The Standard Model at Low Energies

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Effective Field Theories

Field theoretic formulation of the

Quantum Ladder

classical physics

atoms, molecules, condensed matter

nuclear physics

particle physics

. . .

quantum gravity, strings

. . .

Remarks :

- relevant degrees of freedom depend on energy
- energy scales usually well separated

Example :

Hydrogen Atom

$$m_e, \alpha \quad \Rightarrow \quad \text{energy levels}$$

Influence of massive states (up to logarithms) :

$$E_0 = -\frac{1}{2} m_e \alpha^2 \left\{ 1 + O[(\frac{m_e \alpha}{\Lambda})^n] \right\}$$

hyperfine splitting : $\Lambda = m_p$

weak interactions : $\Lambda = M_W$

m_e, α : Low Energy Constants

N.B.: α determined at low energy

e.g., from $(g-2)_e$, Josephson effect, . . .

$\alpha = \alpha_{\text{eff}}(q^2 \simeq 0)$

Lesson :

we don't need to solve quantum gravity to understand the hydrogen atom

Steps of quantum ladder separated by

energy Λ

$E < \Lambda$		$E > \Lambda$
long distances		short distances
effective	QFT	"fundamental"

Remnants of underlying theory

i. Symmetries

examples : spin-statistics connection

(QFT $\to$ QM)

gauge invariance

spontaneously broken symmetries

ii. LECs

incorporate short-distance vestiges

Determination of LECs

- direct matching between eff. and fund. theories (only feasible in weak-coupling regime)
- low-energy phenomenology
- additional input : lattice, models, ...

Applications

1. Parametrization of unknown short-distance structure (Anti-Theory Of Everything)

   examples : low-energy effects of GUTs

   Higgs sector of SM

2. "Fundamental" theory known, but either

not needed	or	not applicable
QED for $E_\gamma \ll m_e$		strong coupling (QCD)
weak interactions		system too complex
$(E \ll M_W)$		(condensed matter)

Classification of EFTs

Criterion : transition from fundamental to effective theory

A. Decoupling of heavy states

$E < \Lambda$: heavy fields "integrated" out

no additional light fields generated

EFT Lagrangian contains only the remaining light fields (masses $\ll \Lambda$)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{n\leq 4} + \sum_{n>4} \frac{1}{\Lambda^{n-4}} \sum_{i_n} g_{i_n} O_{i_n}$$

n : operator dimension (in energy units)

$n_{\text{boson}} = 1, \; n_{\text{fermion}} = 3/2, \; n_{\partial} = 1$

O_{i_n} : monomials in light fields with dim=n

g_{i_n} : dimensionless LECs , expect $\leq O(1)$

$\mathcal{L}_{n\leq 4}$: potentially renormalizable part

Example :

QED for $E \ll m_e$

electrons integrated out $\rightarrow$

only photons left, $\Lambda = m_e$

symmetries :

Lorentz and gauge invariance, C, P

$\rightarrow$ only even powers of $F_{\mu\nu}$, $\tilde{F}_{\mu\nu}$

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c_0}{m_e^2} F_{\mu\nu} \Box F^{\mu\nu} + \frac{c_1}{m_e^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{c_2}{m_e^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \ldots$$

weak coupling $\rightarrow$

perturbative matching possible

to leading order in α:

Euler, Heisenberg

$$c_0 = \frac{\alpha}{60\pi} , \quad c_1 = \frac{\alpha^2}{90} , \quad c_2 = \frac{7\alpha^2}{360}$$

with electrons : Caswell, Lepage, ...

Example :

Standard Model

from experiment : $\Lambda > 100$ GeV

$\Rightarrow \quad \mathcal{L}_{n\leq 4}$ sufficient at present

Remarks :

- "reason" for renormalizability : natural for EFTs of type A
- perturbative expansion competes with $\mathcal{L}_{n>4}$ (asymptotic expansion vs. exact solution)

