(n,n) = G_{00}$ is half the sum of the three independent sectional curvatures of mutually orthogonal planes in space, i.e. orthogonal to n.*

Proof of Proposition 3.5. G_{00} written in components with respect to an adapted orthonormal coordinate frame:

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R \\ &= R_{00} + \frac{1}{2}\left(-R_{00} + \sum_{a=1}^{3} R_{aa}\right) = \frac{1}{2}\left(R_{00} + \sum_{a=1}^{3} R_{aa}\right) \\ &= \frac{1}{2}\left(\sum_{a=1}^{3} R_{0b0b} + \sum_{a,b=1}^{3} R_{baba} - \sum_{a=1}^{3} R_{0a0a}\right) = \frac{1}{2}\sum_{a,b=1}^{3} R_{abab} \end{aligned} \tag{53}$$

It is not difficult to prove directly that the right-hand side of this equation does not depend on which set of three mutually orthogonal planes in $T\Sigma_t$ one evaluates it.

Let us briefly recall the definition of sectional curvature. It assigns to each point $p \in M$ and each non-degenerate 2-dim plane in the tangent space of M at p a number. (A plane is non degenerate if g restricted to that plane is non

degenerate.) That number is the Gauß curvature of the 2-dimensional geodesic sub-manifold tangent to the plane of p.
Let Span $\{X, Y\} \subset T_pM$ be the plane, then

$$Sec(X,Y) = \frac{g(X, R(X,Y)Y)}{g(X,X)g(Y,Y) - [g(X,Y)]^2}. \tag{54}$$

Note that the denominator never vanishes for linearly independent X, Y if the plane they span is non degenerate.

Proposition 3.6. *Expression (54) depends only on Span $\{X, Y\}$ and on the choice $\{X, Y\}$ of the basis.*

Proposition 3.7. *The curvature tensor is determined by the sectional curvatures (The converse is trivially true).*

Replacing R_{abab} in (53) via the Gauß-Codazzi equation (50) gives the following constraint

$$\begin{aligned} 2G(n,n) &= \sum_{a,b=1}^{3} R^{(3)}_{abab} - K_{ab}K^{ab} + (K^a_a)^2 \\ &= R^{(3)} - K_{ab}K^{ab} + (K^a_a)^2 \\ &= R^{(3)} - h^{im}h^{jn}K_{mn}K_{ij} + (h^{mn}K_{mn})^2 \\ &= R^{(3)} - ||K||^2_h + (\mathrm{Tr}_h(K))^2 , \end{aligned} \tag{55}$$

where we have taken into account

$$\sum_{a,b=1}^{3} R^{(3)}_{abab} =: R^{(3)} = \text{Ricci scalar for } h.$$

The second constraint for the mixed components $G_{0b} = G(n, e_b)$ follows immediately from the Codazzi-Mainardi equation (51):

$$G(n, e_b) = R_{0b} = \sum_{a=1}^{3} R_{0aba} = \sum_{a=1}^{3} D_a K_{ab} - D_b \sum_{a=1}^{3} K_{aa}, \tag{56}$$

or rewritten in arbitrary coordinate components:

$$\begin{aligned} G(n, \partial_k) &= h^{mn} D_m K_{nk} - D_k \left(h^{mn} K_{mn}\right) \\ &= D^n K_{nk} - D_k K^n_n. \end{aligned} \tag{57}$$

To derive the dynamical equations we manipulate the components

$$R_{0a0b} = g(n, R(n, e_b)e_a), \tag{58}$$

where from the definition of the Riemann curvature tensor (2)

$$R(n, e_b)e_a = \left(\nabla_n \nabla_{e_b} - \nabla_{e_b} \nabla_n - \nabla_{[n,e_b]}\right) e_a, \tag{59}$$

we arrive at

$$R_{0a0b} = -\left(L_n K\right)_{ab} + K_{ac}K_b^c + a_a a_b + D_a a_b, \tag{60}$$

where we have used

$$\nabla_n \left(g\left(e_a, \nabla_{e_b} n\right)\right) = L_n K_{ab},$$

and introduced the normal acceleration by

$$a := \nabla_n n.$$

Proposition 3.8.

$$R_{0a0b} = -\left(L_n K\right)_{ab} + K_{ac}K_b^c + a_a a_b + D_a a_b.$$

Proof of Proposition 3.8. Calculation with components is most easiest:

$$\begin{aligned} R_{0a0b} &= R_{\mu a \nu b} n^\mu n^\nu = -n^\nu R_{a\mu\nu b} n^\mu = -n^\nu [\nabla_\nu, \nabla_b] n_a \\ &= -n^\nu \left(\nabla_\nu \nabla_b - \nabla_b \nabla_\nu\right) n_a. \end{aligned} \tag{61}$$

Substituting $\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu$ in (61) yields

$$\begin{aligned} R_{0ab0} &= \nabla_b \left(n^\nu \nabla_\nu n_a\right) - \left(\nabla_b n^\nu\right)\left(\nabla_\nu n_a\right) - n^\nu \nabla_\nu \left(K_{ba} - n_b a_a\right) \\ &= \nabla_b a_a - \left(K_b^\nu - n_b a^\nu\right)\left(K_{\nu a} - n_\nu a_a\right) - n^\nu \nabla_\nu K_{ba} + a_b a_a + n_b n^\nu \nabla_\nu a_a. \end{aligned}$$

Having $n_b = 0$, $K_b^\nu n_\nu = 0$ and

$$\begin{aligned} n^\nu \nabla_\nu K_{ba} &= \left(L_n K\right)_{ba} - K_{\mu a} \nabla_b n^\mu - K_{b\mu} \nabla_a n^\mu \\ &= \left(L_n K\right)_{ba} - K_{\mu a} K_b^\mu - K_{b\mu} K_a^\mu. \end{aligned}$$

we arrive at

$$\begin{aligned} R_{0ab0} &= \nabla_b a_a - K_b^\nu K_{\nu a} + 2K_{\mu a} K_b^\mu - \left(L_n K\right)_{ab} + a_a a_b \\ &= -\left(L_n K\right)_{ab} + K_{ac} K_b^c + a_a a_b + D_b a_a. \end{aligned}$$

Now, since we have

$$R_{ab} = -R_{0a0b} + \sum_{c=1}^{3} R_{cacb}, \tag{62}$$

equation (60) with the Gauß-Codazzi equation (50) substituted gives

$$R_{ab} = \left(L_n K\right)_{ab} - K_{ac}K_b^c - a_a a_b - D_a a_b + R_{ab}^{(3)} + K_c^c K_{ab} - K_{ac} K_b^c,$$

or reordered

$$\left(L_n K\right)_{ab} = \left(R_{ab} - R_{ab}^{(3)}\right) + 2K_{ac}K_b^c - K_c^c K_{ab} + a_a a_b + D_a a_b. \tag{63}$$

Equation (63) is *almost* the evolution equation for K we look for. In the following we will reformulate (63) in terms of a coordinate basis and in terms of $\dot{K}$ (introduced analogous to $\dot{h}$ in (41)).
Writing, as usual, $\underline{n} = g(n, \cdot)$, $\underline{a} = g(a, \cdot)$ we prove two Lemmas which we will use later on.

Lemma 3.1.

$$L_n \underline{n} = \underline{a} \tag{64}$$

Proof of Lemma 3.1.

$$\begin{aligned}(L_n \underline{n})(X) &= \nabla_n (g(n,X)) - g(n,[n,X]) \\ &= g(a,X) + g(n, \nabla_n X - [n,X]) \\ &= g(a,X) + g(n, \nabla_X n) \\ &= g(a,X) \quad \forall X.\end{aligned}$$

Lemma 3.2.

$$\left(L_n - \frac{1}{2} i_n \circ \mathrm{d}\right) \mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}} = \mathrm{d}\underline{\mathrm{a}} \wedge \underline{\mathrm{n}}. \tag{65}$$

Proof of Lemma 3.2. Using Lemma 3.1 we can write

$$\begin{aligned}L_n(\mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}}) &= \mathrm{d}\,(\mathrm{L_n}\underline{\mathrm{n}}) \wedge \underline{\mathrm{n}} + \mathrm{d}\underline{\mathrm{n}} \wedge (\mathrm{L_n}\underline{\mathrm{n}}) \\ &= \mathrm{d}\underline{\mathrm{a}} \wedge \underline{\mathrm{n}} + \mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{a}}.\end{aligned} \tag{66}$$

Also, we have

$$\begin{aligned}i_n \mathrm{d}\,(\mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}}) &= i_n\,(\mathrm{d}\underline{\mathrm{n}} \wedge \mathrm{d}\underline{\mathrm{n}}) \\ &= (i_n \mathrm{d}\underline{\mathrm{n}}) \wedge \mathrm{d}\underline{\mathrm{n}} + \mathrm{d}\underline{\mathrm{n}} \wedge\ (\mathrm{i_n}\mathrm{d}\underline{\mathrm{n}})\,.\end{aligned} \tag{67}$$

Now,

$$\begin{aligned}i_n \mathrm{d}\underline{\mathrm{n}} &= (i_n \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i_n})\,\underline{n} \quad (\text{since } i_n n = \text{const.}) \\ &= L_n \underline{n} = \underline{a} \quad (\text{Lemma 3.1}),\end{aligned}$$

so that with (66) we have

$$i_n \mathrm{d}(\mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}}) = 2\underline{\mathrm{a}} \wedge \mathrm{d}\underline{\mathrm{n}}.$$

Hence $-\frac{1}{2} i_n \circ \mathrm{d}(\mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}})$ in (65) substracts the 2^{nd} term in (66) and the identity (65) follows.

