3d GR is a Chern-Simons theory

An identity in three dimensions

$$\int \sqrt{|g|}\left(R + \frac{1}{\ell^2}\right) =$$

$$\int \epsilon_{abc}(R^{ab}e^c + \frac{1}{\ell^2}e^a e^b e^c) =$$

$$\int \left(A_a dA^a + \frac{1}{3}\epsilon_{abc}A^a A^b A^c\right) - \int \left(\bar{A}_a d\bar{A}^a + \frac{1}{3}\epsilon_{abc}\bar{A}^a \bar{A}^b \bar{A}^c\right) + \int dB$$

with

$$\begin{aligned} \frac{1}{\ell}e^a &= A^a - \bar{A}^a, \\ \epsilon^a{}_{bc}w^{bc} &= A^a + \bar{A}^a. \end{aligned}$$

GR in 2+1 dimensions = to two Chern-Simons theories.

Step by step

1. Go from the metric and Christoffel symbol to vielbeins $e^a{}_\mu$ and spin connections $w^a{}_{b\mu}$

$$\begin{aligned} g_{\mu\nu} &= e^a{}_\mu \eta_{ab} e^b{}_\nu \\ \Gamma^\mu{}_{\nu\rho} &= e^\mu{}_a w^a{}_{b\rho} e^b{}_\nu + e^\mu{}_a \partial_\rho e^a{}_\nu \end{aligned}$$

2. In 2+ 1 dimensions

$$w^{ab}{}_\mu = \epsilon^{ab}{}_c\, w^c{}_\mu$$

3. Replace Riemann curvature and torsion by

$$\begin{aligned} R^{\mu\nu}{}_{\alpha\beta} &\rightarrow R^a = dw^a + \frac{1}{2}\epsilon^a{}_{bc} w^b w^c \\ \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu} &\rightarrow T^a = de^a + \epsilon^a{}_{bc} w^b e^c \end{aligned}$$

The dynamics

- Replace Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \rightarrow \quad R^a + \Lambda \epsilon^a{}_{bc} e^b e^c = 0$$
$$\Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu} = 0 \quad \rightarrow \quad T^a = 0$$

- Introduce the Chern-Simons fields

$$A^a = w^a + \frac{1}{\ell} e^a, \qquad \bar{A}^a = w^a - \frac{1}{\ell} e^a,$$

- The above equations get mapped into

$$F^a = dA^a + \frac{1}{2}\epsilon^a{}_{bc} A^b A^c = 0$$
$$\bar{F}^a = d\bar{A}^a + \frac{1}{2}\epsilon^a{}_{bc} \bar{A}^b \bar{A}^c = 0,$$

two copies of the Chern-Simons equations

The last touch

Let J_a be 3 $sl(2,\Re)$ matrices $\quad (\mathrm{Tr}(J_a J_b) = \eta_{ab})$

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

1. Write $A = A^a J_a$ then the action is

$$I[A] = \frac{k}{4\pi} \int \mathrm{Tr} \left(A dA + \frac{2}{3} A^3 \right),$$

2. Its equations of motion

$$F = dA + AA = 0$$

3. Gauge invariance,

$$A \to A' = U^{-1} A\, U + U^{-1} dU$$

In principle, using the gauge freedom we can set A to zero, but

- If there are non-trivial cycles $P e^{\oint A} \neq 1$
- If there are boundaries

Menu

1. Some interesting solutions
2. Chern-Simons in Hamiltonian form,
3. The constraint, gauge transformations
4. The circle, non-trivial states, Kac-Moody algebra
5. Drinfeld - Sokolov, Virasoro algebra
6. Sources, AdS/CFT interpretation, Correlation functions

A family of static solutions on $M = \Re \times \text{disk}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu = 0$$

Introduce coordinates: $x^\mu = \{\Re, \text{disk}\} = \{\, t, r, \varphi\}$

The field: $A_\mu = \{A_0, A_i\} = \{A_0, A_r, A_\varphi\}$

$$\begin{aligned} F_{0r} &= \partial_0 A_r - \partial_r A_0 + A_0 A_r - A_r A_0 = 0 \\ F_{0\varphi} &= \partial_0 A_\varphi - \partial_\varphi A_0 + A_0 A_\varphi - A_\varphi A_0 = 0 \\ F_{r\varphi} &= \partial_r A_\varphi - \partial_\varphi A_r + A_r A_\varphi - A_\varphi A_r = 0 \end{aligned}$$