Decoupling of heavy states

Toy model :

light fields $\varphi_i(x)$, masses m_i

heavy field $H(x)$, mass M

$$\mathcal{L}(H,\varphi_i) = \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2} M^2 H^2 + \mathcal{L}(\varphi_i) + H P(\varphi_i)$$

introduce path integral representation of

generating functional $Z[j_i, J]$

of connected Green functions

$$\exp iZ[j_i, J] = \langle 0 | T \exp i \int d^4x (j_i\varphi_i + JH) | 0 \rangle$$

$$= \int [d\varphi_i dH] \exp i \int d^4x (\mathcal{L}(H,\varphi_i) + j_i\varphi_i + JH)$$

for $E \ll M$: H only as internal lines

$\rightarrow$ set $J = 0$

$$e^{iZ[j_i,0]} = \int [d\varphi_i] e^{i\int d^4x j_i \varphi_i} \int [dH] e^{i\int d^4x \mathcal{L}(H,\varphi_i)}$$

Def.: **effective action $S_{\text{eff}}[\varphi_i]$**

$$e^{iS_{\text{eff}}[\varphi_i]} = \frac{\int [dH] e^{i\int d^4x \mathcal{L}(H,\varphi_i)}}{\int [dH] e^{i\int d^4x \mathcal{L}(H,0)}}$$

containing only light fields φ_i

$$\rightarrow \boxed{e^{iZ[j_i,0]} = \int [d\varphi_i] e^{i(S_{\text{eff}}[\varphi_i] + \int d^4x j_i \varphi_i)}}$$

Calculation of $S_{\text{eff}}[\varphi_i]$

Gaussian integral → exercise

Result :

nonlocal action

$$S_{\text{eff}}[\varphi_i] = \int d^4x \mathcal{L}(\varphi_i)$$

$$-\tfrac{1}{2} \int d^4x d^4y P(\varphi_i(x)) \Delta_F(x-y; M) P(\varphi_i(y))$$

Expansion in powers of M^{-1} (strictly valid only at tree level) :

effective local action

$$S_{\text{eff}}[\varphi_i] = \int d^4x \left\{ \underbrace{\mathcal{L}(\varphi_i) + \frac{1}{2M^2} P(\varphi_i)^2}_{\mathcal{L}_{\text{eff}}(\varphi_i)} \right\} + O(M^{-4})$$

Operator dimension : $n_H = 1 \rightarrow n_{P(\varphi_i)} = 3$

$\Rightarrow \quad \mathcal{L}_{n>4}$ starts with $n = 6$, $\Lambda = M$

Physically relevant example :

Fermi theory of weak interactions

φ_i quarks, leptons

H W, Z

N.B.: more complicated than toy model

B. Spontaneous symmetry breaking

phase transition $\rightarrow$ additional light states : (pseudo) Goldstone bosons

symmetry acts nonlinearly on GBs $\Rightarrow$

- separation between $\mathcal{L}_{n\leq 4}$ and $\mathcal{L}_{n>4}$ meaningless
- EFTs generically nonrenormalizable : perturbative treatment ?

Goldstone Theorem

- existence of GBs
- interactions of GBs vanish for $E_{GB} \rightarrow 0$ (independent of underlying interaction !)

$\Rightarrow$ **systematic low-energy expansion**

Organization of $\mathcal{L}_{\text{eff}}$:
derivatives ($\sim$ momenta) instead of operator dimension (as if $n_{GB} = 0$)

Problem for case B :

phase transition nonperturbative phenomenon

→ perturbative matching impossible in general

number of LECs explodes in higher orders

Examples

- electroweak gauge symmetry
  $SU(2) \times U(1) \to U(1)_{\rm em}$
  $\Lambda = G_F^{-1/2}$, $GBs \to W_{\rm long}, Z_{\rm long}$
  general case : nonlinear realization
  SM : simplest (linear) realization
  (∼ linear σ model)
- chiral symmetry of QCD : $\Lambda \simeq 1$ GeV
  pseudo-GBs : pseudoscalar mesons
- condensed matter :
  magnons, phonons, superconductivity

Spontaneous symmetry breaking

Realization of symmetries in field theory

invariance of the action

classical	quantum
EOM invariant	conserved charge Q
	$[H, Q] = 0$
	symmetric time evolution

Consequence for spectrum

energy eigenstates : $H|n\rangle = E_n|n\rangle$

$$|n_Q\rangle := Q|n\rangle \quad \Rightarrow \quad H|n_Q\rangle = E_n|n_Q\rangle$$

$|n\rangle, |n_Q\rangle$ degenerate

symmetry manifest in states

(rotational symmetry , isospin, . . .)