Proposition 3.9. *The last term in (60) is symmetric.*

$$D_a a_b = D_b a_a. \tag{68}$$

Proof of Proposition 3.9. $\underline{n}$ is, by construction, hypersurface-orthogonal. Hence, by Frobenius' theorem $\mathrm{d}\underline{\mathrm{n}} \wedge \underline{\mathrm{n}} = 0$. The identity (65) then implies $\mathrm{d}\underline{\mathrm{a}} \wedge \underline{\mathrm{n}} = 0$, which is equivalent to

$$P_{\|} \mathrm{d}\underline{\mathrm{a}} = 0 \Leftrightarrow \mathrm{D_a a_b} - \mathrm{D_b a_a} = 0.$$

Consequently each term on the rhs of (60) is symmetric, as it must be. Since the extrinsic curvature K is a spatial covariant tensor field, we define the dotting analogous to (41) and we use Proposition 3.2

$$L_{\partial_t} K = \dot{K} = \alpha L_n K + L_\beta K. \tag{69}$$

Furthermore, by Lemma 3.1, we have with respect to a coordinate basis

$$\begin{aligned} a_k &= \underline{a}(\partial_k) = (L_n \underline{n})(\partial_k) \\ &= -\underline{n}([n, \partial_k]) = -\underline{n}\left(\left[\frac{1}{\alpha}(\partial_t - \beta), \partial_k\right]\right) \\ &= \partial_k \left(\frac{1}{\alpha}\right) \underline{n}(\partial_t) = \frac{1}{\alpha}\partial_k \alpha, \quad (\underline{n}(\partial_t) = -\alpha). \end{aligned} \tag{70}$$

Hence the terms in the evolution equation (63) involving a simplify to,

$$a_m a_n + D_a a_b = \frac{\alpha_{1n}\alpha_{1m}}{\alpha^2} + D_m\left(\frac{\alpha_{1n}}{\alpha}\right) = \frac{1}{\alpha} D_m D_n \alpha. \tag{71}$$

We can now rewrite (63)

$$\dot{K}_{mn} = \alpha\left[R_{mn} - R^{(3)}_{mn} + 2K_{mk}K^k_n - K^k_k K_{mn}\right] + \frac{1}{\alpha} D_m D_n \alpha + (L_\beta K)_{mn}, \tag{72}$$

where we could also write

$$(L_\beta K)_{mn} = \beta^k D_k K_{mn} + K_{kn} D^k \beta_m + K_{mk} D^k \beta_n. \tag{73}$$

This completes the $(3+1)$ - decomposition: We formulated the four constraints (55), (57) and we replaced the six evolution equations of second order ($G^{mn} = \kappa T^{mn}$) by twelve equations of first order, given by (48) and (72). The derived evolution and constraint equations are summarized in Table 2.

3.3. Initial Data Formulation

As explained above, Einstein's equations can be recast into the form of a dynamical system of first order evolution equations (48), (72) with constraints (55), (57). The initial-value formulation of GR can be summarized as follows:

1.) Pick a 3-d manifold Σ with local coordinates $\{x^k\}$.
2.) Pick a Riemannian metric $h_{ij} \in T^*(\Sigma) \otimes T^*(\Sigma)$ and another symmetric covariant tensor field $K_{ij} \in T^*(\Sigma) \otimes T^*(\Sigma)$ so that for given external sources $T_{\perp\perp}$ and $T_{\perp k}$ they satisfy the constraints C_H and C_M (see Table 2).
3.) Pick lapse and shift functions α and β at will and evolve the pair (h, K) according to the evolution equations EV_q and EV_p (see Table 2).

Having done that, the whole construction assures that the so constructed space-time metric

$$\begin{aligned} g = &-\alpha^2(x,t)\mathrm{dt} \otimes \mathrm{dt} \\ &+ h_{mn}(x,t)\left(\mathrm{dx^m} + \beta^{\mathrm{m}}(\mathrm{x,t})\mathrm{dt}\right) \otimes \left(\mathrm{dx^n} + \beta^{\mathrm{n}}(\mathrm{x,t})\mathrm{dt}\right), \end{aligned} \tag{74}$$

solves Einstein's equations.

Summary	
Constraints	Evolution equations
Hamiltonian- or Scalar Constraint C_H	Evolution of q-variable (EV_q)
$K_{mn}K^{mn} - (K^m_m)^2 - \left(R^{(3)} - 2\Lambda\right) = -2\kappa T_{\perp\perp}$	$\dot{h}_{mn} = 2\alpha K_{mn} + (L_\beta h)_{mn}$
Momentum- or Vector Constraint C_M	Evolution of p-variable (EV_p)
$D^m\left(K_{mn} - h_{mn}K^k_k\right) = \kappa T_{\perp k}$	$\dot{K}_{mn} = \alpha\left[2K_{mk}K^k_n - K^k_k K_{mn} - R^{(3)}_{mn}\right] + \frac{1}{\alpha}D_m D_n \alpha + (L_\beta h)_{mn} + \alpha\left(\kappa(T_{mn} - \frac{1}{2}Tg_{mn}) + \Lambda g_{mn}\right)$

Table 2: Summary of the dynamical formulation of Einstein's equations.

3.4. Einstein-Hilbert Action

We conclude this section by writing down explicitly the Hamiltonian which is constructed solely from the canonical variables h_{mn} and π_{mn}. As before, h_{mn} denotes the Riemannian metric on Σ_t and π_{mn} is introduced as its canonical momentum. The Hamiltonian formulation of GR will be discussed in much more detail in the following section.

Proposition 3.10. *The Einstein-Hilbert action*

$$S = \int (R - 2\Lambda)\sqrt{det\, g}\, \mathrm{d}^4\mathrm{x}, \tag{75}$$

takes up to surface terms the following form

$$S = \int \mathcal{L}_{EH}\, d^4x \tag{76}$$

with

$$\mathcal{L}_{EH} = \left\{\left(R^{(3)} - 2\Lambda\right) + K_{mn}K^{mn} - (K^m_m)^2\right\}\alpha\sqrt{h} + \text{ divergences}, \tag{77}$$

where

$$h := det\, h.$$

Proof of Proposition 3.10. To verify this we calculate the $4-d$ Ricci scalar R in (75). Starting with the equation for the Einstein tensor

$$G(n,n) = R(n,n) - \frac{1}{2}\underbrace{g(n,n)}_{=-1}R = R(n,n) + \frac{1}{2}R, \tag{78}$$

we can write

$$R = 2(G(n,n) - R(n,n)), \tag{79}$$

where we know $G(n,n)$ from the Gauß-Codazzi equation (55).
The Ricci tensor $R(n,n)$ follows from

$$\begin{aligned} R(n,n) &= n^\mu n^\nu R^\lambda_{\mu\lambda\nu} \\ &= n^\nu [\nabla_\lambda, \nabla_\nu] n^\lambda \\ &= n^\nu (\nabla_\lambda \nabla_\nu n^\lambda - \nabla_\nu \nabla_\lambda n^\lambda) \\ &= -(\nabla_\lambda n^\nu)(\nabla_\nu n^\lambda) + (\nabla_\nu n^\nu)(\nabla_\lambda n^\lambda) + \nabla_\lambda V^\lambda \\ &= -(K^\nu_\lambda - n_\lambda a^\nu)(K^\lambda_\nu - n_\nu a^\lambda) + (K^\lambda_\lambda)^2 + \nabla_\lambda V^\lambda \\ &= -K^\nu_\lambda K^\lambda_\nu + (K^\lambda_\lambda) + \nabla_\lambda V^\lambda, \end{aligned} \tag{80}$$

where we have defined

$$V^\lambda = (a^\lambda - n^\lambda K^k_k). \tag{81}$$

and used the definition of the curvature tensor (2), integration by parts and $\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu$.
Consequently, the Ricci scalar R reads

$$\begin{aligned} R &= 2(G(n,n) - R(n,n)) \\ &= R^{(3)} + K_{mn}K^{mn} - (K^n_n)^2 - 2\nabla_\lambda V^\lambda. \end{aligned} \tag{82}$$

Substituting (82) in (75) proves the Proposition 3.10.

