Functional Determinants in Quantum Field Theory

Gerald Dunne
Department of Physics, University of Connecticut

May 21, 2009

Abstract

Lecture notes on Functional Determinants in Quantum Field Theory given by Gerald Dunne at the 14th WE Heraeus Saalburg summer school in Wolfersdorf, Thuringia, in September 2008. Lecture notes taken by Babette Döbrich and exercises with solutions by Oliver Schlotterer.

Contents

1 Introduction

1.1 Motivation

Functional determinants appear in a plethora of physical applications. They encode a lot of physical information, but are difficult to compute. It is thus worthwhile to learn under which conditions they can be evaluated and how this can be done efficiently.

For example, in the context of effective actions [1, 2], if one has a sourceless bosonic action reading

$$S[\phi] \quad = \quad \int \mathrm{d}x \; \phi(x) \left(-\Box \; + \; V(x)\right) \phi(x) \; , \tag{1.1}$$

with ϕ being a scalar field, then the Euclidean generating functional is defined as

$$\mathcal{Z} \quad := \quad \int \mathcal{D}\phi \; e^{-S[\phi]} \; . \tag{1.2}$$

The one loop contribution to the effective action is given then in terms of a functional determinant[1]:

$$\Gamma^{(1)}[V] \quad = \quad -\ln(\mathcal{Z}) \quad = \quad \tfrac{1}{2} \; \ln\det\left(-\Box \; + \; V\right) \; . \tag{1.3}$$

Another example for the occurrence of functional determinants is found in tunneling problems and semiclassical physics [3]. There, the strategy is to approximate $\mathcal{Z}$ in Eq.(1.2) by expanding about a known classical solution. Thus, the action

$$S[\phi] \quad = \quad S[\phi_{\mathrm{cl}}] \; + \; \frac{1}{2} \int \mathrm{d}x \int \mathrm{d}y \; \phi(x) \underbrace{\left[\frac{\delta^2 S}{\delta\phi(x)\, \delta\phi(y)} \Big|_{\phi=\phi_{\mathrm{cl}}} \right]}_{K(x,y)} \phi(y) \; + \; \dots \; , \tag{1.4}$$

where the term containing the first order derivative with respect to the field vanishes at ϕ_{cl} since it is a classical solution. The second order derivative, evaluated on the classical solution $\phi = \phi_{\mathrm{cl}}$, results in a kernel K, which constitutes the second derivative of the action with respect to the fields. Thus, the semiclassical approximation to the generating functional Eq. (1.2) reads

$$\mathcal{Z} \quad \approx \quad N \; e^{-S[\phi_{\mathrm{cl}}]} / \sqrt{\det K} \; , \tag{1.5}$$

where N is a normalization factor. We will employ such a semiclassical approximation e.g. in Sect. 7.

Thirdly, functional determinants appear in so-called gap equations [4]. Consider e.g. the Euclidean generating functional $\mathcal{Z}$ for massless fermions with a four-fermion interaction

$$\mathcal{Z} \quad = \quad \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \; \exp\left(- \int \mathrm{d}x \; \left[-\bar{\psi} \, \partial\!\!\!/ \psi \; + \; \frac{g}{2} \, (\bar{\psi}\psi)^2 \right] \right) \; . \tag{1.6}$$

[1]An analogous computational problem arises in statistical mechanics with the Gibbs free energy.

To investigate e.g. if the fermions form a condensate in the ground state with given parameters, one introduces a bosonic condensate field σ and rewrites $\mathcal{Z}$ through a HUBBARD-STRATONOVICH transformation such that the fermions can be integrated out. This gives rise to the generating functional

$$\mathcal{Z} = \int \mathcal{D}\sigma\ e^{-S_{\text{eff}}[\sigma]}\ , \tag{1.7}$$

where

$$S_{\text{eff}}[\sigma] = -\ln\det\big(-\partial\!\!\!/ - i\sigma\big) + \int \mathrm{d}x\ \frac{\sigma^2(x)}{2g^2}. \tag{1.8}$$

In order to find the dominant contribution to the functional integral in Eq. (1.7), one has to solve the "gap equation", $\delta S_{\text{eff}}/\delta\sigma = 0$, which again demands the evaluation of a functional determinant:

$$\sigma(x) = g^2\ \frac{\delta}{\delta\sigma(x)}\ \ln\det(-\partial\!\!\!/ - i\sigma) \tag{1.9}$$

When $\sigma(x)$ is constant this gap equation can be solved, also at finite temperature and nonzero chemical potential, but when $\sigma(x)$ is inhomogeneous, the gap equation requires more sophisticated methods.

As a last example in this list, which could be continued over several pages, we like to mention the appearance of functional determinants in lattice calculations in QCD [5]. An integration over the fermionic fields renders determinant factors in the generating functional[2]

$$\mathcal{Z} = \int \mathcal{D}A\ \det\big(iD\!\!\!/ + m\big)\ e^{-S_{\text{YM}}[A]}\ , \tag{1.10}$$

which are of crucial importance for the dynamics of the gauge fields. The old-fashioned "quenched" approximation of lattice gauge theory involved setting all such fermion determinant factors to 1. Nowadays, the fermion determinant factors are included in "unquenched" computations, taking into account the effect of dynamical fermions.

1.2 Outline

The lectures are organized as follows:

In Sect.2 we will learn how functional determinants relate to ζ-functions and discuss some problems in the corresponding exercises where this relation drastically facilitates the evaluation of spectra of operators. Similarly, in Sect.3, we will discuss the calculation of functional determinants via heat kernels. In the calculation of the spectra, asymptotic series can appear. In Sect.4 we will discuss their origin and learn how to handle them by BOREL summation. Details of the procedure will be exemplified by the EULER-HEISENBERG effective Lagrangian of QED.

[2]Here, the functional determinant appears in the numerator due to the integration over Grassmann fields.

Next, in Sect.5, we will learn about a versatile formalism which will allow us to calculate determinants of one-dimensional operators without knowing the explicit eigenvalues. This method, which is known as the Gel'fand Yaglom formalism, will be extended to higher-dimensional problems that exhibit a radial symmetry in Sect.6. In the last section of these lectures, Sect. 7, we will apply this method to a non-trivial example as we discuss the problem of false vacuum decay.

2 ζ-function Regularization

Zeta function regularization is a convenient way of representing quantum field theoretic functional determinants [6, 7]. Consider an eigenvalue equation for some differential operator $\mathcal{M}$,

$$\mathcal{M}\,\phi_n \quad = \quad \lambda_n\,\phi_n\ , \tag{2.1}$$

where $\mathcal{M}$ could e.g. be a Dirac operator, a Klein-Gordon operator or a fluctuation operator. We define the corresponding ζ-function by

$$\zeta(s) \quad := \quad \mathrm{Tr}\left\{\frac{1}{\mathcal{M}^s}\right\} \quad = \quad \sum_n \frac{1}{\lambda_n^s} \tag{2.2}$$

such that

$$\zeta'(s) \quad = \quad -\sum_n \frac{\ln(\lambda_n)}{\lambda_n^s} \tag{2.3a}$$

$$\zeta'(0) \quad = \quad -\ln\left(\prod_n \lambda_n\right)\ . \tag{2.3b}$$

We therefore obtain a formal definition for the determinant of the operator $\mathcal{M}$ as

$$\det \mathcal{M} \quad := \quad \exp\bigl(-\zeta'(0)\bigr)\ . \tag{2.4}$$

Up to now these are only formal manipulations and the interesting physics lies in understanding how to make this definition both consistent and practically useful. For example, the question of convergence of the ζ-function around $s = 0$ has to be addressed. Typically, convergence is only given in a region where $\Re(s) > \frac{d}{2}$, where d denotes the dimensionality of the space on which the operator $\mathcal{M}$ acts. This is particularly true for second order elliptic operators on d-dimensional manifolds as shown by Weyl [9]. For these operators, the eigenvalues go as

$$\lambda_n^{d/2} \quad \sim \quad \frac{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}+1\right)\,n}{\mathrm{vol}} \qquad \Rightarrow \qquad \lambda_n \quad \sim \quad n^{2/d}\ . \tag{2.5}$$

Then, the corresponding ζ-function behaves as

$$\zeta(s) \sim \sum_n \frac{1}{n^{2s/d}} , \tag{2.6}$$

which converges for $\Re(s) > d/2$. However, since we need to evaluate the ζ-function at $s = 0$, we will have to analytically continue the ζ-function to $s = 0$. In the following, we will discuss some examples of ζ-functions corresponding to spectra that typically appear in physical problems.

2.1 Riemann ζ-function

The simplest representative of the ζ-function family (and one that appears often in computations) is probably the RIEMANN ζ-function which is defined by

$$\zeta_{\mathrm{R}}(s) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^s} , \tag{2.7}$$

corresponding to a spectrum $\lambda_n = n$. This sum is convergent for $\Re(s) > 1$. Note that the spectrum $\lambda_n = n$ arises in the $d = 2$ LANDAU level problem of a charged particle in a uniform magnetic field.

To find another representation for ζ_{R}, recall the definition of the gamma function

$$\Gamma(s) \quad := \quad \int_0^\infty \mathrm{d}t\; t^{s-1}\, e^{-t} . \tag{2.8}$$

As should be familiar, although this integral is only convergent for $\Re(s) > 0$, $\Gamma(s)$ can be analytically continued throughout the complex plane, with simple poles at the non-positive integers. Using Eq. (2.8), we find an integral representation for $\zeta_{\mathrm{R}}(s)$:

$$\zeta_{\mathrm{R}}(s) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1}\, e^{-nt} \tag{2.9}$$

The sum in Eq. (2.9) is just a geometric series. Thus, when $\Re(s) > 1$, we can write

$$\begin{aligned} \zeta_{\mathrm{R}}(s) \quad &= \quad \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1} \sum_{n=1}^{\infty} e^{-nt} \quad = \quad \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1}\, \frac{1}{e^t - 1} \\ &= \quad \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1}\, \frac{e^{-t/2}}{2\sinh(t/2)} . \end{aligned} \tag{2.10}$$

To analytically continue $\zeta_{\mathrm{R}}(s)$ to $s = 0$, we substract the leading small t behaviour of the integrand, and add it back again, evaluating the "added-back" term by virtue of the analytic continuation of the gamma function:

$$\begin{aligned}\zeta_{\mathrm{R}}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-1}\left[\frac{e^{-t/2}}{2\sinh(t/2)} - \frac{1}{t}\right] + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-2}\,e^{-t/2} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-1}\left[\frac{e^{-t/2}}{2\sinh(t/2)} - \frac{1}{t}\right] + \frac{2^{s-1}}{(s-1)} \qquad (2.11)\end{aligned}$$

At $s = 0$, the first term in Eq. (2.11) vanishes, and thus the analytical continuation of the Riemann ζ to zero yields $\zeta_{\mathrm{R}}(0) = -\frac{1}{2}$. Similarly, we find $\zeta'_{\mathrm{R}}(0) = -\frac{1}{2}\ln(2\pi)$.

2.2 Hurwitz ζ-function

As an important generalization of the Riemann zeta function, the Hurwitz zeta function is defined by

$$\zeta_{\mathrm{H}}(s;z) := \sum_{n=0}^{\infty}\frac{1}{(n+z)^s}\ . \qquad (2.12)$$

Thus it relates to the Riemann ζ-function via

$$\zeta_{\mathrm{H}}(s;1) = \zeta_{\mathrm{R}}(s)\ . \qquad (2.13)$$

The Hurwitz zeta function is defined by the sum in (2.12) for $\Re(s) > 1$, but as in the Riemann zeta function case, we can analytically continue to the neighbourhood of $s = 0$ by substracting the leading small t behaviour in an integral representation, and adding this substraction back (again with the help of the gamma function's analytic continuation):

$$\zeta_{\mathrm{H}}(s;z) = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ e^{-2zt}\,t^{s-1}\left(\coth t - \frac{1}{t}\right) + \frac{z^{1-s}}{s-1} \qquad (2.14)$$

Thus, we obtain

$$\zeta_{\mathrm{H}}(0;z) = \frac{1}{2} - z \qquad (2.15a)$$

$$\zeta'_{\mathrm{H}}(0;z) = \ln\Gamma(z) - \frac{1}{2}\ln(2\pi)\ . \qquad (2.15b)$$

To analytically continue $\zeta_{\mathrm{H}}(s;z)$ to the neighbourhood of $s = -1$ (as will be needed below in the physical example of the Euler-Heisenberg effective action), we make a further substraction:

$$\zeta_{\mathrm{H}}(s;z) = \frac{z^{-s}}{2} + \frac{2^{s-1}}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ e^{-2zt}\,t^{s-1}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) + \frac{z^{1-s}}{s-1} + \frac{s\,z^{-1-s}}{12} \qquad (2.16)$$

It then follows that

$$\zeta_{\mathrm{H}}(-1;z) = \frac{z}{2} - \frac{z^2}{2} - \frac{1}{12} \tag{2.17a}$$

$$\zeta'_{\mathrm{H}}(-1;z) = \frac{1}{12} - \frac{z^2}{4} - \zeta_{\mathrm{H}}(-1,z)\,\ln z - \frac{1}{4}\int_0^\infty \frac{\mathrm{d}t}{t^2}\, e^{-2zt}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) . \tag{2.17b}$$

This last expression appears in the Euler-Heisenberg effective action of QED as we will see in the exercises (cf. also Chapter 4).

2.3 Epstein ζ-function

Last, we want to discuss the Epstein ζ-function. It arises, when the eigenvalues λ_n are of the more general form $\lambda_n \sim an^2 + bn + c$. This is e.g. the case in finite temperature field theory, in the context of toroidal compactifications and gravitational effective actions on spaces of constant curvature. For example, on a d-dimensional sphere S^d, the Laplacian has the eigenvalues

$$-\triangle u = \underbrace{n\,(n+d-1)}_{\lambda_n}\, u \tag{2.18}$$

which carry a degeneracy factor of

$$\deg(n;d) = \frac{(2n+d-1)\,(n+d-2)!}{n!\,(d-1)!} . \tag{2.19}$$

Their corresponding ζ-function thus reads

$$\zeta(s) = \sum_n \frac{\deg(n;d)}{\lambda_n^s} , \tag{2.20}$$

and its derivative at $s = 0$ was shown [10] to yield (after appropriate finite substractions)

$$\zeta'(0) = \ln\left(\frac{\Gamma_d^2\left(\frac{d-1}{2}+\alpha\right)\,\Gamma_d^2\left(\frac{d-1}{2}-\alpha\right)}{\Gamma_{d-1}\left(\frac{d-1}{2}+\alpha\right)\,\Gamma\left(\frac{d-1}{2}-\alpha\right)}\right) , \qquad \alpha = i\sqrt{m^2 - (\frac{d-1}{2})^2} . \tag{2.21}$$

The Γ_n denote multiple Γ-functions [11, 12]. As the ordinary Γ-function, they are defined through a functional relation,

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)} \tag{2.22a}$$

$$\Gamma_1(z) = \Gamma(z) \tag{2.22b}$$

$$\Gamma_n(1) = 1 , \tag{2.22c}$$

and their integral representation reads

$$\ln \Gamma_n(1+z) = \frac{(-1)^{n-1}}{(n-1)!} \int_0^z dx\; x\,(x-1)\,(x-2)\,\ldots\,\big(x-(n-2)\big)\,\psi(1+x)\;, \tag{2.23}$$

where $\psi = \Gamma'/\Gamma$.

We now turn back to the Epstein ζ. It can be generalized as follows [6, 12]:

$$\begin{aligned} \zeta_{\mathrm{E}}(s; m^2, \vec{\omega}) &= \sum_{\vec{n}} (m^2 + \omega_1 n_1^2 + \omega_2 n_2^2 + \cdots + \omega_N n_N^2)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1}\, e^{-m^2 t} \sum_{\vec{n}} \exp\Big(-\big(\omega_1 n_1^2 + \omega_2 n_2^2 + \ldots + \omega_N n_N^2\big)\, t\Big) \end{aligned} \tag{2.24}$$

A more useful representation of the function can be obtained by noticing that the modified Bessel function of the second kind can be written as

$$K_{-\nu}(z) = \frac{1}{2} \left(\frac{z}{2}\right)^{-\nu} \int_0^\infty dt\; t^{\nu-1} \exp\left(-t - \frac{z^2}{4t}\right) . \tag{2.25}$$

Hence, by rewriting Eq. (2.24) such that the integration variable t in the second exponential appears in the denominator, we can convert the integral representation of the Epstein ζ-function into a sum over Bessel functions.

Using the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} e^{-\omega n^2 t} = \left(\frac{\pi}{\omega t}\right)^{1/2} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/\omega t}\;, \tag{2.26}$$

the Epstein ζ reads

$$\begin{aligned} \zeta_{\mathrm{E}}(s; m^2, \vec{\omega}) &= \frac{\pi^{N/2}}{\Gamma(s)\sqrt{\omega_1 \ldots \omega_N}} \int_0^\infty dt\; t^{s-1-N/2}\, e^{-m^2 t} \\ &\quad \times \sum_{\vec{n}} \exp\left[-\frac{\pi^2}{t}\left(\frac{n_1^2}{\omega_1} + \frac{n_2^2}{\omega_2} + \ldots + \frac{n_N^2}{\omega_N}\right)\right] . \end{aligned} \tag{2.27}$$

Now, by comparison with (2.25) ζ_{E}, finally reads

$$\begin{aligned} \zeta_{\mathrm{E}}(s; m^2, \vec{\omega}) &= \frac{\pi^{N/2}}{\sqrt{\omega_1 \ldots \omega_N}} \cdot \frac{\Gamma\left(s - \frac{N}{2}\right)}{\Gamma(s)}\, m^{N-2s} \\ &+ \frac{2\pi^s\, m^{\frac{N}{2}-s}}{\Gamma(s)\sqrt{\omega_1 \ldots \omega_N}} \sum_{\vec{n}\neq\vec{0}} \left(\frac{n_1^2}{\omega_1} + \ldots + \frac{n_N^2}{\omega_N}\right)^{\frac{s}{2}-\frac{N}{4}} K_{\frac{N}{2}-s}\left(2\pi\, m \sqrt{\frac{n_1^2}{\omega_1} + \ldots + \frac{n_N^2}{\omega_N}}\right) , \end{aligned} \tag{2.28}$$

where we have isolated the $\vec{n} = \vec{0}$ contribution in the last step. It corresponds to the "zero temperature" result, if we think of the $\vec{n}$-sum as a sum over Matsubara modes.

The crucial point now is that the remaining sum over $\vec{n}$ is rapidly convergent since $K_v(z) \sim e^{-z}$ as $z \to \infty$.

