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BRS Symmetry and Cohomology

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Abstract: The BRS symmetry determines physical states, Lagrange densities and candidate anomalies. It renders gaugefixing unobservable in physical states and is required if negative norm states are to decouple in interacting models. The relevant mathematical structures and the elementary cohomological investigations are presented.

This paper is a slightly enlarged version of the lectures given at the Saalburg Summer School 1995 on "Grundlagen und neue Methoden der Theoretischen Physik". It is meant to give a self contained introduction into a personal view on the subject. In particular the mathematical structure is derived completely with the exception of the cohomology of simple Lie algebras and the covariant Poincaré Lemma which are quoted from the literature.

The first chapter deals with the "raison d'être" of gauge symmetries: the problem to define the subspace of physical states in a Lorentz invariant theory with higher spin. The operator Q_s which characterizes the physical states was found by Becchi, Rouet and Stora as a symmetry generator of a fermionic symmetry, the BRS symmetry, in gauge theories with covariant gauge fixing [1]. For a derivation of the BRS symmetry from the gauge fixing in path integrals the reader should consult [2] or the literature quoted there. The chapter is supplemented by a discussion of free vector fields for gauge parameter $\lambda \neq 1$. This is not a completely trivial exercise [3] and not discussed in detail even in standard references [4] on gauge systems.

The second chapter deals with the requirement that the physical subspace remains physical if interactions are switched on. This restricts the action to be BRS invariant. Consequently the Lagrange density has to satisfy a cohomological equation similar to the physical states. Quantum corrections may violate the requirement of BRS symmetry because the naive evaluation of Feynman diagrams leads to divergent loop integrals which have to be regularized. This regularization can lead to an anomalous symmetry breaking which has to satisfy the celebrated Wess Zumino consistency condition [5] which again is a cohomological equation.

In chapter 3 we study some elementary cohomological problems of a nilpotent fermionic derivative d.

$$d\omega = 0 \quad \omega \bmod d\eta$$

We derive the Poincaré Lemma as the Basic Lemma of all the investigations to come. However, one has to realize that Lagrange densities are defined as functions of the fields and their derivatives and not of coordinates. We investigate differential forms depending on such variables and derive the Algebraic Poincaré Lemma. The relative cohomology, which characterizes Lagrange densities and candidate anomalies, is shown to lead to the descent equations which can again be written compactly as a cohomological problem. The chapter concludes with Künneth's formula which allows to tackle cohomological problems in smaller bits if the complete problem factorizes.

Chapter 4 presents Brandt's formulation [6] of the gravitational BRS transformations. In this formulation the cohomology factorizes and one has to deal only with tensors and undifferentiated ghosts. It is shown that the ghosts which correspond to translations never occur in anomalies, i.e. coordinate transformations are not anomalous.

In Chapter 5 we solve the cohomology of the BRS transformations acting on ghosts and tensors. The tensors have to couple together with the translation ghosts to invariants and also the ghosts for spin and isospin transformations have to couple to invariants. The invariant ghost polynomials generate the Lie algebra cohomology which we quote from the mathematical literature [7]. Moreover the tensors are restricted by the covariant Poincaré Lemma [10]. This lemma introduces the Chern forms which are the BRS transformation of the Chern Simons polynomials.

Chern Simons polynomials and Chern polynomials are the building blocks of the Chern Simons actions in odd dimensions, of topological densities and of the chiral anomalies. They are the subject of the last chapter. We conclude by giving some well known examples of Lagrange densities and anomaly candidates.

The mathematical structures presented in this paper should enable the reader also to understand and participate in the investigation of the master equation which is a still developing field of research [11]. In particular the master equation contains the BRS structures for closed algebras but applies also to open algebras.

Chapter 1

The Space of Physical States

BRS symmetry is indispensable in Lorentz covariant theories with fields with higher spin because it allows to construct an acceptable space of physical states out of the Fock space which contains states with negative norm.

To demonstrate the problem consider the simple example of a massless vectorfield A_m. The action W of the vectorfield $A_m,\ m = 0, 1, 2, 3$ is

$$W[A] = \int d^4x \mathcal{L}(A(x), \partial A(x)) \tag{1.1}$$

$$\mathcal{L}(A, \partial A) = -\frac{1}{4e^2}(\partial_m A_n - \partial_n A_m)(\partial^m A^n - \partial^n A^m) - \frac{\lambda}{2e^2}(\partial_m A^m)^2 \ . \tag{1.2}$$

To avoid technical complications at this stage we consider the case $\lambda = 1$. The general case is discussed at the end of this chapter. We choose to introduce the gauge coupling e here as normalization of the gauge kinetic energies. The equations of motion

$$\frac{\delta W}{\delta A^n(x)} = \frac{1}{e^2}\Box A_n(x) = 0 \quad \Box = \eta^{mn}\partial_m\partial_n = \partial_0{}^2 - \vec{\partial}^2 \tag{1.3}$$

are solved by the free fields

$$A_n(x) = e\int \frac{d^3k}{(2\pi)^3 2k^0}(e^{ikx}a_n^\dagger(\vec{k}) + e^{-ikx}a_n(\vec{k}))_{|_{k^0=\sqrt{\vec{k}^2}}} \ . \tag{1.4}$$

They are quantized by the requirement that the propagator

$$\langle 0|T A_m(x) A_n(0)|0\rangle \tag{1.5}$$

be the Greens function of the Euler Lagrange equation

$$\frac{1}{e^2}\delta_k^m \Box_x \langle 0|T A_m(x) A^n(0)|0\rangle = i\delta^4(x)\delta_k^n \ . \tag{1.6}$$

The creation and annihilation operators $a^\dagger(\vec{k})$ and $a(\vec{k})$ are identified by their commutation relations with the momentum operators P^m

$$\left[P_m, a_n^\dagger(\vec{k})\right] = k_m a_n^\dagger(\vec{k}) \quad \left[P_m, a_n(\vec{k})\right] = -k_m a_n(\vec{k}) \tag{1.7}$$

which follow because by definition P_m generate translations

$$[iP_m, A_n(x)] = \partial_m A_n(x) \ . \tag{1.8}$$

$a_n^\dagger(\vec{k})$ adds and $a_n(\vec{k})$ subtracts energy $k_0 = \sqrt{\vec{k}^2} \geq 0$. Consequently the annihilation operators annihilate the lowest energy state $|0\rangle$ and justify their denomination

$$P_m|0\rangle = 0 \quad a(\vec{k})|0\rangle = 0 \ .$$

For $x^0 > 0$ the propagator (1.5) contains only positive frequencies $e^{-ikx}a_m(\vec{k})$, for $x^0 < 0$ only negative frequencies $e^{ikx}a_m^\dagger(\vec{k})$. These boundary conditions fix the solution to (1.6) to be

$$\langle 0|T A_m(x) A_n(0)|0\rangle = \lim_{\epsilon\to 0+} -i\, e^2\ \eta_{mn} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + i\epsilon} \ . \tag{1.9}$$

Evaluating the p^0 integral for positive and for negative x^0 and comparing with the explicit expression for the propagator (1.5) which results if one inputs the free fields (1.4) one can read off $\langle a_m(\vec{k}) a_n^\dagger(\vec{k}')\rangle$ and the value of the commutator

$$\left[a_m(\vec{k}), a_n^\dagger(\vec{k}')\right] = -\ \eta_{mn}(2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{k}') \ . \tag{1.10}$$

It is inevitable that the Lorentz metric $\eta_{mn} = \mathrm{diag}(1,\text{-}1,\text{-}1,\text{-}1)$ appears in such commutation relations in Lorentz covariant theories with fields with higher spin. The Fock space which results from such commutation relations necessarily contains negative norm states because the Lorentz-metric is indefinite and contains both signs. Consider more specifically the state $|f_0\rangle$

$$|f_0\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} f(\vec{k}) a_0^\dagger(\vec{k})\ |0\rangle \ . \tag{1.11}$$

It has negative norm

$$\langle f_0|f_0\rangle = -\eta_{00}\int \frac{d^3k}{(2\pi)^3 2k^0}|f(\vec{k})|^2 < 0\ .$$

In classical electrodynamics (in the vacuum) one does not have the troublesome amplitude $a_0^\dagger(\vec{k})$. There the wave equation $\Box A_n = 0$ results from Maxwell's equation $\partial^m(\partial_m A_n - \partial_n A_m) = 0$ and the Lorentz condition $\partial_m A^m = 0$. This gauge condition fixes the vectorfield up to the gauge transformation $A_m \to A'_m = A_m + \partial_m C$ where $C(x)$ satisfies the wave equation $\Box C = 0$. In terms of the free fields $A_m(x)$ and $C(x)$

$$C(x) = e\int \frac{d^3k}{(2\pi)^3 2k^0}\left(e^{ikx}c^\dagger(\vec{k}) + e^{-ikx}c(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}} \tag{1.12}$$

one calculates

$$\partial_m A^m = i\,e\int \frac{d^3k}{(2\pi)^3 2k^0}\left(e^{ikx}k^m a_m^\dagger(\vec{k}) - e^{-ikx}k^m a_m(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}}$$

and

$$A'_m - A_m = \partial_m C = i\,e\int \frac{d^3k}{(2\pi)^3 2k^0}\left(e^{ikx}k_m c^\dagger(\vec{k}) - e^{-ikx}k_m c(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}}\ . \tag{1.13}$$

Let us decompose the creation operator $a_m^\dagger(\vec{k})$ into parts in the direction of the lightlike momentum k, in the direction $\bar{k}$ (which is k with reflected 3-momentum)

$$\bar{k}^m = (k^0, -k^1, -k^2, -k^3) \tag{1.14}$$

and in two directions n^i, $i = 1, 2$, which are orthogonal to k and $\bar{k}$. [1]

$$a_m^\dagger(\vec{k}) = \sum_{\tau = k, \bar{k}, 1, 2} \epsilon_m{}^\tau a_\tau^\dagger(\vec{k})\ . \tag{1.15}$$

Explicitly we use polarization vectors ϵ^τ

$$\epsilon_m{}^\tau = \left(\frac{1}{\sqrt{2}}\frac{k_m}{|\vec{k}|}, \frac{1}{\sqrt{2}}\frac{\bar{k}_m}{|\vec{k}|}, n_m^1, n_m^2\right)\quad \tau = k, \bar{k}, 1, 2 \tag{1.16}$$

[1] Disregarding as usual the problem that there are no continuous vectorfields $n^1(\vec{k}), n^2(\vec{k})$ which complete $\frac{\vec{k}}{|\vec{k}|}$ to an orthonormal frame for all directions $\vec{k}$.

with scalar products

$$\epsilon^{\tau} \cdot \epsilon^{\tau'} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{1.17}$$

The field $\partial_m A^m$ contains the amplitudes $a^{\dagger}_{\bar{k}},\ a_{\bar{k}}$. The Lorentz gauge condition $\partial_m A^m = 0$ eliminates these amplitudes in classical electrodynamics.

The fields A'_m and A_m differ in the amplitudes $a^{\dagger}_k,\ a_k$ in the direction of the momentum k. An appropriate choice of the remaining gauge transformation (1.13) cancels these amplitudes.

So in classical electrodynamics $a^{\dagger}_m$ can be restricted to 2 degrees of freedom, the transverse oscillations

$$a^{\dagger}_m(\vec{k}) = \sum_{\tau=1,2} \epsilon_m{}^{\tau} a^{\dagger}_{\tau}(\vec{k}) \ .$$

The corresponding quantized modes generate a positive definite Fock space. We cannot, however, just require $a^{\dagger}_k = 0$ and $a^{\dagger}_{\bar{k}} = 0$ in the quantized theory, this would contradict the commutation relation

$$\left[a_k(\vec{k}), a^{\dagger}_{\bar{k}}(\vec{k}')\right] = -\ (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{k}') \ , \tag{1.18}$$

which does not vanish. To get rid of the troublesome modes we require, rather, that physical states do not contain $a^{\dagger}_k$ and $a^{\dagger}_{\bar{k}}$ modes. A slight reformulation of this condition for physical states leads to BRS symmetry.

To single out a physical subspace of Fock space $\mathcal{F}$ we require that there exists a hermitean operator, the BRS operator,

$$Q_s = Q_s^{\dagger} \tag{1.19}$$

which defines a subspace $\mathcal{N} \subset \mathcal{F}$, the gauge invariant states, by

$$\mathcal{N} = \{|\Psi\rangle : Q_s|\Psi\rangle = 0\} \tag{1.20}$$

This requirement is no restriction at all, each subspace can be characterized as kernel of some hermitean operator.

Inspired by gauge transformations (1.13) we take the operator Q_s to act on one particle states according to

$$Q_s\ a^{\dagger}_m(\vec{k})|0\rangle = k_m c^{\dagger}(\vec{k})|0\rangle \ . \tag{1.21}$$

As a consequence the one particle states generated by $a^\dagger_\tau(\vec{k})$ $\tau = \bar{k}, 1, 2$ belong to $\mathcal{N}$.

$$Q_s \; a^\dagger_\tau(\vec{k})|0\rangle = 0 \quad \tau = \bar{k}, 1, 2 \tag{1.22}$$

The states created by the creation operator $a^\dagger_k$ in the direction of the momentum k are not invariant

$$Q_s \; a^\dagger_k(\vec{k})|0\rangle = \sqrt{2}|\vec{k}|c^\dagger(\vec{k})|0\rangle \; \neq 0$$

and do not belong to $\mathcal{N}$.

The space $\mathcal{N}$ is not yet acceptable because it contains non vanishing zero-norm states

$$|f\rangle = \int \tilde{d}k f(\vec{k}) a^\dagger_{\bar{k}}(\vec{k})|0\rangle \quad \langle f|f\rangle = 0 \text{ because } \left[a_{\bar{k}}(\vec{k}), a^\dagger_{\bar{k}}(\vec{k}')\right] = 0 \; . \tag{1.23}$$

To get rid of these states the following observation is crucial:

Theorem 1.1
Scalar products of gauge invariant states $|\psi\rangle \in \mathcal{N}$ and $|\chi\rangle \in \mathcal{N}$ remain unchanged if the state $|\psi\rangle$ is replaced by $|\psi\rangle + Q_s|\Lambda\rangle$.

Proof:

$$\langle\chi|\left(|\psi\rangle + Q_s|\Lambda\rangle\right) = \langle\chi|\psi\rangle + \langle\chi|Q_s|\Lambda\rangle = \langle\chi|\psi\rangle \tag{1.24}$$

The term $\langle\chi|Q_s|\Lambda\rangle$ vanishes, because Q_s is hermitean and $Q_s|\chi\rangle = 0$.

We arrive at the BRS algebra from the seemingly innocent requirement that $|\psi\rangle + Q_s|\Lambda\rangle$ belongs to $\mathcal{N}$ whenever $|\psi\rangle$ does. The requirement seems natural because $|\psi\rangle + Q_s|\Lambda\rangle$ and $|\psi\rangle$ have the same scalar products with gauge invariant states and therefore cannot be distinguished experimentally. It is, nevertheless, a most restrictive condition, because it requires Q_s^2 to vanish on each state $|\Lambda\rangle$, i.e. Q_s is nilpotent.

$$Q_s^2 = 0 \tag{1.25}$$

We require this relation as defining property of the BRS operator. Then the space $\mathcal{N}$ of gauge invariant states decomposes into equivalence classes

$$|\psi\rangle \sim |\psi\rangle + Q_s|\Lambda\rangle \; . \tag{1.26}$$

These equivalence classes are the physical states.

$$\mathcal{H}_{phys} = \frac{\mathcal{N}}{Q_s\mathcal{F}} = \{|\psi\rangle : Q_s|\psi\rangle = 0 \;, |\psi\rangle \bmod Q_s|\Lambda\rangle\} \tag{1.27}$$

$\mathcal{H}_{phys}$ inherits a scalar product from $\mathcal{F}$ because the scalar product in $\mathcal{N}$ does not depend on the representative of the equivalence class by theorem 1.1.

The construction of $\mathcal{H}_{phys}$ by itself does not guarantee that $\mathcal{H}_{phys}$ has a positive definite scalar product. This will hold only if $\mathcal{F}$ and Q_s are suitably chosen. One has to check this positive definiteness in each model.

In the case at hand, the zero-norm states $|f\rangle$ (1.23) are equivalent to 0 in $\mathcal{H}_{phys}$ if there exists a massless, real field $\bar{C}(x)$

$$\bar{C}(x) = e \int \frac{d^3k}{(2\pi)^3 2k^0} \left(e^{ikx} \bar{c}^\dagger(\vec{k}) + e^{-ikx} \bar{c}(\vec{k}) \right)\Big|_{k^0=\sqrt{\vec{k}^2}} \tag{1.28}$$

and if Q_s transforms the one-particle states according to

$$Q_s \bar{c}^\dagger(\vec{k})|0\rangle = i\sqrt{2}|\vec{k}| a_{\bar{k}}^\dagger(\vec{k})|0\rangle \ . \tag{1.29}$$

For the six one-particle states we conclude that $\bar{c}^\dagger(\vec{k})|0\rangle$ and $a_k^\dagger(\vec{k})|0\rangle$ are not invariant (not in $\mathcal{N}$), $a_{\bar{k}}^\dagger(\vec{k})|0\rangle$ and $c^\dagger(\vec{k})|0\rangle$ are of the form $Q_s|\Lambda\rangle$ and equivalent to 0, the remaining two transverse creation operators generate the physical one particle space with positive norm.

Notice the following pattern: states from the Fock space $\mathcal{F}$ are excluded in pairs from the physical Hilbert space $\mathcal{H}_{phys}$, one state $|n\rangle$ is not invariant

$$Q_s|n\rangle = |t\rangle \neq 0 \tag{1.30}$$

and therefore not contained in $\mathcal{N}$, the other $|t\rangle$ is trivial and equivalent to 0 in $\mathcal{H}_{phys}$ because it is the BRS transformation of $|n\rangle$: $|t\rangle = Q_s|n\rangle$.

The algebra $Q_s^2 = 0$ enforces

$$Q_s|t\rangle = 0 \ . \tag{1.31}$$

If one uses $|t\rangle$ and $|n\rangle$ as basis then Q_s is represented by the matrix

$$Q_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \tag{1.32}$$

This is one of the two possible Jordan block matrices which can represent a nilpotent operator $Q_s^2 = 0$. The only eigenvalue is 0, so a Jordan block consists of a matrix with zeros and with 1 only in the upper diagonal

$$Q_{s\,ij} = \delta_{i+1,j} \ .$$

Because of $Q_s^2 = 0$ the blocks can only have the size 1×1 or 2×2. In the first case the corresponding vector on which Q_s acts is invariant and not trivial and contributes to $\mathcal{H}_{phys}$. The second case is given by (1.32), the corresponding vectors are not physical.

It is instructive to consider the scalar product of the states on which Q_s acts. If it is positive definite then Q_s has to vanish because Q_s is hermitean and can be diagonalized in a space with positive definite scalar product. Thereby the non diagonalizable 2×2 block (1.32) would be excluded. It is, however, in Fock spaces with indefinite scalar product that we need the BRS operator and there it can act nontrivially. In the physical Hilbert space, which has a positive definite scalar product, Q_s vanishes. Nevertheless the existence of the BRS operator Q_s in Fock space severely restricts the possible actions of the models we are going to consider.