We convert the Lagrangian form (75) of the action into the Hamiltonian form by eliminating K_{mn} via equation (46)

$$K_{mn} = \frac{1}{2\alpha}\left(\dot{h}_{mn} - D_m\beta_n - D_n\beta_m\right), \tag{83}$$

and taking h_{mn} and its conjugate momenta

$$\pi^{mn} := \frac{\partial \mathcal{L}_{EH}}{\partial \dot{h}_{mn}} = \sqrt{h}\left(K^{mn} - \frac{1}{2}h^{mn}K^k_k\right), \tag{84}$$

as the canonical variables.
The gravitational Hamiltonian takes the form

$$\begin{aligned} \mathcal{H}_{EH} &= \pi^{mn}\dot{h}_{mn} - \mathcal{L}_{EH} \\ &= \alpha\left\{\frac{1}{\sqrt{h}}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\left(\pi^n_n\right)^2\right) - \sqrt{h}\left(R^{(3)} - 2\Lambda\right)\right\} - 2\beta_m D_n \pi^{mn} \\ &\quad + \text{ divergences}. \end{aligned} \tag{85}$$

Here the first term is quadratic in the canonical momentum π_{mn} and can be interpreted as a "kinetic term", whereas the second term only depends on the metric h_{mn} and can be seen as a "potential term". The third term in (85) is known as the supermomentum of the gravitational field.

Variation of (85) with respect to α (no matter) reproduces the Hamiltonian constraint C_H

$$\frac{1}{\sqrt{h}}\left(\pi^{mn}\pi_{mn}-\frac{1}{2}(\pi^n_n)^2\right)-\sqrt{h}\left(R^{(3)}-2\Lambda\right)=0. \tag{86}$$

and variation with respect to β_m yields the three additional constraints which are combined to form the momentum constraint C_M

$$-2D_m\pi^{mn}=0. \tag{87}$$

4. The Hamiltonian Structure of GR

When reformulating the Lagrangian structure of a mechanical system in terms of the Hamiltonian formalism, the variables are replaced by canonical variables : $(q, \dot{q}) \to (q, p)$. In the canonical formulation of Hamiltonian General Relativity, the corresponding phase space variables are : $(h_{mn}, K_{mn}) \to (h_{mn}, \pi_{mn})$ with

$$\pi_{mn} = \sqrt{h}\left(K^{mn} - \frac{1}{2}h^{mn}K_k^k\right), \tag{88}$$

which is a "q-dependent" combination of velocities. Then the total Hamiltonian is

$$H_{\mathrm{tot}}[h,\pi]_{\alpha,\beta} := S(\alpha) + V(\beta) + \text{boundary terms}. \tag{89}$$

where the dependance in h, π has been omitted. The "Hamiltonian" or "scalar" constraint

$$S(\alpha) = \int_\Sigma \mathrm{d^3x}\,\alpha\left\{\frac{1}{\sqrt{\mathrm{h}}}\left(\pi^{\mathrm{mn}}\pi_{\mathrm{mn}} - \frac{1}{2}(\pi^{\mathrm{m}}_{\mathrm{m}})^2\right) - \sqrt{\mathrm{h}}\left(\mathrm{R}^{(3)} - 2\Lambda\right)\right\}, \tag{90}$$

generates the motion of the space surface Σ in the 4-dimensional space. On the other hand, the "momentum" or"vector" constraint

$$V(\beta) = -2\int_\Sigma \mathrm{d^3x}\,\beta_{\mathrm{n}}\mathrm{D_m}\pi^{\mathrm{mn}}, \tag{91}$$

generates the diffeomorphisms of the 3-dimensional space Σ. The boundary terms must be added according to the requirement that $H[h, \pi]$ must be differentiable, as a functional, with respect to h and π. Otherwise there is no such thing as an Hamiltonian flow. Practically, it means that we have to add surface terms which subtract those that arise from integration by parts. Usually the equations of motion have solutions which are not in the domain where the action is functionally differentiable. For instance, the source-free Maxwell equations admit as solution a constant electric field $E =$ constant while the action does not even exist on this configuration. Another example from General Relativity is the following. The Schwarzschild metric is a solution of Einstein's equations. However the Ricci scalar evaluated on this configuration vanishes : $R = 0$. Consequently the Einstein-Hilbert action is not differentiable at the Schwarzschild solution. We need additional surface terms in the action to produce finite values.

4.1. Functionnal differentiablility with respect to π^{mn}

The vector constraint $V(\beta)$ is a potential problem for the differentiability with respect to π_{mn}. When integrating by part in (91), we pick up a surface term

$$-2\int_{\partial\Sigma} n_m \delta\pi^{mn}\beta_n\,\mathrm{d}\sigma, \tag{92}$$

where n_m is the normal to the surface Σ. The prescription according to this philosophy is to add

$$\int_{\partial\Sigma} 2n_m \pi^{mn} \beta_n \, \mathrm{d}\sigma, \tag{93}$$

which is a term that accounts for the total momentum and total angular momentum at spatial infinity, if

- β_n behaves like a constant translation (e.g. $\beta_n = \text{constant}$) for $r \to \infty$,
- β_n behaves like a constant rotation (e.g. $\beta_n = \epsilon_{nmk}\omega^m x^k$ and $\omega^m =$ constant),

with asymptotically flat metric and asymptotically euclidian motions. These asymptotic Poincaré charges are constants of motion. They are called ADM (Arnowitt-Deser-Misner) momentum and angular momentum. The surface at infinity $\partial\Sigma$ should be interpreted as the large radius limit of a 2-sphere

$$\int_{\partial\Sigma} 2n_m \delta\pi^{mn} \beta_n \, \mathrm{d}\sigma := \lim_{\mathrm{R}\to\infty} \int_{\mathrm{S}^2_\mathrm{R}} 2\mathrm{n_m} \delta\pi^{\mathrm{mn}} \beta_\mathrm{n} \, \mathrm{d}\sigma, \tag{94}$$

where the limit has to be taken only after performing the integration.

4.2. Functionnal differentiablility with respect to h_{mn}

The potential difficulties could arise from $R^{(3)}$ which contains second derivatives with respect to the metric h. In order to take the h_{mn} derivatives of $R^{(3)}$ we need to prove a preliminary result.

Lemma 4.1.

$$h^{mn}\delta R_{mn} = D_k V^k, \tag{95}$$

where

$$V^k = h^{mn}\delta\Gamma^k_{mn} - h^{km}\delta\Gamma^n_{nm}, \tag{96}$$

$$\delta\Gamma^k_{mn} := \frac{1}{2}h^{kj}(-D_j\delta h_{mn} + D_n\delta h_{jm} + D_m\delta h_{nj}). \tag{97}$$

Proof of Lemma 4.1. The proof can be obtained in two ways. The normal Riemannian coordinate system can simplify the manipulations but there is also a coordinate independent proof.

From this result we have

$$h^{mn}\delta\Gamma^k_{mn} = -\frac{1}{2}D^k\delta h + h^{kj}D^n\delta h_{nj}, \tag{98}$$

where δh stands for $h^{mn}\delta h_{mn}$. In a similar way, we obtain

$$h^{km}\delta\Gamma^n_{nm} = \frac{1}{2}D^k\delta h. \tag{99}$$

In consequence, one can express V^k as

$$V^k = h^{kj} D^n \delta h_{nj} - D^k \delta h. \tag{100}$$

Therefore the variation of $\int \alpha\sqrt{h}\, R^{(3)}\, \mathrm{d}^3\mathrm{x}$ leads to the surface integral

$$\int_{\partial\Sigma} \alpha\sqrt{h}\, n^k (D^n \delta h_{nk} - D_k \delta h) = \int \alpha\sqrt{h}\, n^k h^{nm} (D_m \delta h_{nk} - D_k \delta h_{mn}). \tag{101}$$

In order to discuss the asymptotic fall-off behaviour the geometry needs a convenient coordinate system such that the metric takes an asymptotically flat form, and only with respect to this system one can discuss fall-off conditions. The asymptotic fall-off for asymptotically flat configurations are such that

$$h_{mn}(\vec{x}) = \delta_{mn} + \frac{k_{mn}(\vec{x}/r)}{r} + O(1/r^2), \tag{102}$$

$$\pi^{mn}(\vec{x}) = \frac{P_{mn}(\vec{x}/r)}{r^2} + O(1/r^3). \tag{103}$$

Remark 1 The convergence of the boundary term for an asymptotic rotation $\epsilon_{nkl}\omega^k x^l$ is not automatic. A sufficient condition for convergence that includes the description of black holes or stars with non-zero angular momentum is to restrict $P_{mn}(\vec{x}/r)$ to be an even function, i.e. $P_{mn}(\vec{x}/r) = P_{mn}(-\vec{x}/r)$.