3 Heat kernel and heat kernel expansion

3.1 Properties of the heat kernel and relation to the ζ-function

The heat kernel trace of an operator $\mathcal{M}$ is defined as

$$K(t) \quad := \quad \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\} \quad = \quad \sum_n e^{-\lambda_n t} \tag{3.1}$$

and thus it relates to the ζ-function of $\mathcal{M}$ through a MELLIN transform:

$$\zeta(s) \quad = \quad \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t \; t^{s-1}\, K(t) \tag{3.2}$$

The heat kernel trace is connected to the local heat kernel via

$$K(t) \quad = \quad \int \mathrm{d}x \;\, K(t;\vec{x},\vec{x}) \ , \tag{3.3}$$

where

$$K(t;\vec{x},\vec{x}') \quad \equiv \quad \langle\vec{x}|\, e^{-t\mathcal{M}}\, |\vec{x}\rangle \ .$$

The name "heat kernel" follows from its primary application, namely that K should solve a general heat conduction problem with some differential operator $\mathcal{M}$ and given initial condition:

$$\left(\frac{\partial}{\partial t} - \mathcal{M}\right) K(t;\vec{x},\vec{x}') \quad = \quad 0 \tag{3.4a}$$

$$\lim_{t\to 0} K(t;\vec{x},\vec{x}') \quad = \quad \delta(\vec{x}-\vec{x}') \tag{3.4b}$$

The heat equation thus determines, as the name suggests, the distribution of heat over a space-time volume after a certain time. As is well-known, in $\mathbb{R}^d$, the solution to the heat equation with a LAPLACE operator $\mathcal{M} = \triangle$, and initial condition (3.4b), reads

$$K(t;\vec{x},\vec{x}') \quad = \quad \frac{\exp\left(-\frac{(\vec{x}-\vec{x}')^2}{4t}\right)}{(4\pi t)^{d/2}} \ . \tag{3.5}$$

For general differential operators, the heat kernel can only rarely be given in a closed form. However, for practical calculations we can make use of its asymptotic expansions [14], which can turn out to be of great physical interest.

3.2 Heat Kernel expansions

As we discuss in exercise 8.1, for small t, the leading behaviour of the heat kernel reads $K(t) \sim \frac{\mathrm{vol}}{(4\pi t)^{d/2}}$. The subleading small t corrections are encoded in the so-called "heat-kernel-expansion":

$$
\begin{aligned}
K(t;\vec{x},\vec{x}') &\sim \frac{\exp\left(-\frac{(\vec{x}-\vec{x}')^2}{4t}\right)}{(4\pi t)^{d/2}} \sum_k t^k\, \tilde{b}_k(\vec{x},\vec{x}') && \text{(3.6a)}\\
K(t;\vec{x}) \equiv K(t;\vec{x},\vec{x}) &\sim \frac{1}{(4\pi t)^{d/2}} \sum_k t^k\, b_k(\vec{x}) && \text{(3.6b)}\\
K(t) &= \frac{1}{(4\pi t)^{d/2}} \sum_k t^k\, a_k && \text{(3.6c)}
\end{aligned}
$$

Here, the expansion coefficients are related as $b_k(\vec{x}) \equiv \tilde{b}_k(\vec{x},\vec{x}')$ and $a_k \equiv \int \mathrm{d}x\, b_k(\vec{x})$. The k sum ranges over $k = 0, \frac{1}{2}, 1, \frac{3}{2} + \dots$, but the half-odd-integer indexed terms vanish if there is no boundary present. As an example for the local expansion, consider a general Schrödinger operator $\mathcal{M} = -\vec{\nabla}^2 + V(\vec{x})$. In this case, the first few coefficients $b_k(\vec{x})$ read

$$
\begin{aligned}
b_0(\vec{x}) &= 1\\
b_1(\vec{x}) &= -V(\vec{x})\\
b_2(\vec{x}) &= \frac{1}{2}\left(V^2(\vec{x}) - \frac{1}{3}\,\vec{\nabla}^2 V(\vec{x})\right)\\
b_3(\vec{x}) &= -\frac{1}{6}\left(V^3(\vec{x}) - \frac{1}{2}\left(\vec{\nabla}V(\vec{x})\right)^2 - V(\vec{x})\,\vec{\nabla}^2 V(\vec{x}) + \frac{1}{10}\left(\vec{\nabla}^2\right)^2 V(\vec{x})\right) .
\end{aligned}
$$

These are derived in exercise 9.4 for $d = 1$, and see [8] for higher dimensional cases. As one can already see from the first few expansion coefficients, the terms without derivatives exponentiate such that we can rewrite the series of Eq.(3.6b) as

$$
K(t;\vec{x}) \sim \frac{\exp\bigl(-V(\vec{x})\,t\bigr)}{(4\pi t)^{d/2}} \sum_k t^k\, c_k(\vec{x}) . \qquad (3.7)
$$

The expansion coefficients $c_k(\vec{x})$ all include derivatives of $V(\vec{x})$. This modified (partially re-summed) heat kernel expansion is useful in certain physical applications where the derivatives of V may be small, but not necessarily V itself.

On curved surfaces, the heat kernel expansion behaves as [14]

$$
K(t;\vec{x},\vec{x}') \sim \Delta^{\frac{1}{2}}(\vec{x},\vec{x}')\, \frac{\exp\left(-\frac{\sigma(\vec{x},\vec{x}')}{2t}\right)}{(4\pi t)^{d/2}} \sum_k t^k\, d_k(\vec{x},\vec{x}')
$$

with $\sigma(\vec{x},\vec{x}') \equiv \frac{1}{2}\rho^2(\vec{x},\vec{x}')$, where $\rho(\vec{x},\vec{x}')$ denotes the geodesic distance from $\vec{x}$ to $\vec{x}'$. The Δ factor constitutes the Van Vleck determinant,

$$
\Delta(\vec{x},\vec{x}') = \frac{1}{|g(\vec{x})|^{\frac{1}{2}}}\ \det\left(-\sigma_{;\mu\nu}\right)\ \frac{1}{|g(\vec{x}')|^{\frac{1}{2}}} ,
$$

where the corresponding determinant of the metric $g(\vec{x}) \equiv \det g_{\mu\nu}(\vec{x})$ as well as covariant derivatives $\sigma_{;\mu\nu}$ enter.

3.3 Heat kernel expansion in gauge theories

Due to its physical importance, we finally want to present the heat kernel expansion for the functional determinant that appears in the context of the calculation of gauge theory effective actions. Consider e.g. the operator $\mathcal{M} = m^2 - \not{D}^2$; it appears when the fermionic fluctuations are integrated out in some gauge field background. Using (3.2), the log determinant can be expressed in terms of the heat kernel trace of $\mathcal{M}$ as

$$\ln\det\left(m^2 - \not{D}^2\right) = -\int_0^\infty \frac{\mathrm{d}t}{t}\, e^{-m^2 t}\, \mathrm{Tr}\left\{e^{-(-\not{D}^2)t}\right\} . \tag{3.8}$$

The heat kernel trace has an expansion

$$\mathrm{Tr}\left\{e^{-(-\not{D}^2)\,t}\right\} \sim \frac{1}{(4\pi t)^{\frac{d}{2}}} \sum_k t^k\, a_k[F] , \tag{3.9}$$

where the coefficients $a_k[F]$ are now functionals of the field strength $F_{\mu\nu}$. In the non-abelian situation, the first ones read as follows [18, 19]:

$$\begin{aligned} a_0[F] &= \mathrm{vol} \\ a_1[F] &= 0 \\ a_2[F] &= \frac{2}{3}\int \mathrm{Tr}\left\{F_{\mu\nu}^2\right\} \\ a_3[F] &= -\frac{2}{45}\int \mathrm{Tr}\left\{(3D_\nu F_{\lambda\mu})\,(D_\nu F_{\lambda\mu}) - 13i\, F_{\nu\lambda}\, F_{\lambda\mu}\, F_{\mu\nu}\right\} \\ &\ \vdots \end{aligned}$$

The relevant physical quantity is the functional determinant which is normalized with respect to the field free case. A heat kernel expansion in $d = 4$ dimensions leads to

$$\ln\left(\frac{\det\left(m^2 - \not{D}^2\right)}{\det\left(m^2 - \not{\partial}^2\right)}\right) = -\int_0^\infty \frac{\mathrm{d}t}{t}\, \frac{e^{-m^2 t}}{(4\pi t)^2} \sum_{k=3} a_k[F]\, t^k , \tag{3.10}$$

where a_0 drops out[3] since we consider the ratio of the determinants, and a_2 is absorbed by charge renormalization. Thus,

$$\ln\left(\frac{\det\left(m^2 - \not{D}^2\right)}{\det\left(m^2 - \not{\partial}^2\right)}\right) = -\frac{m^4}{(4\pi)^2} \sum_{k=3} \frac{(k-3)!}{m^{2k}}\, a_k[F] , \tag{3.11}$$

[3]For the field-free subtraction the only non-zero expansion coefficient is a_0, since it is independent of the field strength.

this constitutes the large mass expansion of the effective action. At last we would like to note, that high-precision computations for such expressions are obstructed by the fact that the number of terms in the expansion grow approximately factorially with k. Hence, we already have to deal with 300 terms at order $\mathcal{O}(1/m^{12})$.

4 Paradigm: The Euler-Heisenberg effective action

An important and illustrative example is the one-loop contribution to the QED effective action for constant[4] field strength $F_{\mu\nu}$. In this case one can compute all quantities of interest in closed form and gain valuable insight into the zeta function- and heat kernel methods.
The one-loop effective action is essentially given by the log-det of the DIRAC operator, $-i\not{D}+m$, where $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)$, with the corresponding subtraction of the field-free case:

$$\Gamma[A] \quad = \quad -i \ln\det\left(\frac{-i\not{D} + m}{-i\not{\partial} + m}\right) \tag{4.1}$$

In order to evaluate the spectrum of the operators in Eq. (4.1), we first rewrite the argument of the determinant via[5]

$$\begin{aligned} \ln\det\left(-i\not{D} + m\right) &= \frac{1}{2}\Big(\ln\det\left(-i\not{D} + m\right) + \ln\det\left(-i\not{D} + m\right)\Big) \\ &= \frac{1}{2}\Big(\ln\det\left(-i\not{D} + m\right) + \ln\det\left(i\not{D} + m\right)\Big) \\ &= \frac{1}{2}\ln\det\left(\not{D}^2 + m^2\right) = \frac{1}{2}\,\mathrm{Tr}\Big\{\ln\left(\not{D}^2 + m^2\right)\Big\}\,, \end{aligned} \tag{4.2}$$

where we have omitted the field-free subtraction temporarily. The evaluation of Eq. (4.2) for constant, purely magnetic fields is considerably facilitated by the fact that the LANDAU levels appear as part of the spectrum (cf. exercise 9.1). The general result for both constant electric and magnetic fields was given by EULER and HEISENBERG already in 1936 [13] and later rederived by SCHWINGER [47]:

$$\Gamma_R \quad = \quad -\frac{\Omega}{8\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t^3}\, e^{-m^2 t}\left(\frac{e^2\, a\, b\, t^2}{\tanh(ebt)\,\tan(eat)} - 1 - \frac{(a^2-b^2)\, e^2\, t^2}{3}\right) \tag{4.3}$$

We have introduced the LORENTZ invariants

$$a^2 - b^2 \quad = \quad \vec{E}^2 - \vec{B}^2 \quad = \quad -\frac{1}{2}\, F_{\mu\nu}\, F^{\mu\nu} \tag{4.4a}$$

$$a \cdot b \quad = \quad \vec{E}\cdot\vec{B} \quad = \quad -\frac{1}{4}\, F_{\mu\nu}\, \tilde{F}^{\mu\nu}\,. \tag{4.4b}$$

[4] A review on more general but nevertheless soluble cases is e.g. given in [16].
[5] To check the second identity, rewrite the "ln det" into a "Tr ln", insert $\gamma_5\gamma_5 = 1$ and make use of the cyclicity of the trace.

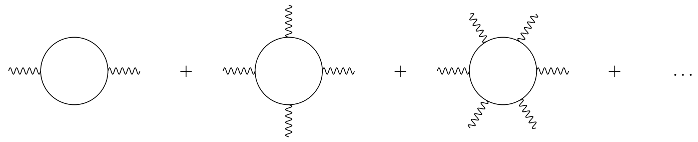

Figure 1: The diagrammatic perturbative expansion in e of the EULER-HEISENBERG effective action. Note that only even numbers of external photon lines appear, due to charge conjugation invariance (FURRY'S theorem).

In (4.3), the first substraction corresponds to substraction of the free-field case, while the second substraction is associated with charge renormalization. The physical consequences of the Lagrangian in Eq. (4.3) become more obvious by a weak field expansion meaning $E\frac{e}{m^2} \equiv \frac{E}{E_{\text{crit}}} \ll 1$, $B\frac{e}{m^2} \equiv \frac{B}{B_{\text{crit}}} \ll 1$.
The total effective Lagrangian then reads in the lowest non-linear order:

$$\mathcal{L} \;=\; \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{2\alpha^2}{45m^4}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\,(\vec{E}\cdot\vec{B})^2\right] + \ldots$$

As one sees, the first contribution to the action is just the classical MAXWELL term. The higher, non-linear terms correspond to light-light scattering (cf. the second diagram in Fig. 1) and thereby imply that the vacuum behaves like a dielectric medium under an external electromagnetic field.
Consider again the integral expression (4.3) in the case of a constant magnetic field. With $a = 0$ and $b = B$, the effective Lagrangian becomes

$$\mathcal{L} \;=\; -\frac{e^2B^2}{8\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t^2}\, e^{-m^2t/eB}\left(\frac{1}{\tanh(t)} - \frac{1}{t} - \frac{t}{3}\right). \tag{4.5}$$

Inserting the series expansion of $\coth t$ into Eq. (4.5) one finds that $\mathcal{L}$ has the expansion

$$\mathcal{L} \;=\; -\frac{m^4}{8\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)\,(2n+3)\,(2n+2)}\left(\frac{2eB}{m^2}\right)^{2n+4}, \tag{4.6}$$

where $\mathcal{B}_{2n}$ are the BERNOULLI numbers. However, the BERNOULLI numbers have the properties of growing exponentially and alternating in sign. So our perturbative series (which we have to all orders!) of the effective Lagrangian in Eq. (4.6) is clearly divergent! As we will see in the following section, we can make sense of this result by the use of BOREL summation.

4.1 Borel summation of the EH perturbative series

The divergent perturbative expansion (4.6) of the EH effective action illustrates some important features of the relation between perturbation theory and non-perturbative physics. A useful framework for this discussion is provided by BOREL summation. To introduce the basic idea, consider the alternating, divergent series

$$f(g) \;=\; \sum_{n=0}^{\infty}(-1)^n\, n!\, g^n \;. \tag{4.7}$$

Let us rewrite the $n!$ in terms of the Γ function as $n! = \int_0^\infty \mathrm{d}t\; e^{-t}t^n$, and suppose for a minute that we could interchange integration and summation just like that. We indicate this questionable procedure by putting a tilde on f. We then get

$$\tilde{f}(g) \;=\; \int\limits_0^\infty \mathrm{d}t\, \sum_{n=0}^{\infty} e^{-t}\, t^n\, (-1)^n\, g^n \;\;, \tag{4.8}$$

where we can evaluate the sum and rescale $t \to t/g$ to find the finite expression

$$\tilde{f}(g) \;=\; \frac{1}{g}\int\limits_0^\infty \mathrm{d}t\, \frac{e^{-t/g}}{1+t} \;\equiv\; \text{BOREL sum of } f \;. \tag{4.9}$$

Note that the integral representation of $\tilde{f}(g)$ in Eq.(4.9) is convergent for all $g > 0$. We then *define* $\tilde{f}(g)$ as the BOREL sum of the divergent series $f(g)$.

At a first glance, our manipulations on f seem to give a rather contradictory result. We started off with a rapidly divergent series and "converted" it to a perfectly well defined integral expression. To make sense of this, let us read the above manipulations backwards: Suppose we have e.g. a non-trivial physical theory with some small coupling g, which is not exactly soluble. A common approach is just to expand the corresponding equations around $g = 0$ in order to be able to calculate some predictions of the theory at all.

What we see from Eqs.(4.5) and (4.6) is that this expansion can actually turn out to be asymptotic, i.e. it diverges for arbitrarily small couplings. Thus a high number of series terms does not necessarily improve the perturbative result. The BOREL summation, which at first seemed rather dubious, is thus to be understood as a *reparation* of asymptotic expansion of the original integral expression and thus does in fact deliver meaningful results. Of course, there is significant mathematical and physical interest in the question of the validity and uniqueness of such a procedure, see [21, 22] for some interesting examples.

In this context, we can even make a very advanced statement. As it turns out, perturbative expansions about a small coupling g in physical examples generally take the form [21]

$$\begin{aligned} f(g) \;&\sim\; \sum_{n=0}^{\infty}(-1)^n\, \underbrace{\alpha^n\, \Gamma(\beta n+\gamma)}_{a_n}\, g^n \\ &\equiv\; \frac{1}{\beta}\int\limits_0^\infty \frac{\mathrm{d}t}{t}\, \left(\frac{1}{1+t}\right)\left(\frac{t}{\alpha g}\right)^{\gamma/\beta} \exp\left[-\left(\frac{t}{\alpha g}\right)^{1/\beta}\right] \;. \end{aligned} \tag{4.10}$$

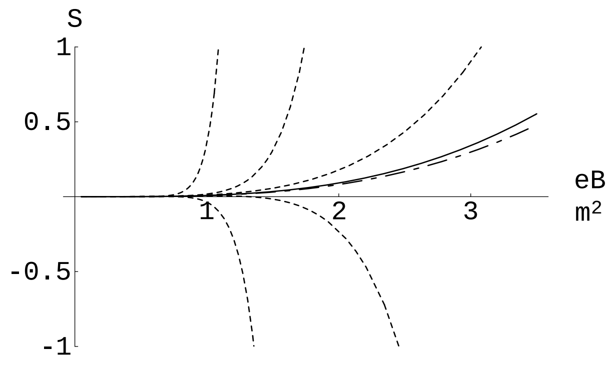

Figure 2: Comparison of the exact EULER-HEISENBERG action S [solid curve] for constant magnetic field (cf. Eq. (4.5)) with the leading BOREL expression [short-long-dash curve], cf. Eq. (4.13), as a function of $\frac{eB}{m^2}$, and successive partial sums from the perturbative series [short-dash curves] (cf. Eq. (4.6)). One can see that the leading BOREL expression is much better than the series expressions for $\frac{eB}{m^2} \geq 1$. Graph taken from [20].

Let us apply this BOREL summation technique to the EULER-HEISENBERG Lagrangian expansion in (4.6). We identify the weak field expansion coefficients as:

$$\begin{aligned} a_n &= \frac{\mathcal{B}_{2n+4}}{(2n+4)\,(2n+3)\,(2n+2)} \\ &= (-1)^{n+1}\,\frac{2}{(2\pi)^{2n+4}}\,\Gamma(2n+2)\,\zeta(2n+4) \\ &\sim (-1)^{n+1}\,\frac{2}{(2\pi)^{2n+4}}\,\Gamma(2n+2)\,\left[1+\frac{1}{2^{2n+4}}+\ldots\right] \end{aligned} \tag{4.11}$$

We have expanded about large n in the last step, and the other parameters of the general expansion read $\alpha = 1/(2\pi)^2$, $\beta = \gamma = 2$ and $g = (2eB/m^2)^2$. Thus, following Eq. (4.10), the BOREL summed expression (4.6) reads

$$\mathcal{L} = \frac{m^4}{8\pi^2}\left(\frac{2eB}{m^2}\right)^4 \underbrace{\frac{-2}{(2\pi)^4}}_{a_0}\,\underbrace{\frac{1}{2}}_{\beta}\int_0^\infty \frac{\mathrm{d}t}{t}\,\frac{1}{1+t}\,\left(\frac{m^2\pi^2}{e^2B^2}\right)\,\exp\left(-\frac{\pi m^2}{eB}\,\sqrt{t}\right) \tag{4.12}$$

which we simplify by virtue of $t^{1/2} \to s/\pi$ to

$$\mathcal{L} = \left(\frac{eB}{2\pi^2}\right)^2 \int_0^\infty \mathrm{d}s\,\frac{s}{\pi^2+s^2}\,e^{-m^2 s/eB}\;. \tag{4.13}$$

Eq. (4.13) is thus the leading term of the BOREL sum of the EULER-HEISENBERG perturbative series.