Wigner-Weyl realization of symmetries

Many examples of different realization

crystal : translation invariance, but only discrete subgroup manifest in states

ferromagnet : rotat. invariance not manifest

SM : γ, W, Z non-degenerate

Relativistic QFT

continuous (internal) symmetry

Noether current $J^\mu(x)$: $\partial_\mu J^\mu = 0$
conserved charge $Q = \int d^3x J^0(x)$
translation invariance : $[Q, P_\mu] = 0$

ground state : $P_\mu|0\rangle = 0$

$$\begin{aligned} ||Q|0\rangle||^2 &= \langle 0|QQ|0\rangle = \int d^3x \langle 0|QJ^0(x)|0\rangle \\ &= \int d^3x \langle 0|Qe^{iPx}J^0(0)e^{-iPx}|0\rangle \\ &= \int d^3x \underbrace{\langle 0|QJ^0(0)|0\rangle}_{\text{x-independent !}} \end{aligned}$$

$\rightarrow$ only 2 possibilities

Goldstone alternative

$Q\|0\rangle = 0$	$\|\|Q\|0\rangle\|\| = \infty$
Wigner–Weyl	Nambu–Goldstone
linear representation	nonlinear realization
degenerate multiplets	massless GBs
exact symmetry	spont. broken symm.

More careful discussion of SSB :

Def.: $Q^V(x^0) = \int_V d^3x J^0(x)$ (finite volume V)

$$\lim_{V\to\infty} [H, Q^V] = 0 \text{ and } \lim_{V\to\infty} ||Q^V|0\rangle|| = \infty$$

Ass.: $\exists$ operator A with

$$\underbrace{\lim_{V\to\infty} \langle 0|[Q^V(x^0), A]|0\rangle}_{\text{order parameter of SSB}} \neq 0$$

only possible for $Q|0\rangle \neq 0$

example : scalar fields $\varphi_i(x)$

$$[Q,\varphi_i] = c_{ij}\varphi_j \text{ with } \langle 0|\varphi_j|0\rangle \neq 0$$

Exercise : Goldstone theorem $\Rightarrow$

$\exists$ massless state $|GB\rangle$ with

$$\langle 0|J^0(0)|GB\rangle\langle GB|A|0\rangle \neq 0$$

necessary and sufficient condition for SSB :

Goldstone matrix element $\langle 0|J^0(0)|GB\rangle \neq 0$

$\Rightarrow$ $|GB\rangle$ same quantum numbers as $J^0(0)|0\rangle$

Remarks :

- $|GB\rangle$ need not correspond to physical particle (gauge symm.)
- $J^0(0)$ (usually) rot. inv. bosonic operator $\Rightarrow$ $|GB\rangle$ spin 0
- discrete SSB does not produce GBs
- nonrelativ. systems (condensed matter):
  $\exists$ excitations with
  $\lim\limits_{k\to 0}\omega(k) = 0 \qquad (k = 2\pi/\lambda)$
  in general : $N_{\text{broken symm.}} \geq N_{\text{GB}}$

Goldstone model

simplest explicit QFT example for SSB

complex scalar field $\phi(x)$

$$\mathcal{L}_{\text{Goldstone}} = \partial_\mu \phi \partial^\mu \phi^\dagger - \lambda(\phi\phi^\dagger - \frac{v^2}{2})^2$$

$$\lambda > 0, \qquad v \text{ real, positive}$$

$U(1)$ symmetry : $\qquad \phi(x) \to e^{i\alpha}\phi(x)$

minimum of (Mexican hat) potential : $\phi\phi^\dagger = \frac{v^2}{2}$
$\Rightarrow$ SSB with

$$\langle 0|R(x)|0\rangle = v, \qquad \langle 0|G(x)|0\rangle = 0$$

$$\phi(x) = (R(x) + iG(x))/\sqrt{2}$$

$$R(x), G(x) \quad \text{hermitian}$$

Spectrum (tree level)

GB field $G(x)$	: $M_G = 0$
scalar field $H(x) = R(x) - v$	: $M_H = \sqrt{2\lambda} v$

Exercise : calculate scattering amplitudes at tree level for $GG \to GG$ and $GH \to GH$

and show that

$$A(GG \to GG) = O(p_G^4), \quad A(GH \to GH) = O(p_G^2)$$

p_G generic momentum of GBs , p_H arbitrary

Why do GBs decouple for $E_{\text{GB}} \to 0$?