Reconsider the doublet (1.30, 1.31): one can easily verify that by suitable choice of $|n\rangle$ and $|t\rangle$ the scalar product if it is non-degenerate can be brought to one of the two standard forms

$$\langle n|n\rangle = 0 = \langle t|t\rangle \quad \langle t|n\rangle = \langle n|t\rangle = 1 \text{ or } (-1) \ . \tag{1.33}$$

This is an indefinite scalar product of Lorentzian type

$$|e_\pm\rangle = \frac{1}{\sqrt{2}}(|n\rangle \pm |t\rangle) \quad \langle e_+|e_-\rangle = 0 \quad \langle e_+|e_+\rangle = -\langle e_-|e_-\rangle = 1 \text{ or } (-1) \ . \tag{1.34}$$

By the definition (1.27) pairs of states with wrong sign norm and with acceptable norm are excluded from the space $\mathcal{H}_{phys}$ of physical states.

Let us close this chapter with a supplement which describes free vector fields for $\lambda \neq 1$. They have to satisfy the equations of motion

$$\frac{1}{e^2}(\Box A_n + (\lambda - 1)\partial_n \partial_m A^m) = 0 \ . \tag{1.35}$$

It is easy to derive from this the necessary condition

$$\Box\Box A_m = 0 \tag{1.36}$$

and its Fourier transformed version

$$(p^2)^2 \tilde{A}_m = 0 \ .$$

From this one can conclude that $\tilde{A}$ vanishes outside the light cone and that the general solution $\tilde{A}$ contains a δ function and its derivative.

$$\tilde{A}_m = a_m(p)\delta(p^2) + b_m(p)\delta'(p^2)$$

However, the derivative of the δ function is ill defined because spherical coordinates $p^2, v, \vartheta, \varphi$ are discontinuous at $p = 0$.

To solve $\Box\Box\phi = 0$ one can restrict $\phi(t,\vec{x})$ to $\phi(t)e^{i\vec{k}\vec{x}}$, the general solution can then be obtained as a wavepacket which is superposed out of solutions of this form. $\phi(t)$ has to satisfy the ordinary differential equation

$$(\frac{d^2}{dt^2} + k^2)^2\phi = 0$$

which has the general solution

$$\phi(t) = (a + bt)e^{ikt} + (c + dt)e^{-ikt} \ .$$

Therefore the equations (1.36) are solved by

$$\begin{aligned} A_n(x) &= \int \frac{d^3k}{(2\pi)^3 2k^0} \left(e^{ikx}a_n^\dagger(\vec{k}) + x^0 e^{ikx} b_n^\dagger(\vec{k}) + \right. \\ &\qquad \left. + \ e^{-ikx}a_n(\vec{k}) + x^0 e^{-ikx} b_n(\vec{k})\right)\Big|_{k^0=\sqrt{\vec{k}^2}} \ . \qquad (1.37) \end{aligned}$$

This equation makes the vague notion $\delta'(p^2)$ explicit. The amplitudes b_n, $b_n^\dagger$ are determined from the coupled equations (1.35).

$$\begin{aligned} A_n(x) &= e \int \frac{d^3k}{(2\pi)^3 2k^0} \left(e^{ikx}a_n^\dagger(\vec{k}) - i\frac{\lambda-1}{\lambda+1}x^0 e^{ikx}\frac{k_n}{k_0}k^m a_m^\dagger(\vec{k}) + \right. \\ &\qquad \left. + \ e^{-ikx}a_n(\vec{k}) + i\frac{\lambda-1}{\lambda+1}x^0 e^{-ikx}\frac{k_n}{k_0}k^m a_m(\vec{k})\right)\Big|_{k^0=\sqrt{\vec{k}^2}} \qquad (1.38) \end{aligned}$$

From (1.8) one can deduce that the commutation relations

$$[P^i, a_m^\dagger(\vec{k})] = k^i a_m^\dagger(\vec{k}) \quad [P^i, a_m(\vec{k})] = -k^i a_m(\vec{k}) \quad i = 1,2,3 \qquad (1.39)$$

and

$$[P_0, a_m^\dagger(\vec{k})] = k_0 a_m^\dagger(\vec{k}) - \frac{(\lambda-1)}{(\lambda+1)}\frac{k_m}{k_0}k^n a_n^\dagger(\vec{k}) \qquad (1.40)$$

have to hold. If we decompose $a^\dagger_m(\vec{k})$ according to (1.15) then we obtain

$$[P_0, a^\dagger_t(\vec{k})] = k_0 a^\dagger_t(\vec{k}) \quad t = 1, 2 \tag{1.41}$$

for the transverse creation operators and also

$$[P_0, a^\dagger_{\bar{k}}(\vec{k})] = k_0 a^\dagger_{\bar{k}}(\vec{k}) \tag{1.42}$$

for the creation operator in direction of $\bar{k}$. For the creation operator in the direction of the four momentum k one gets

$$[P_0, a^\dagger_k(\vec{k})] = k_0 a^\dagger_k(\vec{k}) - 2k_0 \frac{\lambda - 1}{\lambda + 1} a^\dagger_{\bar{k}}(\vec{k}) \ . \tag{1.43}$$

In particular, for $\lambda \neq 1$, $a^\dagger_k(\vec{k})$ does not generate energy eigenstates and the hermitean operator P_0 cannot be diagonalized in Fock space because the commutation relations are given by

$$[P_0, a^\dagger] = M\, a^\dagger$$

with a matrix M which contains a nondiagonalizable Jordan block

$$M \sim k_0 \begin{pmatrix} 1 & -2\frac{\lambda-1}{\lambda+1} \\ 0 & 1 \end{pmatrix} \ . \tag{1.44}$$

That hermitean operators are not guaranteed to be diagonalizable is of course related to the indefinite norm in Fock space. For operators O_{phys} which correspond to measuring devices it is sufficient that they can be diagonalized in the physical Hilbert space. This is guaranteed if $\mathcal{H}_{phys}$ has positive norm. In Fock space it is sufficient that operators O_{phys} commute with the BRS operator Q_s and that they satisfy generalized eigenvalue equations

$$O_{phys}|\psi_{phys}\rangle = c|\psi_{phys}\rangle + Q_s|\chi\rangle \quad c \in \mathbb{R} \tag{1.45}$$

from which the spectrum can be read off.

The Hamilton operator $H = P_0$ which results from the Lagrange density

$$\mathcal{L} = -\frac{1}{4e^2} F_{mn}F^{mn} - \frac{\lambda}{2e^2}(\partial_m A^m)^2 \tag{1.46}$$

$$H = \frac{1}{2e^2}\int d^3x :\Big((\partial_0 A_i)^2 - (\partial_i A_0)^2 - \lambda(\partial_0 A_0)^2 + (\partial_j A_i - \partial_i A_j)^2 + \lambda(\partial_i A_i)^2\Big): \tag{1.47}$$

can be expressed in terms of the creation and annihilation operators.

$$H = \int \frac{d^3k}{(2\pi)^3 2k_0} k_0 \left(\sum_{t=1}^{2} a_t^\dagger a_t - \frac{2\lambda}{\lambda+1} \left(a_k^\dagger a_{\bar{k}} + a_{\bar{k}}^\dagger a_k - 2\frac{\lambda-1}{\lambda+1} a_{\bar{k}}^\dagger a_{\bar{k}} \right) \right) \quad (1.48)$$

H satisfies (1.40) because the creation and annihilation operators fulfil the commutation relations

$$[a_m(\vec{k}), a_n^\dagger(\vec{k}')] = 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\left(-\eta_{mn} + \frac{\lambda-1}{2\lambda k^0}(\eta_{m0}k_n + \eta_{n0}k_m - \frac{k_m k_n}{k^0}) \right) \quad (1.49)$$

which follow from the requirement that the propagator

$$\langle \mathrm{T} A_m(x) A^n(0) \rangle = -ie^2 \lim_{\varepsilon \to 0+} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{(p^2+i\varepsilon)^2} \left(p^2 \delta_m{}^n - \frac{\lambda-1}{\lambda} p_m p^n \right) \quad (1.50)$$

is the Greens function corresponding to the equation of motion (1.35). If one decomposes the creation and annihilation operators according to (1.15) then the transverse operators satisfy

$$[a_i(\vec{k}), a_j^\dagger(\vec{k}')] = 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{ij} \quad i,j \in \{1,2\} \quad (1.51)$$

They commute with the other creation annihilation operators which have the following off diagonal commutation relations

$$[a_{\bar{k}}(\vec{k}), a_k^\dagger(\vec{k}')] = [a_k(\vec{k}), a_{\bar{k}}^\dagger(\vec{k}')] = -\frac{\lambda+1}{2\lambda} 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}') \; . \quad (1.52)$$

The other commutators vanish.

The analysis of the BRS transformations leads again to the result that physical states are generated only by the transverse creation operators.

Chapter 2

BRS symmetry

To choose the physical states one could have proceeded like Cinderella and could pick acceptable states by hand or have them picked by doves. Prescribing the action of Q_s on one particle states (1.21, 1.29) is not really different from such an arbitrary approach. From (1.21,1.29) we know nothing about physical multiparticle states. Moreover we would like to know whether one can switch on interactions which respect our definition of physical states. Interactions should give transition amplitudes which are independent of the choice (1.26) of the representative of physical states. The time evolution should leave physical states physical, otherwise negative norm states could result from physical initial states.

All these requirements can be satisfied if the BRS operator Q_s belongs to a symmetry. We interpret the equation $Q_s^2 = 0$ as a graded commutator, an anticommutator, of a fermionic generator of a Lie algebra

$$\{Q_s, Q_s\} = 0 \ . \tag{2.1}$$

To require that Q_s be fermionic means that the BRS operator transforms fermionic variables into bosonic variables and vice versa. In particular we take $A_m(x)$ to be a bosonic field. Then the fields $C(x)$ and $\bar{C}(x)$ have to be fermionic though they are real scalar fields and carry no spin. They violate the spin statistics relation which requires physical fields with half-integer spin to be fermionic and fields with integer spin to be bosonic. This violation can be tolerated because the corresponding particles do not occur in physical states, they are ghosts. We call $C(x)$ the ghost field and $\bar{C}(x)$ the antighost field. Because the ghost fields C and $\bar{C}$ anticommute they contribute, after introduction of interactions, to loop corrections with the opposite sign as compared to bosonic contributions. The ghosts compensate in loops for the unphysical bosonic degrees of freedom contained in the field $A_m(x)$.

We want to realize the algebra (2.1) as local transformations on fields. Then we have to determine actions which are invariant under these transformations and construct the BRS operator as Noether charge corresponding to this symmetry.

The transformations act on commuting and anticommuting classical variables, the fields, and polynomials in these fields, the Lagrange densities. We write the commutation relation as

$$\phi^i \phi^j = (-1)^{|\phi^i| \cdot |\phi^j|} \phi^j \phi^i =: (-)^{ij} \phi^j \phi^i \tag{2.2}$$

Here we have introduced the grading

$$|\phi^i| = \begin{cases} 0 \text{ if } \phi^i \text{ is bosonic} \\ 1 \text{ if } \phi^i \text{ is fermionic} \end{cases} \tag{2.3}$$

and a shorthand $(-)^{ij}$ for $(-1)^{|\phi^i| \cdot |\phi^j|}$. Because products are understood to be associative monomials get a natural grading

$$|\phi^i \phi^j| = |\phi^i| + |\phi^j| \bmod 2 \ . \tag{2.4}$$

We will consider only polynomials which are sums of monomials with the same grading, these polynomials are graded commutative

$$AB = (-1)^{|A| \cdot |B|} BA \ . \tag{2.5}$$

Transformations and symmetries are operations O acting linearly, i.e. term by term, on polynomials. We consider only operations which map polynomials with a definite grading to polynomials with a definite grading. These operations have a natural grading.

$$\begin{aligned} O(\lambda_1 A + \lambda_2 B) &= \lambda_1 O(A) + \lambda_2 O(B) & (2.6) \\ |O| &= |O(A)| - |A| \bmod 2 & (2.7) \end{aligned}$$

Derivative operators v of first order satisfy in addition a graded Leibniz rule[1]

$$v(AB) = (vA)B + (-)^{|v| \cdot |A|} A(vB) \ . \tag{2.8}$$

[1] More precisely this Leibniz rule defines left derivatives. The left factor A is differentiated without a graded sign.

They are completely determined by their action on the elementary variables ϕ^i: $v(\phi^i) = v^i$, i.e. $v = v^i \partial_i$. The partial derivatives ∂_i are naturally defined by

$$\partial_i \phi^j = \delta_i{}^j . \tag{2.9}$$

They have the same grading as their corresponding variables

$$|\partial_i| = |\phi^i| \quad \partial_i \partial_j = (-)^{ij} \partial_j \partial_i \ . \tag{2.10}$$

The grading of the components v^i results naturally $|v^i| = |v| + |\phi^i| \bmod 2$.

An example of a fermionic derivative is given by the exterior derivative

$$d = dx^m \partial_m \tag{2.11}$$

It transforms coordinates x^m into differentials dx^m which have opposite statistics

$$|dx^m| = |x^m| + 1 \bmod 2 \tag{2.12}$$

and which commute with ∂_n

$$[\partial_n, dx^m] = 0 \ . \tag{2.13}$$

Therefore and because of (2.10) the exterior derivative is nilpotent

$$d^2 = 0 \ . \tag{2.14}$$

Lagrange densities have to be real polynomials to make the resulting S-matrix unitary. This is why we have to discuss complex conjugation. We define conjugation such that hermitean conjugation of a time ordered operator corresponding to some polynomial gives the anti time ordered operator corresponding to the conjugated polynomial. We therefore require

$$\begin{aligned}
(\lambda_1 A + \lambda_2 B)^* &= \lambda_1^* A^* + \lambda_2^* B^* && (2.15)\\
(AB)^* &= B^* A^* = (-)^{|A||B|} A^* B^* && (2.16)\\
|\phi^*| &= |\phi| \ . && (2.17)
\end{aligned}$$

Conjugation preserves the grading and is defined on polynomials whenever it is defined on the elementary variables ϕ^i. It can be used to define conjugation of operations O (they map polynomials to polynomials and have to be distinguished from operators in Fock space).

$$O^*(A) = (-)^{|O||A|} (O(A^*))^* \tag{2.18}$$

This definition ensures that O^* is linear and satisfies the Leibniz rule if O is a first order derivative.

The exterior derivative d is real if the conjugate differentials are given by

$$(dx^m)^* = (-)^{|x^m|} d((x^m)^*) \ . \tag{2.19}$$

The partial derivative with respect to a real fermionic variable is purely imaginary as is the operator δ

$$\delta = x^m \frac{\partial}{\partial(dx^m)} \quad \delta^* = -\delta \ . \tag{2.20}$$

The anticommutator Δ

$$\Delta = \{d, \delta\} = x^m \frac{\partial}{\partial x^m} + dx^m \frac{\partial}{\partial(dx^m)} = N_x + N_{dx} \tag{2.21}$$

which counts the variables x and dx is again real because the definition (2.18) implies

$$(O_1 O_2)^* = (-)^{|O_1||O_2|} O_1^* O_2^* \ . \tag{2.22}$$

Conjugation does <u>not</u> change the order of two operations O_1 and O_2.

We can now define the BRS transformation s. It is a real, fermionic, nilpotent first order derivative.

$$s = s^* \quad |s| = 1 \quad s^2 = 0 \tag{2.23}$$

It acts on Lagrange densities and functionals of fields. Space-time derivatives ∂_m of fields are limits of differences of fields taken at neighbouring arguments. It follows from the linearity of s that it has to commute with space-time derivatives

$$[s, \partial_m] = 0 \ . \tag{2.24}$$

Linearity implies moreover that the BRS transformation of integrals is given by the integral of the transformed integrand. Therefore the differentials dx^m are BRS invariant

$$s(dx^m) = 0 = \{s, dx^m\} \quad ([s, dx^m] = 0 \text{ for fermionic } x^m) \tag{2.25}$$

Taken together the last two equations imply that s and d (2.11) anticommute

$$\{s, d\} = 0 \tag{2.26}$$

In the simplest multiplet s transforms a real anticommuting field $\bar{C}(x) = \bar{C}^*(x)$, the antighost field, into $\sqrt{-1}$ times a real bosonic field $B(x) = B^*(x)$, the auxiliary field. The denominations will be justified once the Lagrange density is given.

$$s\bar{C}(x) = iB(x) \quad sB(x) = 0 \tag{2.27}$$

The BRS transformation which corresponds to abelian gauge transformations acts on a real bosonic vectorfield $A_m(x)$ and a real, fermionic ghost field $C(x)$ by

$$sA_m(x) = \partial_m C(x) \quad sC(x) = 0 \ . \tag{2.28}$$

We can attribute to the fields

$$\phi = \bar{C}, B, A_m, C \tag{2.29}$$

and to s a ghost number

$$\mathrm{gh}(\bar{C}) = -1,\ \mathrm{gh}(B) = 0,\ \mathrm{gh}(A_m) = 0,\ \mathrm{gh}(C) = 1 \ . \tag{2.30}$$

$$\mathrm{gh}(s) = 1 \ . \tag{2.31}$$

We anticipate the analysis of the algebra (2.27, 2.28) and state the result in $D = 4$ dimensions[2]. All invariant, local actions

$$W[\phi] = \int d^4x \mathcal{L}(\phi, \partial\phi, \partial\partial\phi, \ldots) \tag{2.32}$$

with ghostnumber 0 have the form

$$\mathcal{L} = \mathcal{L}_{inv}(F_{mn}, \partial_l F_{mn}, \ldots) + isX(\phi, \partial\phi, \ldots) \ . \tag{2.33}$$

The part $\mathcal{L}_{inv}$ is real, it depends only on the field strengths

$$F_{mn} = \partial_m A_n - \partial_n A_m \tag{2.34}$$

and their partial derivatives. Therefore it is invariant under classical gauge transformations. Typically it is chosen to be

$$\mathcal{L}_{inv} = -\frac{1}{4e^2} F_{mn} F^{mn} \ . \tag{2.35}$$

[2]The result holds more generally in even dimensions. In odd dimensions Chern Simons forms can occur in addition.

The gauge coupling constant e is introduced as normalization of the kinetic energy of the gauge field.

The function $X(\phi, \partial\phi, \ldots)$ is a real, fermionic polynomial with ghostnumber $\mathrm{gh}(X) = -1$. It has to contain a factor $\bar{C}$ and is in the simplest case given by

$$X = \frac{\lambda}{e^2}\bar{C}(-\frac{1}{2}B + \partial_m A^m) \ . \tag{2.36}$$

λ is the gauge fixing parameter. The piece isX contributes the gaugefixing for the vectorfield and contains the action of the ghostfields C and $\bar{C}$.

$$isX = \frac{\lambda}{2e^2}(B - \partial_m A^m)^2 - \frac{\lambda}{2e^2}(\partial_m A^m)^2 - i\frac{\lambda}{e^2}\bar{C}\partial_m\partial^m C \tag{2.37}$$

This Langrange density makes B an auxiliary field, its equation of motion fixes it algebraically $B = \partial_m A^m$. C and $\bar{C}$ are free fields (1.12, 1.28).

To justify the name gauge fixing for the gauge breaking part $-\frac{\lambda}{2e^2}(\partial_m A^m)^2$ of the Lagrange density we show that a change of the fermionic function X cannot be measured in amplitudes of physical states as long as such a change leads only to a differentiable perturbation of amplitudes. This means that gauge fixing and ghostparts of the Lagrange density are unobservable. Only the parameters in the gauge invariant part $\mathcal{L}_{inv}$ are measurable.