Since we are in the fortunate situation to know the exact expression for the effective Lagrangian with constant magnetic field, we can directly compare the exact expression with the perturbative expansion and the leading BOREL sum. In Fig. 2, we give the two approximations and the exact expression for the action as a function of $\frac{eB}{m^2}$. It can be seen that the leading BOREL expression is much better than the perturbative approximation for $\frac{eB}{m^2} \geq 1$. Moreover, Fig. 2 shows that the perturbative approximation becomes increasingly worse for $\frac{eB}{m^2} \gtrsim 1$ with the adding of higher order terms (dashed curves from right to left).

As a final remark we would like to notice that the divergence of the perturbative series (Eq. (4.6)) finds its analogue in the perturbative series for the ZEEMAN effect in quantum mechanics, which also turns out to diverge [23]. After all, the physical situation for these two calculations is closely related. In the first case, the vacuum and thereby the DIRAC sea is perturbed by the external magnetic field, just as the atomic levels are perturbed in the second situation.

4.2 Non-alternating series

We now come to the case of non-alternating series coefficients in Eq. (4.10), and illustrate it again for the instance of the EULER-HEISENBERG Lagrangian. Note to this end, that we can extend our perturbative result for constant magnetic fields without further calculations to the case of a constant electric field by the following observation: Demanding LORENTZ invariance, we know that the effective Lagrangian can only depend on the invariants $I_1 = \vec{B}^2 - \vec{E}^2$ and $I_2 = \vec{E} \cdot \vec{B}$. Thus, by a duality transformation

$$\mathcal{L}(I_1, I_2)\Big|_{\vec{E}=\text{const},\vec{B}=0} = \mathcal{L}(-\vec{E}^2, 0) = \mathcal{L}\big((i\vec{E})^2, 0\big) = \mathcal{L}(I_1, I_2)\Big|_{\vec{E}'=0,\vec{B}'=i\vec{E}} , \tag{4.14}$$

we see that the effective Lagrangian for the constant electric field follows from our result for the constant magnetic field via the substitution $\vec{B} \to i\vec{E}$.

The expansion parameter $g \sim B^2$ in which the field appears quadratically thus changes to $g \sim -E^2$. Following our steps of Sect. 4.1, we see that the BOREL sum for a series for negative expansion parameter (cf. Eq. (4.7)) now becomes[6]

$$f(-g) = \sum_{n=0}^{\infty} n!\, g^n = \frac{1}{g} \int_0^{\infty} \mathrm{d}t\, \frac{e^{-t/g}}{1-t} . \tag{4.15}$$

Even though every term in the series of Eq. (4.15) is real, its BOREL sum has a pole and thereby an imaginary part. With

$$\mathrm{Res}_1\,(f) = -\frac{1}{g}\, e^{-1/g} , \tag{4.16}$$

[6]Here we drop the $\sim$ for the BOREL sum of the function f, since we are now familiar with the justification of this manipulation.

we find that the imaginary part yields

$$\Im f(-g) \;=\; \frac{\pi}{g}\, e^{-1/g}\ . \tag{4.17}$$

Note that this imaginary part is *nonperturbative* in g: it does not appear at any order in perturbation theory. This procedure captures important physics of the EULER-HEISENBERG Lagrangian with a constant electric field,

$$\mathcal{L} \;=\; -\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t^3}\, e^{-m^2 t}\left(\frac{e\,E\,t}{\tan(eEt)} - 1 + \frac{(e\,E\,t)^2}{3}\right)\ , \tag{4.18}$$

whose leading contribution to the imaginary part reads

$$\Im\mathcal{L} \;\sim\; \frac{e^2E^2}{8\pi^3}\, e^{-\pi m^2/eE}\ . \tag{4.19}$$

This can be associated with a pair production[7] rate $w = 2\Im\mathcal{L}$. This result was first calculated by EULER and HEISENBERG, and later reformulated in more modern QED by SCHWINGER and is usually referred to as SCHWINGER pair production. It reflects the fact that the QED vacuum is unstable in the presence of an external electric field, as the electric field can accelerate apart virtual dipole pairs, which can become real asymptotic $e^+$and e^- particles if they gain sufficient energy ($2mc^2$) from the external field. Following our previous analogy with atomic physics, one could say that SCHWINGER pair production corresponds to an ionization process for which some binding energy must be expended.

Note that the above result is non-perturbative in E since it is singular at $E \to 0$. Thus pair creation is a result which cannot be calculated at any order in a series expansion. The pair production rate becomes sizeable at around $E_{\text{crit}} = \frac{m^2}{e} \overset{\text{SI-units}}{=} \frac{m^2c^3}{e\hbar} \approx 1.37 \times 10^{18}\frac{V}{m}$. Though this represents a field strength which is still some orders of magnitude above current experimental possibilities, there are expectations that within a few years, a direct observation of this phenomenon is possible by choice of appropriate non-constant field configurations [25].

4.3 Perturbative vs non-perturbative

We want to conclude this section by recapitulating our findings and adding some final remarks. We have seen that there is no reason to be anxious about the appearance of asymptotic series when calculating functional determinants. Rather, we understand where they come from and how we can deal with them. Most importantly, we have seen that non-alternating perturbative

[7] By definition[47], the effective action Γ it is related to the vacuum persistence amplitude (i.e. the amplitude for the vacuum state persisting under the influence of an external electric field E) through $e^{i\Gamma} = \langle 0|0\rangle_E$ and the probability that the vacuum decays is thus $\mathcal{P} = 1 - \left|\langle 0|0\rangle_E\right|^2 = 1 - e^{-2\Im\Gamma} \approx 2\Im\Gamma$.

series have lost a crucial part of the underlying physics, namely the occurrence of an imaginary contribution related to a decay process. However, this information can be "recovered" by a Borel analysis.

Finally, we note an interesting analogy concerning the question of the convergence (or otherwise) of QED perturbation theory. In fact, Freeman Dyson presented a simple but strong argument against convergence of the QED perturbative series already in 1952 [26]. He argued that for the power series expansion of a function in the fine structure constant $\alpha = \frac{e^2}{\hbar c}$,

$$F(e^2) \;=\; 1 \;+\; a_1\, e^2 \;+\; a_2\, e^4 \;+\; \dots \;, \tag{4.20}$$

to be convergent, it has to hold that the function is analytic in an arbitrarily small radius around the origin $e^2 = 0$ in the complex plane. However, this cannot be the case. Consider $F(-e^2)$, which corresponds to the physics in a world where like charges attract each other. In this world, however, the ordinary vacuum state unstable and is not the state of lowest energy. Spontaneous creation of opposite charges due to quantum fluctuations will lead to an accumulation of like charges in some region as the charges will repel each other and eventually gain enough energy to go on shell. Moreover, this is a process which accelerates quickly as more and more electric charges are created out of the vacuum. Observables such as the electric field strength will thus be rendered infinite. Hence, an integration of the equations of motion in this world thus does not possibly seem to yield an analytic function.

Thus, Dyson suggested, $F(-e^2)$ cannot be analytic and there should instead be a branch cut along the negative e^2 axis. Consequently, the series expansion in Eq. (4.20) can at best be asymptotic.

5 The Gel'fand-Yaglom formalism

5.1 Preliminaries

In Sect. 2 (and particularly in the corresponding exercises) we have learned that for the computation of the determinant of some operator $\mathcal{M}$, it can be convenient to evaluate the derivative of the associated ζ-function at its origin instead of actually performing the sum over the eigenvalues λ_n.

In this section, we will go one step further and discuss how the functional determinant can be evaluated even if the eigenvalues are not known at all. We will see that if we find some function that vanishes exactly at the eigenvalues λ_n of an operator $\mathcal{M}$, we can (numerically) evaluate that function and relate this result to the determinant of $\mathcal{M}$. Here, we will demonstrate the

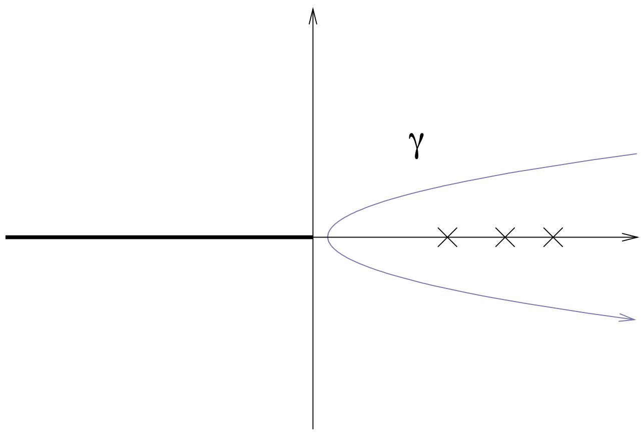

Figure 3: Contour γ in the λ-plane. Notice the branch cut along the negative real axis (thick line).

basic ideas of this relation. A mathematically rigorous and detailed treatment of the derivation is e.g. found in [27, 29, 32].

Thus, let us consider the situation that we do not have the eigenvalues in particular, but that they are rather given as zeros of some function, i.e., the eigenvalues λ_n fulfill

$$\mathcal{F}(\lambda) \quad = \quad 0 \quad \forall\, \lambda = \lambda_n,\ n \in \mathbb{N}\ . \tag{5.1}$$

In this case, the expression

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathcal{F}(\lambda) \quad = \quad \frac{\mathcal{F}'(\lambda)}{\mathcal{F}(\lambda)} \tag{5.2}$$

has poles exactly at λ_n. Moreover it holds, as one can see by expanding Eq. (5.2) about λ_n, that the residue at those poles is 1. We thus can write the ζ-function as

$$\zeta(s) \quad = \quad \frac{1}{2\pi i} \int_\gamma \mathrm{d}\lambda\ \lambda^{-s}\, \frac{\mathrm{d} \ln \mathcal{F}(\lambda)}{\mathrm{d}\lambda}\ , \tag{5.3}$$

where the contour γ is chosen as depicted in Fig. 5.1. Note, that the branch cut which is implied by λ^{-s} is chosen on the negative real λ-axis, as usual.

We now deform the contour $\gamma \to \gamma_-$ such that it encloses the negative real λ-axis, rather than the positive real axis, see Fig. 5.1. When shifting the upper and lower half of the γ_--contour towards the branch cut at the negative real λ-axis, the integrands pick up a phase of $e^{-i\pi s}$ and $e^{i\pi s}$, respectively (see e.g. [28]). Thus Eq. (5.3) becomes

$$\begin{aligned} \zeta(s) \quad &= \quad \frac{1}{2\pi i} \left[e^{-i\pi s} \int\limits_{-\infty}^{0} \mathrm{d}\lambda\ \lambda^{-s}\, \frac{\mathrm{d} \ln \mathcal{F}(\lambda)}{\mathrm{d}\lambda} \ + \ e^{i\pi s} \int\limits_{0}^{-\infty} \mathrm{d}\lambda\ \lambda^{-s}\, \frac{\mathrm{d} \ln \mathcal{F}(\lambda)}{\mathrm{d}\lambda} \right] \\ &= \quad \frac{\sin(\pi s)}{\pi} \int\limits_{0}^{-\infty} \mathrm{d}\lambda\ \lambda^{-s}\, \frac{\mathrm{d} \ln \mathcal{F}(\lambda)}{\mathrm{d}\lambda}\ . \end{aligned} \tag{5.4}$$

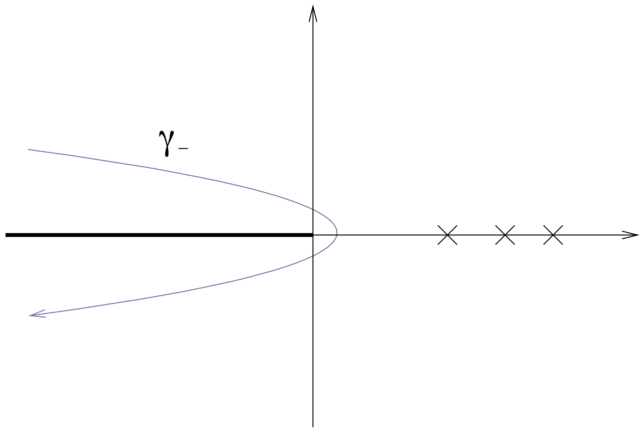

Figure 4: Contour γ_- in the λ-plane. Notice the branch cut along the negative real axis (thick line).

Differentiating Eq. (5.4) with respect to s and setting $s = 0$ yields

$$-\zeta'(0) \quad = \quad -\ln \mathcal{F}(-\infty) \ + \ \ln \mathcal{F}(0) \ , \tag{5.5}$$

since only the total derivative remains as integrand. But we already know from Sect.2, that $\zeta'(0)$ can be related to the spectrum of a general operator $\mathcal{M}$ via

$$\det \mathcal{M} \quad = \quad \exp(-\zeta'(0)) \ . \tag{5.6}$$

Thus, to make use of Eq.(5.5) for the evaluation of the determinant for $\mathcal{M}$, all we need to do now is to find the corresponding function $\mathcal{F}$. Note also that the term $\mathcal{F}(-\infty)$ is typically independent of the details of the potential $V(x)$. Thus, in typical physical problems, where the functional determinants are normalized with respect to the value of the free $[V(\vec{x}) = 0]$ operators, the contribution $\mathcal{F}(-\infty)$ drops out, and we have

$$\ln\left(\frac{\det \mathcal{M}}{\det \mathcal{M}_{\text{free}}}\right) \quad = \quad \ln\left(\frac{\mathcal{F}(0)}{\mathcal{F}_{\text{free}}(0)}\right) \ . \tag{5.7}$$

5.2 One-dimensional Schrödinger operators

In a one-dimensional situation we can use the above considerations to formulate the so-called Gel'fand-Yaglom theorem [31, 33], which gives a very simple way to compute the determinant of a one-dimensional Schrödinger operator.

Let $\mathcal{M} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ be a Schrödinger operator on a finite interval $x \in [0, 1]$. Suppose that we want to solve the eigenvalue equation with Dirichlet boundary conditions

$$\mathcal{M}\,\phi_n \quad = \quad \lambda_n\,\phi_n \ , \qquad \phi_n(0) \quad = \quad \phi_n(1) \quad = \quad 0 \ , \tag{5.8}$$

where $0 < \lambda_1 < \lambda_2 < \dots$ are discrete, non-degenerate eigenvalues that are bounded from below. We can achieve this by contructing an initial value problem with the same operator $\mathcal{M}$, but different boundary conditions:

$$\mathcal{M}\, u_\lambda \;=\; \lambda\, u_\lambda\ , \qquad u_\lambda(0) \;=\; 0\ , \qquad u'_\lambda(0) \;=\; 1 \tag{5.9}$$

This determines the function $u_\lambda(x)$ uniquely[8]. If λ is to be an eigenvalue of the original problem (5.8), then it also holds that $u_\lambda(1) = 0$. So considering $u_\lambda(1)$ as a function of λ, we can set

$$\mathcal{F}(\lambda) \;\equiv\; u_\lambda(1)\ , \tag{5.10}$$

and from our previous considerations (Sect. 5.1) we know that

$$-\zeta'(0) \;=\; \ln\left(\frac{u_{\lambda=0}(1)}{u_{\lambda=-\infty}(1)}\right)\ . \tag{5.11}$$

By means of Eq.(5.6), it thus holds for the operator $\mathcal{M}$, that

$$\frac{\det\left(\mathcal{M}\right)}{\det\left(\mathcal{M}_{\text{free}}\right)} \;=\; \frac{u_{\lambda=0}(1)}{u^{\text{free}}_{\lambda=0}(1)}\ . \tag{5.12}$$

Let us summarize what we have achieved until now. Instead of solving the eigenvalue equation (5.8), we just need to find some function u, such that $\mathcal{M}u = 0$ with the initial values $u(0) = 0$ and $u'(0) = 1$. The determinant of the operator is then given by

$$\det\mathcal{M} \;=\; u(1)\ . \tag{5.13}$$

This is quite an astounding result. It says that to compute the determinant of $\mathcal{M}$, we actually do not have to know any of its eigenvalues.

Let us illustrate this procedure by means of an easy example for which the product over the eigenvalues can also be computed directly. Of course, the usefulness of the Gel'fand Yaglom theorem, however, rather lies in the situations where this is not possible .
Let $\mathcal{M}$ be the Helmholtz operator, with Dirichlet boundary conditions at 0 and L,

$$\mathcal{M} \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2\ , \tag{5.14}$$

where $\mathcal{M}_{\text{free}}$ in this case is simply given by the Laplace operator, $\mathcal{M}_{\text{free}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. For this situation the eigenvalues are well known and we can immediately give the normalized determinant:

$$\frac{\det\left(\mathcal{M}\right)}{\det\left(\mathcal{M}_{\text{free}}\right)} \;=\; \prod_{n=1}^{\infty}\left(\frac{m^2 + \left(\frac{n\pi}{L}\right)^2}{\left(\frac{n\pi}{L}\right)^2}\right) \;=\; \prod_{n=1}^{\infty}\left(1 + \left(\frac{mL}{n\pi}\right)^2\right) \;=\; \frac{\sinh(mL)}{mL} \tag{5.15}$$

[8]The choice of the second condition in Eq. (5.2) is just a choice of normalization for u. A different choice of normalization would render a multiplicative factor in the Gel'fand Yaglom formula [29].

In order to compute this determinant by means of the Gel'fand and Yaglom theorem, we need to solve (5.9) with $\lambda = 0$, for each operator. That is, we solve the two differential equations with initial conditions:

$$-u'' + m^2 u = 0\,, \qquad u(0) = 0\,, \qquad u'(0) = 1 \tag{5.16a}$$

$$-u''_{\text{free}} = 0\,, \qquad u_{\text{free}}(0) = 0\,, \qquad u'_{\text{free}}(0) = 1 \tag{5.16b}$$

The solutions to these equations read $u(x) = \frac{\sinh(mx)}{m}$, and $u_{\text{free}}(x) = x$, thus with Eq. (5.12) we find that

$$\frac{\det(\mathcal{M})}{\det(\mathcal{M}_{\text{free}})} = \frac{u(L)}{u_{\text{free}}(L)} = \frac{\sinh(mL)}{mL}\,, \tag{5.17}$$

which does indeed give exactly the same result as computing directly the product of the eigenvalues in Eq. (5.15). This is a somewhat trivial example, as all the eigenvalues are known, and their product is associated with infinite product representation of the sinh function. But if $\mathcal{M}$ were to include a nontrivial potential, the eigenvalue approach is rarely possible while the Gelfand-Yaglom approach is a simple numerical calculation.

