Black Hole thermodynamics

- Black holes evaporates
- Black holes have a partition function

For a Schwarzschild black hole, the famous Bekenstein-Hawking results are:

$$\begin{aligned} T &= \frac{1}{8\pi M} \\ S &= \frac{A}{4G} = \frac{4\pi r_+^2}{4G} \end{aligned}$$

Note that the entropy grows like M^2 making the partition function unstable.

A nice derivation at $d = 2$..(not done here)

At d=2, the energy momentum tensor has 3 components, and is fully fixed by its conservation equation

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad \text{(2 equations)}$$

plus the trace anomaly

$$g_{\mu\nu} T^{\mu\nu} = \frac{c}{24\pi} R \qquad \text{(1 equation)}$$

Evaluating on a black hole background (including some regularity condition at horizon), $T^{\mu\nu}$ takes the form of a flux at infinity with a temperature equal to Hawking's result.

For the 2+1 black hole (no rotation, reinserting G)

$$ds^2 = -\left(-8GM + \frac{r^2}{\ell^2}\right) dt^2 + \left(-8GM + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2$$

$$\begin{aligned} T &= \frac{r_+}{2\pi\ell^2} \\ S &= \frac{A}{4G} = \frac{2\pi r_+}{4G} \end{aligned}$$

The entropy grows like $\sqrt{M}$, stable.

Quick derivation of black hole thermodynamics parameters

For any black hole of the form,

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + \ldots .$$

where $f(r)$ has a point $r = r_+$ with,

$$f(r_+) = 0, \qquad \left.\frac{df(r)}{dr}\right|_{r_+} \neq 0$$

we can associate a temperature: $T = \frac{1}{4\pi} f'(r_+)$

Indeed, in the vicinity of the horizon $(a \equiv f'(r_+))$

$$f(r) \simeq a\,(r - r_+), \qquad ds^2 \simeq a(r - r_+)dt^2 + \left(\frac{dr}{\sqrt{a(r - r_+)}}\right)^2$$

and doing $d\rho = \frac{dr}{\sqrt{a(r-r_+)}}$ we find

$$ds^2 = \frac{a^2\rho^2}{4}dt^2 + d\rho^2 \quad \Rightarrow \quad \langle t \rangle = \beta = \frac{4\pi}{f'(r_+)}$$

- For the 2+1 black hole

$$f(r) = -8GM + \frac{r^2}{\ell^2}$$

and

$$T = \frac{1}{4\pi} f'(r_+) = \frac{r_+}{2\pi\ell^2}$$

- For the Schwarzschild black hole

$$f = 1 - \frac{2MG}{r}$$

and

$$T = \frac{1}{4\pi}\frac{2MG}{r_+^2} = \frac{1}{8\pi MG}$$

The entropy

If black holes have mass M, and temperature T, there should be an entropy given by first law,

$$dM = TdS$$

For the 2+1 black hole $\left[-8MG + \frac{r_+^2}{\ell^2} = 0,\ T = \frac{r_+}{2\pi\ell^2}\right]$,

$$\begin{aligned} S &= \int \frac{dM}{T} \\ &= \int \frac{d\left(\frac{r_+^2}{8G\ell^2}\right)}{\left(\frac{r_+}{2\pi\ell^2}\right)} \\ &= \frac{2\pi r_+}{4G} \end{aligned}$$

Not convinced? Proving the first law

If black holes are thermodynamical objects, these results should arise directly from the Euclidean General Relativity partition function, interpreted as a thermodynamical free energy:

$$\begin{aligned} e^{-\beta F(\beta)} &= \frac{1}{Z_0} \int Dg \, e^{\frac{1}{16\pi G} \int \sqrt{g} \left(R + \frac{2}{\ell^2} \right)} \\ &\simeq \frac{1}{Z_0} \, e^{\frac{1}{16\pi G} \int \sqrt{g} \left(R + \frac{2}{\ell^2} \right)} \Big|_{on-shell} \end{aligned}$$

This indeed works. Let us evaluate the exponent on the classical solution, properly including Z_0...