A solution to all equations is:

$$A_0 = 0, \qquad A_r = 0, \qquad A_\varphi(\varphi) \neq 0$$

- "Asymptotic" symmetries: $A'_\varphi = A_\varphi + D_\varphi \lambda$ with $\lambda(\varphi)$ maps 'solutions into solutions'.
- Each value for $A_\varphi(\varphi)$ is an independent physical solution.

The Chiral solution

Another, more interesting family of solutions

$$\begin{aligned} F_{0r} &= \partial_0 A_r - \partial_r A_0 + A_0 A_r - A_r A_0 = 0 \\ F_{0\varphi} &= \partial_0 A_\varphi - \partial_\varphi A_0 + A_0 A_\varphi - A_\varphi A_0 = 0 \\ F_{r\varphi} &= \partial_r A_\varphi - \partial_\varphi A_r + A_r A_\varphi - A_\varphi A_r = 0 \end{aligned}$$

$$A_0 = \pm A_\varphi, \qquad A_r = 0, \qquad A_\varphi(t \pm \varphi) \neq 0$$

- Asymptotic symmetry:

$$A'_\varphi = A_\varphi + D_\varphi \lambda$$

with $\lambda(t \pm \varphi)$ maps 'solutions into solutions'.

The set of solutions $A^a_\varphi(t \pm \varphi)$ form an infinite dimensional space with a Poisson bracket structure (Kac-Moody algebra),

$$A^a_\varphi(t \pm \varphi) = \sum_{n=-\infty}^{\infty} T^a_n \, e^{in(t\pm\varphi)}$$

$$[T^a_n, T^b_m] = f^{ab}{}_c \, T^c_{n+m} + k \, n \, g^{ab} \delta_{n+m,0}$$

and energy,

$$E = \int d\varphi \, \mathrm{Tr}(A^2_\varphi)$$

We shall derive this result soon.

Hamiltonian Chern-Simon action

$$A_\mu = \{A_0\,,\,A_i\}, \qquad A_\mu = A^a_\mu\, J_a, \qquad g_{ab} = \mathrm{Tr}(J_a J_b)$$

$$\frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) = \frac{k}{4\pi}\int dt \int d^2x\, \epsilon^{ij}\, g_{ab}\, \left(A^a_i \dot{A}^b_j - A^a_0 F^b_{ij}\right)$$

We deduce:

- the basic Poisson bracket

$$[A^a_i(x), A^b_j(y)] = \frac{2\pi}{k} g^{ab} \epsilon_{ij} \delta^2(x,y), \qquad \epsilon_{ij}\epsilon^{il} = \delta^l{}_i$$

- and constraint

$$F^b_{ij} = 0$$

Boundary details on Functional spaces

According to our general discussion, the constraint generates the gauge transformations via Poisson brackets.
As in any field theory, the generator with a parameter λ^a should then be the integral

$$G_0[\lambda] = \frac{k}{4\pi} \int \lambda_a F^a$$

One should have, then,

$$\delta A_i^a(x) = [A_i^a(x), \frac{k}{4\pi} \int \lambda_b F^b] = D_i \lambda(x)$$

This is far to schematic and not true in general!

Poisson brackets & Functional Variations

$$[A_i^a(x), A_j^b(y)] = \frac{2\pi}{k} g^{ab} \epsilon_{ij} \delta^2(x,y),$$

The Poisson bracket of two functionals $U[A], V[A]$ is

$$[U[A], V[A]] = \frac{2\pi}{k} \int d^2z \, \frac{\delta U[A]}{\delta A_i^a(z)} g^{ab} \epsilon_{ij} \frac{\delta V[A]}{\delta A_j^b(z)}$$