Theorem 2.1
Transition amplitudes of physical states are independent of the gauge fixing within perturbatively connected gauge sectors.

Proof: If one changes X by δX then the Lagrange density and the action change by

$$\delta\mathcal{L} = i\,s\,\delta X \quad \delta W = i\,s\int d^4x\,\delta X \ . \tag{2.38}$$

S-matrix elements of physical states $|\chi\rangle$ and $|\psi\rangle$ change to first order by

$$\delta\langle\chi_{in}|\psi_{out}\rangle = \langle\chi_{in}|i\cdot i\int d^4x\,s\,\delta X|\psi_{out}\rangle \tag{2.39}$$

where $s\,\delta X$ is an operator in Fock space. The transformation $s\,\delta X$ of the operator δX is generated by i times the anticommutator of the fermionic operator δX with the fermionic BRS operator Q_s

$$\langle\chi_{in}|s\int d^4x\,\delta X|\psi_{out}\rangle = \langle\chi_{in}|\{i\,Q_s, \int d^4x\,\delta X\}|\psi_{out}\rangle \ . \tag{2.40}$$

This expression vanishes because $|\chi\rangle$ and $|\psi\rangle$ are physical (1.27) and Q_s is hermitean.

The proof does not exclude the possibility that there exist different sectors of gauge fixing which can be distinguished and cannot be joined by a perturbatively smooth change of parameters.

Using this theorem we can concisely express the restriction which the Lagrange density of a local, BRS invariant action in D dimensions has to satisfy.

It is advantageous to combine $\mathcal{L}$ with the differential $d^D x$ and consider the Lagrange density as a D-form $\omega_D^0 = \mathcal{L} d^D x$ with ghostnumber 0. The BRS transformation of the Lagrange density ω_D^0 has to give a (possibly vanishing) total derivative $d\,\omega_{D-1}^1$.

With this notation the condition for an invariant local action is

$$s\,\omega_D^0 + d\,\omega_{D-1}^1 = 0 \ .$$

It is sufficient to determine this Lagrange density ω_D^0 up to a piece of the form $s\,\eta_D^{-1}$, where η_D^{-1} carries ghostnumber -1. Such a piece contributes only to gaugefixing and to the ghostsector and cannot be observed. It is trivially BRS invariant because s is nilpotent. A total derivative part $d\,\eta_{D-1}^0$ (with $\mathrm{gh}(\eta_{D-1}^0) = 0$) of the Lagrange density contributes only boundary terms to the action and is also neglected. This means that we look for the solutions of the equation

$$s\,\omega_D^0 + d\,\omega_{D-1}^1 = 0 \quad \omega_D^0 \ \bmod \ (s\eta_D^{-1} + d\eta_{D-1}^0) \ . \tag{2.41}$$

This is a cohomological equation and very similar to the equation which determines the physical states (1.27). The equivalence classes of solutions ω_D^0 of this equation span a linear space: the relative cohomology of s mod d with ghost number indicated by the superscript and form degree denoted by the subscript.

If we use a Lagrange density which solves this equation, then the action is invariant under the continuous symmetry $\phi \to \phi + \alpha\, s\,\phi$ with an arbitrary fermionic parameter α. In classical field theory Noether's theorem then guarantees that there exists a current j^m which is conserved as a consequence of the equations of motion. The integral $Q_s = \int d^3x\, j^0$ is constant in time and generates the nilpotent BRS transformations

$$sA = (-)^{|A|}\{A, Q_s\}_{Poisson} \tag{2.42}$$

of functionals $A[\phi, \pi]$ of the phase space variables $\phi^i(x)$ and $\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^i}(x)$ by the graded Poisson bracket

$$\{A, B\}_{Poisson} = \int d^3x \left((-1)^{|i|(|i|+|A|)} \frac{\delta A}{\delta \phi^i(x)} \frac{\delta B}{\delta \pi_i(x)} - (-1)^{|i||A|} \frac{\delta A}{\delta \pi_i(x)} \frac{\delta B}{\delta \phi^i(x)} \right) \quad (2.43)$$

If one investigates the quantized theory then in the simplest of all conceivable worlds the classical Poisson brackets would be replaced by (anti-) commutators of quantized operators. In particular the BRS operator Q_s would commute with the scattering matrix S

$$S = \text{“T}\, e^{i \int d^4x \mathcal{L}_{int}}\text{”} \qquad [Q_s, S] = 0 \quad (2.44)$$

and scattering processes would map physical states unitarily to physical states

$$S \mathcal{H}_{phys} = \mathcal{H}_{phys} \ . \quad (2.45)$$

Classically an invariant action is sufficient to ensure this property. The perturbative evaluation of scattering amplitudes, however, has to face the problem that the S-matrix (2.44) has ill defined contributions from products of $\mathcal{L}_{int}(x_1) \ldots \mathcal{L}_{int}(x_n)$ if arguments x_i and x_j coincide. Though upon integration $\int dx_1 \ldots dx_n$ this is a set of measure zero these products of fields at coinciding space time arguments are the reason for all divergencies which emerge upon the naive application of the Feynman rules. More precisely the S-matrix is a time ordered series in $i \int d^4x \mathcal{L}_{int}$ and a set of prescriptions (indicated by the quotes in (2.44)) to define in each order the products of $\mathcal{L}_{int}(x)$ at coinciding space-time points. To analyze these divergencies it is sufficient to consider only connected diagrams. In momentum space they decompose into products of one particle irreducible n-point functions $\tilde{G}_{1PI}(p_1, \ldots, p_n)$ which define the effective action.

$$\begin{aligned} \Gamma[\phi] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \phi(x_1) \ldots \phi(x_n) \, G_{1PI}(x_1, \ldots, x_n) && (2.46) \\ &= \int d^4x \mathcal{L}_0(\phi, \partial\phi, \ldots) + \sum_{n \geq 1} \hbar^n \Gamma_n[\phi] && (2.47) \end{aligned}$$

To lowest order in $\hbar$ the effective action Γ is given by the classical action $\Gamma_0 = \int d^4x \mathcal{L}_0$. This is a local functional, in particular $\mathcal{L}_0$ is a series in the fields and a polynomial in the partial derivatives of the fields. The Feynman diagrams fix the expansion of the nonlocal effective action $\Gamma = \sum \hbar^n \Gamma_n$ up to local functionals which can be chosen in each loop order. We are free to choose the Lagrange density in each loop order, i.e as a series in $\hbar$.

$$\mathcal{L} = \mathcal{L}_0 + \sum_{n\geq 1} \hbar^n \mathcal{L}_n \tag{2.48}$$

Consider in each loop order the question whether the full effective action is BRS invariant.

$$s\,\Gamma[\phi] = 0$$

To lowest order in $\hbar$ this requires the Lagrange density $\mathcal{L}_0$ to be a solution of (2.41). Assume that one has satisfied $s\,\Gamma[\phi] = 0$ up to n-loop order.

The naive calculation of $n+1$-loop diagrams contains divergencies which make it necessary to introduce a regularization, e.g. the Pauli-Villars regularization, and counterterms (or use a prescription such as dimensional regularization or the BPHZ prescription which is a shortcut for regularization and counterterms). No regularization respects locality, unitarity and symmetries simultaneously, otherwise it would not be a regularization but an acceptable theory. The Pauli-Villars regularization is local. It violates unitarity for energies above the regulator masses and also because it violates BRS invariance. If one cancels the divergencies of diagrams with counterterms and considers the limit of infinite regulator masses then unitarity is obtained if the BRS symmetry guarantees the decoupling of the unphysical gauge modes. Locality was preserved for all values of the regulator masses. What about BRS symmetry?

One cannot argue that one has switched off the regularization and that therefore the symmetry should be restored. There is the phenomenon of hysteresis. For example: if you have a spherically symmetric iron ball and switch on a symmetry breaking magnetic field then the magnetic properties of the iron ball will usually not become spherically symmetric again if the magnetic field is switched off. Analogously in the calculation of Γ_{n+1} we have to be prepared that the regularization and the cancellation of divergencies by counterterms does not lead to an invariant effective action but rather to

$$s\,\Gamma = \hbar^{n+1} a + \sum_{k\geq n+2} \hbar^k a_k\ . \tag{2.49}$$

If the functional a cannot be made to vanish by an appropriate choice of $\mathcal{L}_{n+1}$ then the BRS symmetry is broken by the anomaly a.

Because s is nilpotent the anomaly a has to satisfy

$$s\,a = 0\ . \tag{2.50}$$

This is the celebrated consistency condition of Wess and Zumino [5]. The consistency condition has acquired an outstanding importance because it allows to calculate all possible anomalies a as the general solution to $s\,a = 0$ and to check in each given model whether the anomaly actually occurs. At first sight one would not expect that the consistency equation has comparatively few solutions. The BRS transformation $a = s\Gamma$ of arbitrary functionals Γ satisfies $s\,a = 0$. The anomaly a, however, arises from the divergencies of Feynman diagrams where all subdiagrams are finite and compatible with BRS invariance. These divergencies can be isolated in parts of the n-point functions which depend polynomially on the external momenta, i.e. in local functionals. Therefore it turns out that the anomaly is a local functional.

$$a = \int d^4x \mathcal{A}^1(\phi(x), \partial\phi(x), \ldots) \tag{2.51}$$

The anomaly density $\mathcal{A}^1$ is a series in the fields ϕ and a polynomial in the partial derivatives of the fields comparable to a Lagrange density but with ghost number $+1$. The integrand $\mathcal{A}^1$ represents an equivalence class. It is determined only up to terms of the form $s\mathcal{L}$ because we are free to choose contributions to the Lagrange density at each loop order, in particular we try to choose $\mathcal{L}_{n+1}$ such that $s\mathcal{L}_{n+1}$ cancels $\mathcal{A}^1$ in order to make Γ_{n+1} BRS invariant. Moreover $\mathcal{A}^1$ is determined only up to derivative terms of the form $d\eta^1$.

$\mathcal{A}^1$ transforms into a derivative because the anomaly a satisfies the consistency condition. We combine the anomaly density $\mathcal{A}^1$ with d^Dx to a volume form ω^1_D and denote the ghost numbers as superscripts and the form degree as subscript. Then the consistency condition and the description of the equivalence class read

$$s\omega^1_D + d\omega^2_{D-1} = 0 \quad \omega^1_D \bmod s\eta^0_D + d\eta^1_{D-1}\ . \tag{2.52}$$

This equation determines all possible anomalies and can be analyzed if one is given the field content and the BRS transformations s. Its solutions do not depend on particular properties of the model under consideration.

The determination of all possible anomalies is again a cohomological problem just as the determination of all BRS invariant local actions (2.41) but now with ghost numbers shifted by +1. We will deal with both equations and consider the equation

$$s\omega_D^g + d\omega_{D-1}^{g+1} = 0 \quad \omega_D^g \ \mathrm{mod}\ s\eta_D^{g-1} + d\eta_{D-1}^g \tag{2.53}$$

for arbitrary ghost number g. The form degree is given by the subscript.

Chapter 3

Cohomological Problems

In the preceding chapters we have encountered repeatedly the cohomological problem to solve the linear equation $s\omega = 0$, $\omega \bmod s\eta$, where s is a nilpotent operator $s^2 = 0$. The equivalence classes of solutions ω form a linear space, the cohomology $H(s)$ of s. The equivalence classes of solutions ω_p^g of the problem $s\omega_p^g = -d\omega_{p-1}^{g+1}$ $\omega_p^g \bmod d\eta_{p-1}^g + s\eta_p^{g-1}$, where $s^2 = 0 = d^2 = \{s, d\}$ form the relative cohomology $H_p^g(s|d)$ of s modulo d of ghost number g and form degree p.

Let us start to solve such equations and consider the problem to determine the physical multiparticle states. Multiparticle states can be written as a polynomial P of the creation operators acting on the vacuum

$$P(a_\tau^\dagger, c^\dagger, \bar{c}^\dagger)|0\rangle \quad \tau = k, \bar{k}, 1, 2$$

if one neglects the technical complication that all these creation operators depend on $\vec{k}$ and have to be smeared with normalizable functions. The BRS operator Q_s acts on these states in the same way as the algebraic operation

$$s = \sqrt{2}|\vec{k}|(ia_{\bar{k}}^\dagger \frac{\partial}{\partial \bar{c}^\dagger} + c^\dagger \frac{\partial}{\partial a_k^\dagger}) \tag{3.1}$$

acts on polynomials in commuting and anticommuting variables. For one particle states, i.e. linear homogeneous polynomials P we had concluded that the physical states, the cohomology of Q_s with particle number 1, are generated by the transverse creation operators $a_i^\dagger$, i.e. by variables which are neither generated by s such as $a_{\bar{k}}^\dagger$ or $c^\dagger$ nor transformed such as $\bar{c}^\dagger$ and $a_k^\dagger$.

Let us systematize our notation and denote the variables which are differentiated collectively by x^m and their transformation by dx^m. Then the operator s becomes the nilpotent operator d ((2.11) without reality property). It maps the variables x^m to dx^m with opposite statistics.

$$d = dx^m \frac{\partial}{\partial x^m} \qquad |dx^m| = |x^m| + 1 \mod 2 \tag{3.2}$$

We claim that on polynomials in x^m and dx^m the cohomology of the exterior derivative d is described by the basic lemma.

Theorem 3.1 *Basic Lemma*

$$df(x,dx) = 0 \Leftrightarrow f(x,dx) = f_0 + dg(x,dx)\ . \tag{3.3}$$

f_0 denotes the polynomial which is homogeneous of degree 0 in x^m and dx^m and is therefore independent of these variables.

Applied to the Fock space the Basic Lemma implies that physical n-particle states are generated by polynomials f_0 of creation operators which contain no operators $a_{\bar{k}}^\dagger, a_k^\dagger, c^\dagger, \bar{c}^\dagger$. Physical states are generated from the transverse creation operators $a_i^\dagger,\ i = 1,2$.

To prove the lemma we introduce the operation

$$\delta = x^m \frac{\partial}{\partial(dx^m)}\ . \tag{3.4}$$

The anticommutator Δ of d and δ counts the variables x^m and dx^m.

$$\{d,\delta\} = \Delta = x^m \frac{\partial}{\partial x^m} + dx^m \frac{\partial}{\partial(dx^m)} = N_x + N_{dx} \tag{3.5}$$

From the relation $d^2 = 0$ it follows that d commutes with $\{d,\delta\}$.

$$d^2 = 0 \Rightarrow [d,\{d,\delta\}] = 0 \tag{3.6}$$

Of course we can easily check explicitly that d does not change the number of variables x and dx in a polynomial. We can decompose each polynomial f into pieces f_n of definite homogeneity n in the variables x and dx, i.e. $(N_x + N_{dx})f_n = nf_n$. Using (3.5) we can write f in the following form.

$$\begin{aligned} f &= f_0 + \sum_{n\geq 1} f_n = f_0 + \sum_{n\geq 1}(N_x + N_{dx})\frac{1}{n}f_n \\ &= f_0 + d\left(\delta \sum_{n\geq 1}\frac{1}{n}f_n\right) + \delta\left(d\sum_{n\geq 1}\frac{1}{n}f_n\right) \\ f &= f_0 + d\eta + \delta\chi \end{aligned} \tag{3.7}$$

This is the Hodge decomposition of an arbitrary polynomial in x and dx into a zero mode f_0, a d exact [1] part $d\eta$ and a δ exact part $\delta\chi$. If f solves $df = 0$ then the equations $df_n = 0$ have to hold for each piece df_n separately because d commutes with the number operator Δ. But $df_n = 0$ implies that the last term in the Hodge decomposition, the δ-exact term, vanishes. This proves our lemma. Of course this is not our lemma: it is Poincaré's lemma for forms in a star shaped domain if one writes $\frac{1}{n}$ as $\int_0^1 dt\, t^{n-1}$.

Theorem 3.2 *Poincaré's lemma*

$$df(x,dx) = 0 \Leftrightarrow f(x,dx) = f(0,0) + d\,\delta \int_0^1 \frac{dt}{t}(f(tx,tdx) - f(0,0)) \quad (3.8)$$

In this form the lemma is not restricted to polynomials but applies to all differentiable differential forms f which are defined along all rays tx for $0 \leq t \leq 1$ and all x, i.e. in a star shaped domain. Note that the integral is not singular at $t = 0$.

We chose to present the Poincaré lemma in the algebraic form – though it applies only to polynomials and to analytical functions if one neglects the question of convergence – because we will follow a related strategy to solve the cohomological problems to come: given a nilpotent operator d we inspect operators δ and their anticommutators Δ and try to invert Δ. Only the zero modes of Δ can contribute to the cohomology of d.

We have to generalize Poincaré's lemma because we consider Lagrange densities and more generally forms ω which are series in fields ϕ, polynomials in derivatives of fields $\partial_m\phi, \ldots, \partial_{m_1}\ldots\partial_{m_l}\phi$, polynomials in dx^m and series in the coordinates x^m.

$$\omega = \omega(x, dx, \phi, \partial\phi, \partial\partial\phi, \ldots) \quad (3.9)$$

Such forms occur as integrands of local functionals. Because they depend polynomially on derivatives of fields they contain only terms with a bounded number of derivatives, though there is no bound on the number of derivatives which is common to all forms ω. We call the fields and their derivatives

$$\{\phi\} = \phi, \partial\phi, \partial\partial\phi, \ldots \quad (3.10)$$

the jet variables. Poincaré's lemma does not apply to forms which depend on the coordinates, the differentials and the jet variables. The exceptions are

[1] A polynomial g is called d exact if it is of the form $g = d\eta$ for some polynomial η. The word exact is used as an abbreviation if the nilpotent operator d or δ is evident.

Lagrange densities which lead to nontrivial Euler-Lagrange equations. Then the Lagrange density $\omega = \mathcal{L} d^D x$ cannot be a total derivative $\omega \neq d\eta$ though $d\omega = 0$ because ω is a volume form. Let us prove this result.

The exterior derivative on forms of jet variables differentiates the explicit coordinates x^m, the fields just get an additional label upon differentiation

$$d = dx^m \partial_m \quad \partial_m x^n = \delta_m{}^n \quad \partial_k(\partial_l \dots \partial_m \phi) = \partial_k \partial_l \dots \partial_m \phi \ . \tag{3.11}$$

The fields are assumed to satisfy no differential equation, i.e. the variables $\partial_k \partial_l \dots \partial_m \phi$ are independent up to the fact that partial derivatives commute $\partial_k \dots \partial_m \phi = \partial_m \dots \partial_k \phi$. On such variables we can define the operation t^n

$$\begin{aligned} t^n(x^m) &= 0 \qquad t^n(dx^m) = 0 \\ t^n(\phi) &= 0 \qquad t^n(\partial_{m_1} \dots \partial_{m_l} \phi) = \sum_{i=1}^{l} \delta^n_{m_i} \partial_{m_1} \dots \hat{\partial}_{m_i} \dots \partial_{m_l} \phi \end{aligned} \tag{3.12}$$

The hat $\hat{}$ means omission of the hatted symbol. We define the action of t^n on polynomials in the jet variables by linearity and the Leibniz rule. t^n acts on derivatives of the fields ϕ like a differentiation with respect to ∂_n, i.e. $t^n = \frac{\partial}{\partial(\partial_n)}$. Obviously one gets $[t^m, t^n] = 0$ from this definition. Less trivial is

$$[t^n, \partial_m] = \delta^n_m N_{\{\phi\}} \tag{3.13}$$

$N_{\{\phi\}}$ counts the jet variables $\{\phi\}$. The equation holds for linear polynomials, i.e. for the jet variables and coordinates and differentials, and extends to arbitrary polynomials because both sides of this equation satisfy the Leibniz rule.