As a second paradigmatic example we consider the Pöschl-Teller [32, 34, 35] potential. The corresponding operator reads

$$\mathcal{M} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2 - j\,(j+1)\,\mathrm{sech}^2(x)\,, \tag{5.18}$$

where j takes integer values. The Pöschl-Teller potential has j discrete bound states at $E_l = m^2 - l^2$, where $l = 1 \ldots j$, as well as a continuous spectrum of states, for which the density of states is given by

$$\rho(k) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}k}\,\delta(k) = -\frac{2}{\pi}\sum_{l=1}^{j}\frac{l}{l^2+k^2}\,. \tag{5.19}$$

Here $\delta(k)$ constitutes the phase shift which is induced by a scattering off the potential. Thus, the spectrum of $\mathcal{M}$ is given by

$$\begin{aligned}
\ln\det\mathcal{M} &= \sum_{l=1}^{j}\ln\left(m^2-l^2\right) + \int_0^\infty \mathrm{d}k\;\rho(k)\,\ln\left(k^2+m^2\right)\\
&= \sum_{l=1}^{j}\ln\left(m^2-l^2\right) - \frac{2}{\pi}\sum_{l=1}^{j} l\int_0^\infty \frac{\mathrm{d}k}{l^2+k^2}\;\ln\left(k^2+m^2\right)\\
&= \sum_{l=1}^{j}\ln\left(m^2-l^2\right) - 2\sum_{l=1}^{j}\ln\left(m-l\right)\\
&= \ln\left((-1)^j\,\frac{\Gamma(j-m)\,\Gamma(1+m)}{\Gamma(1-m)\,\Gamma(1+m+j)}\right)\\
&= \ln\left(\frac{\Gamma(m)\,\Gamma(m+1)}{\Gamma(m-j)\,\Gamma(j+m+1)}\right)\,.
\end{aligned} \tag{5.20}$$

It is a simple but instructive numerical exercise to compute det $\mathcal{M}$ by the Gel'fand-Yaglom method, integrating from $x = -L$ to $x = +L$ (for some $L \gg 1$), using the initial value boundary conditions given above, and comparing with the exact expression (5.20). From Eq. (5.20) we see that the determinant vanishes for all integers $m \leq j$. This makes sense, because for such values of m there is a bound state with zero energy: i.e., a "zero mode" makes the determinant vanish. We will learn about the physical background of such zero modes in the following example.

5.3 Sine-Gordon Solitons and zero modes of the determinant

Consider the following scalar Lagrangian in 1+1-dimensional QFT:

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu\phi)^2 \;-\; \underbrace{\frac{2}{g}\,\sin^2\left(\frac{\sqrt{g}}{2}\phi\right)}_{U(\phi)}\;, \qquad \mu = 0,1 \tag{5.21}$$

Suppose that we adopt a semiclassical approximation in order to solve the theory, i.e. we compute the second variation of the potential with respect to the field and expand it around the classical solution. Firstly, to find the classical solution, we compute the field configuration that leads to a static energy E. Writing the energy as

$$E \;=\; \int \mathrm{d}x\,\left(\frac{1}{2}\,\phi'^2 \;+\; U(\phi)\right) \;=\; \int \mathrm{d}x\,\left(\frac{1}{2}\,\left(\phi' \;-\; \sqrt{2U(\phi)}\right)^2 \;+\; \phi'\,\sqrt{2U(\phi)}\right)\;, \tag{5.22}$$

one sees that it is minimized at $\phi' = \sqrt{2U(\phi)}$. For the Sine-Gordon example, the classical solution is thus determined by the first order differential equation

$$\phi' \;=\; \frac{2}{\sqrt{g}}\,\sin\left(\frac{\sqrt{g}}{2}\phi\right)\;, \tag{5.23}$$

which is solved by[9]

$$\phi_{\mathrm{cl}}(x) \;=\; \frac{4}{\sqrt{g}}\,\arctan\bigl(\exp(x)\bigr)\;. \tag{5.24}$$

We now want to compute the fluctuation operator about the classical solution,

$$\mathcal{M} \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \;+\; \left.\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right|_{\phi=\phi_{\mathrm{cl}}}\;, \tag{5.25}$$

which can be evaluated by virtue of Eq. (5.24),

$$\left.\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right|_{\phi=\phi_{\mathrm{cl}}} \;=\; \cos\bigl(\sqrt{g}\phi(x)\bigr)\Bigr|_{\phi=\phi_{\mathrm{cl}}} \;=\; 1 \;-\; 2\,\mathrm{sech}^2(x)\;. \tag{5.26}$$

We see that this corresponds exactly to the Pöschl-Teller [32] potential with $m = 1$ and $j = 1$ (cf. Eq. (5.18)) and thus we already know that the determinant of Eq. (5.25) has a zero

[9] Actually, there appears an integration constant in the exponent. We will discuss this in a few moments.

mode. The corresponding eigenfunction ψ to the eigenvalue 0 is determined by

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi(x) = 0\,, \tag{5.27}$$

where $V(x) = \frac{\mathrm{d}^2 U}{\mathrm{d}\phi^2}\Big|_{\phi=\phi_{\mathrm{cl}}}$. The solution to Eq. (5.27) is given by $\psi = \phi'_{\mathrm{cl}}$, which one can directly verify be insertion of ϕ'_{cl} into Eq. (5.27),

$$-\phi'''_{\mathrm{cl}} + \frac{\mathrm{d}^2 U}{\mathrm{d}\phi^2}\bigg|_{\phi=\phi_{\mathrm{cl}}} \phi'_{\mathrm{cl}} = 0\,, \tag{5.28}$$

where $\phi' = \sqrt{2U(\phi)}$ implies that

$$-\phi'''_{\mathrm{cl}} = -\left(\sqrt{2U(\phi)}\right)''\bigg|_{\phi=\phi_{\mathrm{cl}}} = -\left(\frac{\mathrm{d}U}{\mathrm{d}\phi}\right)'\bigg|_{\phi=\phi_{\mathrm{cl}}} = -\frac{\mathrm{d}^2 U}{\mathrm{d}\phi^2}\,\phi'\bigg|_{\phi=\phi_{\mathrm{cl}}}\,. \tag{5.29}$$

This zero mode actually results from the translational invariance of the classical solution. The general solution to the differential equation (5.23) is given by

$$\phi_{\mathrm{cl}}(x) = \frac{4}{\sqrt{g}}\arctan\big(\exp(x - x_0)\big)\,, \tag{5.30}$$

where x_0 denotes an arbitrary constant. Remember that the fluctuation operator of Eq.(5.25) from the semiclassical approximation results from a Gaussian integration over the fields ϕ, i.e. we use that

$$\int_{-\infty}^{\infty} \mathrm{d}x\; e^{-ax^2} = \sqrt{\frac{\pi}{a}}\,, \tag{5.31}$$

where a in our functional integration yields the determinant of the fluctuation operator. However, if we integrate over a field configuration that has zero eigenvalue, or zero a in the above terminology, it should give us simply a volume factor corresponding to integration over the kink location x_0. I.e. in quantizing around the classical solution, cf. Eq. (1.5), we implicitly assumed that no zero modes exist.

Thus, what we really need to compute rather than just the determinant of the fluctuation operator, is

$$\sqrt{\frac{S_{\mathrm{cl}}}{2\pi}}\left[\det{}'\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 U}{\mathrm{d}\phi^2}\bigg|_{\phi=\phi_{\mathrm{cl}}}\right)\right]^{-\frac{1}{2}}\,, \tag{5.32}$$

where the prime shall denote that occurring zero modes have been removed and the determinant picks up a volume factor of $(S_{\mathrm{cl}}/2\pi)^{1/2}$ for each zero mode, cf. [42]. In practical calculations to find the determinant without zero modes, one uses a small parameter k^2, in order to move the zero mode away from zero:

$$\frac{\det(\mathcal{M} + k^2)}{\det(\mathcal{M}_{\mathrm{free}} + k^2)} \sim k^2\,\frac{\det'(\mathcal{M})}{\det(\mathcal{M}_{\mathrm{free}})}\,, \qquad k^2 \to 0 \tag{5.33}$$

In fact, there is a quick way to compute $\det'(\mathcal{M})$: see [30] for one-dimension, and [44, 41] for the multidimensional radial case.

5.4 Gel'fand Yaglom with generalized boundary conditions

In our calculations using the GEL'FAND YAGLOM theorem, we have so far only considered DIRICHLET boundary conditions for the operator $\mathcal{M}$. Clearly, it would be very helpful to extend our findings to non-DIRICHLET boundary conditions. This has been done [30]. Here we simply quote the results.

Consider again the eigenvalue equation for the Schrödinger operator:

$$-u''_\lambda \;+\; V\,u_\lambda \;=\; \lambda\,u_\lambda \tag{5.34}$$

By defining $v_\lambda \equiv u'_\lambda$, we can rewrite Eq.(5.34) in first order matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ V-\lambda & 0 \end{pmatrix}\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} \tag{5.35}$$

Using 2×2 matrices M and N, we can then implement generalized boundary conditions by demanding

$$M\begin{pmatrix} u_\lambda(0) \\ v_\lambda(0) \end{pmatrix} \;+\; N\begin{pmatrix} u_\lambda(1) \\ v_\lambda(1) \end{pmatrix} \;=\; 0\,, \tag{5.36}$$

for appropriate choice of M and N. E.g., we have

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \qquad \leftrightarrow \quad \text{Dirchlet b.c.'s} \tag{5.37a}$$

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad \leftrightarrow \quad \text{Neumann b.c.'s} \tag{5.37b}$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad \leftrightarrow \quad \text{periodic b.c.'s} \tag{5.37c}$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \leftrightarrow \quad \text{antiperiodic b.c.'s}\,. \tag{5.37d}$$

The GEL'FAND-YAGLOM theorem for generalized boundary conditions then reads [29]

$$\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \;+\; V(x)\right) \;=\; \det\left(M \;+\; N\begin{pmatrix} u_{(1)}(L) & u_{(2)}(L) \\ u'_{(1)}(L) & u'_{(2)}(L) \end{pmatrix}\right)\,, \tag{5.38}$$

where $u_{(1)}$ and $u_{(2)}$ define two linearly independent solutions to

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \;+\; V(x)\right)u_{(i)} \;=\; 0 \tag{5.39}$$

with initial value conditions

$$u_{(1)}(0) \;=\; 1\,, \qquad u'_{(1)}(0) \;=\; 0 \tag{5.40a}$$

$$u_{(2)}(0) \;=\; 0\,, \qquad u'_{(2)}(0) \;=\; 1\,. \tag{5.40b}$$

Thus again, we have reduced the problem of computing an infinite-dimensional determinant to the evaluation of a 2×2 matrix determinant whose entries are obtained simply by the (numerical) integration of two initial value problems. At last we would like to mention that the method of Gel'fand and Yaglom also generalizes to coupled systems of ordinary differential equations [29] and to general Sturm-Liouville operators [30].

6 Radial Gel'fand-Yaglom formalism in higher dimensions

6.1 d-dimensional radial operators

In the previous section we have discussed the very versatile Gel'fand-Yaglom formalism which is applicable for computing the determinant of several types of one-dimensional differential operators. It is natural to ask if and how our findings translate to higher-dimensional problems. Unfortunately a corresponding theorem for such a general class of differential operators in higher dimensions is not known. However, for radially symmetric problems the formalism of the previous section can be extended to arbitrarily many dimensions [41].
Consider the d-dimensional eigenvalue problem

$$\Big(-\triangle \;+\; V(r)\Big)\,\Psi(x) \;=\; \lambda\,\Psi(x)\,, \tag{6.1}$$

where $\triangle$ constitutes the Laplace operator in d dimensions. Due to the radial symmetry of the potential in Eq. (6.1), the eigenfunctions Ψ can be given as linear combinations of hyperspherical harmonics

$$\Psi(r,\vec{\theta}) \;=\; \frac{1}{r^{(d-1)/2}}\,\psi_{(l)}(r)\,Y_{(l)}(\vec{\theta})\,, \tag{6.2}$$

where the $\psi_{(l)}$ are solutions to the radial equation

$$\mathcal{M}_{(l)}\,\psi_{(l)}(r) \;:=\; \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} \;+\; \frac{(l+\frac{d-3}{2})\,(l+\frac{d-1}{2})}{r^2} \;+\; V(r)\right)\psi_{(l)}(r) \;=\; \lambda\,\psi_{(l)}(r)\,. \tag{6.3}$$

The radial eigenfunctions $\psi_{(l)}$ come in $d \geq 2$ with a degeneracy factor of

$$\deg(l;d) \;=\; \frac{(2l+d-2)\,(l+d-3)!}{l!\,(d-2)!}\,, \tag{6.4}$$

which e.g. in three dimensions results in a factor of $\deg(l;3) = 2l+1$, which is familiar from standard quantum mechanical problems. Note also, that for large l the degeneracy factor behaves as

$$\deg(l;d) \quad \sim \quad l^{d-2} \,. \tag{6.5}$$

We now want to discuss how to evaluate the determinant of the radial operator $\mathcal{M}_{(l)}$ as defined in Eq. (6.3). From the previous section, adapted to a radial STURM-LIOUVILLE form, we know that we can solve, instead of explicitly calculating the eigenvalues of the operator $\mathcal{M}_{(l)}$, the initial value problem and obtain

$$\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}^{\text{free}} + m^2\right)} \quad = \quad \frac{\phi_{(l)}(R)}{\phi_{(l)}^{\text{free}}(R)} \,. \tag{6.6}$$

Here $\phi_{(l)}$ and $\phi_{(l)}^{\text{free}}$ are again the eigenfunctions of the operator corresponding to the eigenvalue zero and obey the usual initial value conditions:

$$\left(\mathcal{M}_{(l)} + m^2\right)\phi_{(l)} \quad = \quad 0\,, \qquad \phi_{(l)} \quad \sim \quad r^{l+\frac{d-1}{2}}\,, \quad r \to 0 \tag{6.7a}$$

$$\left(\mathcal{M}_{(l)}^{\text{free}} + m^2\right)\phi_{(l)}^{\text{free}} \quad = \quad 0\,, \qquad \phi_{(l)}^{\text{free}} \quad \sim \quad r^{l+\frac{d-1}{2}}\,, \quad r \to 0 \tag{6.7b}$$

Here, $\mathcal{M}_{(l)}^{\text{free}}$ constitutes the operator in Eq. (6.12) without the potential term $V(r)$; m^2 is not to be considered a part of $V(r)$ in the following. The eigenfunctions of $\mathcal{M}_{(l)}^{\text{free}}$ are known to read

$$\phi_{(l)}^{\text{free}}(r) \quad = \quad \frac{\Gamma\left(l+\frac{d}{2}\right)\sqrt{r}}{\left(\frac{m}{2}\right)^{l+\frac{d}{2}-1}}\, I_{l+\frac{d}{2}-1}(mr) \quad \sim \quad r^{l+(d-1)/2}\,, \quad r \to 0 \,, \tag{6.8}$$

where the I denote the BESSEL functions of complex arguments $I_n(r) = i^{-n} J_n(ir)$.

In physical situations the outer DIRICHLET boundary is often at $R = \infty$. Hence, one can evaluate Eq. (6.6) and let $R \to \infty$ in the end. However, since the eigenfunctions $\phi_{(l)}^{(\text{free})} \sim e^{mr} \stackrel{r\to\infty}{\longrightarrow} \infty$, it is numerically favorable not to evaluate $\phi_{(l)}$ and $\phi_{(l)}^{\text{free}}$ separately, but instead to immediately calculate their ratio. Defining

$$\mathcal{R}_{(l)}(r) \quad := \quad \frac{\phi_{(l)}(r)}{\phi_{(l)}^{\text{free}}(r)} \,, \tag{6.9}$$

we can bring Eq. (6.6) into the form

$$\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}^{\text{free}} + m^2\right)} \quad = \quad \mathcal{R}_{(l)}(\infty) \,. \tag{6.10}$$

The corresponding differential equation for $\mathcal{R}_{(l)}$ can be obtained from the differential equations for $\phi_{(l)}$ and $\phi_{(l)}^{\text{free}}$ (cf. Eq. (6.7b)), since the exact expression for $\phi_{(l)}^{\text{free}}$ is known. It yields for each partial wave l

$$-\mathcal{R}_{(l)}''(r) \;-\; \left(\frac{1}{r} \;+\; \frac{2I_{l+\frac{d}{2}-1}'(r)}{2I_{l+\frac{d}{2}-1}(r)}\right)\mathcal{R}_{(l)}' \;+\; V(r)\,\mathcal{R}_{(l)}(r) \quad = \quad 0\,, \tag{6.11}$$

with initial conditions $\mathcal{R}_{(l)}(0) = 1$ and $\mathcal{R}'_{(l)}(0) = 0$.

All in all we see that computing the determinant of a partial differential operator that is radially separable is straightforward for any given partial wave l with the Gel'fand Yaglom method. The determinant of the operator normalized with respect to the potential-free case is given through Eq. (6.10), where the function $\mathcal{R}_{(l)}$ is obtained uniquely from the differential Eq. (6.11) with given boundary conditions. However, as we will see shortly, though solving the problem for each partial wave is rather easy, combining them is not.

6.2 Example: 2-dimensional Helmholtz problem on a disc

Let us demonstrate the findings of the previous section by means of an explicit example, analogous to the 1-dimensional example in Sec.5.2. Consider a 2-dimensional Helmholtz problem on a disc of radius R, i.e. we want to calculate

$$\det\left(\mathcal{M}_{(l)} + m^2\right) = \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right)}{r^2} + m^2\right) \tag{6.12}$$

with Dirichlet boundary conditions at 0 and R. The generalized radial Gel'fand Yaglom theorem of Eq. (6.6) translates in this this situation to

$$\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}\right)} = \frac{\psi_{(l)}(R)}{\psi^{\text{free}}_{(l)}(R)} , \tag{6.13}$$

thus the free operator $\mathcal{M}^{\text{free}}_{(l)}$ in this case just corresponds to $\mathcal{M}_{(l)}$, since the mass-term here takes the place of the missing potential $V(r)$. Thus, the solutions to the Gelfand-Yaglom initial value problem are

$$\psi_{(l)}(r) = \frac{l!\,\sqrt{r}\,I_l(mr)}{\left(\frac{m}{2}\right)^l} \tag{6.14a}$$

$$\psi^{\text{free}}_{(l)}(r) = r^{l+\frac{1}{2}} . \tag{6.14b}$$

Consequently, Eq. (6.13) evaluates to

$$\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}\right)} = \frac{l!\,I_l(mR)}{\left(\frac{mR}{2}\right)^l} . \tag{6.15}$$

We can check on this result by directly computing the determinant from the eigenvalues, which are known for this situation. Due to the imposed Dirichlet boundary conditions, it must hold for each eigenvalue λ that $J_l(\sqrt{\lambda}R) = 0$, and thus the eigenvalues λ just read $j^2_{(l),n}/R^2$, where $j^2_{(l),n}$ are the zeros of the Bessel function, which are only known numerically. Thus, we find for a fixed l that

$$\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}\right)} = \prod_{n=1}^{\infty}\left(\frac{m^2 + \frac{j_{(l),n}^2}{R^2}}{\frac{j_{(l),n}^2}{R^2}}\right) = \prod_{n=1}^{\infty}\left(1 + \left(\frac{mR}{j_{(l),n}^2}\right)^2\right)$$
$$= \frac{l!\, I_l(mR)}{\left(\frac{mR}{2}\right)^l}\,, \tag{6.16}$$

which corresponds to the result of Eq. (6.15) as expected. In the last step of (6.16) we used the product representation of the Bessel function I_l. Note the interesting fact that there is a simple expression for the determinant, even though there is no known explicit expression for the eigenvalues.

In Eq. (6.15) and Eq. (6.16) we have thus found the determinant for each partial wave. The general result to the Helmholtz problem, however, is given by a sum over all of them. Thus we would naively guess (which will turn out to be wrong)

$$\ln\frac{\det(-\triangle + m^2)}{\det(-\triangle)} \stackrel{?}{=} \sum_{l=0}^{\infty}\deg(l;2)\,\ln\frac{\det\left(\mathcal{M}_{(l)} + m^2\right)}{\det\left(\mathcal{M}_{(l)}\right)}$$
$$= \ln\bigl(I_0(ml)\bigr) + \sum_{l=1}^{\infty} 2\,\ln\left(\frac{l!\, I_l(mR)}{\left(\frac{mR}{2}\right)^l}\right)\,. \tag{6.17}$$

We have put a question mark on the first equals sign to indicate that the sum on the right hand sign of equation (6.17) is divergent because (for fixed mR)

$$\ln\left(\frac{l!\, I_l(mR)}{\left(\frac{mR}{2}\right)^l}\right) \sim \frac{1}{l}\,. \tag{6.18}$$

However, this divergence should not be too much of a surprise, since in $d > 1$ we know that we should regularize and renormalize the determinant [38].