Hence, we need to compute the functional derivative of

$$G_0[\lambda] = \frac{k}{4\pi} \int d^2z \, \lambda_a \, \epsilon^{ij} F^a_{\ ij}$$

$$
\begin{aligned}
\delta G_0[\lambda] &= \frac{k}{4\pi}\int \lambda_a\, \delta F^a \\
&= \frac{k}{2\pi}\int \lambda_a\, D\delta A^a \\
&= \frac{k}{2\pi}\int D\left(\lambda_a \delta A^a\right) - \frac{k}{2\pi}\int D\lambda_a\ \delta A^a \\
&= \frac{k}{2\pi}\oint \lambda_a\ \delta A^a \qquad\quad - \frac{k}{2\pi}\int D\lambda_a\ \delta A^a \\
&= \delta\left(\frac{k}{2\pi}\oint \lambda_a\ A^a\right) - \frac{k}{2\pi}\int D\lambda_a\ \delta A^a
\end{aligned}
$$

Passing the boundary term to the other side we find a functional with wwll defined functional variations

$$
\begin{aligned}
\Rightarrow \delta\left(G_0[\lambda] - \frac{k}{2\pi}\oint \lambda_a\ A^a\right) &= -\frac{k}{2\pi}\int D\lambda_a\ \delta A^a \\
\frac{\delta G[\lambda]}{\delta A^a_i} &= -\frac{k}{2\pi} D\lambda_a
\end{aligned}
$$

The generator is the NOT a constraint!

The functional that has well defined functional derivatives is:

$$
\begin{aligned}
G[\lambda] &\equiv \frac{k}{4\pi}\int \lambda_a F^a - \frac{k}{2\pi}\oint \lambda_a\ A^a \neq 0, \\
\frac{\delta G[\lambda]}{\delta A_i^a} &= -\frac{k}{4\pi}\epsilon^{ij}\ D_i \lambda_a
\end{aligned}
$$

and generates the expected transformation

$$
\delta A_i^a = [A_i^a, G[\lambda]] = \frac{4\pi}{k}\frac{\delta G[\lambda]}{\delta A_i^a} = -D_i \lambda^a
$$

Gauge Farm

$$G[\lambda] = G_0[\lambda] + Q[\lambda], \qquad Q[\lambda] = -\frac{k}{2\pi}\oint \lambda_a A^a$$

- If λ_a is such that $Q = 0$, this is a "proper gauge transformation" that does not change the state.
- If λ_a is such that $Q \neq 0$, this is an "improper gauge transformation" that **does** change the state.

The field $A_\varphi(\varphi)$ requires a non-zero Q transformation to change its values: hence $A_\varphi(\varphi)$ is observable.

The Hamiltonian and Energy: Chiral boundary conditions

The action must also have we-defined functional variations :

$$I[A_i^a, A_0] \;\; = \;\; \frac{k}{4\pi} \int dt \int d^2x \, \epsilon^{ij} \, g_{ab} \left(A_i^a \dot{A}_j^b - A_0^a F_{ij}^b \right) + B$$

- Using the Chiral condition $A_0 = A_\varphi$.

$$\frac{k}{2\pi} \oint A_0 \delta A_\phi = \delta \left(\frac{k}{4\pi} \oint A_\phi^2 \right)$$

- The Hamiltonian is then

$$H = \int , g_{ab} \, A_0^a F^b + \frac{k}{4\pi} \oint A_\phi^2$$

and we identify the energy of configurations (chiral b.c.),

$$E = \frac{k}{4\pi} \oint A_\phi^2$$

Q is conserved.... it is the Noether charge associated to a symmetry

Algebra of generators. The Kac-Moody algebra

$$
\begin{aligned}
\left[G[\lambda], G[\rho]\right] &= \frac{4\pi}{k} \int \frac{\delta G[\lambda]}{\delta A^a(x)} g^{ab} \frac{\delta G[\rho]}{\delta A^b(x)} \\
&= \frac{k}{4\pi} \int D\lambda_a D\rho^a \\
&= -\frac{k}{4\pi} \int \lambda_a DD\rho^a + \frac{k}{4\pi} \int D\left(\lambda_a D\rho^a\right) \\
&= \frac{k}{4\pi} \int f_{abc}\, \lambda^b \rho^c\, F^a + \frac{k}{4\pi} \oint \lambda_a D\rho^a \\
&= \frac{k}{4\pi} \int f_{abc}\, \lambda^b \rho^c\, F^a + \frac{k}{4\pi} \oint f_{abc}\, \lambda^b \rho^c\, A^a + \frac{k}{4\pi} \oint \lambda_a d\rho^a \\
&= G[f_{abc}\, \lambda^b \rho^c] + \frac{k}{4\pi} \oint \lambda_a d\rho^a
\end{aligned}
$$