To determine the cohomology of $d = dx^m \partial_m$ we consider separately forms ω with a fixed form degree p

$$N_{dx} = dx^m \frac{\partial}{\partial(dx^m)} \quad N_{dx}\omega = p\,\omega \tag{3.14}$$

which are homogeneous of degree N in $\{\phi\}$. We assume $N > 0$, the case $N = 0$ is covered by Poincaré's lemma (theorem 3.2).

Consider the operation

$$b = t^m \frac{\partial}{\partial(dx^m)} \tag{3.15}$$

and calculate its anticommutator with the exterior derivative d as an exercise in graded commutators:

$$\begin{aligned}\{b,d\} &= \{t^m\frac{\partial}{\partial(dx^m)},dx^n\}\partial_n - dx^n[t^m\frac{\partial}{\partial(dx^m)},\partial_n] \\ &= t^m{\delta_m}^n\partial_n - dx^n{\delta_n}^m N\frac{\partial}{\partial dx^m} \\ &= \partial_n t^n + {\delta_n}^n N - N\,N_{dx}\ .\end{aligned} \tag{3.16}$$

So we get

$$\{d,b\} = N(D-N_{dx}) + P_1\ . \tag{3.17}$$

$D = \delta_n^n$ is the dimension of the manifold, the operator P_1 is given by

$$P_1 = \partial_k t^k\ . \tag{3.18}$$

Consider more generally the operations P_n

$$P_n = \partial_{k_1}\ldots\partial_{k_n}t^{k_1}\ldots t^{k_n} \tag{3.19}$$

which take away n derivatives and redistribute them afterwards. For each polynomial ω in the jet variables there exists a $\bar{n}(\omega)$ such that

$$P_n\omega = 0\ \forall n \geq \bar{n}(\omega) \tag{3.20}$$

because each monomial of ω has a bounded number of derivatives. Using the commutation relation (3.13) one proves the recursion relation

$$P_1P_k = P_{k+1} + kNP_k \tag{3.21}$$

which can be used iteratively to express P_k in terms of P_1 and N

$$P_k = \prod_{l=0}^{k-1}(P_1 - lN)\ . \tag{3.22}$$

Using the argument (3.6) that a nilpotent operation commutes with all its anticommutators we conclude from (3.17)

$$[d, N(D-N_{dx}) + P_1] = 0\ . \tag{3.23}$$

Therefore $d\omega = 0$ implies $d(P_1\omega) = 0$ and from (3.22) we conclude $d(P_k\omega) = 0$ by induction. We use the relation (3.17) to express these closed forms $P_k\omega$ as exact forms up to terms $P_{k+1}\omega$.

$$
\begin{aligned}
d(b\omega) &= P_1\omega + N(D-p)\omega \\
d(bP_k\omega) &= P_1 P_k\omega + N(D-p)P_k\omega \\
&= P_{k+1}\omega + kNP_k\omega + N(D-p)P_k\omega \\
d(bP_k\omega) &= P_{k+1}\omega + N(D-p+k)P_k\omega \quad k = 0,1,\ldots
\end{aligned} \tag{3.24}
$$

If $p < D$ then we can solve for ω in terms of exact forms $d(b\omega)$ and $P_1\omega$ which can be expressed as exact form and a term $P_2\omega$ and so on. This recursion terminates because $P_n\omega = 0 \; \forall n \geq \bar{n}(\omega)$ (3.20). Explicitly we have for $p < D$ and $N > 0$:

$$
d\omega = 0 \Rightarrow \quad \omega = d\left(b \sum_{k=0}^{\bar{n}(\omega)} \frac{(-)^k}{N^{k+1}} \frac{(D-p-1)!}{(D-p+k)!} P_k\omega \right) = d\eta \; . \tag{3.25}
$$

To complete the investigation of the cohomology of d we have to consider volume forms $\omega = \mathcal{L} d^D x$. We treat separately pieces $\mathcal{L}_N$ which are homogeneous of degree $N > 0$ in the jet variables $\{\phi\}$. These pieces can be written as

$$
\begin{aligned}
N\mathcal{L}_N &= \phi^i \frac{\partial \mathcal{L}_N}{\partial \phi^i} + \partial_m \phi^i \frac{\partial \mathcal{L}_N}{\partial(\partial_m \phi^i)} + \ldots \\
&= \phi^i \frac{\hat{\partial} \mathcal{L}_N}{\hat{\partial} \phi^i} + \partial_m X_N^m \quad X_N^m = \phi^i \frac{\partial \mathcal{L}_N}{\partial(\partial_m \phi^i)} + \ldots \; .
\end{aligned} \tag{3.26}
$$

Here we use the notation

$$
\frac{\hat{\partial} \mathcal{L}}{\hat{\partial} \phi^i} = \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_m \frac{\partial \mathcal{L}}{\partial(\partial_m \phi^i)} + \ldots \tag{3.27}
$$

for the Euler derivative of the Lagrange density with respect to ϕ^i. The dots denote terms which come from higher derivatives. The derivation of (3.26) is analogous to the derivation of the Euler Lagrange equations from the action principle. Eq.(3.26) implies that the volume form $\omega_N = \mathcal{L}_N d^D x$ is an exact term and a piece proportional to the Euler derivatives

$$
\mathcal{L}_N d^D x = \frac{1}{N} \phi^i \frac{\hat{\partial} \mathcal{L}_N}{\hat{\partial} \phi^i} d^D x + d\left(\frac{1}{N} X_N^m \frac{\partial}{\partial(dx^m)} d^D x \right) \tag{3.28}
$$

If we combine this equation with Poincaré's lemma (theorem 3.2) and with (3.25), combine terms with different degrees of homogeneity N and different form degree p we obtain the Algebraic Poincaré Lemma for forms of the coordinates, differentials and jet variables

Theorem 3.3 *Algebraic Poincaré Lemma*

$$d\omega(x, dx, \{\phi\}) = 0 \Leftrightarrow \omega(x, dx, \{\phi\}) = const + d\eta(x, dx, \{\phi\}) + \mathcal{L}(x, \{\phi\})d^D x \tag{3.29}$$

The Lagrange form $\mathcal{L}(x, \{\phi\})d^D x$ is trivial, i.e. of the form $d\eta$, if and only if its Euler derivatives with respect to all fields vanish identically in the fields.

The Algebraic Poincaré Lemma does not hold if the base manifold is not starshaped or if the fields ϕ take values in a topologically nontrivial target space. In these cases the operations $\delta = x\frac{\partial}{\partial(dx)}$ and $b = t^n\frac{\partial}{\partial(dx^n)}$ cannot be defined because a relation like $x \cong x + 2\pi$, which holds for the coordinates on a circle, would lead to the contradiction $0 \cong 2\pi\frac{\partial}{\partial(dx)}$. Here we restrict our investigations to topologically trivial base manifolds and topologically trivial target spaces. It is the topology of the invariance groups and the Lagrangean solutions in the Algebraic Poincaré lemma which give rise to a nontrivial cohomology of the exterior derivative d and the BRS transformation s.

The Algebraic Poincaré lemma is modified if the jet space contains in addition variables which are space time constants. This occurs for example if one treats rigid transformations as BRS transformations with constant ghosts C, i.e. $\partial_m C = 0$. If these ghosts occur as variables in forms ω then they are not counted by the number operators N which have been used in the proof of the Algebraic Poincaré Lemma and can appear as variables in $const = f(C)$, in η and in $\mathcal{L}$.

We are now prepared to investigate the relative cohomology and derive the so called descent equations. We recall that we deal with two nilpotent derivatives, the exterior derivative d and the BRS transformation s, which anticommute which each other

$$d^2 = 0 \quad s^2 = 0 \quad \{s, d\} = 0\ . \tag{3.30}$$

s leaves the form degree N_{dx} invariant, d raises it by 1

$$[N_{dx}, s] = 0 \quad [N_{dx}, d] = d\ . \tag{3.31}$$

We consider the equation

$$s\,\omega_D + d\,\omega_{D-1} = 0 \quad \omega_D \mod (s\eta_D + d\eta_{D-1})\ . \tag{3.32}$$

The subscript denotes the form degree. The relative cohomology (3.32) relates forms of different ghost number

$$gh(\omega_D) = gh(\omega_{D-1}) - 1 = gh(\eta_D) + 1 = gh(\eta_{D-1}) \ . \tag{3.33}$$

Let us derive the descent equations as a necessary consequence of (3.32). We apply s and use (3.30)

$$0 = s(s\,\omega_D + d\,\omega_{D-1}) = sd\,\omega_{D-1} = d(-s\,\omega_{D-1}) \ . \tag{3.34}$$

By the Algebraic Poincaré Lemma (3.3) $-s\,\omega_{D-1}$ is of the form $const + d\eta(\{\phi\}) + \mathcal{L}(\{\phi\})d^Dx$. The piece $\mathcal{L}(\{\phi\})d^Dx$ has to vanish because ω_{D-1} has form degree $D-1$ and if $D > 1$ then also the piece $const$ vanishes because ω_{D-1} contains $D-1 > 0$ differentials and is not constant. Therefore we conclude

$$s\,\omega_{D-1} + d\,\omega_{D-2} = 0 \quad \omega_{D-1} \mod (s\eta_{D-1} + d\eta_{D-2}) \tag{3.35}$$

where we denoted η by ω_{D-2} to indicate its form degree. Adding to ω_{D-1} a piece of the form $s\eta_{D-1} + d\eta_{D-2}$ changes ω_D only within its class of equivalent representatives. Therefore ω_{D-1} is naturally a representative of an equivalence class. From (3.32) we have derived (3.35) which is nothing but (3.32) with form degree lowered by 1. Iterating the arguments we lower the form degree step by step and obtain the descent equations

$$s\,\omega_i + d\,\omega_{i-1} = 0 \quad i = D, D-1, \ldots, 1 \quad \omega_i \mod (s\eta_i + d\eta_{i-1}) \tag{3.36}$$

until the form degree drops to zero. It cannot become negative. For $i = 0$ one has

$$s\,\omega_0 = 0 \quad \omega_0 \mod s\eta_0 \ . \tag{3.37}$$

A more careful application of the Algebraic Poincaré Lemma would only have allowed to conclude

$$s\ \omega_0 = const \ .$$

If, however, the BRS transformation is not spontaneously broken i.e. if $s\phi_{|(\phi=0)} = 0$ then $s\ \omega_0$ has to vanish. This follows most easily if one evaluates both sides of $s\ \omega_0 = const$ for vanishing fields. We assume for the following that the BRS transformations are not spontaneously broken. We will exclude from our considerations also spontaneously broken rigid symmetries. There we cannot apply these arguments because then $s\phi_{|(\phi=0)} = C$ gives ghosts which are space time constant and one can have $s\,\omega_0 = const = f(C) \neq 0$.

Actually the descent equations (3.36,3.37) are just another cohomological equation for a nilpotent operator $\tilde{s}$ and a form $\tilde{\omega}$

$$\tilde{s} = d + s \quad \tilde{s}^2 = 0 \tag{3.38}$$

$$\tilde{\omega} = \sum_{i=0}^{D} \omega_i \tag{3.39}$$

$$\tilde{s}\,\tilde{\omega} = 0 \quad \tilde{\omega} \bmod \tilde{s}\tilde{\eta}\ . \tag{3.40}$$

The fact that $\tilde{s}$ is nilpotent follows from (3.30). The descent equations (3.36, 3.37) imply $\tilde{s}\,\tilde{\omega} = 0$. The equivalence class of $\tilde{\omega}$ is given by $\tilde{s}(\sum_i \eta_i)$. So (3.40) is a consequence of the descent equations. On the other hand if (3.38) holds then the equation (3.40) implies the descent equations. This follows if one splits $\tilde{s}$, $\tilde{\omega}$ and $\tilde{\eta}$ with respect to the form degree (3.31).

Let us formulate this result as

Theorem 3.4
If $\tilde{s} = s + d$ is a sum of two fermionic operators where s preserves the form degree and d raises it by one, then $\tilde{s}$ is nilpotent if and only if s and d are nilpotent and anticommute.

Each solution $(\omega_0, \dots, \omega_D)$, $\omega_i \bmod s\ \eta_i + d\ \eta_{i-1}$ of the descent equations (3.36, 3.37) with nilpotent, anticommuting operators s and d corresponds one to one to an element $\tilde{\omega}$ of the cohomology $H(\tilde{s}) = \{\tilde{\omega} :\ \tilde{s}\,\tilde{\omega} = 0 \quad \tilde{\omega} \bmod \tilde{s}\tilde{\eta})\}$. ω_i are the parts of $\tilde{\omega}$ with form degree i.

The formulation of the descent equations as a cohomological problem of the operator $\tilde{s}$ has several virtues. The solutions to $\tilde{s}\,\tilde{\omega} = 0$ can obviously be multiplied to obtain further solutions. Phrased mathematically they form an algebra, not just a vector space. More importantly for the BRS operator in gravitational Yang Mills theories we will find that the equation $\tilde{s}\,\tilde{\omega} = 0$ can be cast into the form $s\,\omega = 0$ by a change of variables, where s is the original BRS operator. This equation has to be solved anyhow as part of the descent equations. Once one has solved it one can recover the complete solution of the descent equations, in particular one can read off ω_D as the D form part of $\tilde{\omega}$. These virtues justify to consider with $\tilde{\omega}$ a sum of forms of different form degrees which in traditional eyes would be considered to add peaches and apples.

As the last subject of this chapter we study the action of a nilpotent derivative d on a product $A = A_1 \times A_2$ of vectorspaces (algebras) which are separately invariant under d

$$dA_1 \subset A_1 \quad dA_2 \subset A_2 \ . \tag{3.41}$$

Künneth's theorem states that the cohomology $H(A, d)$ of d acting on A is given by the product of the cohomology $H(A_1, d)$ of d acting on A_1 and $H(A_2, d)$ of d acting on A_2.

Theorem 3.5 *Künneth-formula*
Let $d = d_1 + d_2$ be a sum of nilpotent differential operators which leave their vectorspaces A_1 and A_2 invariant

$$d_1 A_1 \subset A_1 \quad d_2 A_2 \subset A_2 \tag{3.42}$$

and which are defined on the product $A = A_1 \times A_2$ by the Leibniz rule

$$d_1(kl) = (d_1 k)l \quad d_2(kl) = (-)^{|k|} k(d_2 l) \quad \forall k \in A_1,\, l \in A_2 \ . \tag{3.43}$$

Then the cohomology $H(A, d)$ of d acting on A is the product of the cohomologies of d_1 acting on A_1 and d_2 acting on A_2

$$H(A_1 \times A_2, d_1 + d_2) = H(A_1, d_1) \times H(A_2, d_2) \tag{3.44}$$

To prove the theorem we consider an element $f \in H(d)$

$$f = \sum_i k_i l_i \tag{3.45}$$

given as a sum of products of elements $k_i \in A_1$ and $l_i \in A_2$. Without loss of generality we assume that the elements k_i are taken from a basis of A_1 and the elements l_i are taken from a basis of A_2.

$$\sum c_i k_i \;=\; 0 \Leftrightarrow c_i = 0 \;\forall\; i \tag{3.46}$$

$$\sum c_i l_i \;=\; 0 \Leftrightarrow c_i = 0 \;\forall\; i \tag{3.47}$$

Otherwise one has a relation like $l_1 = \sum_i' \alpha_i l_i$ or $k_1 = \sum_i' \beta_i k_i$, where $\sum'$ does not contain $i = 1$, and can rewrite f with fewer terms $f = \sum_i' (k_i + \alpha_i k_1) \cdot l_i$ or $f = \sum_i' k_i \cdot (l_i + \beta_i l_1)$. We can even choose $f \in H(d)$ in such a manner that the elements k_i are taken from a basis of a complement to the space $d_1 A_1$. In other words we can choose f such that no linear combination of the elements k_i combines to a d_1-exact form.

$$\sum_i c_i k_i = d_1 g \Leftrightarrow d_1 g = 0 = c_i \ \forall \ i \tag{3.48}$$

Otherwise we have a relation like $k_1 = \sum_i' \beta_i k_i + d_1 \kappa$, where $\sum'$ does not contain $i = 1$, and we can rewrite $f \in H(d)$ up to an irrelevant piece $d(\kappa \ l_1)$ $f = \sum_i' k_i \cdot (l_i + \beta_i l_1) - (-)^{|\kappa|} \kappa \ d_2 l_1 + d(\kappa \ l_1)$ with elements $k_i' = \kappa, k_2, \ldots$. We can iterate this argument until no linear combination of the elements k_i' combines to a d_1-exact form.

By assumption f solves $df = 0$ which implies

$$\sum_i \Big((d_1 k_i) l_i + (-)^{k_i} k_i (d_2 l_i) \Big) = 0 \ . \tag{3.49}$$

In this sum $\sum_i (d_1 k_i) l_i$ and $\sum_i (-)^{k_i} k_i (d_2 l_i)$ have to vanish separately because the elements k_i are linearly independent from the elements $d_1 k_i \in d_1 A_1$. $\sum_i (d_1 k_i) l_i = 0$, however, implies

$$d_1 k_i = 0 \tag{3.50}$$

because the elements l_i are linearly independent and $\sum_i (-)^{k_i} k_i (d_2 l_i) = 0$ leads to

$$d_2 l_i = 0 \tag{3.51}$$

analogously. So we have shown

$$df = 0 \Rightarrow f = \sum_i k_i l_i + d\chi \text{ where } d_1 k_i = 0 = d_2 l_i \ \forall i \ . \tag{3.52}$$

Changing k_i and l_i within their equivalence class $k_i \bmod d_1 \kappa_i$ and $l_i \bmod d_2 \lambda_i$ does not change the equivalence class $f \bmod d\chi$:

$$\sum_i (k_i + d_1 \kappa_i)(l_i + d_2 \lambda_i) = \sum_i k_i l_i + d \ \sum_i \Big(\kappa_i (l_i + d_2 \lambda_i) + (-)^{k_i} k_i \lambda_i \Big) \tag{3.53}$$

Therefore $H(A, d)$ is contained in $H_1(A_1, d_1) \times H_2(A_2, d_2)$. The inclusion $H_1(A_1, d_1) \times H_2(A_2, d_2) \subset H(A, d)$ is trivial. This concludes the proof of the theorem.