To see this more clearly, consider a different operator which we discussed already: the normalized determinant of $(i\not{D} + m)$. Formally, we can write

$$\frac{\det(i\not{D} + m)}{\det(i\not{\partial} + m)} = \det\left(\frac{i\not{\partial} + m + \not{A}}{i\not{\partial} + m}\right) = \det\left(1 + G\not{A}\right)\,, \tag{6.19}$$

where G denotes the Green's function $(i\not{\partial}+m)^{-1}$. Defining $G\not{A} \equiv T$, the functional determinant can be rewritten as

$$\det(1 + T) = \exp\Bigl(\mathrm{Tr}\bigl\{\ln(1 + T)\bigr\}\Bigr) = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\,\mathrm{Tr}\bigl\{T^k\bigr\}\right)\,. \tag{6.20}$$

For some lower order k's, the traces in Eq. (6.20) can diverge, depending on the dimension. Thus, what one does is to regularize the determinant by dropping the divergent diagrams and defining the regularized determinant [37, 38]:

$$\det\left(1 + T\right)_n \quad := \quad \exp\left(\sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k}\,\mathrm{Tr}\Big\{T^k\Big\}\right) \tag{6.21}$$

This *mathematical* definition of a finite determinant (dating back to work of Poincaré and Hilbert) can be made *physically* relevant by the procedure of renormalization.

6.3 Renormalization

In order to see how we can do the renormalization for the sum over the partial waves, Eq. (6.17), let us review briefly how we derived the expression for the computation of the determinant after all. In Sect. (5.1), we used that if we found for an operator $\mathcal{M}$ with eigenvalues λ_n a function $\mathcal{F}(\lambda)$ that fulfilled $\mathcal{F}(\lambda) = 0$ exactly at the eigenvalues λ_n, then we could relate this function to the ζ-function through

$$\zeta(s) \quad = \quad \frac{\sin(\pi s)}{\pi} \int_0^{-\infty} \mathrm{d}\lambda\ \lambda^{-s}\,\frac{\mathrm{d}\ln\mathcal{F}(\lambda)}{\mathrm{d}\lambda}\ . \tag{6.22}$$

We then applied our results of Sect.(2) and linked the derivative of this ζ-function at the origin to the determinant of $\mathcal{M}$ (cf. Eq. (2.4)). *Formally*, in the setup a radially separable, d-dimensional operator this procedure lead to

$$-\zeta'(0) \quad = \quad \ln\left(\frac{\det\left(\mathcal{M} + m^2\right)}{\det\left(\mathcal{M}^{\mathrm{free}} + m^2\right)}\right) \quad = \quad \sum_{l=0}^{\infty} \deg(l;d)\,\ln\left(\frac{\phi_{(l)}(\infty)}{\phi^{\mathrm{free}}_{(l)}(\infty)}\right)\ , \tag{6.23}$$

where the $\phi_{(l)}$ were the eigenfunctions of $\mathcal{M}$ to the eigenvalue 0 with appropriate initial value conditions as discussed in Sec. 6.1.

However, as we saw in the previous section at the instance of the two-dimensional Helmholtz problem on a disc, the sum over the partial waves in Eq. (6.23) diverges and thus what we need is to do the analytic continuation of the ζ-function to $s = 0$ more carefully. A simple approach [41] is to use the Jost function [40, 39] of scattering theory for the analytic continuation of $\zeta(s)$, since its asymptotics are well known.

Let us consider the scattering problem for the l^{th} partial wave,

$$\mathcal{M}_{(l)}\,\phi_{(l)} \quad = \quad k^2\,\phi_{(l)}\ . \tag{6.24}$$

The solution to Eq. (6.24) without a scattering potential reads

$$\phi^{\mathrm{free}}_{(l)}(r) \quad = \quad \sqrt{\frac{\pi k r}{2}}\,J_{l+\frac{d}{2}-1}(kr)\ , \tag{6.25}$$

where the now the BESSEL-J functions appear in contrast to Eq. (6.8), since the "m^2" of the previous section has been replaced by "$-k^2$". Including a (radial) potential term, the regular solution $\phi_{(l)}$ is determined by an integral equation of the form

$$\phi_{(l)}(r) \;=\; \phi_{(l)}^{\text{free}}(r) \;+\; \int_0^r \mathrm{d}r' \; G \, V \, \phi_{(l)}(r') \; , \tag{6.26}$$

where G defines the Green's function corresponding to the operator $\mathcal{M}_{(l)}$.
The asymptotic behaviour of the $\phi_{(l)}$ defines the JOST function $\mathcal{J}_{(l)}$

$$\phi_{(l)}(r) \;\sim\; \frac{i}{2}\left[\mathcal{J}_{(l)}(k)\, h_{(l)}^{(-)}(r) \;+\; \mathcal{J}_{(l)}^{\star}(k)\, h_{(l)}^{(+)}(r)\right] \; , \quad r \to \infty \; , \tag{6.27}$$

where the $h_{(l)}^{(\pm)}$ are the HANKEL functions that behave like $h_{(l)}^{(\pm)}(r) \sim e^{\pm ikr}$ for large r. Ultimately, we want the asymptotic behaviour of these functions for $k \to im$, since this relates the solution of the above scattering problem to the solution for the massive radial operator. We thus write the asymptotic behaviour for $\phi_{(l)}$ and $\phi_{(l)}^{\text{free}}$ as

$$\phi_{(l),ik}(r) \;\sim\; \mathcal{J}_{(l)}(ik)\, e^{kr} \; , \quad r \to \infty \tag{6.28a}$$

$$\phi_{(l),ik}^{\text{free}}(r) \;\sim\; \mathcal{J}_{(l)}^{\text{free}}(ik)\, e^{kr} \; , \quad r \to \infty \; . \tag{6.28b}$$

Consequently, the ratio between the two eigenfunctions gives

$$\frac{\phi_{(l),ik}(r)}{\phi_{(l),ik}^{\text{free}}(r)} \;\sim\; \frac{\mathcal{J}_{(l)}(ik)}{\mathcal{J}_{(l)}^{\text{free}}(ik)} \;\equiv\; f_{(l)}(k) \; , \quad r \to \infty \; , \tag{6.29}$$

where $f_{(l)}$ constitutes the "normalized JOST function". Thus, for each partial wave, we can rewrite Eq. (6.23) by analytic continuation of the JOST function:

$$\frac{\phi_{(l),m}(r)}{\phi_{(l),m}^{\text{free}}(r)} \;=\; \frac{\det\left(\mathcal{M}_{(l)} \;+\; m^2\right)}{\det\left(\mathcal{M}_{(l)}^{\text{free}} \;+\; m^2\right)} \;=\; f_{(l)}(im) \tag{6.30}$$

The point is now as follows: From Eq. (6.26), we know that the regular solution ϕ is given through an integral equation. This LIPPMANN-SCHWINGER integral equation for the $\phi_{(l),ik}$ leads to an iterative expansion for the JOST functions $f_{(l)}$. In terms of $\ln f_{(l)}$ it reads

$$\begin{aligned} \ln f_l(ik) \;=\; & \int_0^\infty \mathrm{d}r \; r\, V(r)\, K_\nu(kr)\, I_\nu(kr) \\ & - \int_0^\infty \mathrm{d}r \; r\, V(r)\, K_\nu^2(kr) \int_0^r \mathrm{d}r' \; r'\, V(r')\, I_\nu^2(kr') \;+\; \mathcal{O}(V^3) \; . \end{aligned} \tag{6.31}$$

We have defined the convenient short-hand notation

$$\nu \;:=\; l \;+\; \frac{d}{2} \;-\; 1 \; . \tag{6.32}$$

Thus, just as in dimensional regularization, we can add and subtract enough terms of the asymptotic behaviour $f_{(l)}^{\mathrm{asym}}$ to make the sum on the right hand side of Eq. (6.23) finite. We write this as

$$\begin{aligned} \zeta(s) \;=\; & \frac{\sin(\pi s)}{\pi} \sum_{l=0}^{\infty} \deg(l;d) \int_0^{\infty} \mathrm{d}k \, \frac{1}{(k^2-m^2)^s} \frac{\partial}{\partial k} \left(\ln f_{(l)}(ik) \,-\, \ln f_{(l)}^{\mathrm{asym}}(ik) \right) \\ & + \frac{\sin(\pi s)}{\pi} \sum_{l=0}^{\infty} \deg(l;d) \int_0^{\infty} \mathrm{d}k \, \frac{1}{(k^2-m^2)^s} \frac{\partial}{\partial k} \left(\ln f_{(l)}^{\mathrm{asym}}(ik) \right) . \end{aligned} \tag{6.33}$$

The number of terms that needs to be included in $f_{(l)}^{\mathrm{asym}}$ of course depends on the dimension d. As it turns out, in $d = 2$ it is sufficient to subtract just the first term, and in $d = 4$ dimensions it suffices to subtract the first two terms. The results in $d = 2, 4$ dimensions are [41]

$$\begin{aligned} \ln\left(\frac{\det\left(\mathcal{M} \,+\, m^2\right)}{\det\left(\mathcal{M}^{\mathrm{free}} \,+\, m^2\right)} \right)\Bigg|_{d=2} \;=\; & \ln f_{(0)}(im) \,+\, 2\sum_{l=1}^{\infty} \left[\ln f_{(l)}(im) \,-\, \frac{1}{2l} \int_0^{\infty} \mathrm{d}r \; r\, V(r) \right] \\ & + \int_0^{\infty} \mathrm{d}r \; r\, V(r) \left[\ln\left(\frac{\mu r}{2}\right) \,+\, \gamma \right] \end{aligned} \tag{6.34a}$$

$$\begin{aligned} \ln\left(\frac{\det\left(\mathcal{M} \,+\, m^2\right)}{\det\left(\mathcal{M}^{\mathrm{free}} \,+\, m^2\right)} \right)\Bigg|_{d=4} \;=\; & \sum_{l=0}^{\infty} (l+1)^2 \Bigg(\ln f_{(l)}(im) \,-\, \frac{1}{2(l+1)} \int_0^{\infty} \mathrm{d}r \; r\, V(r) \\ & + \frac{1}{8\,(l+1)^3} \int_0^{\infty} \mathrm{d}r \; r^3\, V \left(V \,+\, 2m^2\right) \Bigg) \\ & - \frac{1}{8} \int_0^{\infty} \mathrm{d}r \; r^3\, V \left(V \,+\, 2m^2\right) \left[\ln\left(\frac{\mu r}{2}\right) \,+\, \gamma \,+\, 1 \right] , \end{aligned} \tag{6.34b}$$

where μ is the renormalization scale (in the $\overline{\mathrm{MS}}$ renormalization scheme) and γ denotes the Euler-Mascheroni constant. These expressions generalize the Gelfand-Yaglom result to higher dimensions, when the operator is radially seperable. One computes $f_{(l)}(im)$ by a numerical integration, and the log determinant is rendered finite and renormalized by the simple subtractions indicated in (6.34a) and (6.34b).

We want to conclude this section with two remarks. Firstly we mention that another way to deal with the divergent sum over the partial waves l, is to introduce a cutoff at some large L and treat the remaining terms with a radial WKB approximation, as we will illustrate in the next section. Secondly we note, as already discussed in 1-dimensional situations, zero modes of the determinant can appear, if the problem exhibits a translational invariance. This will also be the case in the problem which we discuss in the following.

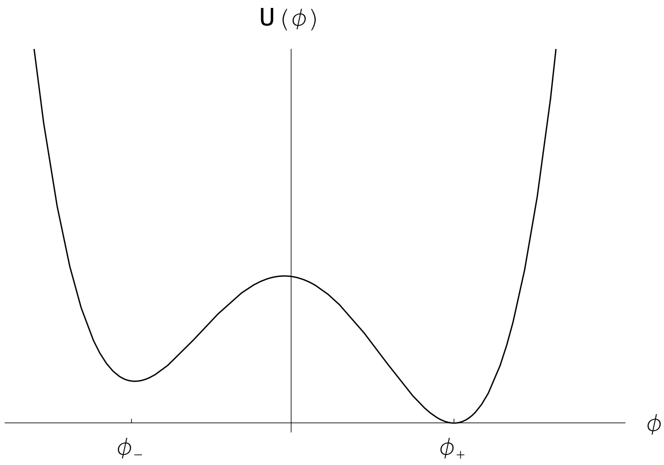

Figure 5: Field potential $U(\phi)$ showing the true and false vacua, ϕ_+ and ϕ_-, respectively. Graph taken from [44].

7 False vacuum decay

7.1 Preliminaries

Consider an asymmetric potential $U(\phi)$ with a global and a local minimum at ϕ_- and ϕ_+, respectively (see Fig. 5). The wells shall be separated by an energy gap $U(\phi_+) - U(\phi_-) = \epsilon$. Then, ϕ_- obviously constitutes only a metastable state, the false vacuum, which will eventually decay by tunneling into the lower state at ϕ_+, which is the true vacuum state.

In the following we want to consider the decay rate Γ of the false vacuum per volume V. Evoking a semiclassical approximation, we expect the generic form

$$\gamma \;:=\; \frac{\Gamma}{V} \;=\; A\, e^{-B/\hbar}\,, \tag{7.1}$$

as the calculation of the decay rate constitutes a tunneling problem.

A useful analogy which is often employed in this context is the process of nucleation, see e.g. [42]. Consider e.g. a superheated fluid: thermodynamic fluctuations in the fluid will cause the creation of bubbles of vapor in the fluid. If the bubbles are too small then the surface tension will let them shrink again to nothing. However, if the bubble is large enough, the gain in volume energy will compensate for the increase in surface tension and the bubble will grow until the entire fluid is vaporized. Similarly one can picture bubbles of true vacuum that form inside the false vacuum state, due to quantum rather than thermal fluctuations, and eventually eat up the false vacuum.

This picture gives us a first estimate for the coefficient B in Eq. (7.1), which is just the total action that is adopted. The action in four dimensions is given by a volume and a surface term with volume energy $E_{\rm V}$ and surface energy $E_{\rm S}$, respectively, which differ in sign. Thus in the limit of a thin bubble wall,

$$S_{\rm tot} \;=\; -\frac{\pi^2}{2}\, R^4\, E_{\rm V} \;+\; 2\pi^2\, R^3\, E_{\rm S}\ , \tag{7.2}$$

which is minimized by $R_{\rm crit} = 3E_{\rm S}\,/E_{\rm V}$. Therefore

$$S_{\rm tot}(R_{\rm crit}) \;=\; \frac{27\pi^2}{2}\,\frac{E_{\rm S}^4}{E_{\rm V}^3}\ , \tag{7.3}$$

approximates the coefficient B in the limit of vanishing thickness of the "bubble wall", i.e. for small ϵ. But to compute the prefactor A in (7.1) we need a more detailed approach.

7.2 The classical bounce solution

We now turn back to field theory and discuss the computation of the decay rate γ for a scalar field ϕ in $d = 4$. Working in a Euclidean formulation, we start from the generating functional

$$\mathcal{Z} \;=\; \int \mathcal{D}\phi\; e^{-S[\phi]/\hbar}\ . \tag{7.4}$$

A potential as shown in Fig. 5 is at least of fourth order in the field. Thus, the problem cannot be solved exactly. We therefore adopt a semiclassical approximation and expand ϕ about the classical solution:

$$S[\phi] \;=\; S[\phi_{\rm cl}] \;+\; \frac{1}{2}\int {\rm d}x \int {\rm d}y\; \phi(x)\, \frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)}\bigg|_{\phi=\phi_{\rm cl}} \phi(y) \;+\; \dots\ . \tag{7.5}$$

The integration over the fields can then be performed which leads us to

$$\mathcal{Z} \;\approx\; \frac{e^{-S[\phi_{\rm cl}]}}{\sqrt{\det\big(-\Box + U''(\phi_{\rm cl})\big)}}\ . \tag{7.6}$$

The decay rate γ is now given by [42, 44]

$$\gamma \;=\; \frac{\Gamma}{V} \;=\; \left(\frac{S_{\rm cl}[\Phi_{\rm cl}]}{2\pi}\right)^2 \left|\frac{\det'\big(-\Box + U''(\Phi_{\rm cl})\big)}{\det\big(-\Box + U''(\Phi_{-})\big)}\right|^{-1/2} e^{-S_{\rm cl}[\Phi_{\rm cl}]}\ , \tag{7.7}$$

where we have introduced a new field variable Φ, which we will discuss below. The prefactor in Eq. (7.7) corresponds to the "volume" factor which comes along with the removal of the four zero modes. (The zero modes correspond to the translational invariance with respect to the bubble's location.) The normalization to the "free" solution here translates to a normalization with the fluctuation operator evaluated at the false vacuum, since we will quantize around the fluctuations about the false vacuum solution.

All in all, one sees that the computation of γ requires a computation of $S_{\mathrm{cl}}[\phi_{\mathrm{cl}}]$ on the one hand, and the evaluation of the determinant prefactor on the other. Furthermore, it turns out that $\Phi_{\mathrm{cl}}(r)$ is *radial* [42]. Thus, the fluctuation operator is radial, and is of the form discussed in Sec. 6. For the prefactor we can apply our results from the previous section and determine the determinant with the generalized theorem of GEL'FAND and YAGLOM [43, 44].

We consider the classical action in Euclidean space-time:

$$S_{\mathrm{cl}}[\phi] \;=\; \int \mathrm{d}^4x \, \left(\frac{1}{2}\,(\partial_\mu\phi)^2 \;+\; U(\phi)\right) \tag{7.8}$$

A canonical choice for the antisymmetric double well potential as shown in Fig.5, which is often used in literature [42], is the quartic potential

$$U(\phi) \;=\; \frac{\lambda}{8}\,\left(\phi^2 \,-\, a^2\right)^2 \;-\; U_{\mathrm{asym}}(\phi)\;. \tag{7.9}$$

Without the U_{asym} contribution, U represents just two symmetric potential wells localized at $\phi = \pm a$. Choosing e.g.

$$U_{\mathrm{asym}}(\phi) \;=\; \frac{\epsilon}{2a}\,(\phi \,-\, a) \tag{7.10}$$

breaks the symmetry of the potential and yields two non-degenerate vacua at $\phi = \phi_\pm$. To lowest order in ϵ we find that the minima $U(\phi_\pm)$ lie at $\phi_\pm = \pm a$; also it holds that $U(\phi_+) - U(\phi_-) \approx \epsilon$ for small ϵ, thus in this limit, which is known as the "thin-wall" limit, ϵ is a measure for the energy difference between the false and the true vacuum.