different choice of fields (polar decomposition)

$$\phi(x) = \frac{1}{\sqrt{2}}[h(x) + v]e^{ig(x)/v}$$

inserting into $\mathcal{L}_{\text{Goldstone}}$ $\Rightarrow$

$$\begin{aligned} \mathcal{L}_{\text{Goldstone}} &= \frac{1}{2}(\partial_\mu g)^2 + \frac{1}{2v^2}(h^2 + 2vh)(\partial_\mu g)^2 \\ &+ \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2 h^2 - \frac{\lambda}{4}(h^4 + 4vh^3) \end{aligned}$$

Haag–Borchers: fields G, H and g, h lead to
different Green functions (in general)
but identical S-matrix elements

Observations

- spectrum unchanged

  GB field $g(x)$: $M_g = 0$

  scalar field $h(x)$: $M_h = M_H = \sqrt{2\lambda}v$

- only derivative couplings for Goldstone field g

$$\Rightarrow \qquad \lim_{p_{GB}\to 0} A = 0$$

Reason

$$p_{GB} = 0 \leftrightarrow g(x) = \text{ constant}$$

$U(1)$ invariance of $\mathcal{L}_{\text{Goldstone}}$:

$$\phi(x) = \frac{1}{\sqrt{2}}[h(x)+v]e^{ig(0)/v} \simeq \phi(x) = \frac{1}{\sqrt{2}}[h(x)+v]$$

$\Rightarrow$ constant GB field decouples !

General property of GBs :

S-matrix elements vanish if all $p_{GB} \to 0$

Nonlinear realization

SSB for general nonabelian symmetry

Explicit example :

linear σ model

$$\mathcal{L}_\sigma = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right) - \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2 - v^2\right)^2$$

$$= \frac{1}{2}\partial_\mu\Phi^\top\partial_\mu\Phi - \frac{\lambda}{4}\left(\Phi^\top\Phi - v^2\right)^2$$

$$\Phi^\top = (\pi_1, \pi_2, \pi_3, \sigma)$$

- invariance group of $\mathcal{L}_\sigma$: $G = O(4)$
- degenerate vacua at $\Phi^\top\Phi = v^2$

possible choice of ground state :

$$\langle 0|\Phi^\top(x)|0\rangle = (0, 0, 0, v)$$

invariance group of the vacuum : $H = O(3)$

Fluctuations (fields) along minimum of potential

$\Rightarrow$ x-dependent $O(4)$ transformations

general form

$$\Phi(x) = \underbrace{e^{i\phi^a(x)X_a} e^{i\sigma^i(x)H_i}}_{L(\phi^a,\sigma^i)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$$

$$H_i \in \mathcal{H} \quad (i = 1,2,3) \ , X_a \in \mathcal{G} \setminus \mathcal{H} \quad (a = 1,2,3)$$

coset space G/H

$$L(\phi^a, \sigma^i) \equiv L(\phi^a, 0) \in G/H$$

possible (standard) parametrization : $\sigma^i = 0$

$\phi^a(x)$: Goldstone fields

(zero curvature fluctuations)

How do $\phi^a(x)$ transform under G ?

G : general (compact) Lie group (dim $= N_G$)

GB fields : $\phi^\top = (\phi_1, \ldots, \phi_n)$

most general transformation :

$$g \in G : \quad \phi \longrightarrow \phi' = f(g, \phi)$$

vacuum : $\phi = 0$

left invariant by $h \in G$ with $f(h, 0) = 0$

group property :

$$f(g_2, f(g_1, \phi)) = f(g_2 g_1, \phi)$$

with $f(h_1, 0) = f(h_2, 0) = 0$

$$\longrightarrow f(h_2 h_1, 0) = f(h_2, f(h_1, 0)) = 0$$

$\Rightarrow$ invariance group of the vacuum : $H \subset G$

$g_1 = h \in H$, $g_2 = g \notin H \to \phi = f(g, 0) \neq 0$

$$f(gh, 0) = f(g, \underbrace{f(h, 0)}_{0}) = f(g, 0)$$

$\Rightarrow$ mapping $G/H \longrightarrow \phi = f(g, 0)$

claim : mapping is one-to-one

if $f(g_1, 0) = f(g_2, 0) \quad \to g_1^{-1} g_2 =: h_{12} \in H$

$\to g_2 = g_1 h_{12} \longrightarrow g_1 = g_2$ as elements of G/H

Conclusion

$n = \dim G/H = N_G - N_H$ Goldstone fields

"parametrize" coset space

Furthermore :