Chapter 4

BRS algebra of Gravitational Yang Mills Theories

Gauge theories such as gravitational Yang Mills theories rely on tensor analysis [1]. The set of tensors is a subalgebra of the polynomials in the jet variables.

$$(Tensors) \subset (Polynomials(\phi, \partial\phi, \partial\partial\phi, \ldots)) \tag{4.1}$$

The covariant operations Δ_M which are used in tensor analysis

$$\Delta_M : (Tensors) \to (Tensors) \tag{4.2}$$

map tensors to tensors and satisfy the Leibniz rule (2.8). These covariant operations have a basis consisting of the covariant space time derivatives $D_a,\ a = 0, \ldots, D-1$ and spin and isospin transformations δ_I, which correspond to a basis of the Lie algebra of the Lorentz group and of the gauge group, and - if one considers supergravitational theories - the covariant spinor derivatives $D_\alpha,\ D_{\dot\alpha}$.

$$(\Delta_M) = (D_a,\ \delta_I,\ D_\alpha, D_{\dot\alpha}) \tag{4.3}$$

The space of covariant operations is closed with respect to graded commutation

$$[\Delta_M, \Delta_N] := \Delta_M\Delta_N - (-)^{MN}\Delta_N\Delta_M = \mathcal{F}_{MN}{}^K\Delta_K \ . \tag{4.4}$$

The structure functions $\mathcal{F}_{MN}{}^K$ are also tensors. Some of these structure functions have purely numerical values as for example the structure constants

[1] This chapter is nothing but a slightly simplified version of [6].

of the spin and isospin Lie algebra

$$[\delta_I, \delta_J] = f_{IJ}{}^K \delta_K \tag{4.5}$$

or the matrix elements of representations of the Lorentz algebra

$$[\delta_{[a,b]}, D_c] = -(G_{[a,b]})_c{}^d D_d = \eta_{ca} D_b - \eta_{cb} D_a \tag{4.6}$$

or constant torsion in superspace. Other components of the tensors $\mathcal{F}_{MN}{}^K$ are given by the Riemann curvature, the Yang Mills field strength and in supergravity the Rarita Schwinger field strength and auxiliary fields of the supergravitational multiplet. We use the word field strength also to denote collectively the Riemann curvature and the Yang Mills field strength.

The commutator algebra (4.4) implies a generalized Jacobi identity

$$\sum_{cyclic(MNP)} sign(MNP)[\Delta_M, [\Delta_N, \Delta_P]] = 0 \tag{4.7}$$

which is the first Bianchi identity for the structure functions $\mathcal{F}_{MN}{}^K$

$$\sum_{cyclic(MNP)} sign(MNP)(\Delta_M \mathcal{F}_{NP}{}^K - \mathcal{F}_{MN}{}^L \mathcal{F}_{LP}{}^K) = 0\ . \tag{4.8}$$

It involves the sum over the cyclic permutations of M, N, P. If the algebra contains fermionic covariant derivatives then there are additional signs $sign(MNP)$ for each odd permutation of indices of fermionic covariant derivatives.

The covariant operations are not defined on arbitrary polynomials of the jet variables. In particular one cannot realize the commutator algebra (4.4) on connections, on ghosts or on auxiliary fields.

To keep the discussion simple we will not consider fermionic covariant derivatives in the following. Then the commutator algebra (4.4) has more specifically the structure

$$\begin{aligned}
[D_a, D_b] &= -T_{ab}{}^c D_c - F_{ab}{}^I \delta_I \quad \text{torsion and field strength} && (4.9)\\
[\delta_I, D_a] &= -G_{I\,a}{}^b D_b \quad \text{representation matrices} && (4.10)\\
[\delta_I, \delta_J] &= f_{IJ}{}^K \delta_K \quad \text{structure constants .} && (4.11)
\end{aligned}$$

We will simplify this algebra even more and choose the spin connection by the requirement that the torsion vanishes.

The field content ϕ of gravitational Yang Mills theories consists of ghosts C^N, antighosts $\bar{C}^N$, auxiliary fields B^N, gauge potentials (connections) ${A_m}^N$ $m = 0, \ldots, D-1$ and elementary tensor fields T. The gauge potentials, ghosts and auxiliary fields are real and correspond to a basis of the covariant operations Δ_M, i.e. there are connections, ghosts and auxiliary fields for translations (covariant space time derivatives), for Lorentz transformations and for isospin transformations. Matter fields are tensors and denoted by T.

$$\phi = \{C^N, \bar{C^N}, B^N, {A_m}^N, T\} \tag{4.12}$$

We define the BRS transformation on the antighosts and the auxiliary fields by

$$s\bar{C}^N = iB^N \quad sB^N = 0 \ . \tag{4.13}$$

The BRS transformation of tensors is given by a sum of covariant operations with ghosts as coefficients [6]

$$sT = C^N \Delta_N T \ . \tag{4.14}$$

Moreover we consider the exterior derivative $d = dx^m \partial_m$. We require that the action of partial derivatives ∂_m on tensors can be expressed as a linear combination of covariant operations. The expansion coefficients introduced in this way turn out to be the connections or gauge fields.

$$dT = dx^m \partial_m T = dx^m {A_m}^N \Delta_N T = A^N \Delta_N T \ . \tag{4.15}$$

If we use the connection one forms

$$A^N = dx^m {A_m}^N \tag{4.16}$$

introduced in the last equation then s and d act on tensors in a strikingly similar way: sT contains ghosts C^N where dT contains (composite) connection one forms A^N.

Let us check that (4.15) is nothing but the usual definition of covariant derivatives. We spell out the sum over covariant operations and denote the connection ${A_m}^a$ by ${e_m}^a$, the vielbein.

$$\partial_m = {A_m}^N \Delta_M = {e_m}^a D_a + {A_m}^I \delta_I \tag{4.17}$$

If the vielbein has an inverse ${E_a}^m$, which we take for granted like the rest of the world,

$${e_m}^a {E_a}^n = {\delta_m}^n \tag{4.18}$$

then we can solve for the covariant space time derivative and obtain the usual expression

$$D_a = E_a{}^m(\partial_m - A_m{}^I\delta_I) \ . \tag{4.19}$$

We require that s and d anticommute and be nilpotent (3.30). This fixes the BRS transformation of the ghosts and the connection and identifies the curvature and field strength. In particular $s^2 = 0$ implies

$$0 = s^2 T = s(C^N\Delta_N T) = (sC^N)\Delta_N T - C^N s(\Delta_N T) \ . \tag{4.20}$$

$\Delta_N T$ is a tensor so

$$C^N s(\Delta_N T) = C^N C^M \Delta_M \Delta_N T = \frac{1}{2} C^N C^M [\Delta_M, \Delta_N] T \ . \tag{4.21}$$

The commutator is given by the algebra (4.4) and we conclude

$$0 = (sC^N - \frac{1}{2} C^K C^L \mathcal{F}_{LK}{}^N)\Delta_N T \quad \forall T \ . \tag{4.22}$$

This means that the operation $(sC^N - \frac{1}{2}C^K C^L \mathcal{F}_{LK}{}^N)\Delta_N$ vanishes. The covariant operations Δ_N are understood to be linearly independent. Therefore sC^N is fixed.

$$sC^N = \frac{1}{2} C^K C^L \mathcal{F}_{LK}{}^N \tag{4.23}$$

The BRS transformation of the ghosts is given by a polynomial which is quadratic in the ghosts with expansion coefficients given by the structure functions $\mathcal{F}_{LK}{}^N$. s transforms the algebra of polynomials generated by ghosts (not derivatives of ghosts) and tensors into itself (4.14, 4.23).

The requirement that s and d anticommute fixes the transformation of the connection.

$$\begin{aligned} 0 &= \{s, d\}T = s(A^N\Delta_N T) + d(C^N\Delta_N T) \\ &= (sA^N)\Delta_N T - A^N C^M \Delta_M \Delta_N T + (dC^N)\Delta_N T - C^N A^M \Delta_M \Delta_N T \\ &= (sA^N + dC^N - A^K C^L \mathcal{F}_{LK}{}^N)\Delta_N T \quad \forall T \end{aligned}$$

So we conclude

$$sA^N = -dC^N + A^K C^L \mathcal{F}_{LK}{}^N \tag{4.24}$$

for the connection one form A^N. For the gauge field $A_m{}^N$ we obtain [2]

$$sA_m{}^N = \partial_m C^N - A_m{}^K C^L \mathcal{F}_{LK}{}^N \tag{4.25}$$

[2] Anticommuting dx^m through s changes the signs.

The BRS transformation of the connection contains the characteristic inhomogeneous piece $\partial_m C^N$.

$d^2 = 0$ identifies the field strength as curl of the connection.

$$\begin{aligned}
0 &= d^2 T = dx^m dx^n \partial_m \partial_n T = dx^m dx^n \partial_m ({A_n}^N \Delta_N T) \\
&= dx^m dx^n \Big[(\partial_m {A_n}^N) \Delta_N T + {A_n}^N \partial_m (\Delta_N T) \Big] \\
&= dx^m dx^n \Big[(\partial_m {A_n}^N) \Delta_N T + {A_n}^N {A_m}^M \Delta_M \Delta_N T \Big]
\end{aligned}$$

Therefore

$$0 = \partial_m {A_n}^K - \partial_n {A_m}^K + {A_m}^M {A_n}^N {\mathcal{F}_{MN}}^K \tag{4.26}$$

We split the summation over $M\,N$, employ the definition of the vielbein

$$\begin{aligned}
0 = \partial_m {A_n}^K - \partial_n {A_m}^K &+ {e_m}^a {e_n}^b {\mathcal{F}_{ab}}^K + {e_m}^a {A_n}^I {\mathcal{F}_{aI}}^K \\
&+ {A_m}^I {e_n}^a {\mathcal{F}_{Ia}}^K + {A_m}^I {A_n}^J {\mathcal{F}_{IJ}}^K
\end{aligned}$$

and solve for ${\mathcal{F}_{ab}}^K$, $K \in (a, [a,b], i)$.

$$\begin{aligned}
{\mathcal{F}_{ab}}^K &= -{E_a}^m {E_b}^n \Big(\partial_m {A_n}^K - \partial_n {A_m}^K + {e_m}^c {A_n}^I {\mathcal{F}_{cI}}^K \\
&\qquad + {A_m}^I {e_n}^c {\mathcal{F}_{Ic}}^K + {A_m}^I {A_n}^J {\mathcal{F}_{IJ}}^K \Big)
\end{aligned} \tag{4.27}$$

The structure functions

$${F_{ab}}^K = -{\mathcal{F}_{ab}}^K \tag{4.28}$$

are the torsion[3] ${T_{ab}}^c$, if $K = c$ corresponds to space-time translations, the Riemann curvature ${R_{ab}}^{cd}$, if $K = [cd]$ corresponds to Lorentz transformations, and the Yang Mills field strength ${F_{ab}}^i$, if $K = i$ ranges over isospin indices. The formula applies, however, also to supergravity, which has a more complicated algebra (4.4). It allows in a surprisingly simple way to identify the Rarita Schwinger field strength ${\Psi_{ab}}^\alpha$ when $K = \alpha$ corresponds to supersymmetry transformations.

The formulas

$$sT = C^N \Delta_N T \quad dT = A^N \Delta_N T \tag{4.29}$$

for the nilpotent, anticommuting operations s and d not only encrypt the basic geometric structures. They allow also to prove easily that the cohomologies of s and $s + d$ acting on tensors and ghosts (not on connections,

[3] We require ${T_{ab}}^c = 0$ which amounts to a choice of the spin connection.

derivatives of ghosts, auxiliary fields and antighosts) differ only by a change of variables. Let us inspect $(s+d)T$.

$$\tilde{s}T = (s+d)T = (C^N + A^N)\Delta_N T = \tilde{C}^N \Delta_N T \tag{4.30}$$

where

$$\tilde{C}^N = C^N + A^N = C^N + dx^m A_m{}^N \tag{4.31}$$

The $\tilde{s}$-transformation of tensors is obtained from the s-transformation by replacing the ghosts C by $\tilde{C}$.

The $\tilde{s}$-transformation of $\tilde{C}$ follows from $\tilde{s}^2 = 0$ and the transformation of tensors (4.30) by the same arguments which determined sC from $s^2 = 0$ and from (4.14) and led to (4.23). So we obtain

$$\tilde{s}\tilde{C}^N = \frac{1}{2}\tilde{C}^K \tilde{C}^L \mathcal{F}_{LK}{}^N \ . \tag{4.32}$$

This is just the tilded version of (4.23). Define the map ρ to substitute ghosts C by $\tilde{C}$ in arbitrary polynomials P of ghosts and tensors.

$$P(\tilde{C},T) = \rho \circ P(C,T) \quad \rho = \exp(A\frac{\partial}{\partial C}) \tag{4.33}$$

Taken together (4.30, 4.32) and (4.14, 4.23) imply

$$\tilde{s} \circ \rho = \rho \circ s \tag{4.34}$$

From this equation one easily concludes the following theorem.

Theorem 4.1
Let s be the BRS operation in gravitational theories. A form $\omega(C,T)$ solves $s\,\omega(C,T) = 0$ if and only if $\omega(\tilde{C},T)$ solves $\tilde{s}\,\omega(\tilde{C},T) = 0$.

If we combine this result with theorem (3.4) then the solutions to the descent equations can be found from the cohomology of s if we can restrict the jet variables to ghosts and tensors.

Actually we can make this restriction if the base manifold and the target space of the fields have trivial topology. This follows because the algebra of jet variables is a product of algebras on which $\tilde{s}$ acts separately. Using Künneth's formula (theorem 3.5) we can then determine nontrivial Lagrange densities and anomaly candidates as solutions of $\tilde{s}\omega(\tilde{C},T) = 0$ and by determination of the cohomology of d in the base manifold and of $\tilde{s}$ in the target manifold.

To establish this result we prove the following theorem:

Theorem 4.2
The algebra A of series in x^m and the fields ϕ (4.12) and of polynomials in dx^m and the partial derivatives of the fields is a product algebra

$$A = A_{\tilde{C},T} \times \prod_l A_{u_l, \tilde{s}u_l} \tag{4.35}$$

where the variables u_l are given by the following set

$$(u_l) = \Big(x^m, e_m{}^a, A_m{}^I, \partial_{(m_k} \dots \partial_{m_1} A_{m_0)}{}^N, \bar{C}^N, \partial_{m_k} \dots \partial_{m_1} \bar{C}^N\Big) \tag{4.36}$$

for $k = 1, 2, \dots$. $\tilde{s}$ acts on each factor A_i separately $\tilde{s}\, A_i \subset A_i$.

In (4.35) the braces around indices denote symmetrization. The subscript of the algebras denote the generating elements e.g. $A_{e_m{}^a, \tilde{s}e_m{}^a}$ is the algebra of series in the vielbein $e_m{}^a$ and in $\tilde{s}e_m{}^a$. $\tilde{s}$ leaves $A_{u_l, \tilde{s}u_l}$ invariant by construction because of $\tilde{s}^2 = 0$.

To prove the theorem we inspect the variables u_l and $\tilde{s}u_l$ to lowest order in the differentials and fields. [4] In lowest order the variables $\tilde{s}u_l$ are given by

$$(\tilde{s}\, u_l) \approx \Big(dx^m, \partial_m C^a, \partial_m C^I, \partial_{m_k} \dots \partial_{m_0} C^N, iB^N, i\partial_{m_k} \dots \partial_{m_1} B^N\Big) \tag{4.37}$$

We recall that to lowest order the covariant derivatives of the field strengths are given by

$$(T) \approx \Big(E_{a_k}{}^{m_k} \dots E_{a_0}{}^{m_0} \partial_{m_k} \dots \partial_{[m_1} A_{m_0]}{}^N,\ k = 1, 2, \dots\Big)\ . \tag{4.38}$$

The brackets denote antisymmetrization of the enclosed indices. In linearized order we find all jet variables as linear combinations of the variables $\tilde{C}, T, u_l$ and $\tilde{s}u_l$: the symmetrized derivatives of the connections belong to (u_l), the antisymmetrized derivatives of the connections belong to the field strengths listed as T. The derivatives of the vielbein are slightly tricky. The symmetrized derivatives are contained in $\partial_{(m_k} \dots \partial_{m_1} A_{m_0)}{}^N$ for $N = a$. The antisymmetrized derivatives are in one to one correspondence to the spin connection $\omega_{k\ [a,b]}$ ($A_k{}^I$ for $I = [ab]$). We choose the spin connection $\omega_{ma}{}^b$ and a symmetric affine connection $\Gamma_{mn}{}^l = \Gamma_{nm}{}^l$ not to be elementary variables and determine them from the equations $D_a e_n{}^b = 0$ and $T_{ab}{}^c = 0$. This

[4] We do not count powers of the vielbein $e_m{}^a$ or its inverse. Derivatives of the vielbein, however, are counted.

choice does not restrict the validity of our investigation because a different choice amounts only to the introduction of additional tensor fields.

$$\partial_m e_n{}^c - \partial_n e_m{}^c = \omega_{ma}{}^c e_n{}^a - \omega_{na}{}^c e_m{}^a \tag{4.39}$$

$$\begin{aligned}\omega_{k\ [a,b]} \;=\;& \frac{1}{2}\Big(E_a{}^m \eta_{bc}(\partial_k e_m{}^c - \partial_k e_m{}^c) - E_b{}^m \eta_{ac}(\partial_k e_m{}^c - \partial_k e_m{}^c) - \\ & - E_a{}^m E_b{}^n(\partial_m e_n{}^c - \partial_n e_m{}^c) e_k{}^d \eta_{cd}\Big)\end{aligned} \tag{4.40}$$

We conclude that the transformation of the jet variables to the variables $\left(\tilde{C}, T, u_l, \tilde{s}u_l\right)$ has the structure

$$\phi'^i = M^i{}_j \phi^j + O^i(\phi^2) \tag{4.41}$$

where M is an invertible matrix.

$$\phi^i = M^{-1}{}^i{}_j(\phi'^j - O^j(\phi^2)) \tag{4.42}$$

Consider an element of the algebra A generated by the jet variables. We show that it can be written as an element of $A_{\tilde{C},T} \times \prod_l A_{u_l,\tilde{s}u_l}$. This holds trivially for the variables x^m and dx^m which coincide with x^m and $\tilde{s}x^m$. For the remaining variables we neglect in a first step all differentials in (4.42). Concentrate on the terms with the highest derivatives in the expression for each ϕ^i. The terms $O(\phi^2)$ contain only lower derivatives. Therefore, using (4.42), we can recursively substitute in a polynomial in ϕ the highest derivative terms by ϕ'^i variables. This changes the expression for the lower derivative terms. Then substitute the second highest derivative terms. They can be expressed in terms of ϕ'^i with changed terms with third highest derivatives and so on. Therefore each polynomial in ϕ can be written in terms of ϕ'^i. In a second step we take into account the differentials which come into play because we use the variables $\tilde{s}u_l$ and therefore $O^i(\phi^2)$ contains also the variables dx^m combined also with higher derivatives than ϕ^i. Given an arbitrary differential form ω we apply our substitution procedure first to the zero form. It can be expressed as zero form in the variables ϕ'^i but the 1-form part has changed. The substitution procedure applied to this 1-form part expresses it in terms of ϕ'^i and changes the 2-form and so on. We iterate the substitution until we reach $D+1$-forms which vanish. Then we have expressed the elements of the algebra A of the jet variables in terms of the product algebra $A_{\tilde{C},T} \times \prod_l A_{u_l,\tilde{s}u_l}$. This completes the proof of the theorem.□

By Künneth's theorem (theorem 3.5) the cohomology of $\tilde{s}$ acting on the algebra A of the jet variables is given by the product of the cohomologies of $\tilde{s}$ acting on the ghost tensor algebra $A_{\tilde{C},T}$ and on the algebras $A_{u_l,\tilde{s}u_l}$

$$H(A,\tilde{s}) = H(A_{\tilde{C},T},\tilde{s}) \times \prod_l H(A_{u_l,\tilde{s}u_l},\tilde{s}) \ . \qquad (4.43)$$

By the Basic Lemma (theorem 3.3) the cohomology of d acting on an algebra $A_{x,dx}$ of differential forms $f(x,dx)$ which depend on generating and independent variables x and dx is given by numbers f_0. Exchanging the names d by $\tilde{s}$ and x, dx by $u_l, \tilde{s}u_l$ one can copy the Basic Lemma and conclude that the cohomology $H(A_{u_l,\tilde{s}u_l},\tilde{s})$ is given by numbers. One can apply this argument if the variables u_l and $\tilde{s}u_l$ are independent and not subject to constraints.