Remark 2 Since the covariant derivative differs only from the partial derivative by the Christoffel symbols which are linear in the first order derivative of h_{mn},

$$D\delta h = \partial \delta h + \text{terms} \sim \frac{1}{r^2}\delta h, \tag{104}$$

but $\frac{1}{r^2}\delta h \sim \frac{1}{r^3}$ and hence no contribution is expected in the surface integral in the limit $r \to \infty$.
If we apply the second remark to the equation (101), we obtain

$$\int_{\partial\Sigma} \alpha\sqrt{h}\, n^k (D^n \delta h_{nk} - D_k \delta h). \tag{105}$$

However, asymptotically $D^n \delta h_{nk}$ can be replaced by $\partial^n \delta h_{nk}$ and the surface term can be written as a total variation

$$\delta \int_{\partial\Sigma} \alpha\sqrt{h}\, n^k (\partial^n h_{nk} - \partial_k h_{nn}). \tag{106}$$

Finally the term to be added to $S(\alpha)$ in order to establish functional differentiability with respect to h_{mn} is

$$S_{ADM} = \int_{S^2_R \to \infty} n^k (\partial_n h_{nk} - \partial_k h_{nn})\, \mathrm{d}\sigma. \tag{107}$$

The so-called "ADM energy" has to be evaluated in an asymptotically euclidian coordinate system. S_{ADM} is proportional to the total energy of the configuration given by the canonical variable (h, π).

Example The Schwarzschild solution can be written in isotropic coordinates as

$$g = -\left[\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right]^2 dt^2 + \left[1 + \frac{m}{2r}\right]^4 \vec{dx} \cdot \vec{dx}, \tag{108}$$

which is spatially conformally flat. The spatial metric is $h_{mn} = \delta_{mn}(1 + \frac{m}{2r})^4$. The evaluation of the corresponding ADM mass gives $S_{ADM} = 16\pi m$ and allows to identify m as a mass. The gravitational mass is proportional to the binding energy, in particular to the gravitational binding energy. We have $S_{ADM} = 16\pi m = 16\pi \frac{GM}{c^2}$ and consequently we define $M_{ADM} = \frac{c^2}{16\pi G} S_{ADM}$. Let us note the efficiency of this mass concept : only the total amount of energy is proportional to the mass.

Theorem 1. *If (h, π) satisfies the constraints then $M_{ADM}[h, \pi] \geq 0$. Furthermore, if $M_{ADM}[h, \pi] = 0$ then the initial data evolve into flat Minkowski space (or portions thereof).*

Finally, we note that the functional ADM mass

$$\int_{S^2(\infty)} n^k (\partial_m h_{mk} - \partial_k h_{mm})\, \mathrm{d}\sigma, \tag{109}$$

seems to depend only on h and not on π. However this is deceptive since h is tight to π by the constraints. Note that the ADM functional is only meant to be evaluated on those 3-metrics which satisfy the constraints for some momentum π. It is easy to come up with regular metrics that formally give rise to negative ADM masses. The positive mass theorem tells us that those metrics can never satisfy the constraints, whatever choice for π is made (compare Exercise 9).

4.3. The Poisson Structure of the Constraints

One can prove quite easily

$$\Big\{h_{mn}, V(\beta)\Big\} = (L_\beta h)_{mn}, \tag{110}$$

$$\Big\{\pi^{mn}, V(\beta)\Big\} = (L_\beta \pi)^{mn}, \tag{111}$$

which shows that the vector constraints generates the infinitesimal spatial diffeomorphisms. More tedious is to calculate the Poisson brackets of constraints. For the constraints, we have

$$\Big\{V(\beta), V(\beta')\Big\} = V\big([\beta, \beta']\big), \tag{112}$$

$$\Big\{V(\beta), S(\alpha)\Big\} = S\big(\beta(\alpha)\big), \tag{113}$$

$$\Big\{S(\alpha), S(\alpha')\Big\} = V\big(\alpha\ \mathrm{grad}_h(\alpha') - \alpha'\ \mathrm{grad}_h(\alpha)\big), \tag{114}$$

where $\mathrm{grad}_h := (h^{mn}\partial_n\alpha)\partial_m$ which depends on h. Let us elaborate on the structure of the constraints. The first Poisson bracket shows that the $V(\beta)$ form a Lie-subalgebra. However the second bracket states that this subalgebra does *not* form an ideal. Finally the third bracket shows that the whole set of constraints does *not* form a Lie-algebra. These facts often lead to the the statement that the algebra does not close, or is soft or open. The proper statement is that the constraints form a Lie-algebroid.
Let us focus on the third bracket. The fact that there is no $S(\alpha'')$ at the *rhs* of the third relation tells us that going with α_1 from Σ to Σ_1 and with α_2 from Σ to Σ_{12} results in the same hypersurface than first applying the normal deformation with $\alpha_2 : \Sigma \to \Sigma_2$ and then with $\alpha_1 : \Sigma_2 \to \Sigma_{21}$: that this is a diffeomorphism of $\Sigma_{12} \to \Sigma_{21}$.

$$\begin{tikzcd}
& \Sigma_2 \arrow[r, "(\alpha_1,\beta_1)"] & \Sigma_{21} \arrow[dd, "(\alpha',\beta')"] \\
\Sigma \arrow[ur, "(\alpha_2,\beta_2)"] \arrow[dr, "(\alpha_1,\beta_1)"'] & & \\
& \Sigma_1 \arrow[r, "(\alpha_2,\beta_2)"'] & \Sigma_{12}
\end{tikzcd}$$

Figure 3: The diagram commutes.

General Remark. These three relations are *universal* in the sense that they are the same for all diffeomorphism invariant theories if put into Hamiltonian form. In particular, they are independent of having started from Einstein-Hilbert action. What is particular to our constraints $V(\beta)$ and $S(\alpha)$ is their dependence on just h and π.
To illustrate this point we can derive these relations from pure geometric classification. Consider the space of space-like embeddings $\Sigma \hookrightarrow M$ that we denote $\mathrm{Emb}(\Sigma, M)$. The obvious left-action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(\Sigma, M)$ is defined as follows: let y^μ be the local coordinates on M, which are x^i on Σ, then $y^\mu(x^i)$ belongs to $\mathrm{Emb}(\Sigma, M)$. Let $X(V)$ be a vector field on $\mathrm{Emb}(\Sigma, M)$ corresponding to the vector field V on M, which looks like

$$X(V) = \int_\Sigma \mathrm{d^3x\, V^\mu\big(y(x)\big)} \frac{\delta}{\delta \mathrm{y^\mu(x)}}. \tag{115}$$

Then it is easy to show that the infinitesimal version of the left action of Diff(M) on Emb(Σ, M) is

$$\big[X(V), X(V')\big] = X([V, V']). \tag{116}$$

But now we can decompose $X(V)$ in a point dependent way into "normal" and "tangential" components with respect to the embedding:

$$X(\alpha, \vec{\beta}) = \int_\Sigma \mathrm{d^3x} \left(\alpha(\mathrm{x})\mathrm{n}^\mu[\mathrm{y}](\mathrm{x}) + \beta^{\mathrm{m}}(\mathrm{x})\mathrm{y}^\mu_{,\mathrm{m}}(\mathrm{x})\right) \frac{\delta}{\delta \mathrm{y}^\mu(\mathrm{x})}. \tag{117}$$

The Lie-bracket becomes

$$\big[X(\alpha_1, \beta_1), X(\alpha_2, \beta_2)\big] = -X(\alpha', \beta'), \tag{118}$$

where

$$\alpha' = \beta_1(\alpha_2) - \beta_2(\alpha_1), \tag{119}$$
$$\beta' = [\beta_1, \beta_2] + \sigma\left(\alpha_1 \mathrm{grad}_h(\alpha_2) - \alpha_2 \mathrm{grad}_h(\alpha_1)\right), \tag{120}$$

and σ is defined via the signature of g : $(-\sigma, +, +, +)$. This is this the same result obtained previously up to an overall minus sign. The reason of this is easily seen. Let us consider a simpler case: if T acts on coordinates by $T : \vec{x} \to \vec{x} + \vec{a}$ then $(Tf)(\vec{x}) = f(\vec{x} - \vec{a})$. The generator takes the form $T = -\frac{d}{dx}$ with a judicious minus sign. In the same way the generator of the transformations which fulfills the right relations is $-X$, and this explains the overall minus (anti-Lie homomorphism).
Actually it is possible to reconstruct the Einstein's equations from the kinematics of hypersurfaces in 4-spaces (geometrodynamics).