We rewrite the potential of Eq. (7.9), by expanding the field ϕ about the false vacuum $\phi = \phi_- + \chi$. This yields for terms up to dimension four the potential

$$U(\chi) \;=\; \frac{m^2}{2}\,\chi^2 \;-\; \eta\,\chi^3 \;+\; \frac{\lambda}{8}\,\chi^4\;. \tag{7.11}$$

Rescaling $\chi \longrightarrow \frac{m^2}{2\eta}\Phi$ and $x \longrightarrow \frac{x}{m}$ results in an action in terms of dimensionless quantities

$$S_{\mathrm{cl}} \;=\; \left(\frac{m^2}{4\eta^2}\right)\int \mathrm{d}^4x\,\left[\frac{1}{2}\,(\partial_\mu\Phi)^2 \;+\; \frac{1}{2}\,\Phi^2 \;-\; \frac{1}{2}\,\Phi^3 \;+\; \frac{\alpha}{8}\,\Phi^4\right]\,, \tag{7.12}$$

where $\alpha = \frac{\lambda m^2}{4\eta^2}$. Equation (7.12) defines the field Φ which appears in the decay rate (7.7) and explains, why the determinant prefactor of the decay rate is normalized with respect to the potential at Φ_-: The shape of the potential is now such that $U(\Phi_-) = 0$.

In Fig. 6 the potential $U[\Phi]$ is given for various values of α. We see, that α determines the shape of the potential: $\alpha = 1$ corresponds to degenerate levels and as we move α away from 1, the potential trough widens. For a semiclassical approximation to be justified, we need that $S_{\mathrm{cl}} \gg \hbar$ and thus we demand $m \gg \eta$ in Eq. (7.12).

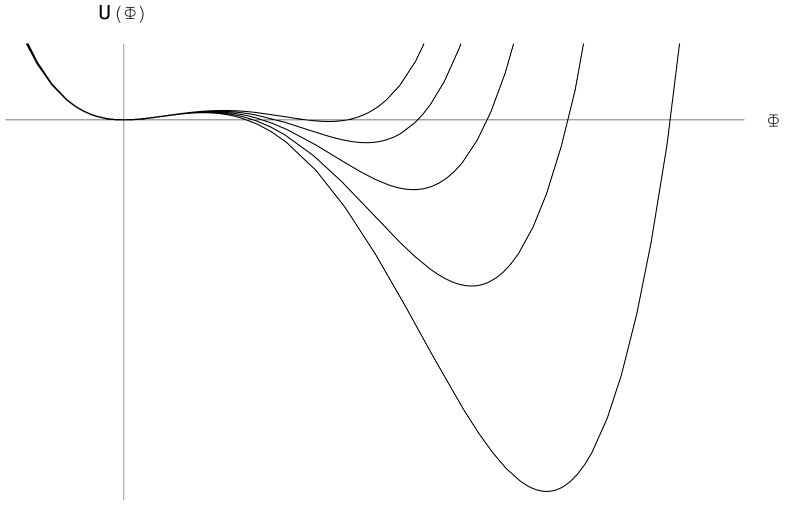

Figure 6: Plots of the rescaled potential, $U(\Phi) = \frac{1}{2}\Phi^2 - \frac{1}{2}\Phi^3 + \frac{\alpha}{8}\Phi^4$, for $\alpha = 0.6, 0.7, 0.8, 0.9, 0.99$. As α approaches 1, the vacua become degenerate. Figure taken from [44].

The classical solution $\Phi_{\rm cl}$, also referred to as "the bounce solution" [42], is a function of the radius $r = \sqrt{x_\mu x_\mu}$. It determines the behaviour of the exponential in the expression for the decay rate (7.7) and is a solution to the differential equation

$$\Phi'' + \frac{3}{r}\,\Phi' - \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi} = 0 \tag{7.13}$$

and thus here

$$-\Phi'' - \frac{3}{r}\,\Phi' + \Phi - \frac{3}{2}\,\Phi^2 + \frac{\alpha}{2}\,\Phi^3 = 0 \tag{7.14}$$

with boundary conditions $\Phi(\infty) = \Phi_- = 0$ and $\Phi'(0) = 0$. Equation (7.14) cannot be solved analytically and thus one resorts to a numerical evaluation, which, however is also non-trivial. The idea is to study Φ for the respective limits $r \to 0$ and $r \to \infty$ and thereafter to match the solutions for intermediate r. For large r, the higher powers of Φ vanish more quickly due to the imposed boundary conditions and thus the differential equation (7.14) simplifies to

$$-\Phi'' - \frac{3}{r}\,\Phi' + \Phi \approx 0\,, \tag{7.15}$$

which is solved by

$$\Phi_{\rm cl} \sim \Phi_\infty \frac{K_1(r)}{r}\,, \tag{7.16}$$

with some numerical coefficient Φ_∞. In the limit of $r \to 0$, one can approximate the classical solution by the polynomial

$$\Phi_{\rm cl} \sim \Phi_0 + \frac{r^2}{16}\left(2\,\Phi_0 - 3\,\Phi_0^2 + \alpha\,\Phi_0^3\right)\,, \tag{7.17}$$

which solves Eq. (7.14) as $r \to 0$, as one can easily convince oneself.

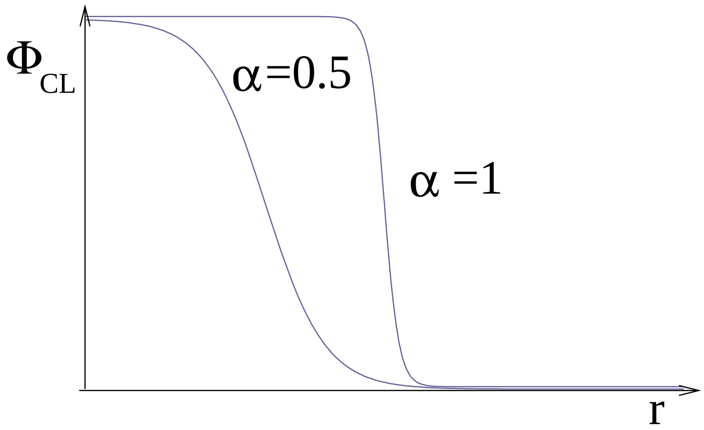

Figure 7: Qualitative behaviour of $\Phi_{\rm cl}(r)$ for two exemplary values of α. For $\alpha \approx 1$, there is a sharp "boundary" at $r_0 \approx \frac{1}{1-\alpha}$. At lower values of the distance $r < r_b$, the classical field approximates the value $\Phi_{\rm cl} \approx \Phi_0$. For $r > r_b$, the classical field approaches zero $\Phi_{\rm cl} \approx 0$. Thus for $\alpha \approx 1$, the thin wall approximation is justified. However, for values $\alpha < 1$, the boundary smears out and the thin wall approximation is no longer valid.

It turns out that a practical parametrization for the solution in the entire r-range is $\Phi_{\rm cl} \sim \tanh(r - r_0)$ as $\alpha \to 1$. For smaller values of α, the drop-off of the tanh at r_0 is less pronounced, cf. Fig. 7.2.

7.3 Computing the determinant factor with radial Gel'fand Yaglom

We now turn to the evaluation of the determinant prefactor in Eq. (7.7). In this context it is crucial that the classical bounce solution Φ is a function of the radius only, and thus we can decompose the fluctuation operator into partial waves of degeneracy $(l+1)^2$. We adopt the language of Sect. 6 and write the fluctuation operator about the classical solution as $\mathcal{M}_{(l)}$, and the "normalization" which in this case constitutes the fluctuations about the state of false vacuum Φ_-

$$\mathcal{M}_{(l)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l\,(l+2)}{r^2} + 1 + V(r) \tag{7.18a}$$

$$\mathcal{M}_{(l)}^{\rm free} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l\,(l+2)}{r^2} + 1 \tag{7.18b}$$

with potential $V = (\mathrm{d}^2U/\mathrm{d}^2\Phi)\big|_{\Phi_{\rm cl}} - 1$. Plugging in the parametrization for the classical solution $\Phi_{\rm cl} \sim \tanh(r - r_0)$ in the thin wall limit $\alpha \approx 1$, the potential becomes

$$V(r) \approx \frac{11}{2} - \frac{3}{2}\,\mathrm{sech}^2(r - r_0)\ , \tag{7.19}$$

and thus constitutes a PÖSCHL-TELLER potential which we have already encountered several times before. In particular, we already suspect the appearance of zero modes. Note that the

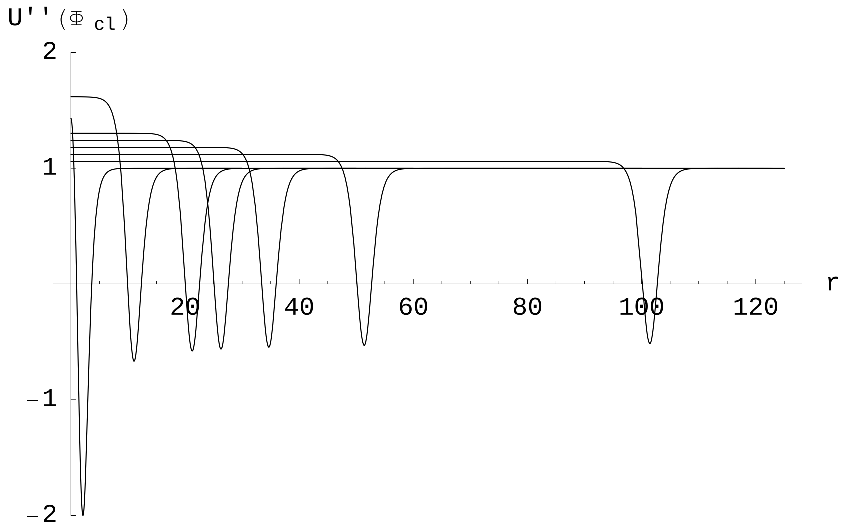

Figure 8: Plots of the fluctuation potential $U''(\Phi_{\rm cl}(r))$ for various values of α : $\alpha = 0.5$, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99, with the binding well of the potential appearing farther to the right for increasing α. Observe that as $\alpha \to 1$, the potential $U''(\Phi_{\rm cl}(r))$ is localized at $r \sim \frac{1}{1-\alpha}$, and is approximated well by the analytic form in (7.19). Figure taken from [44].

fluctuations $\Phi_{\rm cl}$ are localized at the boundary, cf. Fig. 8.
With the corresponding degeneracy factor $(l+1)^2$, we can directly use the GEL'FAND YAGLOM result for the partial waves,

$$\frac{\det\left(\mathcal{M}_{(l)}\right)}{\det\left(\mathcal{M}^{\rm free}_{(l)}\right)} \;=\; \left(\frac{\phi_{(l)}(\infty)}{\phi^{\rm free}_{(l)}(\infty)}\right)^{(l+1)^2} \;=\; \left(\mathcal{R}_{(l)}(\infty)\right)^{(l+1)^2} \tag{7.20}$$

Again, we introduce the function $\mathcal{R}_{(l)}$ whose numerical evaluation is favorable due to the exponential growth of the $\phi^{\rm (free)}_{(l)}$ for large r . The $\mathcal{R}_{(l)}$ now obey the differential equation

$$-\mathcal{R}''_{(l)}(r) \;-\; \left(\frac{1}{r} \;+\; \frac{2I'_{l+\frac{d}{2}-1}(r)}{2I_{l+\frac{d}{2}-1}(r)}\right)\mathcal{R}'_{(l)}(r) \;+\; V(r)\,\mathcal{R}_{(l)}(r) \;=\; 0 \tag{7.21}$$

with initial conditions $\mathcal{R}_{(l)}(0) = 1$ and $\mathcal{R}'_{(l)}(0) = 0$ (cf. Sect. 6), which can be straightforwardly evaluated numerically. As it turns out, the solutions for the partial waves can be classified the in the following way:

- $\mathcal{R}_{(l=0)}(\infty) < 0$: The $l = 0$-solution constitutes the negative mode which is non-degenerate and thus yields the unique decay mode. As a side remark we would like to note that it can be shown that in flat space the negative mode is always non-degenerate. On curved surfaces, however, it is possible that the negative modes become degenerate and the decay problem becomes much more interesting [45, 46].

- $\mathcal{R}_{(l=1)}(\infty) = 0$: As discussed earlier, the zero mode arises due to the translational invariance of the problem. Note, that the zero mode is fourfold degenerate, corresponding to the translational invariance along the three spatial and the Euclidean time direction. For the evaluation of the decay rate γ, this zero mode needs to be removed. In this particular situation, the prescription for the removal of the zero mode, Eq. (5.33), yields [44]:

$$\left(\frac{S_{\mathrm{cl}}[\Phi_{\mathrm{cl}}]}{2\pi}\right)^2 \left(\frac{\det'\left(\mathcal{M}_{(1)}\right)}{\det\left(\mathcal{M}_{(1)}^{\mathrm{free}}\right)}\right)^{-1/2} = \left[\frac{\pi}{2}\,\Phi_\infty\left(\Phi_0 - \frac{3}{2}\,\Phi_0^2 + \frac{\alpha}{2}\,\Phi_0^3\right)\right]^2 \tag{7.22}$$

- $\mathcal{R}_{(l\geq 2)}(\infty) > 1$: The partial waves $l \geq 2$, yield positive eigenvalues and are $(l+1)^2$-fold degenerate. As discussed in the previous section, due to the divergence of the sum over the partial waves, a renormalization scheme has to be employed.

7.4 Overall result

The final result for the determinant prefactor is obtained by summing over the contributions of the partial waves. Using the result (6.34b) we obtain:

$$\begin{aligned}\ln\left(\frac{\det'\left(\mathcal{M}_{(1)}\right)}{\det\left(\mathcal{M}_{(1)}^{\mathrm{free}}\right)}\right) &= \ln|\mathcal{R}_0(\infty)| - 4\ln\left[\frac{\pi}{2}\,\Phi_\infty\left(\Phi_0 - \frac{3}{2}\,\Phi^2 + \frac{\alpha}{2}\,\Phi_0^3\right)\right] \\ &+ \sum_{l=2}^{\infty}(l+1)^2\left[\ln\mathcal{R}_{(l)}(\infty) - \frac{1}{2\,(l+1)}\int_0^\infty \mathrm{d}r\; r\; V(r) + \frac{1}{8\,(l+1)^3}\int_0^\infty \mathrm{d}r\; r^3\, V(r)\left(V(r)+2\right)\right] \\ &- \frac{3}{2}\int_0^\infty \mathrm{d}r\; r\; V(r) + \frac{1}{8}\int_0^\infty \mathrm{d}r\; r^3\, V(r)\left(V(r)+2\right)\left[\frac{1}{2} - \gamma - \ln\left(\frac{r}{2}\right)\right]\end{aligned} \tag{7.23}$$

The last two terms constitute the counter terms obtained by $\overline{\mathrm{MS}}$-regularization and an on-shell renormalization has been employed. A check on this result is the thin wall limit $\alpha = 1$, which is the limiting case of the Pöschl-Teller potential. The numerical agreement in this limit is excellent [44].

8 First exercise session

8.1 Small t behaviour of the heat kernel

The large n behaviour of eigenvalues of a second order elliptic differential operator on a d dimensional manifold can be approximated by Weyl's estimate:

$$\lambda_n^{d/2} \sim \frac{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}+1\right)\,n}{\mathrm{vol}} + \ldots \tag{8.1}$$

Use this to show that

$$K(t) \quad \sim \quad \frac{\mathrm{vol}}{(4\pi t)^{d/2}} \; . \tag{8.2}$$

Solution: According to K's definition, we first of all have

$$\begin{aligned} K(t) \quad &= \quad \sum_{n=0}^{\infty} e^{-\lambda_n t} \quad = \quad \sum_{n=0}^{\infty} e^{-4\pi t \left(\frac{\Gamma(d/2+1)\, n}{\mathrm{vol}}\right)^{2/d}} \\ &=: \quad \sum_{n=0}^{\infty} e^{-An^{2/d}} \; . \end{aligned} \tag{8.3}$$

Now, one can approximate the sum by an integral due to the Euler Maclaurin formula,

$$\sum_{n=0}^{\infty} f(n) \quad = \quad \int_0^{\infty} \mathrm{d}n \; f(n) \; + \; \frac{1}{2} \left(f'(\infty) \; - \; f'(0)\right) \; + \; \ldots \, , \tag{8.4}$$

which implies

$$\begin{aligned} K(t) \quad &\approx \quad \int_0^{\infty} \mathrm{d}n \; e^{-An^{2/d}} \quad = \quad \frac{d}{2} \int_0^{\infty} \mathrm{d}t \; t^{d/2-1} \; e^{-At} \quad = \quad \frac{d}{2\, A^{d/2}}\, \Gamma\left(d/2\right) \\ &= \quad \frac{\mathrm{vol}}{(4\pi t)^{d/2}} \cdot \underbrace{\frac{d}{2} \cdot \frac{\Gamma\left(d/2\right)}{\Gamma\left(d/2+1\right)}}_{1} \quad = \quad \frac{\mathrm{vol}}{(4\pi t)^{d/2}} \; . \end{aligned} \tag{8.5}$$

8.2 Heat kernel for Dirichlet- and Neumann boundary conditions

Consider the differential operator

$$\mathcal{M} \quad := \quad -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \; + \; m^2 \, , \qquad x \in [0, L] \; . \tag{8.6}$$

(a) Compute $\int_0^\infty \mathrm{d}t \, \mathrm{Tr}\left\{e^{-\mathcal{M}t}\right\}$ for Dirichlet- and Neumann boundary conditions.

(b) Compare the result of (a) with the heat kernel expansion to 1st subleading order.

hint:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \; + \; a^2} \quad = \quad \frac{\pi}{2a} \coth(\pi a) \; - \; \frac{1}{2a^2} \tag{8.7}$$

Solution:

(a) Eigenfunctions to $\mathcal{M}$ are given as

$$\begin{aligned} u_n(x) \quad &:= \quad \sin\left(\tfrac{n\pi x}{L}\right) \; , \quad n = 1, 2, \ldots \quad : \; \text{Dirichlet boundary conditions} \\ v_n(x) \quad &:= \quad \cos\left(\tfrac{n\pi x}{L}\right) \; , \quad n = 0, 1, \ldots \quad : \; \text{Neumann boundary conditions} \end{aligned} \tag{8.8}$$

with eigenvalues

$$\lambda_n \;=\; \frac{n^2\pi^2}{L^2} \;+\; m^2\,, \qquad \begin{array}{ll} n = 1, 2, \ldots & : \text{ DIRICHLET boundary conditions} \\ n = 0, 1, \ldots & : \text{ NEUMANN boundary conditions} \end{array} \;. \tag{8.9}$$

Starting from the spectrum (8.9), one can derive the LAPLACE transform of the associated heat kernel:

$$\begin{aligned}
\int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\}\Big|_{\text{Dirichlet}} &= \int_0^\infty \mathrm{d}t\ \sum_{n=1}^\infty e^{-\left(\frac{n^2\pi^2}{L^2}+m^2\right)t} \;=\; \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t\ e^{-\left(\frac{n^2\pi^2}{L^2}+m^2\right)t} \\
&= \sum_{n=1}^\infty \frac{1}{\frac{n^2\pi^2}{L^2} \;+\; m^2} \;=\; \frac{L^2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2 \;+\; \left(\frac{mL}{\pi}\right)^2} \\
&= \frac{L}{2m}\,\coth(mL) \;-\; \frac{1}{2m^2} && (8.10a) \\
\int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\}\Big|_{\text{Neumann}} &= \frac{L^2}{\pi^2}\sum_{n=0}^\infty \frac{1}{n^2 \;+\; \left(\frac{mL}{\pi}\right)^2} \\
&= \int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\}\Big|_{\text{Dirichlet}} \;+\; \frac{L^2}{\pi^2}\cdot\frac{1}{n^2 \;+\; \left(\frac{mL}{\pi}\right)^2}\Big|_{n=0} \\
&= \frac{L}{2m}\,\coth(mL) \;+\; \frac{1}{2m^2} && (8.10b)
\end{aligned}$$

(b) Recall the heat kernel expansion for the Laplacian $\mathcal{M}_0 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$

$$K(t) \;\equiv\; \mathrm{Tr}\Big\{e^{-\mathcal{M}_0 t}\Big\} \;\sim\; \frac{1}{\sqrt{4\pi t}}\Big(b_0 \;+\; \sqrt{t}\,b_{1/2} \;+\; t\,b_1 \;+\; \ldots\Big) \tag{8.11}$$

with $b_0 = \text{volume} = L$. Then,

$$\int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\} \;=\; \int_0^\infty \mathrm{d}t\ e^{-m^2 t}\,K(t) \;\sim\; \frac{b_0}{2m} \;+\; \frac{b_{1/2}}{2m^2} \;+\; \ldots \tag{8.12}$$

Compare this with the results from (a):

$$\int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\}\Big|_{\text{Dirichlet}} \;=\; \frac{L}{2m}\,\coth(mL) \;-\; \frac{1}{2m^2} \;\sim\; \frac{L}{2m} \;-\; \frac{1}{2m^2} \;+\; \ldots \tag{8.13a}$$

$$\int_0^\infty \mathrm{d}t\ \mathrm{Tr}\Big\{e^{-\mathcal{M}t}\Big\}\Big|_{\text{Neumann}} \;=\; \frac{L}{2m}\,\coth(mL) \;+\; \frac{1}{2m^2} \;\sim\; \frac{L}{2m} \;+\; \frac{1}{2m^2} \;+\; \ldots \tag{8.13b}$$

The ... represent exponentially suppressed terms. So in both cases, $b_0 = L$, as expected, and $b_{1/2} = \mp 1$ (with minus sign for DIRICHLET boundary conditions).