Whether the variables $u_l, \tilde{s}u_l$ are subject to constraints is a matter of choice of the theory which one considers. This choice influences the cohomology. For example, one could require that two coordinates x^1 and x^2 satisfy $(x^1)^2 + (x^2)^2 = 1$ because one wants to consider a theory on a circle. Then the differential $d(\arctan\frac{y}{x}) = d\varphi$ is closed ($dd\varphi = 0$) but not exact, because the angle φ is not a function on the circle, $d\varphi$ is just a misleading notation for a one form which is not d of a function φ. In this example the periodic boundary condition $\varphi \sim \varphi + 2\pi$ gives rise to a nontrivial cohomology of d acting on φ and $d\varphi$. Nontrivial cohomologies also arise if the fields take values in nontrivial spaces. For example if in nonlinear sigma models one requires scalar fields ϕ^i to take values on a sphere $\sum_{i=1}^{n+1} {\phi^i}^2 = 1$ then the volume form $d^n\phi$ is nontrivial. More complicated is the case where scalar fields are restricted to take values in a general coset G/H. Also the relation

$$\det e_m{}^a \neq 0 \qquad (4.44)$$

restricts the vielbeine to take values in the group $GL(D)$ of invertible real $D \times D$ matrices. This group has a nontrivial cohomology.

For several reasons we choose to neglect the cohomologies coming from a nontrivial topology of the base manifold with coordinates x^m or the target space with coordinates ϕ or $e_m{}^a$.

We have to determine the cohomology of $\tilde{s}$ on the ghost tensor variables anyhow and start with this problem. To obtain the complete answer we can determine the cohomology of the base space and the target space in a second step which we postpone.

One can also legitimately argue that perturbation theory replaces fields by deviations from a ground state and thereby replaces the target space by its tangent space with a trivial cohomology.

Canonical quantization does not respect inequalites like $x \neq 0$. If there exists a conjugate variable p with $[x, p] = -i$ and if the unitary operators $U(y) = e^{iyp}$ exist for all real numbers y then the spectrum of x extends over the real line including $x = 0$. So how does one control the more complicated inequality $\det e_m{}^a \neq 0$ after quantization?

Whether one accepts these arguments is a matter of choice until the physical differences of different choices are calculated and tested in nature. We choose to investigate topologically trivial base manifolds and target spaces. We combine eq. (4.43) with theorem (3.4) and theorem (4.1) and conclude

Theorem 4.3

If the target space and the base manifold have trivial topology then the nontrivial solutions of the descent equations in gravitational theories are in one to one correspondence to the nontrivial solutions $\omega(C, T)$ of the equation $s\,\omega = 0$. The relative cohomology (3.32) is given by the D-form parts of the forms $\omega(C + A, T) \bmod \tilde{s}\eta$.

ω depends only on the ghosts, not on their derivatives. Therefore the ghost number of ω is bounded by the number of translation ghosts and the number of ghosts for spin and for isospin transformations $D + \frac{D(D-1)}{2} + \dim(G)$. If we take the D-form part of $\omega(C + A, T)$ then D differentials dx^m rather than ghosts have to be picked. Therefore the ghost number of nontrivial solutions of the relative cohomology is bounded by $\frac{D(D-1)}{2} + \dim(G)$. This argument, however, does not apply if there are commuting ghosts for supersymmetry transformations.

From this theorem one can conclude that in an appropriate basis of variables anomaly candidates can be chosen such that they contain no ghosts

$$C^m = C^a E_a{}^m \tag{4.45}$$

of coordinate transformations or in other words that coordinate transformations are not anomalous. This result holds if one uses the variables

$$\hat{C}^I = C^I - C^a E_a{}^m A_m{}^I \qquad \hat{C}^m = C^a E_a{}^m \tag{4.46}$$

as ghost fields. This choice is not very suitable if one wants to split the algebra of $\tilde{s}$ and therefore we have preferred not to work with $\hat{C}^I$. But this choice arises naturally if one enlarges the BRS transformation of Yang Mills theories to allow also general coordinate transformations. In our formulation the BRS transformation is given by

$$sT = C^N \Delta_N T = C^a E_a{}^m (\partial_m - A_m{}^I \delta_I) T + C^I \delta_I T. \tag{4.47}$$

In the basis of $\hat{C}^m$, $\hat{C}^I$ this is a shift term $C^m \partial_m T$ and the BRS transformation of a Yang Mills theory

$$sT = \hat{C}^m \partial_m T + \hat{C}^I \delta_I T \ . \tag{4.48}$$

The variables $\hat{C}^m$, $\hat{C}^I$ change very simply under the substitution of C by $C + A$.

$$\hat{C}^m (C + A) = \hat{C}^m + dx^m \quad \hat{C}^I (C + A) = \hat{C}^I \tag{4.49}$$

If one expresses a form $\omega(C+A, T)$ by ghost variables $\hat{C}^m$, $\hat{C}^I$ then ω depends on dx^m only via the combination $\hat{C}^m + dx^m$. The D form part ω_D originates from a coefficient function multiplying

$$(\hat{C}^1 + dx^1)(\hat{C}^2 + dx^2) \ldots (\hat{C}^D + dx^D) = (dx^1 dx^2 \ldots dx^D + \ldots) \ . \tag{4.50}$$

This coefficient function cannot contain a translation ghost $C^m = \hat{C}^m$ because $\hat{C}^m$ enters only in the combination $\hat{C}^m + dx^m$ and $D + 1$ factors of $\hat{C}^m + dx^m$ vanish.

Chapter 5

BRS cohomology on ghosts and tensors

In the last chapter the problem to determine Lagrange densities and anomaly candidates has been reduced to the calculation of the cohomology of s acting on tensors and ghosts. Let us recall this transformation s explicitly [1]

$$\begin{aligned} sT &= (C^a D_a + C^I \delta_I) T && (5.1) \\ sC^a &= C^I C^b G_{Ib}{}^a && (5.2) \\ sC^I &= -\frac{1}{2} C^K C^L f_{KL}{}^I + \frac{1}{2} C^a C^b F_{ab}{}^I \; . && (5.3) \end{aligned}$$

The BRS transformation

$$s = s_0 + s_1 + s_2 \qquad (5.4)$$

consists of a nilpotent part s_0

$$\begin{aligned} s_0 T &= C^I \delta_I T && (5.5) \\ s_0 C^a &= C^I G_{Ib}{}^a C^b && (5.6) \\ s_0 C^I &= -\frac{1}{2} C^K C^L f_{KL}{}^I \; , && (5.7) \end{aligned}$$

which does not increase the number of translation ghosts C^a, and of parts s_1

$$s_1 T = C^a D_a T \quad s_1 C^a = 0 \quad s_1 C^I = 0 \qquad (5.8)$$

[1] We use a spin connection which makes $T_{ab}{}^c$ vanish.

and s_2

$$s_2 T = 0 \quad s_2 C^a = 0 \quad s_2 C^I = \frac{1}{2} C^a C^b F_{ab}{}^I \ , \tag{5.9}$$

which increase the number of translation ghosts by 1 and 2. The fact that s_0^2 vanishes follows easily if one splits

$$s^2 = s_0^2 + \{s_0, s_1\} + (\{s_0, s_2\} + s_1^2) + \{s_1, s_2\} + s_2^2 = 0 \tag{5.10}$$

into pieces which raise the number of translation ghosts by 0,1,2,3,4. These different pieces vanish separately.

s_0 acts on tensors and ghosts exactly like the BRS transformation in Yang Mills theories – if one interprets the s_0 transformation of the translation ghosts C^a as the BRS transformation of an additional tensor.

Let us split each solution $\omega(C,T)$ of $s\omega = 0$ into pieces ω_n which are homogeneous of degree n in translation ghosts

$$\omega = \omega_{\underline{n}} + \sum_{n>\underline{n}} \omega_n + s\eta \ . \tag{5.11}$$

We call the pieces ω_n ghosts forms of degree n. Let us concentrate on the ghost form $\omega_{\underline{n}}$ with the lowest degree in C^a. It belongs to the s_0 cohomology, i.e. it satisfies

$$s_0 \omega_{\underline{n}} = 0 \quad \omega_{\underline{n}} \bmod s_0 \eta_{\underline{n}} \ . \tag{5.12}$$

The equation $s_0\omega_{\underline{n}} = 0$ is the piece with degree $\underline{n}$ in the equation $s\omega = 0$. A piece $s_0\eta_{\underline{n}}$ can be neglected because it is of the form $s\eta_{\underline{n}}$ up to pieces with higher degree in C^a which can be absorbed in a redefined sum $\sum_{n>\underline{n}} \omega_n$. Therefore to each element ω of the s cohomology there corresponds an element $\omega_{\underline{n}}$ of the s_0 cohomology. We choose η such that $\underline{n}$ becomes maximal. Then this correspondence is unique.

To determine ω we hunt down $\omega_{\underline{n}}$ and determine the s_0 cohomology. We proceed as in the derivation of the Basic Lemma and investigate the anticommutator of s_0 with other femionic operations. Here we employ the partial derivatives with respect to the isospin ghosts C^I. These anticommutators coincide with the generators δ_I of isospin transformations

$$\delta_I = \{s_0, \frac{\partial}{\partial C^I}\} \tag{5.13}$$

which on the ghosts are represented by G_I and the adjoint representation

$$\delta_I C^a = G_{Ib}{}^a C^b \quad \delta_I C^J = f_{KI}{}^J C^K \ . \tag{5.14}$$

Eq. (5.13) is easily verified on the elementary variables C^a, C^I and T. It extends to arbitrary polynomials because both sides of the equation are linear operators with the same product rule.

The isospin transformations commute with s_0 because each anticommutator $\{s_0, \delta\}$ of a nilpotent s_0 commutes with s_0 no matter what δ is (3.6).

$$[\delta_I, s_0] = 0 \tag{5.15}$$

The representation of the isospin transformations on the algebra of ghosts and tensors is completely reducible because the isospin transformations belong to a semisimple group or to abelian transformations which decompose the algebra into polynomials of definite charge and definite dimension. Therefore the following theorem applies.

Theorem 5.1

If the representation of δ_I is completely reducible then each solution of $s_0\omega = 0$ is δ_I invariant up to an irrelevant piece.

$$s_0\omega = 0 \Rightarrow \omega = \omega_{inv} + s_0\eta \quad \delta_I\omega_{inv} = 0 \ . \tag{5.16}$$

The theorem is proven by the following arguments. The space

$$Z = \{\omega : \ s_0\omega = 0\} \tag{5.17}$$

is mapped by isospin transformations to itself ($s_0(\delta_I\omega) = \delta_I s_0\omega = 0$), i.e. $\delta_I Z \subset Z$. Z contains the subspace of elements which can be written as isospin transformations applied to some other elements $\kappa^I \in Z$

$$Z_\delta = \{\omega \in Z : \ \omega = \delta_I(\kappa^I) \quad s_0\kappa^I = 0\} \ . \tag{5.18}$$

Z_δ is mapped by isospin transformations to itself. A second invariant subspace is given by Z_{inv}, the subspace of δ_I invariant elements

$$Z_{inv} = \{\omega \in Z : \ \delta_I\omega = 0 \ \} \ . \tag{5.19}$$

If the representation of δ_I is completely reducible then the space Z is spanned by $Z_{inv} \oplus Z_\delta \oplus Z_{comp}$ with a complement Z_{comp} which is also mapped to itself. This complement, however, contains only $\omega = 0$ because if there were a nonvanishing element $\omega \in Z_{comp}$ it would not be invariant because it is not from Z_{inv}. ω would be mapped to $\delta_I\omega \in Z_\delta$ and Z_{comp} would not be an invariant subspace.

$$Z = Z_{inv} \oplus Z_\delta \tag{5.20}$$

Each ω which satisfies $s_0\omega = 0$ can therefore be decomposed as

$$\omega = \omega_{inv} + \delta_I \kappa^I \quad s_0 \kappa^I = 0 \ . \tag{5.21}$$

We replace δ_I by $\{s_0, \frac{\partial}{\partial C^I}\}$ (5.13), use $s_0\kappa^I = 0$ and verify the theorem.

$$\omega = \omega_{inv} + s_0 \eta \quad \eta = \frac{\partial}{\partial C^I}\kappa^I \tag{5.22}$$

The theorem restricts nontrivial solutions to $s_0\omega = 0$ to spin and isospin invariant combinations.

We can exploit this theorem a second time and conclude that the translation ghosts C^a and the tensors T occur only in invariant combinations and that the ghosts C^I of spin and isospin transformations couple separately to invariants. This follows from the peculiar form of s_0 which is given by $C^I\delta_I$ if it acts on translation ghosts and tensors and by $\frac{1}{2}C^I\delta_I$ if it acts on the ghosts C^I of spin and isospin transformations.

$$s_0 = C^I \delta_I - s_c \tag{5.23}$$

s_c transforms only spin and isospin ghosts

$$s_c T = 0 \quad s_c C^a = 0 \quad s_c C^I = -\frac{1}{2} C^K C^L f_{KL}{}^I \ . \tag{5.24}$$

The equations $s_0\omega_{inv} = 0$ and $\delta_I \omega_{inv} = 0$ imply

$$s_c \omega_{inv} = 0 \ . \tag{5.25}$$

The anticommutator

$$\delta_{C^I} = \{s_c, \frac{\partial}{\partial C^I}\} \tag{5.26}$$

generates the adjoint transformations of the spin and isospin ghosts.

$$\delta_{C^I} C^J = f_{KI}{}^J C^K \quad \delta_{C^I} T = 0 \quad \delta_{C^I} C^a = 0 \tag{5.27}$$

It can be used to express s_c in the forms

$$s_c = \frac{1}{2} C^I \delta_{C^I} = \frac{1}{2} \delta_{C^I} C^I \tag{5.28}$$

which are both valid because $f_{IJ}{}^I = 0$ in Lie algebras which consist of simple and abelian factors.

By theorem (5.1) one can conclude from $s_c\omega_{inv} = 0$ that ω_{inv} consists of a part $\omega_{inv|inv}$ which is invariant under δ_{C^I} and a piece $(s_c\eta)_{inv}$ which is also s_0 exact because $(s_c\eta)_{inv}$ is δ_I invariant. Therefore ω is of the form

$$\omega = f(\theta_\alpha(C^I), I_\tau(C^a, T)) + s_0\eta.$$

where $\theta_\alpha(C^I)$ and $I_\tau(C^a, T)$ are invariant functions. A contribution $s_c\eta$ to θ_α changes f only by an irrelevant piece because $s_c\eta\, I(C^a, T) = s_0(\eta\, I)$. We can therefore state:

Theorem 5.2
An element ω of the algebra of ghosts and tensors satisfies $s_0\omega = 0$ if and only if it is of the form

$$\omega = f(\theta_\alpha(C^I), I_\tau(C^a, T)) + s_0\eta \tag{5.29}$$

where $I_\tau(C^a, T)$ are invariant functions and where the invariant functions $\theta_\alpha(C^I)$ $\alpha = 1, \ldots, r$ generate the Lie algebra cohomology

$$s_c\Theta(C^I) = 0 \Leftrightarrow \Theta(C^I) = \Phi(\theta_1(C), \ldots, \theta_r(C)) + s_c\eta(C^I)\ . \tag{5.30}$$

ω is trivial if and only if f vanishes.

The solutions of $s_c\Theta(C) = 0$ are given by the δ_{C^I} invariant polynomials $\Theta(C^I)$. Obviously these invariant polynomials satisfy $s_c\Theta = 0$ and they are nontrivial because all trivial solutions $s_c\eta$ are contained in Z_δ as $s_c = \frac{1}{2}\delta_{C^I}C^I$ (5.28) shows. Similarly the equation $s_0 = \delta_I C^I - \frac{1}{2}\delta_{C^I}C^I$ shows that f is nontrivial because it is invariant under δ_I and δ_{C^I} and cannot be expressed as a sum of terms of the form $\delta_I\kappa^I$ or $\delta_{C^I}\kappa^I$.

The space of invariant polynomials can be determined separately for each factor of the Lie algebra. The general solution for the product algebra can then be obtained with Künneth's formula (theorem 3.5).

The following results for simple Lie algebras can be found in the mathematical literature [7] or in translations into a language which a (german) physicist is used to [8]. For a simple Lie algebra G the dimension of the space of invariant polynomials $\Theta(C)$ is 2^r where r is the rank of G. These invariant polynomials are generated by r primitive polynomials $\theta_\alpha(C)$, $\alpha = 1, \ldots, r$ which cannot be written as a sum of products of other invariant polynomials. They have odd ghost number $\mathrm{gh}(\theta_\alpha(C)) = 2m(\alpha) - 1$ and therefore are fermionic. They can be obtained from traces of suitable matrices M_i which represent the Lie algebra and are given with a suitable normalization by

$$\theta_\alpha(C) = \frac{(-)^{m-1} m!(m-1)!}{(2m-1)!} tr(C^i M_i)^{2m-1} \quad m = m(\alpha) \quad \alpha = 1, \dots, r \ . \tag{5.31}$$

The number $m(\alpha)$ is the degree of homogeneity of the corresponding Casimir invariant $I_\alpha(X)$

$$I_\alpha(X) = tr(X^i M_i)^{m(\alpha)} \ . \tag{5.32}$$

These Casimir invariants generate all invariant functions of a set of commuting variables X^i which transform as an irreducible multiplet under the adjoint representation.

The degrees $m(\alpha)$ for the classical simple Lie algebras are given by

$$\begin{array}{llll} SU(n+1) & A_n & m(\alpha) = \alpha + 1 & \alpha = 1, \dots, n \\ SO(2n+1) & B_n & m(\alpha) = 2\alpha & \alpha = 1, \dots, n \\ SP(2n) & C_n & m(\alpha) = 2\alpha & \alpha = 1, \dots, n \\ SO(2n) & D_n & m(\alpha) = 2\alpha & \alpha = 1, \dots, n-1 \quad m(n) = n \end{array} \tag{5.33}$$

With the exception of the last primitive element of $SO(2n)$ the matrices M_i are the defining representation of the classical Lie algebras. The last primitive element θ_n and the last Casimir invariant I_n of $SO(2n)$ are constructed from the spin representation Γ_i. Up to normalization they are given by

$$\theta_n \sim \varepsilon_{a_1 b_1 \dots a_n b_n} (C^2)^{a_1 b_1} \dots (C^2)^{a_{n-1} b_{n-1}} C^{a_n b_n} \quad I_n \sim \varepsilon_{a_1 b_1 \dots a_n b_n} X^{a_1 b_1} \dots X^{a_n b_n} \ .$$

If n is even then the primitive element θ_n of $SO(2n)$ is degenerate in ghost number with $\theta_{\frac{n}{2}}$.

The primitive elements for the exceptional simple Lie algebras G_2, F_4, E_6, E_7, E_8 can also be found in the literature [9]. Their explicit form is not important for our purpose. In each case the Casimir invariant with lowest degree m is quadratic ($m = 2$).