Theorem 2. *[5] In four space-time dimensions (Lorentzian for $\sigma = 1$ and Euclidian for $\sigma = -1$) the most general functional $H[h, \pi](\alpha, \beta)$ (i.e. on the phase space of 3-metrics h) satisfying the conditions (112), (113), (114) and subject to the restriction that $\{H(\alpha = 0, \beta), h\} = L_\beta h$ and $\{H(\alpha, \beta = 0), h\} = 2\alpha L_n K$, is given by that of General Relativity with κ and λ as free parameters, i.e.*

$$H[h, \pi](\alpha, \beta) = S_\alpha[h, \pi] + V_\beta[h, \pi], \tag{121}$$

where

$$V_\beta[h, \pi] = -2 \int_\Sigma \mathrm{d^3x}\, \mathrm{h_{kn}} \beta^{\mathrm{k}} \mathrm{D_m} \pi^{\mathrm{mn}}, \tag{122}$$

$$S_\alpha[h, \pi] = \int_\Sigma \mathrm{d^3x}\, \alpha \left\{ 2\kappa \mathrm{G_{ijnm}} \pi^{\mathrm{ij}} \pi^{\mathrm{nm}} - \frac{\sqrt{\mathrm{h}}}{2\kappa}(\mathrm{R} - 2\Lambda) \right\}, \tag{123}$$

with the (1+5)- Lorentzian metric

$$G_{ijnm} = \frac{1}{2\sqrt{h}}\left(h_{in}h_{jm} + h_{im}h_{jn} - h_{ij}h_{nm}\right), \tag{124}$$

which is known as DeWitt metric.
(A proof can be found in [5].)

5. Foliation Condition and Construction of Vacuum Initial Data

As stated in section 3.3, the minimal vacuum initial data set in GR consists of a differentiable, spatial 3-manifold Σ (the initial spacelike hypersurface), a Riemannian spatial 3-metric h on Σ (the induced metric on the initial hypersurface) and another symmetric second rank tensor field K on Σ (the extrinsic curvature of Σ). In the presence of sources our initial data would include the energy density ρ and the momentum density j^a.
The configuration variable h and the velocity variable K have to be chosen in such a way that the constraints (55) and (57) are fulfilled. Restricting the initial data by the scalar and vector constraints remove the gauge freedom in the dynamical theory: the lapse function α is related to time rescaling and the shift 3-vector β generates spatial diffeomorphisms of Σ. Lapse and shift can be seen as the gauge potentials that have to be fixed to find a solution to the initial value problem.
In this section we will simplify the constraint equations assuming *time-symmetry* and we will present Brill waves and (Multi) Schwarzschild black hole space-times as simple, non-trivial solutions to the vacuum constraint equations.
Note in the following we shall work with the pair (h, K) rather than with the (h, π) variables used in the last section.

5.1. Foliation Condition

Note that step 2.) in the initial value formulation of GR (see section 3.3) *does not* impose any topological constraints on the choice of $\Sigma \in M$: All topologies Σ allow *some* initial data; the Gauß-Codazzi (50) and the Codazzi-Mainardi (51) equations *do not* impose topological restrictions [6]. But an useful gauge condition on the slices Σ of space-time is to take spatial slices $\Sigma_t \in M_i$ such that the trace of the extrinsic curvature $h^{mn}K_{mn}$ is a spatial constant (not in time) on each Σ_t. This is the so-called constant mean curvature gauge. Here we shall look at vanishing mean curvature:

$$h^{mn}K_{mn} = \mathrm{Tr}_h(K) \stackrel{!}{=} 0, \qquad (125)$$

which is known as the *maximal slicing condition*.

Lemma 5.1. *If $\Sigma \hookrightarrow M$ satisfies the maximal slicing condition $\mathrm{tr_h}(\mathrm{K}) = 0$ then the volume of Σ in M is stationary with respect to normal deformations.*

Proof of Lemma 5.1. The 3-dimensional volume functionals for domains in Σ are given by

$$\mathrm{Vol_h}(\Sigma) = \int_\Sigma \mathrm{d^3x}\,\sqrt{\mathrm{h}}\,.$$

Taking the variation with respect to h yields

$$\delta \mathrm{Vol}_h(\Sigma) = \frac{1}{2}\int_\Sigma \mathrm{d^3x}\,\sqrt{\mathrm{h}}\,\mathrm{h^{nm}}\delta \mathrm{h_{nm}}.$$

Using Proposition 3.2 and equation (43), we can write

$$\delta h_{nm} = (L_{\alpha n} h)_{nm} = 2\alpha K_{nm}.$$

Hence the variation is

$$\delta \mathrm{Vol_h}\,(\Sigma) = \int \mathrm{d^3x}\,\sqrt{\mathrm{h}}\,\alpha\,\mathrm{h^{mn}K_{mn}}.$$

This vanishes for all $\alpha : \Sigma \to \mathbb{R}$ iff the maximal slicing condition $h^{mn}K_{mn} = 0$ is imposed. In a Lorentzian space-time the volume must be maximal rather than minimal (like for geodesics), hence one speaks of the maximal slicing condition.

Let us now choose a maximal slice $\Sigma \subset M$ and derive a condition on α that assures the preservation of (125) in time. First we calculate the time dependance of $h^{mn}K_{mn}$:
We have

$$\begin{aligned} L_n(h^{mn}K_{mn}) &= -h^{ni}h^{mj}(L_n h_{ij})K_{nm} + h^{mn}L_n K_{mn} \\ &= -2K_{nm}K^{nm} - h^{nm}R_{0n0m} + K_{nm}K^{nm} + h^{nm}\left(a_n a_m + D_n a_m\right), \end{aligned}$$

where we used (43) and (60) to replace $L_n h_{ij}$ and $L_n K_{mn}$, respectively. Hence

$$L_n(h^{mn}K_{mn}) = -R_{00} - K_{nm}K^{nm} + \frac{1}{\alpha}\Delta\alpha, \tag{126}$$

where we used (71) and $\Delta := D_i D^i$. Summarized, the time evolution of $h^{mn}K_{mn}$ is given by

$$L_{\partial_t}(h^{mn}K_{mn}) = \partial_t(h^{mn}K_{mn}) = \left(\Delta - R_{00} - K_{nm}K^{nm}\right)\alpha + L_\beta(h^{nm}K_{nm}),$$

where the "dotting" is defined analogous to (69).
So given that initially $h^{mn}K_{mn} = 0$, we preserve maximality (125) provided that we choose α according to the equation

$$\mathcal{O}\alpha = 0, \tag{127}$$

with the elliptical operator

$$\mathcal{O} := \Delta - R_{00} - K_{nm}K^{nm}, \tag{128}$$

where $R_{00} = R_{\perp\perp} = Ric(n,n)$ and n is normal to the surface.
Assuming the strong energy condition

$$\begin{aligned} R_{\perp\perp} &= \kappa\left(T_{\perp\perp} + \frac{1}{2}T^\mu_\mu\right) \\ &= \frac{1}{2}\kappa\left(T_{\perp\perp} - \sum_{a=1}^{3} T_{aa}\right) \geq 0, \end{aligned}$$

(i.e. $\rho \geq 3p/c^2$ for a perfect fluid), we can verify that (127) implies

$$\Delta\alpha = \underbrace{(R_{\perp\perp} + K_{nm}K^{nm})}_{\geq 0}\alpha, \tag{129}$$

for any smooth function α in the kernel of $\mathcal{O}$. Consequently, α cannot have a positive local maximum or a negative local minimum on Σ. In a vacuum space-time, we can rewrite the elliptical operator $\mathcal{O}$ in equation (127) purely in terms of the intrinsic geometry of Σ

$$\mathcal{O} = \Delta - R^{(3)}, \tag{130}$$

where we used the scalar constraint (55) to replace $K_{nm}K^{nm}$ in equation (128) by $K_{nm}K^{nm} = R^{(3)}$.
The use of the maximal slicing gauge condition (125) is as follows: Since it preserves maximality of Σ, it will avoid Σ running into regions of strong spatial compression, i.e. it has a *singularity avoiding character*. This property turns it into a often used gauge choice in numerical GR.
Note that not all vacuum space-times admit maximal slices [6]; there exists space-times which *have not maximal* initial data sets.