Algebra of generators. The Kac-Moody algebra

$$
\begin{aligned}
\big[G[\lambda], G[\rho]\big] &= \int \frac{4\pi}{k} \frac{\delta G[\lambda]}{\delta A^a{}_i(x)} \epsilon_{ij} \eta^{ab} \frac{\delta G[\rho]}{\delta A^b{}_j(x)} \\
&= \frac{k}{4\pi} \int d^2x \; \eta^{ab} \underbrace{\epsilon_{ij} \epsilon^{in} \epsilon^{jm}}_{\epsilon^{nm}} (D_n \lambda_a)(D_m \rho_b) \\
&= \frac{k}{4\pi} \int d^2x \; \epsilon^{nm} \partial_n (\lambda_a D_m \rho^a) - \frac{k}{4\pi} \int d^2x \; \epsilon^{nm} \lambda_a D_n D_m \rho^a \\
&= \frac{k}{4\pi} \oint_{r\to\infty} d\varphi \; \lambda_a D_\varphi \rho^a - \frac{k}{4\pi} \int d^2x \; \frac{1}{2} \lambda_a \epsilon^a{}_{bc} F^b{}_{ij} \, \rho^c \epsilon^{ij} \\
&= \frac{k}{4\pi} \oint_{r\to\infty} d\varphi \; \lambda_a \partial_\varphi \rho^a - \frac{k}{4\pi} \oint_{r\to\infty} d\varphi \, f_{abc} \, \lambda^a \rho^c \, A^b{}_\varphi + \\
&\qquad + \frac{k}{8\pi} \int d^2x \, \epsilon_{abc} \lambda^a \rho^c \epsilon^{ij} F^b{}_{ij} \quad , \\
&= G[f^a{}_{bc} \, \lambda^b \lambda^c] + \frac{k}{4\pi} \oint d\varphi \; \lambda_a \partial_\varphi \rho^a
\end{aligned}
$$

Conformal field theory and Virasoro operators

The simplest conformal field theory

$$I[\phi] = \int \frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial\bar{z}} dz d\bar{z} = \int \partial\phi\bar{\partial}\phi \, d^2 z$$

$$\delta\phi = -\epsilon(z)\partial\phi \quad \Rightarrow \quad T(z) = \frac{1}{2}(\partial\phi)^2 \quad \Rightarrow \quad \bar{\partial}T = 0$$

In Laurent modes,

$$T(z) = \sum_{n\in Z} \frac{L_n}{z^{n+2}}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \qquad c = 1$$

Quick derivation: treat $\bar{z}$ as "time". Basic 'equal time' Poisson bracket is

$$[\partial\phi(z,\bar{z}), \phi(w,\bar{z})] = \delta(z-w)$$

For any CFT,

- Variation of T

$$\delta T(z) = 2\partial\epsilon(z)T(z) + \epsilon(z)\partial T(z) - \frac{c}{12}\partial^3\epsilon(z)$$

- Quantum Operator Product Expansion

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \frac{c/2}{(z-w)^4} + \text{reg terms}$$

The central charge c measures the number of degrees of freedom: $c = 1$, one boson, $c = 2$, two bosons....

- We shall extract these formulae from Chern-Simons theory, and compute for GR, the Brown-Henneaux central charge

$$c = \frac{3\ell}{2G}$$

The sl(2) Chern-Simons theory

The field $A_\varphi(z)$ for sl(2) has 3 components

$$A_\varphi = \begin{pmatrix} u(z) & v(z) \\ p(z) & -u(z) \end{pmatrix}, \qquad z = t - \varphi.$$

A very interesting truncation $v(z) = 1, u(z) = 0$

$$A_\varphi = \begin{pmatrix} 0 & 1 \\ \frac{6}{c} T(z) & 0 \end{pmatrix}$$

AdS boundary conditions!