Notice that the leading (volume) term is independent of the boundary conditions while the next (boundary) term is sensitive to the boundary conditions.

8.3 Casimir effect

Compute the Casimir force between 2 parallel plane mirrors of area A and distance L using

$$\frac{\langle E\rangle}{A} \;=\; \mathrm{Tr}\left\{\frac{\hbar\omega}{2}\right\}, \qquad \omega \;=\; c\cdot\sqrt{\vec{k}_\perp^2 \;+\; \left(\frac{n\pi}{L}\right)^2}\,. \tag{8.14}$$

Express in terms of the Riemann zeta function.

Solution: Let us start computing the trace over the frequency operator: The momentum components parallel to the mirrors - denoted as $\vec{k}_\perp$ - are both continuous and can be integrated over, the remaining k component has to be discretized due to the boundary conditions on electromagnetic field modes on the mirrors. The trace also involves a factor of 2 in order to take both photon polarizations into account:

$$\begin{aligned}
\mathrm{Tr}\left\{\frac{\hbar\omega}{2}\right\} &= \frac{\hbar}{2}\cdot 2\cdot\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^2}\sum_{n=1}^{\infty} c\cdot\sqrt{\vec{k}_\perp^2 \;+\; \left(\frac{n\pi}{L}\right)^2} \\
&= \frac{\hbar c}{4\pi^2}\sum_{n=1}^{\infty}\int \mathrm{d}^2k_\perp\,\frac{1}{\Gamma(-1/2)}\int_0^\infty\frac{\mathrm{d}t}{t^{3/2}}\,e^{-\left(k_\perp^2+\frac{n^2\pi^2}{L^2}\right)t} \\
&= \frac{\hbar c}{4\pi^2\,\Gamma(-1/2)}\sum_{n=1}^{\infty}\int_0^\infty\frac{\mathrm{d}t}{t^{3/2}}\,e^{-\frac{n^2\pi^2 t}{L^2}}\cdot\frac{\pi}{t} \\
&= \frac{\hbar c}{4\pi\,\Gamma(-1/2)}\sum_{n=1}^{\infty}\int_0^\infty\frac{\mathrm{d}t}{t^{5/2}}\,e^{-\frac{n^2\pi^2 t}{L^2}} \\
&= \frac{\hbar c}{4\pi\,\Gamma(-1/2)}\sum_{n=1}^{\infty}\Gamma(-3/2)\left(\frac{n^2\pi^2}{L^2}\right)^{3/2} \\
&= \frac{\hbar c}{4\pi}\,\underbrace{\frac{\Gamma(-3/2)}{\Gamma(-1/2)}}_{-2/3}\,\frac{\pi^3}{L^3}\underbrace{\sum_{n=1}^{\infty}n^3}_{\zeta(-3)} \\
&= -\frac{\hbar c\,\pi^2\,\zeta(-3)}{6L^3}
\end{aligned} \tag{8.15}$$

From the first to the second line, we used a standard trick in field theory due to Dyson and others

$$\left(k^2 \;+\; m^2\right)^s \;=\; \frac{1}{\Gamma(-s)}\int_0^\infty\frac{\mathrm{d}t}{t^{s+1}}e^{-(k^2+m^2)t} \tag{8.16}$$

which can best be verified by reverse calculation and the further Γ function identity

$$\int_0^\infty \mathrm{d}t\;e^{-\alpha t}\,t^{s-1} \;=\; \frac{\Gamma(s)}{\alpha^s}\,. \tag{8.17}$$

Now, the CASIMIR force is the negative gradient of the energy (density):

$$F = -\frac{\partial}{\partial L}\,\mathrm{Tr}\left\{\frac{\hbar\omega}{2}\right\} = -\frac{\hbar c\,\pi^2\,\zeta(-3)}{2\,L^4} = -\frac{\hbar c\,\pi^2}{240\,L^4} \tag{8.18}$$

Since F has negative sign, the force between the plates is attractive. This prediction was made in the 50's due to the understanding of QED and recently confirmed experimentally.

9 Second exercise session

9.1 The Euler Heisenberg effective action

For 4 dimensional fermions in a constant B field, the relevant operator $m^2 + \not{D}^2$ has eigenvalues

$$\lambda_n^{\pm} = m^2 + \vec{k}_\perp^2 + eB\,(2n+1\pm 1)\ . \tag{9.1}$$

Compute the zeta function

$$\zeta(s) = \mathrm{Tr}\left\{\frac{\mu^{2s}}{\lambda^s}\right\} \tag{9.2}$$

and hence

$$-\zeta'(0) = \tfrac{1}{2}\,\ln\det\left(m^2 + \not{D}^2\right)\ , \tag{9.3}$$

renormalized at $\mu = m$.

Solution: Firstly, we have

$$\begin{aligned}
\zeta(s) &= \underbrace{\frac{eB}{2\pi}}_{\text{degeneracy}} \cdot \int\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{n=0}^{\infty}\sum_{\pm}\frac{\mu^{2s}}{\left(k^2 + m^2 + eB(2n+1\pm 1)\right)^s}. \\
&= \frac{eB\,\mu^{2s}}{(2\pi)^3}\sum_{n=0}^{\infty}\sum_{\pm}\frac{\pi}{\left(m^2 + eB(2n+1\pm 1)\right)^{s-1}}\cdot\frac{1}{(s-1)} \\
&= \frac{\pi\,eB\mu^{2s}}{(2\pi)^3}\cdot\frac{1}{(s-1)}\cdot\frac{1}{(2eB)^{s-1}}\sum_{n=0}^{\infty}\sum_{\pm}\frac{1}{\left(n + \frac{1}{2} \pm \frac{1}{2} + \frac{m^2}{2eB}\right)^{s-1}} \\
&= \frac{e^2B^2}{2\pi^2}\cdot\frac{1}{(s-1)}\cdot\left(\frac{\mu^2}{2eB}\right)^s\left(\zeta_H\left(s-1;\frac{m^2}{2eB}\right) - \left(\frac{m^2}{2eB}\right)^{1-s}\right)
\end{aligned} \tag{9.4}$$

using

$$\int \mathrm{d}^2k\,\frac{1}{(k^2 + a^2)^s} = \frac{\pi}{(s-1)\,a^{2(s-1)}}\ . \tag{9.5}$$

In computing the derivative, it is useful to introduce the shorthand $z := \frac{m^2}{2eB}$,

$$\begin{aligned}\zeta'(s) &= \frac{e^2B^2}{2\pi^2}\left\{\frac{-1}{(s-1)^2}\left(\frac{\mu^2}{2eB}\right)^s \zeta_H(s-1;z) + \frac{1}{s-1}\left(\frac{\mu^2}{2eB}\right)^s \zeta'_H(s-1;z)\right. \\ &\quad + \frac{1}{s-1}\ln\left(\frac{\mu^2}{2eB}\right)\left(\frac{\mu^2}{2eB}\right)^s \zeta_H(s-1;z) \\ &\quad \left. + \frac{z}{2(s-1)^2}\left(\frac{\mu^2}{m^2}\right)^s - \frac{z}{2(s-1)}\ln\left(\frac{\mu^2}{m^2}\right)\left(\frac{\mu^2}{m^2}\right)^s\right\}. \end{aligned} \tag{9.6}$$

To obtain the determinant of the DIRAC operator, we finally evaluate (9.6) at $s = 0$ and $\mu = m$:

$$\begin{aligned}\zeta'(0)\Big|_{\mu=m} &= \frac{e^2B^2}{2\pi^2}\Big\{-\zeta_H(-1;z) - \zeta'_H(-1;z) - \ln z\,\zeta_H(-1;z) + z/2\Big\} \\ &= \frac{e^2B^2}{2\pi^2}\left\{\left(\underline{-\frac{z}{2}} + \frac{z^2}{2} + \underline{\frac{1}{12}}\right) \underline{- \ln z\,\zeta_H(-1;z)} + \underline{\frac{z}{2}}\right. \\ &\quad \left. - \left(\underline{\frac{1}{12}} - \frac{z^2}{4} \underline{- \ln z\,\zeta_H(-1;z)} - \frac{1}{4}\int_0^\infty \frac{\mathrm{d}t}{t^2}\, e^{-2zt}\left[\coth t - \frac{1}{t} - \frac{t}{3}\right]\right)\right\} \\ &= \frac{e^2B^2}{2\pi^2}\left\{\frac{3\,z^2}{4} + \frac{1}{4}\int_0^\infty \frac{\mathrm{d}t}{t^2}\, e^{-\frac{m^2t}{eB}}\left[\coth t - \frac{1}{t} - \frac{t}{3}\right]\right\} \end{aligned} \tag{9.7}$$

The underlined terms have cancelled, and the remaining $\frac{3e^2B^2z^2}{8\pi^2}$ term can be dropped as it is independent of the applied field B. Thus,

$$\mathcal{L}_{\text{spinor}} = -\frac{e^2B^2}{8\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t^2}\, e^{-\frac{m^2}{eB}t}\left(\coth t - \tfrac{1}{t} - \tfrac{t}{3}\right). \tag{9.8}$$

9.2 Solitons in 1+1 dimensional ϕ^4 theory

Consider a scalar field ϕ in the quartic potential

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2. \tag{9.9}$$

(a) find a solution ϕ_{cl} to the first order BOGOMOLNY equation $\phi' = \sqrt{2U(\phi)}$

(b) find the fluctuation operator

$$\mathcal{M} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\Big|_{\phi=\phi_{\text{cl}}} \tag{9.10}$$

Solution:

(a) We will solve the ordinary differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{\mu^2}{\lambda}\right) \tag{9.11}$$

using separation of variables:

$$\sqrt{\frac{\lambda}{2}}\int\limits_{x_0}^{x} \mathrm{d}x' = \int\limits_0^{\phi_{\mathrm{cl}}} \frac{\mathrm{d}\phi}{\phi^2 - \mu^2/\lambda} = \frac{\sqrt{\lambda}}{\mu}\int\limits_0^{\psi_{\mathrm{cl}}} \frac{\mathrm{d}\psi}{\psi^2 - 1}$$
$$= -\frac{\sqrt{\lambda}}{\mu}\,\mathrm{artanh}\,\psi\Big|_0^{\psi_{\mathrm{cl}}}$$
$$= -\frac{\sqrt{\lambda}}{\mu}\,\mathrm{artanh}\,\frac{\sqrt{\lambda}\,\phi}{\mu}\Big|_0^{\phi_{\mathrm{cl}}} \tag{9.12}$$

The classical solution is thus given by

$$\phi_{\mathrm{cl}}(x) = -\frac{\mu}{\sqrt{\lambda}}\,\tanh\left(\frac{\mu\,(x - x_0)}{\sqrt{2}}\right)\,, \tag{9.13}$$

which is a "kink" localized at $x = x_0$.

(b) Now we have to plug the classical solution (9.13) into the potential's second derivative

$$\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2} = 3\,\lambda\,\phi^2 - \mu^2 \tag{9.14}$$

in order to get the fluctuation operator

$$\mathcal{M} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3\,\lambda\,\frac{\mu^2}{\lambda}\,\tanh^2\left(\frac{\mu\,(x - x_0)}{\sqrt{2}}\right) - \mu^2$$
$$= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mu^2}{2}\left\{4 - 6\,\mathrm{sech}^2\left(\frac{\mu\,(x - x_0)}{\sqrt{2}}\right)\right\}\,. \tag{9.15}$$

One can regard $\mathcal{M}$ as a Schrödinger operator for a particle moving in a well potential of Pöschl Teller form. The spectrum consists of two discrete eigenvalues and a continuous part $\{E \geq 2\mu^2\}$. Due to translation invariance, corresponding to the parameter x_0, we know in advance that there must be a zero mode, and one can indeed show that one of $\mathcal{M}$'s eigenvalues is zero.

9.3 Deriving series from functional determinants

(a) using the determinant

$$\ln\left(\frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2\right)}{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)}\right) = \ln\left(\frac{\sinh m}{m}\right) \tag{9.16}$$

for Dirichlet boundary conditions, compute the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\,. \tag{9.17}$$

(b) using the $d = 2$ radial result for the HELMHOLTZ operator on the disc of radius $R = 1$,

$$\ln\left(\frac{\det\left(-\mathcal{M}_{(\ell)} + m^2\right)}{\det\left(-\mathcal{M}_{(\ell)}\right)}\right) = \ln\left(\frac{\ell!\, J_\ell(m)}{(m/2)^\ell}\right) , \qquad J_\ell(m) = \sum_{k=0}^{\infty} \frac{(m/2)^{\ell+2k}}{k!\,(\ell+k)!} , \tag{9.18}$$

compute

$$\sum_{n=1}^{\infty} \frac{1}{j^2_{(\ell),n}} , \qquad j_{(\ell),n} \equiv n'\text{th zero of BESSEL function } J_\ell . \tag{9.19}$$

Solution: Express the determinants in terms of the eigenvalues and compare with the results given by (9.16) and (9.18):

(a) Eigenvalues $\lambda_n = n^2\pi^2$ of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ (with $n = 1, 2, ...$) imply

$$\begin{aligned} \frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2\right)}{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)} &= \prod_{n=1}^{\infty} \frac{\lambda_n + m^2}{\lambda_n} = \prod_{n=1}^{\infty} \frac{n^2\pi^2 + m^2}{n^2\pi^2} \\ &= \prod_{n=1}^{\infty} \left(1 + \left(\frac{m}{n\pi}\right)^2\right) \\ &= 1 + \frac{m^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} + \mathcal{O}(m^4) . \end{aligned} \tag{9.20}$$

On the other hand, the expansion of the sinh representation gives

$$\frac{\sinh(m)}{m} = \frac{1}{m} \sum_{n=1}^{\infty} \frac{m^{2n-1}}{(2n-1)!} = 1 + \frac{m^2}{6} + \mathcal{O}(m^4) . \tag{9.21}$$

By equating the m^2 coefficients of (9.20) and (9.21), we find

$$\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} , \tag{9.22}$$

which is EULER'S famous result.

(b) Again, start with the product representation of the determinants in terms of eigenvalues $\lambda_{\ell,n} = j^2_{(\ell),n}$ on the unit disc:

$$\begin{aligned} \frac{\det\left(-\mathcal{M}_{(\ell)} + m^2\right)}{\det\left(-\mathcal{M}_{(\ell)}\right)} &= \prod_{n=1}^{\infty} \frac{\lambda_{\ell,n} + m^2}{\lambda_{\ell,n}} = \prod_{n=1}^{\infty} \frac{j^2_{(\ell),n} + m^2}{j^2_{(\ell),n}} \\ &= \prod_{n=1}^{\infty} \left(1 + \left(\frac{m}{j^2_{(\ell),n}}\right)^2\right) \\ &= 1 + m^2 \sum_{n=1}^{\infty} \frac{1}{j^2_{(\ell),n}} + \mathcal{O}(m^4) \end{aligned} \tag{9.23}$$

By virtue of (9.18), this is equal to

$$\frac{\ell!}{(m/2)^\ell}\sum_{k=0}^{\infty}\frac{(m/2)^{\ell+2k}}{k!\,(\ell+k)!} \;=\; 1+\frac{\ell!}{(\ell+1)!}\,\frac{(m/2)^{\ell+2}}{(m/2)^\ell}+\mathcal{O}(m^4) \;=\; 1+\frac{m^2}{4\,(\ell+1)}+\mathcal{O}(m^4)\,, \tag{9.24}$$

so matching m^2 coefficients of (9.23) and (9.24) yields

$$\sum_{n=1}^{\infty}\frac{1}{j^2_{(\ell),n}} \;=\; \frac{1}{4\,(\ell+1)}\,. \tag{9.25}$$

Notice that even though there is no simple formula for the $j_{(\ell),n}$, this sum takes a very nice form.