5.2. Construction of Time-Symmetric Initial Data

Simple initial data are *time-symmetric* ones, i.e. pairs (h, K) with vanishing extrinsic curvature,

$$K_{mn} = 0. \tag{131}$$

In this case the hypersurface Σ is *totally geodesic* in M; which means that geodesics starting *in* and *tangential to* $\Sigma \subset M$ will remain in Σ. This follows from (22) which simplifies for $K = 0$ to

$$\nabla_X Y = D_X Y, \tag{132}$$

for all spatial vectors X, Y. Consequently, given a curve with tangent vector field γ' over γ, then

$$\begin{aligned} \nabla_{\gamma'}\gamma' &= 0 \quad \text{in } M, \\ D_{\gamma'}\gamma' &= 0 \quad \text{in } \Sigma, \end{aligned}$$

and γ is a geodesic in Σ given that it is a geodesic in M. Imposing a totally geodesic Σ (131) and performing a maximal development (127) of the initial data set also shows that there is an isometry of a neighborhood U of Σ in M which has the points of Σ as fixed points and interchanges the sides of Σ in M – a *time-reversal isometry*. Hence we speak of a time-symmetric initial data set, since the forward evolution of such data is isometric to the backward evolution.
For time-symmetric vacuum data, the diffeomorphism constraints (57) are trivially satisfied, since they are linear in K: $K = \pi = 0$. Also the total linear and angular momentum necessarily vanish. The Hamiltonian constraint C_H

$$K_{nm}K^{nm} - (h^{nm}K_{nm})^2 - (R^{(3)} - 2\Lambda) = -2\kappa T_{\perp\perp},$$

requires for $\Lambda = 0, T_{\mu\nu} = 0$ that (Σ, h) is of intrinsic vanishing scalar curvature

$$R^{(3)}(h) = 0. \tag{133}$$

Consequently, the number of independent curvature components reduces from 6 to 5.

To solve (133), we make the following *Ansatz*

$$h = \Phi^4 h', \tag{134}$$

where Φ is an overall conformal factor and the fourth power was chosen just for convenience. We note the conformal transformation law for the Ricci-scalar $R^{(3)}$

$$R^{(3)}(\Phi^4 h') = -8\Phi^{-5}\left(\Delta_{h'} - \frac{1}{8}R^{(3)}(h')\right)\Phi := -8\Phi^{-5}C_{h'}\Phi \stackrel{!}{=} 0, \tag{135}$$

where $\Delta_{h'}$ denotes the Laplacian for the metric h'.

We are interested in C^2- solutions with $\Phi > 0$ and where (Σ, h) has no boundaries at finite distances, that is Σ should be topologically complete in the metric topology defined through h (By Hopf-Rinow-DeRahm Theorem [7] this is equivalent to (Σ, h) being geodesically complete). Hence Σ will have a finite number of asymptotically flat ends, to each of which we can associate a ADM mass.

5.2.1. Example 1: "Brill Waves" (axisymmetric)

"Brill waves" [8] are the simplest, asymptotically flat, non-trivial solutions to $C_{h'}\Phi = 0$ on $\Sigma = \mathbb{R}^3$ describing localized gravitational waves of non-zero total ADM energy. We use an axially symmetric 3-metric h' of the following form

$$h' = e^{\lambda q(z,\rho)}(dz^2 + d\rho^2) + \rho^2 d\varphi^2, \tag{136}$$

where the profile function q is required to show for $r \to \infty$ a r^{-2}- fall-off behavior and in its first derivatives a fall-off like r^{-3} to ensure asymptotically flatness. Regularity on the axis requires $q = \partial_\rho q = 0$ along the z-axis. λ is a constant introduced to parameterize the overall amplitude.

As stated above, the metric h' is taken as the conformally transformed to the metric h (134) so that the metric h' satisfies the time-symmetry constraint (135) of the vacuum Einstein equations. Inserting (136) in (135) leads to

$$\left(\Delta_f + \frac{\lambda}{4}\Delta^{(2)}q\right)\Phi = 0, \tag{137}$$

where the *flat* Laplacian is given by

$$\Delta_f = \partial_\rho^2 + \partial_z^2 + \frac{1}{\rho^2}\partial_\varphi^2, \tag{138}$$

and

$$\Delta^{(2)} = \partial_\rho^2 + \partial_z^2. \tag{139}$$

Now, for any given function q there exists a critical value $\epsilon > 0$, such that for $0 < \lambda < \epsilon$ a unique solution for Φ exists.

5.2.2. Example 2: Time-Symmetric Conformally Flat Data for Black Holes

We can already generate time-symmetric initial data sets for the Einstein vacuum equations that can represent black hole space-times by assuming a flat h' in (134). Since the 3-metric h in (134) is then conformally flat there exists a coordinate system such that

$$h_{mn} = \Phi^4 \delta_{mn}. \tag{140}$$

Inserting (140) into the time-symmetry constraint equation (133) results in the harmonic equation

$$\Delta_f \Phi = 0, \tag{141}$$

where the flat Laplacian Δ_f is given by

$$\Delta_f = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{142}$$

Schwarzschild Data. An immediate, non-trivial solution to (141) for Σ with two ends, i.e. for $\Sigma = \mathbb{R}^3 - \{0\}$ and $\Phi(r \to \infty) = 1$ can be found with

$$\Phi(\vec{x}) = 1 + \frac{m/2}{r}, \tag{143}$$

and hence the 3-metric h in (140) reads

$$h = \left(1 + \frac{m/2}{r}\right)^4 \delta, \tag{144}$$

where the mass m has to be chosen positive and the Euclidean metric δ in the spatial part (144) of the Schwarzschild solution in isotropic coordinates is given by

$$\delta = \left(dr^2 + r^2 \mathrm{d}\Omega^2\right), \tag{145}$$

with $\mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$.

Note that there are two isometries I and I' of the 3-metric h defined in (144)

$$I(r, \theta, \varphi) := \left(\frac{(m/2)^2}{r}, \theta, \varphi\right), \tag{146a}$$

$$I'(r, \theta, \varphi) := \left(\frac{(m/2)^2}{r}, \pi - \theta, \varphi + \pi\right). \tag{146b}$$

Here I describes a reflection at a 2-sphere with radius $r = m/2$ (see Fig. 4), whereas I' contains an additional antipodal reflection. The map I (146a) has a fixed point set $S = \{\vec{x} \,|\, r = m/2\}$ and hence S is totally geodesic, which can be proven using the uniqueness of geodesics γ starting on and tangentially to S. S being totally geodesic implies it is a minimal surface and therefore an *apparent* horizon.

The map I' has no fixed points, hence we can construct a manifold Σ' with only one isometric end by taking

$$\Sigma' = \Sigma / I'. \tag{147}$$

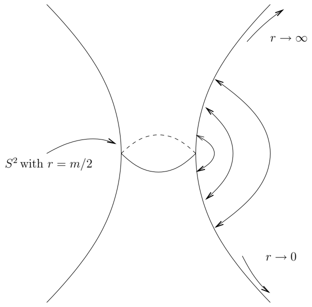

Figure 4: Inversion I at $r = m/2$ at the 2-sphere.

This can be thought of as cutting away the region $r < m/2$ and identifying antipodal points on the S^2- boundary $r = m/2$. In this way one obtains a manifold with one end ($r \to \infty$) and a compact interior. Topologically Σ' is the real projective space $\mathbb{R}P^3$ minus a point:

$$\Sigma' \simeq \mathbb{R}P^3 - \{\text{point}\}. \tag{148}$$

Let us conclude this section with the following theorem:

Theorem 3. *Each 3-manifold Σ can carry some initial data (h, K) that satisfy the Hamiltonian (55) and diffeomorphism (57) constraints. However, special initial data sets like sets which admit maximal slices do impose topological constraints.*

Multi-Schwarzschild data. We can now generalize the results of the last section to obtain time-symmetric initial data for n black hole space-times. Taking $\Sigma = \mathbb{R}^3 - \{\vec{c}_1, \ldots, \vec{c}_n\}$ we make the following *Ansatz* for the conformal factor Φ in (144) for n black holes

$$\Phi(\vec{x}) = 1 + \sum_{i=1}^{n} \frac{a_i}{r_i}, \tag{149}$$

where $r_i := ||\vec{x} - \vec{c}_i||$, $\vec{c}_i$ is the "location" of the i'th black hole and the positive parameter a_i describes the strength of the i'th pole. (149) is a solution to (141). Σ has $n+1$ asymptotically flat ends at $\vec{c}_1, \ldots, \vec{c}_n$ and $\vec{x} \to \infty$. At each end we can calculate an ADM mass: The masses m_i of the i'th black hole at $\vec{c}_i$ are given by

$$m_i = 2a_i(1 + \chi_i), \quad \text{where } \chi_i := \sum_{j \neq i} \frac{a_j}{r_{ij}}, \quad \text{and } r_{ij} := ||\vec{c}_i - \vec{c}_j||, \tag{150}$$

and the total mass M at $\vec{x} \to \infty$ reads

$$M = 2 \sum_{i=1}^{n} a_i. \tag{151}$$

The binding energy ΔM simplifies in leading order to the Newtonian expression

$$\Delta M := M - \sum_{i=1}^{n} m_i = -2\sum_{i=1}^{n} a_i \chi_i = -2\sum_{i=1}^{n}\sum_{j\neq i} \frac{a_i a_j}{r_{ij}} < 0. \tag{152}$$