Let is look at "asymptotic symmetries", transformations $\delta A_\varphi = D\lambda$ with

$$\lambda(z) = \begin{pmatrix} q_1(z) & q_2(z) \\ q_3(z) & -q_1(z) \end{pmatrix}$$

that preserve that truncation. See Maple file

AdS_3/CFT_2. Sources, correlation functions & OPE

The AdS/CFT proposal:

$$\int_{\text{CFT}} D\Phi \; e^{I_{\text{CFT}}[\Phi]+\int J\mathcal{O}} = \int_{\text{AdS}} D\phi \; e^{I_{\text{AdS}}[\phi]}$$

The field ϕ in the bulk carries both the source J and the operator $\mathcal{O}$ (its vev).

$$\begin{aligned}
\phi(x,\rho) &= \rho^{\sigma}(\phi_0(x) + \rho\,\phi_1(x) + \rho^2\phi_2(x) + \cdots) \\
\delta I_{\text{AdS}}[\phi] &= \int_{\text{AdS}} (eom)\delta\phi \; + \; \int_{\partial\text{AdS}} \phi_2(x)\delta\phi_0(x) \\
\phi_0(x) &= J(x), \\
\phi_2(x) &= \langle\mathcal{O}(x)\rangle
\end{aligned}$$

The classical equations determine $\phi_2(\phi_0)$ and hence

$$\langle\mathcal{O}(x_n)...\mathcal{O}(x_1)\rangle = \frac{\delta}{\delta\phi_0(x_n)}\cdots\frac{\delta}{\delta\phi_0(x_2)}\phi_2(\phi_0(x_1))$$

Chern-Simons vevs

For Chern-Simons, we look again at the truncated field

$$A_z = \begin{pmatrix} 0 & 1 \\ \frac{6}{c} T(z, \bar{z}) & 0 \end{pmatrix}$$

and turn on the other sector ($\bar{z}$ shows up)

$$A_{\bar{z}} = \begin{pmatrix} \alpha(z, \bar{z}) & \mu(z, \bar{z}) \\ \beta(z, \bar{z}) & -\alpha(z, \bar{z}) \end{pmatrix}$$

The variation of the Chern-Simons action gives

$$\delta I_{CS} = (\text{eom}) + T\delta\mu$$

- T identified with a vev
- μ identified with a source

And the Chern-Simons equations do give $T(\mu)$. See Maple.

Turning on a source μ coupled to T in a CFT

$$e^{W(\mu)} = \int D\phi e^{I[\phi]+\int \mu T}$$

$$\begin{aligned} \frac{\delta W(\mu)}{\delta \mu(z)} &= \langle T(z) \rangle \\ \left.\frac{\delta^2 W(\mu)}{\delta \mu(z) \delta \mu(w)}\right|_{\mu=0} &= \langle \frac{\delta T(z)}{\delta \mu(w)} \rangle = \langle T(z) T(w) \rangle \end{aligned}$$

and we can compute $\frac{\delta T(z)}{\delta \mu(w)}$ because the Chern-Simons equations give us a relation $T(\mu)$

$$\bar{\partial} T = -\frac{c}{12} \partial^3 \mu + \mu \partial T + 2 \partial \mu T$$

The trick is the following:

$$\bar{\partial}\frac{1}{z-w} = 2\pi\delta^{(2)}(z-w), \qquad \frac{1}{z-w} = \frac{2\pi}{\bar{\partial}}\delta^{(2)}(z-w)$$

Let us take the derivative $\frac{\delta}{\delta\mu(w)}$ (and evaluate at $\mu = 0$) of the Ward identity

$$\bar{\partial}T = -\frac{c}{12}\partial^3\mu + \mu\partial T + 2\partial\mu T$$

$$\bar{\partial}\frac{\delta T}{\delta\mu(w)} = -\frac{c}{12}\partial^3\delta^{(2)}(z-w) + \delta^{(2)}(z-w)\partial T + 2\partial\delta^{(2)}(z-w)T(z)$$

Dividing by $\bar{\partial}$ we get

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \frac{c/2}{(z-w)^4}$$