9.4 Schrödinger resolvent

For a SCHRÖDINGER operator $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)$, the diagonal resolvent is given by

$$R(x;\lambda) \;:=\; \langle x|\,\frac{1}{H\,-\,\lambda}\,|x\rangle \;=\; \frac{\psi_1(x)\,\psi_2(x)}{W(x)} \tag{9.26a}$$

$$W(x) \;:=\; \psi_1'(x)\,\psi_2(x)\;-\;\psi_1(x)\,\psi_2'(x)\,. \tag{9.26b}$$

(a) show that R satisfies

$$-2\,R\,R''\;+\;(R')^2\;+\;4\,R^2\,(V\,-\,\lambda)\;=\;1 \tag{9.27}$$

(b) use (a) to derive the heat kernel expansion

$$R(x;-\lambda) \;=\; \int_0^\infty \mathrm{d}t\; e^{-\lambda t}\,\langle x|\,e^{-Ht}\,|x\rangle \tag{9.28}$$

Solution:

(a) It is a special property of one dimension that the Green's function to the $H-\lambda$ operator can be expressed as a product of two solutions $\psi_{1,2}$ to the eigenvalue equation $-\psi_i''+V\psi_i = \lambda\psi_i$. Let us start by showing that the Wronskian $W(x)$ does not depend on x:

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}x} \;&=\; \psi_1''(x)\,\psi_2(x)\;+\;\underbrace{\psi_1'(x)\,\psi_2'(x)\;-\psi_1'(x)\,\psi_2'(x)}_{0}\;-\;\psi_1(x)\,\psi_2''(x)\\ &=\; \big((V\,-\,\lambda)\,\psi_1\big)\,\psi_2(x)\;-\;\psi_1(x)\,\big((V\,-\,\lambda)\,\psi_2\big)\\ &=\;0\end{aligned} \tag{9.29}$$

Given (9.29), it is easy to compute the derivatives

$$
\begin{aligned}
R' &= \frac{\psi_1' \psi_2 + \psi_2' \psi_1}{\psi_1' \psi_2 - \psi_2' \psi_1} \\
R'' &= 2(\lambda - V) R + \frac{2 \psi_1' \psi_2'}{W} ,
\end{aligned} \tag{9.30}
$$

then one can conclude that

$$
-2 R R'' + (R')^2 + 4 R^2 (V - \lambda) = \frac{(\psi_1' \psi_2 - \psi_1 \psi_2')^2}{W^2} = 1 . \tag{9.31}
$$

(b) The non-linear equation (9.31) can be rendered linear by deriving with respect to x:

$$
\begin{aligned}
0 &= -2 R' R'' - 2 R R''' + 2 R' R'' + 8 R R' (V - \lambda) + 4 R^2 V' \\
&= -2 R \Big(R''' - 4 R' (V - \lambda) - 2 R V' \Big)
\end{aligned} \tag{9.32}
$$

We want to extract the expansion coefficients $b_k(x)$ in the heat kernel ansatz from (9.32)

$$
K(t, x, x) = \frac{1}{\sqrt{4\pi t}} \sum_k t^k b_k(x) , \tag{9.33}
$$

for that purpose, we have to convince ourselves that the matrix element $\langle x | e^{-Ht} | x \rangle$ actually correspond to K. Let ϕ_n denote the H eigenfunctions with energy λ_n, then:

$$
\begin{aligned}
K(t, x, x) &:= \sum_n e^{-\lambda_n t} \phi_n(x) \phi_n^*(x) = \sum_n \langle x | \psi_n \rangle \langle \psi_n | e^{-Ht} | x \rangle \\
&= \langle x | e^{-Ht} | x \rangle
\end{aligned} \tag{9.34}
$$

Now, (9.33) and (9.34) allow to express R as

$$
R(x; -\lambda) = \int_0^\infty \mathrm{d}t \, \frac{e^{-\lambda t}}{\sqrt{4\pi t}} \sum_k t^k b_k(x) = \frac{1}{\sqrt{4\pi}} \sum_k \frac{\Gamma(k + 1/2)}{\lambda^{k+1/2}} b_k(x) \tag{9.35}
$$

which we can plug into (9.32) (with $\lambda \mapsto -\lambda$) to derive a recurrence relation for the b_k:

$$
\begin{aligned}
0 &= R'''(x; -\lambda) - 4 R'(x; -\lambda) \big(V(x) + \lambda \big) - 2 V'(x) R(x; -\lambda) \\
&= \sum_k \frac{\Gamma(k + 1/2)}{\lambda^{k+1/2}} \Big(b_k'''(x) - 4 b_k'(x) \big(V(x) + \lambda \big) - 2 b_k(x) V'(x) \Big) \\
&= \sum_k \Bigg\{ \frac{\Gamma(k + 1/2)}{\lambda^{k+1/2}} \Big(b_k'''(x) - 4 b_k'(x) V(x) - 2 b_k(x) V'(x) \Big) \\
&\qquad - \frac{4 \Gamma(k + 3/2)}{\lambda^{k+1/2}} b_{k+1}'(x) \Bigg\} \\
&= \sum_k \frac{\Gamma(k + 1/2)}{\lambda^{k+1/2}} \Big\{ b_k'''(x) - 4 b_k'(x) V(x) - 2 b_k(x) V'(x) - 4 \big(k + \tfrac{1}{2} \big) b_{k+1}'(x) \Big\}
\end{aligned} \tag{9.36}
$$

Each b_{k+1} is determined in terms of the preceding coefficient b_k as

$$b'_{k+1}(x) \quad = \quad \frac{1}{4k+2}\,\Big(b'''_k(x) \;-\; 4\,V(x)\,b'_k(x) \;-\; 2\,V'(x)\,b_k(x)\Big)\;. \tag{9.37}$$

Starting from $b_0 = 1$, one obtains

$$\begin{aligned} b'_1 \quad &= \quad \frac{1}{2}\,\Big(-2\,V'\cdot 1\Big) \quad \Rightarrow \quad b_1 \quad = \quad -V \\ b'_2 \quad &= \quad -\frac{1}{6}\Big(V''' \;-\; 4\,V\,V' \;-\; 2\,V'\,V\Big) \quad = \quad -\frac{1}{6}\,\big(V'' \;-\; 3\,V^2\big)' \\ &\Rightarrow \quad b_2 \quad = \quad \frac{V^2}{2} \;-\; \frac{V''}{6}\;. \end{aligned} \tag{9.38}$$

10 Extra exercise: $\zeta_{\mathbf{R}}(-3)$

Compute $\zeta_{\mathrm{R}}(-3)$.

Solution: In the lectures, we derived the integral representation of the Riemann zeta function which converges for $\Re\{s\} > 1$,

$$\zeta_{\mathrm{R}}(s) \quad = \quad \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}\,\frac{e^{-t/2}}{2\,\sinh(t/2)}\;. \tag{10.1}$$

This expression can be analytically continued to the neighbourhood of $s = 0$ by subtracting, and adding back, the leading small t behaviour of $\frac{1}{2\sinh(t/2)} \sim \frac{1}{t}$. Moreover, we use the analytic continuation of the Γ function to evaluate the term which is added back. In other words, we write

$$\begin{aligned} \zeta_{\mathrm{R}}(s) \quad &= \quad \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}\,e^{-t/2}\left\{\frac{1}{2\sinh(t/2)} \;-\; \frac{1}{t}\right\} \;+\; \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}\,\frac{e^{-t/2}}{t} \\ &= \quad \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}\,e^{-t/2}\left\{\frac{1}{2\sinh(t/2)} \;-\; \frac{1}{t}\right\} \;+\; 2^{s-1}\,\frac{\Gamma(s-1)}{\Gamma(s)} \\ &= \quad \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}\,e^{-t/2}\left\{\frac{1}{2\sinh(t/2)} \;-\; \frac{1}{t}\right\} \;+\; \frac{2^{s-1}}{s-1}\;. \end{aligned} \tag{10.2}$$

At $s = 0$, the first term vanishes because the integral is finite while $\frac{1}{\Gamma(0)} = 0$. Therefore, $\zeta_{\mathrm{R}}(0) = -\frac{1}{2}$.

Now, to analytically continue $\zeta_{\mathrm{R}}(s)$ to $s = -3$, we need to subtract the first 3 terms of the small t behaviour:

$$\frac{1}{2\sinh(t/2)} \quad \sim \quad \frac{1}{t} \;-\; \frac{t}{24} \;+\; \frac{7\,t^3}{5760} \;+\; \mathcal{O}(t^5) \tag{10.3}$$

Thus, a more globally alternative to (10.2) is given by

$$\zeta_{\rm R}(s) = \frac{1}{\Gamma(s)} \int\limits_0^\infty \mathrm{d}t \; t^{s-1}\, e^{-t/2} \left\{ \frac{1}{2\sinh(t/2)} - \left(\frac{1}{t} - \frac{t}{24} + \frac{7\,t^3}{5760} \right) \right\}$$
$$+ \frac{1}{\Gamma(s)} \int\limits_0^\infty \mathrm{d}t \; t^{s-1}\, e^{-t/2} \left(\frac{1}{t} - \frac{t}{24} + \frac{7\,t^3}{5760} \right) . \tag{10.4}$$

The "added-back" term is

$$\frac{1}{\Gamma(s)} \left(2^{s-1}\,\Gamma(s-1) - \frac{2^{s+1}}{24}\,\Gamma(s+1) + \frac{7\cdot 2^{s+3}}{5760}\,\Gamma(s+3) \right)$$
$$= \frac{2^{s-1}}{(s-1)} - \frac{2^{s+1}\,s}{24} + \frac{7\cdot 2^{s+3}\,s\,(s+1)\,(s+2)}{5760} \tag{10.5}$$

For $s = -1, -2, -3$, the first term in (10.4) vanishes because the integral is finite. Therefore, we simply evaluate the contribution from the second one:

$$\begin{aligned}
\zeta_{\rm R}(-1) &= \frac{2^{s-1}}{s-1} - \frac{2^{s+1}\,s}{24} + \frac{7\cdot 2^{s+3}\,s\,(s+1)\,(s+2)}{5760}\Big|_{s=-1} \\
&= -\frac{1}{8} + \frac{1}{24} + 0 = -\frac{1}{12} \\
\zeta_{\rm R}(-2) &= \frac{2^{s-1}}{s-1} - \frac{2^{s+1}\,s}{24} + \frac{7\cdot 2^{s+3}\,s\,(s+1)\,(s+2)}{5760}\Big|_{s=-2} \\
&= -\frac{1}{24} + \frac{1}{24} + 0 = 0 \\
\zeta_{\rm R}(-3) &= \frac{2^{s-1}}{s-1} - \frac{2^{s+1}\,s}{24} + \frac{7\cdot 2^{s+3}\,s\,(s+1)\,(s+2)}{5760}\Big|_{s=-3} \\
&= -\frac{1}{64} + \frac{1}{32} - \frac{7}{960} = \frac{1}{120}
\end{aligned} \tag{10.6a}$$

By the way, there is another way of evaluating the RIEMANN zeta functions at non-positive integers based on BERNOULLI numbers B_n,

$$\frac{t\,e^t}{e^t - 1} = \sum_{n\in\mathbb{N}_0} \frac{B_n}{n!}\, t^n , \tag{10.7}$$

namely:

$$\zeta_R(-n) = -\frac{B_{n+1}}{n+1} \tag{10.8}$$

Knowledge of the first BERNOULLI numbers reproduces the results of (10.6a) and (10.2):

n	B_n	$\zeta_{\rm R}(-n)$
0	1	$-\frac{1}{2}$
1	$\frac{1}{2}$	$-\frac{1}{12}$
2	$\frac{1}{6}$	0
3	0	$\frac{1}{120}$
4	$-\frac{1}{30}$	0

The last entry follows from the general identity $\zeta_{\rm R}(-2n) = 0$ for any $n \geq 1$.

A Further reading

For the sake of clarity, we summarize the continuative literature for the respective subjects which could only be addressed briefly it these lectures.

- Sect.3: Techniques of heat kernel expansions for different field types and various boundary conditions are reviewed in "Heat kernel expansion: User's manual" by Vassilevich [14]. This review is available on arXiv.

- Sect.4: The original derivation of the Euler-Heisenberg effective action is available in an english translation on the arXiv [13], a more modern derivation was later given by Schwinger [47]. A pedagogical review on the extension to inhomogeneous and non-abelian background fields, as well as higher order loop effective actions is found in [16], for constant gravitational curvature in de Sitter and anti de Sitter spaces, see [17]. The book of Carl Bender and Steven Orszag [24] provides a very didactic introduction to asymptotic expansions in general and Borel summation, the application to Euler-Heisenberg effective actions is covered in [15]. More elaborate discussion on perturbative and non-perturbative physics can be found in [21] and in [20], which also include some historical remarks. It is also very worthwile to read Dyson's "proof" of the divergence of perturbative series in QED [26], as he manages to present his reasoning using just two formulas.

- Sect.5: A very readable introduction to the formalism of Gel'fand and Yaglom was written by Klaus Kirsten [27]. There, the crucial steps of the derivation of the formalism and its application are demonstrated by means of easy examples.

- Sect.6: The extension of the one-dimensional result of Gel'fand and Yaglom is covered in "Functional determinants for radial operators" [41], available on arXiv.

- Sect.7: A very pedagocical introduction to the scope of "False vacuum decay" is given in the Chapter "The uses of instantons in Coleman's "Aspects of Symmetry" [42]. The calculations given in the present lectures using the radial Gel'fand Yaglom formalism are taken from "Beyond the thin-wall approximation: Precise numerical computation of prefactors in false vacuum decay" [44]. See also the work of Baacke and Lavrelashvili [43].

References

[1] R. Jackiw, "Functional evaluation of the effective potential," Phys. Rev. D **9**, 1686 (1974).

[2] J. Iliopoulos, C. Itzykson and A. Martin, "Functional Methods And Perturbation Theory," Rev. Mod. Phys. **47**, 165 (1975).

[3] S. Coleman, *Aspects of Symmetry* (Cambridge Univ. Press).

[4] V. A. Miransky, *Dynamical symmetry breaking in quantum field theories* (World Scientific, Singapore, 1993).

[5] M. Luscher, "The Schroedinger functional in lattice QCD with exact chiral symmetry," JHEP **0605**, 042 (2006) [arXiv:hep-lat/0603029].

[6] E. Elizalde et al., *Zeta Regularization Techniques with Applications* (World scientific, Singapore, 1994) .

[7] S. W. Hawking, "Zeta Function Regularization Of Path Integrals In Curved Space-Time," Comm. Math. Phys. **55**, 133 (1977);

[8] See Appendix B of: S.Wilk, Y. Fujiwara and T.A.Osborn, "N Body Green's functions and their semiclassical expansion", Phys. Rev. A **24**, 2187 (1981).

[9] H. Weyl, "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen ", Math. Ann. 71, 441 (1912).

[10] S. Minakshisundaram, "Zeta functions on the sphere", J. Ind. Math. Soc. 13, 41 (1949); S. Minakshisundaram, Å Pleijel, "Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds", Canadian J. Math. 1, (1949). 242–256.

[11] I. Vardi, "Determinants of Laplacians and multiple gamma functions", SIAM J. Math. Anal. 19, 493 (1988).

[12] J. S. Dowker and K. Kirsten, "The Barnes zeta-function, sphere determinants and Glaisher-Kinkelin-Bendersky constants," Anal. Appl. **3**, 45 (2005) [arXiv:hep-th/0301143].

[13] W. Heisenberg and H. Euler, "Consequences of Dirac's theory of positrons," Z. Phys. **98**, 714 (1936); English translation at [arXiv:physics/0605038].

[14] D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388**, 279 (2003) [arXiv:hep-th/0306138].

[15] G. V. Dunne and T.M. Hall, "Borel Summation of the Derivative Expansion", Phys. Rev., D 60, 065002 (1999) .

[16] G. V. Dunne, "Heisenberg-Euler effective Lagrangians: Basics and extensions," in *Ian Kogan Memorial Collection, 'From Fields to Strings: Circumnavigating Theoretical Physics'*, M. Shifman et al (ed.) vol. 1 pp 445-522 (World Scientific, 2005), [arXiv:hep-th/0406216].

[17] A. Das, G. V. Dunne, "Large-order perturbation theory and de Sitter/anti-de Sitter effective actions", Phys. Rev. D 74, 044029 [arXiv:hep-th/0607168].

[18] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "Calculations In External Fields In Quantum Chromodynamics. Technical Review," Fortsch. Phys. **32**, 585 (1984).

[19] O. K. Kwon, C. Lee and H. Min, "Massive field contributions to the QCD vacuum tunneling amplitude," Phys. Rev. D **62**, 114022 (2000) [arXiv:hep-ph/0008028].

[20] G. V. Dunne, "Perturbative – nonperturbative connection in quantum mechanics and field theory" [arXiv:hep-th/0207046], in proceedings of CAQCD 2002, Arkady Fest, M. Shifman (Ed.) (World Scientific Singapore, 2002).

[21] J.C. Le Guillou and J. Zinn-Justin (Eds.), *Large-order Behaviour of Perturbation theory*, (North-Holland, Amsterdam, 1990).

[22] J. P. Boyd, "The Devil's Invention: Asymptotic, Superasymptotic, and Hyperasymptotic Series", Acta Appl. Math. **56**, 1 (1999).

[23] H. J. Silverstone, "High-order perturbation theory and its application to atoms in strong fields", in *Atoms in Strong Fields*, C. A. Nicolaides *et al* (Eds) (Plenum Press, NY, 1990).

[24] C. M. Bender, S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers", (Springer, Berlin, 1999).

[25] R. Schützhold, H. Gies and G. V. Dunne, "Dynamically assisted Schwinger mechanism", Phys. Rev. Lett. **101**, 130404 (2008), [arXiv:hep-th/0807.0754].

[26] F. J. Dyson, "Divergence of Perturbation Theory in Quantum Electrodynamics", Phys. Rev. **85**, 631 (1952).

[27] K. Kirsten, *Spectral Functions in Mathematics and Physics*, (Chapman-Hall, 2001), see also K. Kirsten & P. Loya, "Computation of determinants using contour integrals", Am.J.Phys.**76**, 60-64 (2008), arXiv:0707.3755 [hep-th].

[28] E. Freitag, R. Busam, Complex Analysis, (Springer, Berlin, 2006).

[29] K. Kirsten and A. J. McKane, "Functional determinants by contour integration methods," Annals Phys. **308**, 502 (2003) [arXiv:math-ph/0305010].

[30] K. Kirsten and A. J. McKane, "Functional determinants for general Sturm-Liouville problems," J. Phys. A **37**, 4649 (2004) [arXiv:math-ph/0403050].

[31] I. M. Gelfand and A. M. Yaglom, "Integration In Functional Spaces And It Applications In Quantum Physics," J. Math. Phys. **1**, 48 (1960).

[32] H. Kleinert, "Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets," (World Scientific, Singapore, 2004).

[33] S. Levit and U. Smilansky, "A theorem on infinite products of eigenvalues of Sturm-Liouville type operators", Proc. Am. Math. Soc. **65**, 299 (1977).

[34] P. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill), Chapter 12.

[35] S. Flügge, *Practical Quantum Mechanics*, (Springer, Berlin, 1974).

[36] G. V. Dunne, "Functional determinants in Quantum Field Theory", J.Phys.A**41**, 304006 (2008), [arXiv:hep-th/0711.1178].

[37] A. W. Wipf, "Tunnel Determinants," Nucl. Phys. B **269**, 24 (1986).

[38] B. Simon, "Notes on Infinite Determinants of Hilbert Space Operators", Adv. in Math., 24, 244 (1977).

[39] J. R. Taylor, *Scattering Theory,* (Wiley, New York, 1972).

[40] V. de Alfaro and T. Regge, *Potential scattering,* (Wiley, New York, 1972).

[41] G. V. Dunne, K. Kirsten, "Functional determinants for radial operators", J.Phys.A**39**, 11915-11928 (2006), [arXiv:hep-th/0607066].

[42] S. R. Coleman, "The Fate Of The False Vacuum. 1. Semiclassical Theory," Phys. Rev. D **15**, 2929 (1977) [Erratum-ibid. D **16**, 1248 (1977)]; C. G. Callan and S. R. Coleman, "The Fate Of The False Vacuum. 2. First Quantum Corrections," Phys. Rev. D **16**, 1762 (1977); S. R. Coleman, "The Uses Of Instantons," *Lectures delivered at 1977 International School of Subnuclear Physics, Erice: The Whys of subnuclear physics*, Edited by A. Zichichi, (Plenum Press, 1979).

[43] J. Baacke and G. Lavrelashvili, “One-loop corrections to the metastable vacuum decay,” Phys. Rev. D **69**, 025009 (2004) [arXiv:hep-th/0307202].

[44] G. V. Dunne, H. Min, ”Beyond the thin-wall approximation: Precise numerical computation of prefactors in false vacuum decay”, Phys.Rev. D**72** (2005) 125004, [arXiv:hep-th/0511156].

[45] G.V. Dunne and Q. Wang, “Fluctuations about Cosmological Instantons”, Phys.Rev. D **74,** 024018 (2006), [arXiv:hep-th/0605176].

[46] J. Hackworth and E. Weinberg, “Oscillatory Bounce Solutions and vacuu, tunneling in de Sitter spacetime”, Phys.Rev. D **71,** 044014 (2005), [arXiv:hep-th/0410142]; A. Brown and E. Weinberg, “Thermal derivation of the Coleman-DeLuccin tunneling prescription”, Phys.Rev. D **76,** 064003 (2007), [arXiv:hep-th/0706.1573].

[47] J. S. Schwinger, “On gauge invariance and vacuum polarization,” Phys. Rev. **82**, 664 (1951).