Note that the area A_i of the n minimal surfaces of the manifold $\Sigma = \mathbb{R}^3 - \{\vec{c}_1, \ldots, \vec{c}_n\}$ is bounded below by

$$A_i > 16\pi\,(2a_i)^2, \tag{153}$$

where the rhs of (153) corresponds to the minimal area in the metric (144) for one hole with parameter $m = 2a_i$:

$$\tilde{h} = \left(1 + \frac{a_i}{r_i}\right)^4 \delta, \tag{154}$$

which is strictly smaller than

$$h = \left(1 + \sum_{i=1}^{n} \frac{a_i}{r_i}\right)^4 \delta. \tag{155}$$

Penrose inequality implies an upper bound for the area

$$A_i \leq 16\pi m_i^2 \stackrel{(150)}{=} 6\pi(2a_i)^2(1+\chi_i)^2. \tag{156}$$

For time-symmetric initial data minimal surfaces correspond to apparent horizons. The area of the apparent horizon is a lower bound for the area of an event horizon for that hole. The total area of the event horizon cannot decrease by Hawking's Theorem. Hence we can put an upper bound on the energy released e.g. in the form of gravitational waves when n black holes merge into a single final one that eventually becomes static. Its event horizon is bounded below by

$$A_f \geq 16\pi M_f^2, \tag{157}$$

which corresponds to a stationary black hole of mass

$$M_f = \left(\frac{A_f}{16\pi}\right)^{\frac{1}{2}} = 2\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}. \tag{158}$$

Hence the mass/energy-loss is bounded by

$$\Delta M := M - M_f \leq 2\left\{\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\right\}. \tag{159}$$

Setting all a_i equal to a (159) simplifies to

$$\Delta M \leq 2na\left(1 - \frac{1}{\sqrt{n}}\right),$$

or

$$\frac{\Delta M}{M} \leq \left(1 - \frac{1}{\sqrt{n}}\right).$$

Hence for two black holes we can approximate the efficiency of energy extraction (gravitational waves) of the merging black hole by

$$\frac{\Delta M}{M} < \left(1 - \frac{1}{\sqrt{2}}\right) \approx 29\%.$$

6. Exercises

6.1. Problem Set 1

Let us begin by clarifying notations. Let $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$ be vector fields. Then

$$\nabla_X Y = X^\mu (\nabla_\mu Y^\nu) \partial_\nu =: X^\mu Y^\nu_{;\mu} \partial_\nu,$$

i.e. $\nabla_\mu Y^\nu$ is the ν-component of the vector field $\nabla_{\partial_\mu} Y$.

Exercise 1. *Let X, Y and Z be three vector fields. Prove that*

$$\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z = X^\mu Y^\nu [\nabla_\mu, \nabla_\nu] Z^\lambda \partial_\lambda,$$

and thus

$$[\nabla_\mu, \nabla_\nu] Z^\lambda = R^\lambda_{\sigma\mu\nu} Z^\sigma.$$

Exercise 2. *Prove*

$$R_{0a0b} = -(L_n K)_{ab} + K^c_a K_{cb} + a_a a_b + D_a a_b.$$

Hint:

$$R_{0a0b} = R_{\mu a \nu b} n^\mu n^\nu = n^\nu [\nabla_\nu, \nabla_b] n_a.$$

Exercise 3. *Prove that*

$$\begin{aligned} \overset{\nabla}{R}(X,Y)Z =& \overset{D}{R}(X,Y)Z + (\nabla_X n) K(Y,Z) - (\nabla_Y n) K(X,Z) \\ &+ n\left((D_X K)(Y,Z) - (D_Y K)(X,Z)\right), \end{aligned}$$

where X, Y, Z are any spatial vector fields.
Recall that from the above identity the Gauss-Codazzi (50) and Codazzi-Mainardi (51) evolution equations follow.

Exercise 4. *Show that Maxwell's equations are the Hamilton's equations for*

$$H = \int_\Omega d^3x \left\{ \frac{1}{2}\left(\vec{E}^2 + (\vec{\nabla} \times \vec{A})^2\right) + \phi(\rho - \vec{\nabla} \cdot \vec{E}) - \vec{A} \cdot \vec{j} \right\},$$

where $p = -\vec{E}$ is the conjugate momentum of $q = \dot{\vec{A}}$ and ϕ is the scalar potential which plays the role of a Lagrange multiplier.

- *What motions are generated by $\phi(\rho - \vec{\nabla} \cdot \vec{E})$?*
- *Show that $-\vec{E}(x)$ is the L^2-orthogonal of $\dot{\vec{A}}$ to gauge orbits when $\rho = 0$.*
- *Restore gauge invariance on $\partial\Omega$ by introducing new degrees of freedom on $\partial\Omega$ assuming $(\vec{n} \times \vec{B})|_{\partial\Omega} = 0$.*

6.2. Solutions to Problem Set 1

Solution 1. *Let us detail the calculation: we need to use*

$$\nabla_X \nabla_Y Z = X^\mu Y^\nu_{,\mu} \nabla_\nu Z + X^\mu Y^\nu \nabla_\mu \nabla_\nu Z,$$

and

$$[X,Y] = X^\mu Y^\nu_{,\mu} \partial_\nu - Y^\mu X^\nu_{,\mu} \partial_\mu.$$

Hence one can finally obtain the solution

$$\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z = X^\mu Y^\nu [\nabla_\mu, \nabla_\nu] Z^\lambda \partial_\lambda,$$

while the last statement follows from the definitions (2) and (3).

Solution 2. *We first remark that*

$$R_{0a0b} = R_{\mu a \nu b} n^\mu n^\nu = -n^\nu \big(\nabla_\nu \nabla_b - \nabla_b \nabla_\nu\big) n_a. \tag{160}$$

Now we make use of:

$$\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu, \tag{161}$$

with the normal acceleration $a_\nu = (\nabla_\mu n_\nu) n^\mu$. The rhs of the equation (160) becomes

$$\begin{aligned} &\nabla_b (n^\nu \nabla_\nu n_a) - (\nabla_b n^\nu)(\nabla_\nu n_a) - n^\nu \nabla_\nu (K_{ba} - n_b a_a) \\ &= \nabla_b a_a - (K^\nu_b - n_b a^\nu)(K_{\nu a} - n_\nu a_a) - n^\nu \nabla_\nu K_{ba} + a_b a_a + n_b n^\nu \nabla_\nu a_a. \end{aligned}$$

Some simplifications arise because the vector field n has no spatial component: $n_b = 0$ and $K^\nu_b n_\nu = 0$. Finally, we obtain for (160)

$$R_{0a0b} = \nabla_b a_a - K^\nu_b K_{\nu a} - n^\nu \nabla_\nu K_{ba} + a_b a_a. \tag{162}$$

Using the formula for the Lie derivative of a tensor field:

$$\begin{aligned} (L_n K)_{ab} &:= (\nabla_n K)_{ab} + K_{\mu b} \nabla_a n^\mu + K_{a\mu} \nabla_b n^\mu \\ &= (\nabla_n K)_{ab} + K_{\mu b}(K^\mu_a - n_a a^\mu) + K_{a\mu}(K^\mu_b - n_b a^\mu) \\ &= (\nabla_n K)_{ab} + 2 K_{\mu b} K^\mu_a, \end{aligned} \tag{163}$$

we obtain

$$(\nabla_n K)_{ab} = (L_n K)_{ab} - 2 K_{\mu b} K^\mu_a.$$

Finally, we substitute this last result in (162) and we get the desired answer:

$$R_{0a0b} = -(L_n K)_{ab} + K^c_a K_{cb} + a_a a_b + D_a a_b.$$

(see Proposition 3.4).

Solution 3. *We begin by rewriting the Riemann curvature tensor in terms of the covariant derivative D on the spatial slice and the extrinsic curvature K using:*

$$\nabla_X Z = D_X Z + nK(X, Z).$$

Recall that

$$\overset{\nabla}{R}(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z.$$

We get after an elementary algebra:

$$\begin{aligned}\overset{\nabla}{R}(X,Y)Z = &\left(D_X D_Y - D_Y D_X - D_{[X,Y]}\right) Z \\ &+ D_X\left(nK(Y,Z)\right) - D_Y\left(nK(X,Z)\right) \\ &+ n\left(K(X, D_Y Z) - nK(Y, D_X Z)\right) \\ &- n\left(K(X,n)K(Y,Z) - K(Y,n)K(X,Z)\right).\end{aligned}$$

A simple rearrangement produces

$$\begin{aligned}\overset{D}{R}(X,Y)Z = &(D_X n + nK(X,n))K(Y,Z) - (D_Y n + nK(Y,n))K(X,Z) \\ &+ n\left(\nabla_X K(Y,Z) - \nabla_Y K(X,Z)\right),\end{aligned}$$

which leads to the final result (see Proposition 3.4).