- The black hole has $R = -\frac{6}{\ell^2}$ and $\sqrt{g} = r$
- The on-shell action is

$$
\begin{aligned}
I &= -\frac{1}{16\pi G} \int \sqrt{g} \left(R + \frac{2}{\ell^2} \right) \\
&= -\frac{1}{16\pi G} \int_0^{\beta} dt \int_{r_+}^{\infty} dr \int_0^{2\pi} d\varphi \, r \left(\frac{-4}{\ell^2} \right) \\
&= \frac{1}{2G\ell^2} \int_0^{\beta} dt \int_{r_+}^{\infty} dr \, r
\end{aligned}
$$

This clearly diverges, we need to regularize and subtract the AdS background

Integrating to a large radius L, and subtracting the AdS action,

$$I - I_0 = \frac{1}{2G\ell^2}\int_0^{\beta} dt \int_{r_+}^{L} dr\, r - \frac{1}{2G\ell^2}\int_0^{\beta_0} dt \int_{0}^{L} dr\, r$$

- For AdS the radial coordinate runs from the origin
- Ar large $r = L$, both metrics must coincide with their **proper** periods at infinity calibrated (the same temperature),

$$\left(-8MG + \frac{L^2}{\ell^2}\right)\beta^2 = \left(1 + \frac{L^2}{\ell^2}\right)\beta_0^2$$

This condition does not imply $\beta_0 = \beta$ (even for large L),

$$
\begin{aligned}
I - I_0 &= \frac{1}{4G\ell^2}\left[\beta(L^2 - r_+^2) - \beta_0 L^2\right) \\
&= \frac{\beta}{4G\ell^2}\left[L^2 - r_+^2 - L^2\sqrt{\frac{-r_+^2/\ell^2 + L^2/\ell^2}{1 + L^2/\ell^2}}\right]_{L\to\infty} \\
&= \frac{\beta}{4G\ell^2}\left(-\frac{1}{2}r_+^2\right) + \mathcal{O}(1/L)
\end{aligned}
$$

The free energy is then,

$$F(\beta) = -\frac{\pi^2\ell^2}{2G\beta^2}$$

Now, from the known thermodynamical expressions

- The energy $M = U$,

$$M = F + \beta\frac{\partial F}{\partial\beta} = \frac{\pi^2\ell^2}{2G\beta^2} \quad \rightarrow \quad 8GM = \frac{r_+^2}{\ell^2}$$

- The entropy

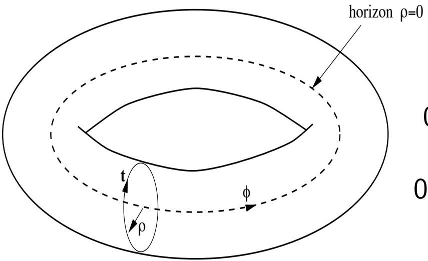
$$S = \beta^2\frac{\partial F}{\partial\beta} = \frac{\pi^2\ell^2}{G\beta} = \frac{2\pi r_+}{4G}$$

Could we not do this calculation in a easier way?

Yes, we can. The magic is provided, again, by the Chern-Simons action,

$$I[A_\mu] = \frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right), \qquad A = A_\mu dx^\mu \in \mathcal{G}.$$

We shall do a much more general calculation: Evaluate the Chern-Simons action for any Lie algebra on a solid torus:



$0 \leq t < \beta$, contractible loop

$0 \leq \phi < 2\pi$, non-contractible loop

The three dimensional spacetime topology can be seen as a

$$\text{torus} \times \Re^+ = \text{disc} \times S_1.$$

The crucial point is that the radial dependence is a gauge transformation. If A is the field representing the black hole, then a new field A_0 can be built,

$$A_0 = U^{-1}(r)\, A\, U(r) + U^{-1}(r) dU(r)$$

such that A_0 does not depend on r.

- Performing this transformation is equivalent to work in the 'radial' gauge

$$A_r = 0$$

- Inserting the solution in the action will produce no IR divergencies

- Some care is necessary with possible singularities.
  A constant field is not necessarily regular!

Interesting (not zero), regular, solutions on the solid torus

Interesting solutions $A_\mu = \{A_t, A_r, A_\phi\} \in SL(N, \Re)$ must satisfy:

1. The Chern-Simons equations of motion $F_{\mu\nu} = 0$
2. Must have a **non-trivial** holonomy along ϕ:

   $$Pe^{\oint A_\phi d\phi} \neq 1$$

   If this holonomy was trivial, the solution can be set to zero by a gauge transformation.
3. Must have a **trivial** holonomy along t.

   $$Pe^{\oint A_t dt} = 1.$$

   If this holonomy is not trivial, the field will be singular.