For a one dimensional abelian Lie algebra the ghost C is invariant under the adjoint transformation. It generates the invariant polynomials $\Theta(C) = a + bC$ which span a 2^r dimensional space where $r = 1$ is the rank of the abelian Lie algebra. The generator θ of this algebra of invariant polynomials has odd ghost number $\mathrm{gh}(C) = 2m - 1$ with $m = 1$.

$$\theta(C) = C \tag{5.34}$$

The Casimir invariant I of the one dimensional, trivial adjoint representation acting on a bosonic variable X is homogeneous of degree $m=1$ in X and is simply given by X itself.

$$I(X) = X \ . \tag{5.35}$$

If the Lie algebra is a product of simple and abelian factors then the list of primitive elements θ_α and the list of the Casimir invariants I_α are the union of the respective lists of the factors of the Lie algebra.

Polynomials of r anticommuting variables θ_α span a 2^r dimensional space, which theoretical physicists would call a superspace. The statement that the primitive elements $\theta_\alpha(C)$ span the space of δ_I invariant polynomials in the anticommuting ghosts

$$\delta_I \Theta(C) = 0 \Rightarrow \Theta(C) = \Phi(\theta_1(C), \ldots, \theta_r(C)) \tag{5.36}$$

asserts that the Lie algebra cohomology is given by

$$s_c \Theta(C) = 0 \Leftrightarrow \Theta(C) = \Phi(\theta_1(C), \ldots, \theta_r(C)) + s_c \eta \ . \tag{5.37}$$

Because the space of these invariant functions is 2^r dimensional there are no algebraic relations among the functions $\theta_\alpha(C)$ apart from the anticommutation relations which result from their odd ghost number.

$$\Theta(C) = \Phi(\theta_1(C), \ldots, \theta_r(C)) = 0 \Leftrightarrow \Phi(\theta_1, \ldots, \theta_r) = 0 \tag{5.38}$$

The Casimir invariants $I_\alpha(X)$ generate the space of δ_I invariant polynomials in commuting variables X which transform under the adjoint representation

$$\delta_I P(X) = 0 \Rightarrow P(X) = f(I_1(X), \ldots, I_r(X)) \ . \tag{5.39}$$

If there are no algebraic relations among the variables X apart from their commutation relations then there is no algebraic relation among the Casimir invariants $I_\alpha(X)$ up to the fact that the I_α commute [7].

$$P(X) = f(I_1(X), \ldots, I_r(X)) = 0 \Leftrightarrow f(I_1, \ldots, I_r) = 0 \tag{5.40}$$

Theorem (5.2) describes all solutions $\omega_{\underline{n}}$ of the equation $s_0 \omega_{\underline{n}} = 0$. This equation is the part of $s\omega = 0$ with lowest degree in the translation ghosts. In degree $\underline{n}+1$ the equation $s\omega = 0$ imposes the restriction

$$s_1 \omega_{\underline{n}} + s_0 \omega_{\underline{n}+1} = 0 \ . \tag{5.41}$$

We choose $\omega_{\underline{n}} = f(\theta_\alpha(C), I_\tau(C^a, T))$. Then $s_1\omega_{\underline{n}}$ is δ_I invariant and not of the form $s_0\eta$ because s_1 (5.8)

$$s_1 T = C^a D_a T \quad s_1 C^a = 0 \quad s_1 C^I = 0 \tag{5.42}$$

maps invariant functions I_τ of tensors and translation ghosts to invariant functions. Therefore $s_1\omega_{\underline{n}}$ has to vanish because it is not of the form $s_0\eta$.

We can require more restrictively that $\omega_{\underline{n}}$ is an element of the s_1 cohomology after we consider the following argument. A contribution to $\omega_{\underline{n}}$ of the form $s_1\eta(\theta_\alpha, I_\tau)$ can be written as $s\eta - s_2\eta$ because $s_0\eta$ vanishes (η is δ_I invariant). $s\eta$ changes $\omega = \omega_{\underline{n}} + \ldots$ only by an irrelevant piece. $s_2\eta$ can be absorbed in the parts $\ldots$ with higher ghost degree. Therefore we can neglect contributions $s_1\eta(\theta_\alpha, I_\tau)$ to $\omega_{\underline{n}}$.

$$s_1\omega_{\underline{n}} = 0 \quad \omega_{\underline{n}} \bmod s_1\eta_{inv} \tag{5.43}$$

The operation s_1 acting on *invariant* functions is nilpotent because (5.5,5.9,5.10)

$$s_1^2 + \{s_0, s_2\} = 0 = s_1^2 + F^I\hat{\delta}_I \tag{5.44}$$

where F^I is the ghost two form

$$F^I = \frac{1}{2}C^a C^b F_{ab}{}^I \tag{5.45}$$

and $\hat{\delta}_I$ generates the adjoint transformation of translation ghosts and tensors

$$\hat{\delta}_I T = \delta_I T \quad \hat{\delta}_I C^a = \delta_I C^a \quad \hat{\delta}_I C^J = 0 \ . \tag{5.46}$$

s_1 acting on invariant functions is the exterior derivative $d = dx^m\partial_m$ in disguise. It does not differentiate the translation ghosts, the relation $s_1(C^a) = 0$ corresponds to the relation $d(dx^m) = 0$. An invariant ghost form of degree $l = \underline{n}$ is given by

$$\omega(C, T) = \frac{1}{l!}C^{a_1}\ldots C^{a_l}\omega_{a_1\ldots a_l}(T) \tag{5.47}$$

where the components $\omega_{a_1\ldots a_l}$ belong to an isospin invariant Lorentz tensor which transforms as indicated by the index picture. s_1 acts on ω (5.8) by

$$s_1\omega = \frac{1}{(l+1)!}C^{a_1}\ldots C^{a_{l+1}}\sum_{cyclic(1,2,\ldots,l+1)} sign(cyclic) D_{a_1}\omega_{a_2\ldots a_{l+1}} \ . \tag{5.48}$$

If we convert the index picture from Lorentz indices to space time indices by help of the vielbein $e_m{}^a$ and its inverse $E_a{}^m$ and define the space time covariant derivative D_m and the ghosts C^m by

$$D_{a_0}\omega_{a_1\dots a_l} = E_{a_0}{}^{m_0}E_{a_1}{}^{m_1}\dots E_{a_l}{}^{m_l}D_{m_0}\omega_{m_1\dots m_l} \quad C^m = C^aE_a{}^m \tag{5.49}$$

then s_1 acts on forms ω

$$\omega(C,T) = \frac{1}{l!}C^{m_1}\dots C^{m_l}\omega_{m_1\dots m_l}(T) \tag{5.50}$$

in the same way as the exterior covariant derivative $D = dx^mD_m$. Only the name of the differential dx^m is changed to C^m.

$$s_1\omega = \frac{1}{(l+1)!}C^{m_1}\dots C^{m_{l+1}} \sum_{cyclic(1,2,\dots,l+1)} sign(cyclic)D_{m_1}\omega_{m_2\dots m_{l+1}} \; . \tag{5.51}$$

s_1 simplifies on δ_I invariant forms even more because one can neglect the isospin transformations in the covariant derivatives $D_a = e_a{}^m(\partial_m - A_m{}^I\delta_I)$. The spin connection $\omega_{ma}{}^b$ in the covariant derivative is exchanged for the symmetric Christoffel symbol

$$\Gamma_{mn}{}^k = \frac{1}{2}g^{kl}\left(\partial_m g_{nl} + \partial_n g_{ml} - \partial_l g_{mn}\right) \quad \Gamma_{mn}{}^k = \Gamma_{nm}{}^k \quad g_{mn} = e_m{}^ae_n{}^b\eta_{ab} \tag{5.52}$$

if the Lorentz vector indices $a, b, \dots$ are traded for tangent space indices $m, n, \dots$. The contributions of these Christoffel symbols vanish if s_1 is applied to an invariant form because all tangent space indices are contracted with anticommuting ghosts C^m, e.g.

$$\begin{aligned} s_1C^n\omega_n &= s_1C^b\omega_b = -C^bC^aD_a\omega_b = C^mC^nD_m\omega_n \\ &= C^mC^n(\partial_m\omega_n - \Gamma_{mn}{}^l\omega_l) = C^mC^n\partial_m\omega_n \end{aligned}$$

Therefore s_1 acts on invariant ghost forms in the same way as the exterior derivative $d = dx^m\partial_m$ acts on differential forms.

The cohomology of d acting on the jet variables is given by the Algebraic Poincaré Lemma (theorem (3.3)). This lemma, however, does not apply here because among the tensors there are the field strengths on which the derivatives do not act freely, i.e. with no constraint apart from the fact that they commute, but subject to the Bianchi identities

$$\sum_{cyclic} D_aF_{bc} = 0 \; . \tag{5.53}$$

These constraints on the action of the derivatives change the cohomology of d. It is given by the Covariant Poincaré Lemma [10].

Theorem 5.3 *Linearized Covariant Poincaré Lemma*
Consider functions $\mathcal{L}$ and differential forms ω and η which depend on coordinates, linearized field strengths ${F_{mn}}^i = \partial_m {A_n}^i - \partial_n {A_m}^i$ and their derivatives, which are restricted by $\sum_{cyclic} \partial_k {F_{mn}}^i = 0$, and on other fields ψ and their derivatives. If ω satisfies $d\omega = 0$ then it can be written as a sum of a volume form $\mathcal{L} d^D x$ and a polynomial $P(F)$ in the field strength two forms $F^i = \frac{1}{2} dx^m dx^n {F_{mn}}^i$ and an exact form $d\eta$.

$$
\begin{aligned}
&d\ \omega\left(x^m, dx^m, F_{mn}, \partial_{(k}F_{m)n}, \ldots, \psi, \partial_k\psi, \ldots, \partial_{(k}\ldots\partial_{l)}\psi\right) = 0 \Leftrightarrow \\
&\omega = \mathcal{L}(x^m, F_{mn}, \partial_{(k}F_{mn)}, \ldots, \psi, \partial_k\psi, \ldots) d^D x + P(F) + d\eta \qquad (5.54)
\end{aligned}
$$

The Lagrange density $\mathcal{L} d^D x$ cannot be written as $P(F) + d\eta$ if its Euler derivatives (3.27) with respect to ψ and ${A_n}^i$ do not vanish

$$
\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} \neq 0 \quad \text{or} \quad \partial_m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}{F_{mn}}^i} \neq 0 \ . \qquad (5.55)
$$

A nonvanishing polynomial $P(F)$ cannot be written as d of a form η which depends on field strengths and fields ψ and their derivatives because η would have to contain at least one connection ${A_m}^i$ without derivative.

The theorem can be extended to cover Lorentz and isospin invariant Lagrange densities depending on the (nonlinear) field strengths, other tensors and their covariant derivatives.

Theorem 5.4 *Covariant Poincaré Lemma*
Consider δ_I invariant functions $\mathcal{L}$ and differential forms ω and η which depend on field strengths ${F_{mn}}^I$ and their covariant derivatives, which are restricted by $\sum_{cyclic} D_k {F_{mn}}^I = 0$, and on other fields ψ and their covariant derivatives. If ω satisfies $d\omega = 0$ then it can be written as a sum of a volume form $\mathcal{L} d^D x$ and an invariant polynomial $P(F)$ in the field strength two forms $F^I = \frac{1}{2} dx^m dx^n {F_{mn}}^I$ and an exact form $d\eta$.

$$
\begin{aligned}
&d\ \omega\left(dx^m, F_{mn}, D_{(k}F_{m)n}, \ldots, \psi, D_k\psi, \ldots, D_{(k}\ldots D_{l)}\psi\right) = 0 \Leftrightarrow \\
&\omega = \mathcal{L}(F_{mn}, D_{(k}F_{m)n}, \ldots, \psi, \ldots) d^D x + P(F) + d\eta \qquad (5.56)
\end{aligned}
$$

The Lagrange density $\mathcal{L}d^Dx$ cannot be written as $P(F) + d\eta$ if its Euler derivatives with respect to ψ and $A_n{}^I$ do not vanish

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} \neq 0 \quad or \quad D_m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}F_{mn}{}^I} \neq 0 \ . \tag{5.57}$$

We call the invariant polynomials $P(F^I)$ Chern forms. They are polynomials in commuting variables, the field strength two forms F^I which transform as an adjoint representation of the Lie algebra. These invariant polynomials are generated by the elementary Casimir invariants $I_\alpha(F^I)$. The Chern forms enlarge the cohomology of the exterior derivative if it acts on tensors rather than on jet variables. They comprise all topological densities which one can construct from connections for the following reason. If a functional is to contain only topological information its value must not change under continuous deformation of the fields. Therefore it has to be gauge invariant and invariant under general coordinate transformations. If it is a local functional it is the integral over a density which satisfies the descent equation and which can be obtained from a solution to $s\omega = 0$. If this density belongs to a functional which contains only topological information then the value of this functional must not change even under arbitrary differentiable variations of the fields, i.e. its Euler derivatives with respect to the fields must vanish. Therefore the integrand must be a total derivative in the space of jet variables. But it must not be a total derivative in the space of tensor variables because then it would be constant and contain no information at all. Therefore, by theorem (5.4), all topological densities which one can construct from connections are given by Chern polynomials in the field strength two form.

Theorem (5.4) describes also the cohomology of s_1 acting on invariant ghost forms because s_1 acts on invariant ghost forms (5.50) exactly like the exterior derivative d acts on differential forms. We have to allow, however, for the additional variables $\theta_\alpha(C)$ in $\omega_{\underline{n}}$. They generate a second, trivial algebra A_2 and can be taken into account by Künneth's theorem (theorem (3.5)). If we neglect the trivial part $s_1\eta_{inv}$ then the solution to (5.44) is given by

$$\omega_{\underline{n}} = \mathcal{L}(\theta_\alpha(C), T)C^1C^2\dots C^D + P(\theta_\alpha(C), I_\alpha(F)) \tag{5.58}$$

The δ_I invariant Lagrange ghost density satisfies already the complete equation $s\omega(C,T) = 0$ because it is a D ghost form. The solution to $\tilde{s}\tilde{\omega} = 0$ is given by $\tilde{\omega} = \omega(C + A, T)$ and the Lagrange density and the anomaly candidates are given by the part of $\tilde{\omega}$ with $d^D x$. The coordinate differentials come from $C^a + dx^m e_m{}^a$ [2]. If one picks the D form part then one gets

$$dx^{m_1} \dots dx^{m_D} e_{m_1}{}^1 \dots e_{m_D}{}^D = \det(e_m{}^a) d^D x \quad \det(e_m{}^a) =: \sqrt{g} \qquad (5.59)$$

Therefore the solutions to the descent equations of Lagrange type are given by

$$\omega_D = \mathcal{L}(\theta_\alpha(C), T)\sqrt{g} d^D x \ . \qquad (5.60)$$

They are constructed in the well known manner from tensors T, including field strengths and covariant derivatives of tensors, which are combined to a Lorentz invariant and isospin invariant Lagrange function. This composite scalar field is multiplied by the density $\sqrt{g}$. Integrands of local gauge invariant actions are obtained from this formula by restricting ω_D to vanishing ghost number. Then the variables $\theta_\alpha(C)$ do not occur. We indicate the ghost number by a superscript and have

$$\omega_D^0 = \mathcal{L}(T)\sqrt{g} d^D x \ . \qquad (5.61)$$

Integrands of anomaly candidates are obtained by choosing D forms with ghost number 1. Only abelian factors of the Lie algebra allow for such anomaly candidates because the primitive invariants θ_α for nonabelian factors have at least ghost number 3.

$$\omega_D^1 = \sum_i C^i \mathcal{L}_i(T)\sqrt{g} d^D x \ . \qquad (5.62)$$

The sum ranges over all abelian factors of the gauge group. Anomalies of this form actually occur as trace anomalies or β functions if the isospin algebra contains dilatations.

This completes the discussion of Lagrange densities and anomaly candidates coming from the first term in (5.58).

[2] We can use the ghosts variables C or $\hat{C}$ (4.46). The expessions remain unchanged because they are multiplied by D translation ghosts.

Chapter 6

Chiral anomalies

It remains to investigate solutions which correspond to

$$\omega_{\underline{n}} = P(\theta_\alpha(C), I_\alpha(F)) \ . \tag{6.1}$$

Ghosts C^I for spin and isospin transformations and ghost forms F^I generate a subalgebra which is invariant under s and takes a particularly simple form if expressed in terms of matrices $C = C^I M_I$ and $F = F^I M_I$ which represent the Lie algebra. For nearly all algebraic operations it is irrelevant that F is a composite field. The transformation of C (5.3) can be read as definition of $F = sC + C^2$ and $s^2 = 0$ determines the transformation of F which is given by the adjoint transformation. One calculates

$$sF = sC\,C - C\,sC = (F - C^2)C - C(F - C^2)$$

$$sC = -C^2 + F \quad sF = FC - CF \ . \tag{6.2}$$

If one changes the notation and replaces s by $d = dx^m \partial_m$ and C by $A = dx^m A_m{}^I M_I$ then the same equations are the definition of the field strengths in Yang Mills theories and their Bianchi identities. The equations are valid whether the anticommuting variables C and the nilpotent operation s are composite or not. [1]

The Chern polynomials I_α satisfy $sI_\alpha = 0$ because they are invariant under adjoint transformations. All I_α are trivial i.e. of the form sq_α. To show this explicitly we define a one parameter deformation $F(t)$ of F

$$F(t) = tF + (t^2 - t)C^2 = t\,sC + t^2 C^2 \quad F(0) = 0 \quad F(1) = F \tag{6.3}$$

[1] This does not mean that there are no differences at all. For example the product of $D + 1$ matrix elements of the one form matrix A vanish.

which allows to switch on F.

All invariants I_α can be written as $tr(F^{m(\alpha)})$ with suitable representations M_I. We rewrite $tr(F^m)$ in an artificially more complicated form

$$tr(F^m) = \int_o^1 dt \frac{d}{dt} tr(F(t)^m) = m \int_0^1 dt\, tr\left((sC + 2tC^2)F(t)^{m-1}\right) \ .$$

The integrand coincides with

$$\begin{aligned} s\, tr\,(CF(t)^{m-1}) &= tr\,((sC)F(t)^{m-1} - tC[F(t)^{m-1}, C]) \\ &= tr(sCF(t)^{m-1} + 2tC^2F(t)^{m-1}) \ . \end{aligned}$$

The Chern form I_α is the s transformation of the Chern Simons form q_α, these forms generate a subalgebra.

$$sq_\alpha = I_\alpha \quad sI_\alpha = 0 \tag{6.4}$$

$$q_\alpha = m \int_0^1 dt\ tr\left(C\left(tF + (t^2 - t)C^2\right)^{m-1}\right) \quad m = m(\alpha) \tag{6.5}$$

Using the binomial formula and

$$\int_0^1 dt\, t^k (1-t)^l = \frac{k!\, l!}{(k+l+1)!}$$

the t-integral can be evaluated. It gives the combinatorial coefficients of the Chern Simons form.

$$q_\alpha(C, F) = \sum_{l=0}^{m-1} \frac{(-)^l m!(m-1)!}{(m+l)!(m-l-1)!} tr_{sym}\left(C(C^2)^l (F)^{m-l-1}\right) \tag{6.6}$$

It involves the traces of completely symmetrized products of the l factors C^2, the $m-l-1$ factors F and the factor C. The part with $l = m-1$ has form degree 0 and ghost number $2m-1$ and agrees with θ_α

$$q_\alpha(C, 0) = \frac{(-)^{m-1} m!(m-1)!}{(2m-1)!} trC^{2m-1} = \theta_\alpha(C) \ . \tag{6.7}$$

Each polynomial $\omega_{\underline{n}} = P(\theta_\alpha(C), I_\alpha(F))$ defines naturally a form

$$\omega(C, F) = P(q_\alpha(C, F), I_\alpha(F)) \tag{6.8}$$

which coincides with $\omega_{\underline{n}}$ in lowest form degree.

$$\omega(q_\alpha(C,F),I_\alpha(F)) = \omega_{\underline{n}}(\theta_\alpha(C),I_\alpha(F)) + \dots \tag{6.9}$$

On such forms s acts simply as the operation $s = I_\alpha \frac{\partial}{\partial q_\alpha}$.

$$s\omega = I_\alpha \frac{\partial}{\partial q_\alpha} P(q_\alpha, I_\alpha)_{|q_\alpha(C,F),I_\alpha(F)} \tag{6.10}$$

The basic lemma (3.3) implies that among the polynomials $P(q,I)$ the only nontrivial solutions of $sP = 0$ are independent of q and I. (Introduce the operation $r = q_\alpha \frac{\partial}{\partial I_\alpha}$ and observe that the anticommutator $\{s,r\}$ gives the counting operator N of the variables q and I.)

$$s\omega(q,I) = 0 \Leftrightarrow \omega(q,I) = \omega_0 + s\chi(q,I) \tag{6.11}$$

Though correct this result is misleading because we are not looking for solutions of $s\omega(q,I) = 0$ but have to solve $s\omega(q(C,F),I(F)) = 0$. This equation has more solutions than (6.11) because F is a two form and $D+1$ forms vanish.