Solution 4. *The equations of motion are*

$$\dot{\vec{A}} = \{\vec{A}(\vec{x}), H\} = -\vec{E}(\vec{x}) - \vec{\nabla}\phi, \tag{164}$$

$$-\dot{\vec{E}}(\vec{x}) = \{-\vec{E}(\vec{x}), H\} = \vec{j}(\vec{x}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{A}). \tag{165}$$

The conjugate momentum of $\vec{A}$ is $\vec{E} = -\dot{\vec{A}} - \vec{\nabla}\Phi$. In the absence of charge density $\vec{\nabla} \cdot \vec{E} = 0$. This is the same thing as saying

$$\int_\Omega \vec{E} \cdot \vec{\nabla}\Theta \, \mathrm{d}^3 x = 0 = \langle \vec{E}, \vec{\nabla}\Theta \rangle_{L^2},$$

for every Ω of compact support. The term $\Phi(\rho - \vec{\nabla} \cdot \vec{E})$ generates local gauge transformations only if the electric field falls off fast enough at infinity $E \to 1/r^2$ and the Lagrange multiplier goes to zero at infinity: $\Phi \to 0$. If Φ tends to a constant at $r \to \infty$, there is an obstruction for the gauge transformation. The solution to this issue is to add additional degrees of freedom at the boundary at infinity. We define:

$$\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R},$$
$$f : \partial\Omega \times \mathbb{R} \to \mathbb{R},$$

and we add to the action the kinetic term

$$\int \mathrm{dt} \int_{\partial\Omega} \dot{\lambda} \mathrm{f} \, \mathrm{d}\omega,$$

and the boundary term that compensates for the gauge transformation

$$\int \mathrm{dt} \int_{\partial\Omega} \Phi \left(\vec{E} \cdot \vec{n} - f \right) \mathrm{d}\omega.$$

The new momentum corresponds now to the flux at infinity. (For further reading on the importance of surface contributions to the action at spatial infinity in QED: J.L. Gervais and D. Zwanziger, Derivation from first Principles of the infrared Structure of Quantum Electrodynamics, Phys. Lett. **B94** *(1980) 389.).*

Exercise 5. *Prove if*

$$\left(G_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T_{\mu\nu}\right) n^{\mu} n^{\nu} = 0,$$

in a globally hyperbolic space-time for all possible embeddings (n is the normal to Σ_t), then $g_{\mu\nu}$ satisfies the Einstein equations.

Exercise 6. *Show that*

$$\begin{aligned}\left\{h_{mn}, V(\beta)\right\} &= \left(L_{\beta} h\right)_{mn}, \\ \left\{\pi^{mn}, V(\beta)\right\} &= \left(L_{\beta} \pi\right)^{mn}.\end{aligned}$$

Exercise 7. *Show that the diffeomorphism contraint $V_{\beta} = 0$ is equivalent to the statement that the momentum vanishes in the directions generated by diffeomorphism on Σ (analog to the Gauß constraint in electrodynamics).*

Exercise 8. *Show that the functional gradient of a diffeomorphism-invariant functional $S[g]$ with metric g satisfies*

$$\nabla_{\mu} \left(\frac{\delta S}{\delta g_{\mu\nu}} \right) = 0.$$

Exercise 9. *Consider the 3-metric*

$$h_{nm} = \begin{cases} \left(1 - \frac{m}{2r}\right)^4 d\vec{x} \cdot d\vec{x} & \textit{for } r > R > \frac{m}{2} \\ \textit{smooth continuation} & \textit{for } r < R, \end{cases}$$

where $m > 0$.
Show $M_{ADM} < 0$.
What is about the positive-mass theorem?

Solution 5. *Note that the Hamiltonian constraint C_H is equivalent to $G(n,n) = 0$, where G is the Einstein tensor and n denotes the normal vector to the hypersurface Σ_t (The momentum constraint is equivalent to $G(n,X) = 0$, where X is any tangent vector to the hypersurface).*

$$\underbrace{G_{\mu\nu} + \lambda g_{\mu\nu} - \kappa T_{\mu\nu}}_{\mathcal{O}} = 0.$$

$$\begin{aligned} \mathcal{O}(n,n) &= 0, \\ \mathcal{O}(n+\epsilon e, n+\epsilon e) &= 0, \end{aligned}$$

$$\epsilon O(e,n) + \epsilon' O(e,e) = 0 \Rightarrow O(n,e) = 0,\, O(e,e) = 0.$$

Solution 6. *To verify the Poisson structure of the constraints*

$$\begin{aligned} \left\{h_{mn}, V(\beta)\right\} &= \frac{\partial V}{\partial \pi} = D_m\beta_n + D_n\beta_m = \left(L_\beta h\right)_{nm}, \\ \left\{\pi^{mn}, V(\beta)\right\} &= -\frac{\partial V}{\partial h_{mn}} = \left(L_\beta \pi\right)^{mn}, \end{aligned}$$

rewrite the diffeomorphism constraint $V(\beta)$ as follows:

$$\begin{aligned} V(\beta) &= -2\int \mathrm{d}^3x\, \beta_n D_m \pi^{nm} \\ &= 2\int \mathrm{d}^3x\, \left(D_m\beta_n\right)\pi^{nm}, \qquad L_\beta h_{nm} = D_{(m}\beta_{n)} \\ &= 2\int \mathrm{d}^3x\, \frac{1}{2}\left(L_\beta h\right)_{mn}\pi^{mn} \\ &= \int \mathrm{d}^3x\, L_\beta\left(h_{mn}\pi^{mn}\right) - \int \mathrm{d}^3x\, h_{mn}\left(L_\beta\pi\right)^{mn} \\ &= \int \mathrm{d}^3x\, \left(h_{mn}\pi^{mn}\beta^k\right)_{,k} - \int \mathrm{d}^3x\, h_{mn}\left(L_\beta\pi\right)^{mn}. \end{aligned}$$

Assuming a β of compact support, the Poisson- brackets follow.

Solution 7. *We have to show that the diffeomorphism constraint $D_m\pi^{mn} = 0$ implies that π annihilates all directions generated by spatial diffeomorphisms on Σ and vice versa. Note that all the variations in h induced by spatial diffeomorphisms are $L_\beta h, V_\beta$.*
We apply π to $L_\beta h$:

$$\int_\Sigma \pi^{mn}(x)\underbrace{\left(L_\beta h(x)\right)_{mn}}_{2D_{(m}\beta_{n)}}\mathrm{d}^3x = 0 \quad \forall\beta \Leftrightarrow D_m\pi^{mn} = 0.$$

Hence, the Hamiltonian dynamics takes place on $T^*(Riem(\Sigma))$, but not all momenta in the fibre $T^*_n(Riem(\Sigma))$ are allowed.

Solution 8. *Let S be a diffeomorphism invariant functional of $g_{\mu\nu}$.*

$$\begin{aligned}
\delta_\beta S &= \int_\Omega \mathrm{d}^4x \, \frac{\delta S}{\delta g_{\mu\nu}(x)} \underbrace{L_\beta g_{\mu\nu}(x)}_{2\nabla(_\mu \beta_\nu)} \\
&= -2 \int \mathrm{d}^4x \, \nabla_\mu \left(\frac{\delta S}{\delta g_{\mu\nu}(x)} \right) \beta_\nu = 0. \quad \forall \beta \text{ of compact support.} \\
&\Rightarrow \nabla_\mu \left(\frac{\delta S}{\delta g_{\mu\nu}} \right) = 0
\end{aligned}$$

Solution 9. *Consider metric $m > 0$ (Schwarzschild metric with negative mass)*

$$h_{nm} = \begin{cases} \left(1 - \frac{m}{2r}\right)^4 d\vec{x} \cdot d\vec{x} & \text{for } r > R > \frac{m}{2} \\ \text{smooth continuation} & \text{for } r < R. \end{cases}$$

$M_{ADM} < 0$ obvious.

Theorem 4. *The ADM-mass is ≥ 0 for any (h, π) satisfying the constraints. There exist metrics h, for which no π can be found such that (h, π) satisfies the constraints.*

Hence, there exist classically forbidden regions in the configuration space. This result is expressed in the metric formulation of Canonical Quantum Gravity (Wheeler DeWitt equation). (For further reading: B.S. DeWitt, Quantum Theory of Gravity. 1. The Canonical Theory, Phys. Rev. **160** *(1967) 1113.).*

Acknowledgements

We would like to thank all the organizers and lecturers of the 16th Saalburg Summer School on “Fundamentals and New Methods in Theoretical Physics” for making this event such a successful one. Proofreading by André Großardt is gratefully acknowledged.

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