Building the general solution in radial gauge $A_r = 0$

- $F_{\mu\nu} = 0$ in the gauge $A_r = 0$ imply

$$A_t(t,\phi), \quad A_\phi(t,\phi), \quad \partial_t A_\phi - \partial_\phi A_t + [A_t, A_\phi] = 0.$$

- Furthermore, for black holes, we consider static and spherically symmetric fields. That is, we take A_t, A_ϕ to be constant matrices. The equations reduce to:

$$[A_t, A_\phi] = 0$$

In summary, our game will be to find constant $SL(N, \Re)$ matrices A_ϕ, A_t that commute, and satisfy the holonomy conditions.

$$Pe^{\oint A_\phi d\phi} = e^{2\pi A_\phi} \neq 1, \qquad Pe^{\oint A_t dt} = e^{A_t} = 1$$

Chemical potentials

For a given A_ϕ, we seek A_t such that

$$[A_t, A_\phi] = 0$$

The Cayley-Hamilton theorem imply

$$A_t = \sigma_2 A_\phi + \sigma_3 A_\phi^2 + \cdots + \sigma_N A_\phi^{N-1} - \text{Trace}$$

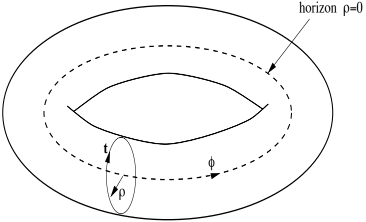
<u>Example $N = 3$.</u> The condition $Pe^{\oint A_t dt} = 1$ becomes:

$$\begin{aligned} 0 &= -16\,\sigma_3^3\,Q_2^3 + 72\,\sigma_3(1+\sigma_2)^2\,Q_2^2 + 54\,\sigma_3^2\,(1+\sigma_2)\,Q_3\,Q_2 \\ &\quad +27\,\sigma_3^3\,Q_3^2 + 27\,(1+\sigma_2)^3\,Q_3\,, \\ 8\,\pi^2 &= 4\,Q_2\,(1+\sigma_2)^2 + 6\,Q_3\,(1+\sigma_2)\sigma_3 + \frac{8}{3}\,Q_2^2\,\sigma_3^2\,. \end{aligned}$$

where $Q_2 = \text{Tr}(A_\phi^2)$, $Q_3 = \text{Tr}(A_\phi^3)$. These 2 equations relate Q_2, Q_3 with σ_2, σ_3, just like the M relates to β in the metric formulation.

Evaluating the action: Angular quantization

Consider the angle ϕ as 'time':



The foliation is regular everywhere

$$
\begin{aligned}
I &= \int (p_i \partial_\phi q^i - \text{constraints}) + B_\infty \\
&= 0 + B_\infty
\end{aligned}
$$

- Spherical symmetry ($\partial_\phi q = 0$) plus the constraints make the bulk part equal to zero.
- We are left with a term at infinity, easily calculated.

Chern-Simons action in angular quantization.

A 2+1 angular decomposition: $x^\mu = \phi, x^\alpha$.

$$\begin{aligned} I_{CS} &= \frac{k}{4\pi}\int \epsilon^{\sigma\nu\rho}\mathrm{Tr}\left(A_\sigma\partial_\rho A_\nu + \frac{2}{3}A_\sigma A_\nu A_\rho\right) \\ &= \frac{k}{4\pi}\int \epsilon^{\alpha\beta}\mathrm{Tr}\left(A_\alpha\partial_\phi A_\beta + A_\phi F_{\alpha\beta}\right) - \frac{k}{4\pi}\int_\infty dt d\phi\, \mathrm{Tr}(A_t A_\phi) \\ &= \quad 0 \quad - \frac{k}{2}\,\mathrm{Tr}(A_t A_\phi). \end{aligned}$$

The last 'touch' is to add a boundary term to get the right Legendre transformation,

$$W(\sigma_2, \sigma_3) = I_{CS} - 2k\sigma_3 Q_3$$

$$\delta W = kQ_2\delta\sigma_2 + kQ_3\delta\sigma_3$$

$$
\begin{aligned}
W(\sigma_2, \sigma_3) &= \frac{k}{4\pi}\int\left(AdA + \frac{2}{3}A^3\right) - \left(2k\sigma_3 Q_3\right) \\
&= 2k\left(3\sigma_2\sigma_3 Q_3 + \frac{4}{3}\sigma_3^2 Q_2^2 + 2(\sigma_2^2 - 1)Q_2 - \sigma_3 Q_3\right).
\end{aligned}
$$

- Consistency check (a fantastic partial derivative exercise)

$$
\frac{\partial W}{\partial \sigma_2} = kQ_2, \qquad \frac{\partial W}{\partial \sigma_3} = kQ_3
$$

These equations are the analog of

$$
M = F + \beta\frac{\partial F}{\partial \beta}
$$

in the metric formulation.

Tomorrow...

Phase transitions in black hole thermodynamics

- Hawking - Page
- First order and second order (higher spin fields)