$0 \xrightarrow{g} \phi = f(g, 0)$ inhom. transformation

no G-invariant mass term possible

$\Rightarrow$ Goldstone fields are massless

Lie algebra of G

$$[H_i, H_j] = i c_{ijk} H_k \qquad \text{subgroup } H$$

$$[H_i, X_a] = i c_{iab} X_b \qquad X_a \text{ repr. of } H$$

$$[X_a, X_b] = i c_{abi} H_i + i c_{abc} X_c$$

Remarks :

- G compact, H subgroup : $c_{iaj} = -c_{ija} = 0$
- $c_{abc} = 0$ in many cases (e.g., for chiral symmetry)

G transformation on coset space

standard choice : $L(\phi) = e^{i\phi^a X_a} \in G/H$

action of G

$$L(\phi) \xrightarrow{g \in G} gL(\phi) = L(\phi')h(g,\phi)$$

$h(g,\phi) \in H$: compensator (field)

for $g \in H$: $g = e^{i\varepsilon^i H_i}$

$$\begin{aligned} gL(\phi) &= \underbrace{e^{i\varepsilon^i H_i} e^{i\phi^a X_a} e^{-i\varepsilon^i H_i}}_{e^{i\phi'^a X_a}} e^{i\varepsilon^i H_i} \\ &= L(\phi')g \end{aligned}$$

$\rightarrow$ trivial compensator $h(g) = g$ and

$$\begin{aligned} e^{i\varepsilon^i H_i} X_a e^{-i\varepsilon^i H_i} &= X_b R_X(\varepsilon)_{ba} \\ \phi'^b &= R_X(\varepsilon)_{ba}\phi^a \end{aligned}$$

linear representation of H on ϕ

for $g \notin H$:

nonlinear realization $\phi \to \phi'$

nontrivial compensator $h(g,\phi)$

G transformation of Goldstone fields

$$L(\phi) \xrightarrow{g} L(\phi') = gL(\phi)h(g,\phi)^{-1}$$

Geometry of G/H

vielbein e connection ω

compact formulation : differential forms

Lie-algebra valued 1-form

$$L(\phi)^{-1}dL(\phi) = L^{-1}\frac{\partial L}{\partial \phi^a}d\phi^a = \omega(\phi) + e(\phi)$$
$$= \omega^i H_i + e^a X_a = \left(\omega^i_a H_i + e^b_a X_b\right) d\phi^a$$

Exercise : $L(\phi) \xrightarrow{g} L(\phi')$ implies

$e(\phi') = \quad h(g,\phi)e(\phi)h(g,\phi)^{-1}$ vielbein

$\omega(\phi') = \quad h\omega h^{-1} + hdh^{-1}$ connection

$\Rightarrow$ basic ingredients for effective Lagrangians

Coupling to non-Goldstone fields

Ass.: fields ψ carry

linear representation of H

$$\psi \xrightarrow{h\in H} \psi' = R_\psi(h)\psi$$

$\Rightarrow$ realization of G

$$\psi \xrightarrow{g\in G} \psi' = h_\psi(g,\phi)\psi$$

$$h_\psi(h,0) \equiv R_\psi(h)$$

However :

$$\partial_\mu\psi'(x) \neq h_\psi(g,\phi(x))\partial_\mu\psi(x)$$

$\Rightarrow$ introduce covariant derivative $\nabla = d + \omega$

explicitly : $\omega = \underbrace{\omega_a^i H_i}_{\omega_a} d\phi^a$

$$\nabla_\mu\psi = \left(\partial_\mu + \omega_a \frac{\partial\phi^a}{\partial x^\mu}\right) \xrightarrow{g} h_\psi(g,\phi)\nabla_\mu\psi$$

$\Rightarrow$ all ingredients ready for construction of

G-invariant $\mathcal{L}_{\text{eff}}(\phi,\psi)$