Let us investigate the question whether a differential form $\omega(q(C,F),I(F))$ or $s\omega(q(C,F),I(F))$ vanishes. To answer this question one can neglect the two forms F contained in $q(C,F)$ because in lowest form degree q_α coincides with $\theta_\alpha(C)$ (6.7) and the variables θ_α satisfy no algebraic identities apart from the the fact that they anticommute. We count the lowest form degree with the counting operator

$$N_{form} = \sum_\alpha 2m(\alpha) I_\alpha \frac{\partial}{\partial I_\alpha} \tag{6.12}$$

and decompose the polynomials $\omega(q,I)$ into parts ω_f with definite lowest form degree.

$$\omega(q,I) = \sum_f \omega_f(q,I) \qquad N_{form}\omega_f = f\omega_f \tag{6.13}$$

Obviously f is even and $\omega_f(q(C,F),I(F))$ vanishes if $f > D$.

$$\prod (I_{\alpha_i}(F))^{\beta_i} = 0 \text{ if } \sum 2\beta_i m(\alpha_i) > D \tag{6.14}$$

Because $s = \sum_\alpha I_\alpha \frac{\partial}{\partial q_\alpha}$ replaces Chern Simons forms q_α, which have vanishing lowest form degree, by I_α with form degree given by $2m(\alpha)$, the form

$s\omega_f$ consists of pieces with lowest form degree given by $f + 2m(1), f + 2m(2), \ldots, f + 2m(\text{rank})$, i.e. the increase of lowest form degree depends on the degree $m(\alpha)$ of the variables q_α, I_α. The action of s becomes more transparent if we group these variables according to their degree $m(\alpha)$ and split ω_f into pieces $\omega_{f,m}$ which depend only on variables with $m(\alpha) \geq m$.

Let us introduce the notation x_m for the variables q_α and I_α with a fixed $m(\alpha) = m$.

$$\{x_m\} = \{q_\alpha, I_\alpha : m(\alpha) = m\} \tag{6.15}$$

For some values of m the set $\{x_m\}$ may be empty. We denote the highest degree by $\bar{m}$.

The part of a polynomial $P(x_1, x_2, \ldots, x_{\bar{m}})$ which depends on x_1 is given by

$$P_1 = P(x_1, x_2, \ldots, x_{\bar{m}}) - P(0, x_2, \ldots, x_{\bar{m}}),$$

the rest is independent of x_1 can be decomposed into a part which depends on x_2 and a rest which is independent of x_1 and x_2

$$P_2 = P(0, x_2, x_3, \ldots, x_{\bar{m}}) - P(0, 0, x_3, \ldots, x_{\bar{m}})$$

and so on.

$$\begin{aligned} P_m &= P(0, \ldots, 0, x_m, x_{m+1}, \ldots, x_{\bar{m}}) - P(0, \ldots, 0, 0, x_{m+1}, \ldots, x_{\bar{m}}) && (6.16) \\ P_{\bar{m}+1} &= P(0, \ldots, 0) && (6.17) \\ P &= \sum_m P_m && (6.18) \end{aligned}$$

We call this decomposition of polynomials a level decomposition. The space $\mathcal{P}_m$ of polynomials at level m consists of all polynomials $P(x_m, x_{m+1}, \ldots, x_{\bar{m}})$ modulo polynomials $Q(x_{m+1}, \ldots, x_{\bar{m}})$. s does not mix levels (6.10) and raises the form degree of polynomials at level m by at least $2m$. We consider each P_m separately.

For $m \leq \bar{m}$ we decompose the space $\mathcal{P}_m$ of polynomials P_m à la Hodge (3.7) with the operators [2]

$$s_m = \sum_{m(\alpha)=m} I_\alpha \frac{\partial}{\partial q_\alpha} \qquad r_m = \sum_{m(\alpha)=m} q_\alpha \frac{\partial}{\partial I_\alpha} \tag{6.19}$$

into $\mathcal{S}_m = s_m \mathcal{P}_m$ and $\mathcal{R}_m = r_m \mathcal{P}_m$

$$\mathcal{P}_m = \mathcal{S}_m \oplus \mathcal{R}_m \tag{6.20}$$

[2] Hopefully the s_m are not confused with s_0, s_1, s_2 defined in (5.5–5.9).

and write $P_m = S_m + R_m$ as an s_m exact piece S_m and an r_m exact piece R_m

$$P_m = S_m + R_m \quad S_m = s_m \rho \quad R_m = r_m \sigma \tag{6.21}$$

Without loss of generality we can take ρ from $\mathcal{R}_m$ and σ from $\mathcal{S}_m$.

The piece S_m can be rewritten as a trivial contribution to ω and a part which lies in $\mathcal{R}_m$ because $\rho \in \mathcal{R}_m$

$$s_m \rho = s\rho - \sum_{m' \geq m+1} s_{m'} \rho = s\rho + \rho' \quad \rho' \in \mathcal{R}_m \tag{6.22}$$

Eq. (6.22) holds because

$$s = \sum_m s_m \tag{6.23}$$

and $s_{m'}\rho = 0$ for $m' < m$ and $s_{m'}\rho \in \mathcal{R}_m$ for $m' > m$. Therefore we can restrict P_m to the r_m exact part.

$$P_m = r_m \sigma \quad \sigma \in \mathcal{S}_m \tag{6.24}$$

Such a polynomial $P_m(q, I)$, however, cannot be made to satisfy $sP_m = 0$ as a polynomial in q_α, I_α

$$sP_m = s_m r_m \sigma + \sum_{m' \geq m+1} s_{m'} r_m \sigma = N_m \sigma + \sum_{m' \geq m+1} s_{m'} r_m \sigma \ . \tag{6.25}$$

The pieces $s_m r_m \sigma$ and the sum have to vanish separately because the sum lies in $\mathcal{R}_m$. Moreover because $s_m \sigma = 0$ we can replace $s_m r_m$ by the anticommutator $\{s_m, r_m\}$ which counts the variables at level m

$$N_m = \{s_m, r_m\} = \sum_{m(\alpha)=m} I_\alpha \frac{\partial}{\partial I_\alpha} + q_\alpha \frac{\partial}{\partial q_\alpha} \tag{6.26}$$

and maps $\mathcal{S}_m$ invertibly to itself. Therefore $sP_m = 0$ has only the trivial solution $P_m = 0$.

As pointed out, however, it is the form $\omega(C, F) = P_m(q_\alpha(C, F), I_\alpha(F))$ which has to satisfy $s\omega = 0$, and not the polynomial $P_m(q_\alpha, I_\alpha)$. If the polynomial $s\omega(q, I)$ does not vanish then the form $s\omega$ vanishes if and only if its lowest form degree is larger than D. We obtain therefore the solutions ω if we take $\sigma \in \mathcal{S}_m$ and restrict it in addition to be composed of monomials with sufficiently many factors I_α such that the form degree D' of σ lies above D and the form degree of $\omega = r_m \sigma$ starts below $D + 1$. This restriction can easily be formulated if we split $\mathcal{S}_m$ into spaces $\mathcal{S}_{m,D'}$ with definite and even degrees D' with the number operator N_{form} (6.12).

$$\mathcal{S}_m = \sum_{D'} \mathcal{S}_{m,D'} \quad P \in \mathcal{S}_{m,D'} \Leftrightarrow P \in \mathcal{S}_m \wedge N_{form} P = D'P \tag{6.27}$$

Because each term in $\mathcal{S}_m$ contains at least one factor I_α with $m(\alpha) = m$ the degrees D' are not smaller than $2m$. σ has to be taken from $\mathcal{S}_{m,D'}$ where D' is restricted by

$$D' - 2m \leq D < D' \tag{6.28}$$

to obtain a nonvanishing solution

$$\omega = (r_m \sigma)_{|q_\alpha(C,F),I_\alpha(F)} \quad \sigma \in \mathcal{S}_{m,D'} \tag{6.29}$$

which satisfies $s\omega = 0$ because the number of translation ghosts in $s\omega$ is at least D' and larger than D.

If we want to obtain a solution ω with a definite ghost number then we have to split the spaces $\mathcal{S}_{m,D'}$ with the ghost counting operator N_C

$$N_C = \sum_\alpha \left(2m(\alpha) I_\alpha \frac{\partial}{\partial I_\alpha} + (2m(\alpha) - 1) q_\alpha \frac{\partial}{\partial q_\alpha} \right) \tag{6.30}$$

N_C counts the total ghost number of translation ghosts, Lorentz ghosts and isospin ghosts and splits $\mathcal{S}_{m,D'}$ into eigenspaces $\mathcal{S}_{m,D',G}$ with total ghost number G

$$\mathcal{S}_{m,D'} = \sum_G \mathcal{S}_{m,D',G} \quad P \in \mathcal{S}_{m,D',G} \Leftrightarrow P \in \mathcal{S}_{m,D'} \wedge N_C P = GP \tag{6.31}$$

The total ghost number of $\omega = r_m \sigma$ is G if $\sigma \in \mathcal{S}_{m,D',G+1}$ because r_m lowers the total ghost number by 1.

We obtain the long sought solutions ω^g_D of the relative cohomology (2.53) which for $g = 0$ gives Lagrange densities of invariant actions (2.41) and for $g = 1$ gives anomaly candidates (2.52) if we substitute in ω the ghosts C by ghosts plus connection one forms $C + A$ and if we pick the part with D differentials. Therefore the total ghost number G of σ has to be chosen to be $G = g + D + 1$ to obtain a solution ω which contributes to ω^g_D. If the ghost variables $\hat{C}$ (4.46) are used to express ω then ω^g_D is simply obtained if all translation ghosts C^m are replaced by dx^m and the part with the volume element $d^D x$ is taken.

$$\begin{aligned} \omega(C,F) &= (r_m\sigma)_{|q_\alpha(C,F),I_\alpha(F)} \quad \sigma \in \mathcal{S}_{m,D',g+D+1} && (6.32)\\ \omega(C,F) &= f(\hat{C}^m,\hat{C}^i,F^i) && (6.33)\\ \omega^g_D &= f(dx^m,\hat{C}^i,\frac{1}{2}dx^m dx^n F_{mn}{}^i)_{|\text{D form part}} && (6.34) \end{aligned}$$

These formulas end our general discussion of the BRS cohomology of gravitational Yang Mills theories. The general solution of the consistency equations can is a linear combination of the Lagrangean solutions and the chiral solutions.

Let us conclude by spelling out the general formula for $g = 0$ and $g = 1$. If $g = 0$ then σ can contain no factors q_α because the complete ghost number $G \geq D'$ is not smaller than the ghost number D' of translation ghosts. D' has to be larger than D (6.28) and not larger than $G = g + D + 1 = D + 1$ which leaves $D' = D+1$ as only possibility. D' is even (6.12), therefore chiral contributions to Lagrange densities occur only in odd dimensions.

If, for example $D = 3$, then σ is an invariant 4 form.

For $m = 1$ such a form is given by $\sigma = F_i F_j a^{ij}$ with $a^{ij} = a^{ji} \in \mathbb{R}$ if the isospin group contains abelian factors with the corresponding abelian field strength F_i and i and j enumerate the abelian factors. σ lies in $\mathcal{S}_1$ because $\sigma = s(q_i F_j a^{ij})$. The form $\omega = r_1\sigma = 2q_i F_j a^{ij}$ yields the gauge invariant abelian Chern Simons action in 3 dimensions which is remarkable because it cannot be constructed from tensor variables alone and because it does not contain the metric.

To construct ω^0_3 one has to express $q(C) = C$ by $C = \hat{C} + C^m A_m$. Then one has to replace all translation ghosts by differentials dx^m and to pick the volume form. One obtains

$$\omega^0_{3\,abelian} = dx^m A_{m\,i} dx^k dx^l F_{kl\,j} a^{ij} = \varepsilon^{klm} A_{m\,i} F_{kl\,j} a^{ij} d^3x \ . \qquad (6.35)$$

For $m = 2$ the form $\sigma = trF^2$ of each nonabelian factor contributes to the nonabelian Chern Simons form. One has $I_1 = trF^2 = sq_1$, so $\sigma \in \mathcal{S}_2$ as required. ω is directly given by the Chern Simons form q_1(6.6)

$$\omega = tr(CF - \frac{1}{3}C^3) \qquad (6.36)$$

The corresponding Lagrange density is

$$\omega^0_{3\,nonabelian} = tr(AF - \frac{1}{3}A^3) = \frac{1}{2}\sum_I (A_m{}^I F_{kl}{}^I - \frac{1}{3}A_m{}^I A_k{}^J A_l{}^K f_{JK}{}^I)\varepsilon^{mkl} d^3x \tag{6.37}$$

Chiral anomalies are obtained if one looks for solutions ω^1_D with ghost number $g = 1$. This fixes $G = D + 2$ and because G is not less than $D' > D$ we have to consider the cases $D' = D + 1$ and $D' = D + 2$.

The first case can occur in odd dimensions only, because D' is even, and only if the level m, the lowest degree occuring in σ, is 1 because the missing total ghost number $D + 2 - D'$, which is not carried by $I_\alpha(F)$, has to be contributed by one Chern Simons polynomial q_α with $2m(\alpha) - 1 = 1$, i.e. with $m(\alpha) = 1$. Moreover $\sigma \in \mathcal{S}_1$ and therefore has the form

$$\sigma = \sum_{ij\ abelian} a^{ij}(I_\alpha) q_i I_j \quad a^{ij} = -a^{ji} \tag{6.38}$$

where the sum runs over the abelian factors and the form degrees contained in the antisymmetric a^{ij} and in the abelian $I_j = F_j$ have to add up to $D+1$. In particular this anomaly can occur only if the gauge group contains at least two abelian factors because a^{ij} is antisymmetric. In $D = 3$ dimensions a^{ij} is linear in abelian field strengths and one has

$$\sigma = \sum_{ijk\ abelian} a^{ijk} q_i I_j I_k \quad a^{ijk} = a^{ikj} \quad \sum_{cyclic} a^{ijk} = 0 \tag{6.39}$$

This leads to

$$\omega = r_1\sigma = \sum_{ijk\ abelian} b^{ijk} q_i q_j I_k = \sum_{ijk\ abelian} b^{ijk} C_i C_j F_k \quad b^{ijk} = -a^{ijk} + a^{jik} \tag{6.40}$$

and the candidate anomaly is

$$\omega^1_3 = 2\sum_{ijk\ abelian} b^{ijk}\hat{C}_i A_j F_k = \sum_{ijk\ abelian} b^{ijk}\hat{C}_i A_{m\,j} F_{rs\,k}\varepsilon^{mrs} d^3x\ . \tag{6.41}$$

If one considers $g = 1$ and $D = 4$ then $D' = 6$ because it is bounded by $G = D + 1 + g = D + 2$, larger than D and even. This leaves $D' = G$ as only possibility, so the total ghost number is carried by the translation ghosts contained in $\sigma = \sigma(I_\alpha)$ which is a cubic polynomial in the field strength two forms F. Abelian two forms can occur in the combination

$$\sigma = \sum_{ijk\ abelian} d^{ijk} F_i F_j F_k \tag{6.42}$$

with completely symmetric coefficients d^{ijk}. One checks that these polynomials lie in $\mathcal{S}_1$. They lead to the abelian anomaly

$$\omega^1_{4\,abelian} = \frac{3}{4}\sum_{ijk:\,abelian} d^{ijk}\hat{C}_i F_{mn\ j} F_{rs\ k}\varepsilon^{mnrs}d^4x = 3\sum_{ijk:\,abelian} d^{ijk}\hat{C}_i\, d(A_j dA_k) \tag{6.43}$$

Abelian two forms F_i can also occur in σ multiplied with $tr(F_k)^2$ where i enumerates abelian factors and k nonabelian ones. The mixed anomaly which corresponds to

$$\sigma = \sum_{ik} c^{ik}F_i tr_k(F^2) \tag{6.44}$$

is very similar in form to the abelian anomaly

$$\omega^1_{4\,mixed} = -\frac{1}{4}\sum_{ik} c^{ik}\hat{C}_i(\sum_I F_{mn}{}^I F_{rs}{}^I)_k\varepsilon^{mnrs}d^4x\ . \tag{6.45}$$

The sum, however extends now over abelian factors enumerated by i and nonabelian factors enumerated by k. Moreover we assumed that the basis, enumerated by I, of the simple Lie algebras is chosen such that $trM_IM_J = -\delta_{IJ}$ holds for all k. Phrased in terms of dA the mixed anomaly differs from the abelian one because the nonabelian field strength contains also A^2 terms[3].

$$\omega^1_{4\,mixed} = \sum_{ik} c^{ik}\hat{C}_i\, tr_k\, d\left(AdA + \frac{2}{3}A^3\right) \tag{6.46}$$

The last possibility to construct a polynomial σ with form degree $D' = 6$ is given by the Chern form $tr(F)^3$ itself. Such a Chern polynomial with $m = 3$ exists for classical algebras only for the algebras $SU(n)$ for $n \geq 3$ (5.33) [4]. In particular the Lorentz symmetry in $D = 4$ dimensions is not anomalous. The form ω which corresponds to the Chern form is the Chern Simons form

$$\omega(C, F) = tr\left(CF^2 - \frac{1}{2}C^3F + \frac{1}{10}C^5\right)\ . \tag{6.47}$$

The nonabelian anomaly follows after the substitution $C \to C + A$ and after taking the volume form

$$\begin{aligned}\omega^1_{4\,nonabelian} &= tr(\hat{C}F^2 - \frac{1}{2}(\hat{C}A^2F + A\hat{C}AF + A^2\hat{C}F) + \frac{1}{2}\hat{C}A^4)\\ &= tr\left(\hat{C}d\left(AdA + \frac{1}{2}A^3\right)\right)\ .\end{aligned} \tag{6.48}$$

[3]The trace over an even power of one form matrices A vanishes.

[4]The Lie algebra $SO(6)$ is isomorphic to $SU(4)$.

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