The Hamiltonian formulation of gauge theories

$$I[p,q] = \int dt \left[p_i \dot{q}^i - H(p,q) \right] \qquad \left[\begin{array}{l} \dot{q}^i = \frac{\partial H}{\partial p_i} = [q^i, H] \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} = [p_i, H] \end{array} \right]$$

1. Symplectic geometry, Hamilton-Jacobi theory,...

2. The first (general) quantization method

$$[p_i, q^j] = i\hbar \delta^j{}_i, \qquad H(p,q)$$

3. The Energy $E = H(p,q)$ functional is built-in in the formalism.

Symmetries

Symmetries are displacements $\delta_s q^i$ such that, for all $q^i(t)$,

$$I[q^i + \delta_s q^i] - I[q^i] = \int dt \frac{d}{dt} B_s(q^i, \delta_s q^i) \tag{1}$$

On-shell variations have the property that, for all $\delta q^i(t)$,

$$I[q^i_{os} + \delta q^i] - I[q^i_{os}] = \int dt \frac{d}{dt} (p_i \delta q^i) \tag{2}$$

- In (1), q^i is arbitrary while δq^i is restricted.
- In (2), q^i is restricted while δq^i is arbitrary.

Replacing $q^i = q^i_{os}$ in (1), and $\delta q^i = \delta_s q^i$ in (2), the left hand sides are equal. Subtracting yields Noether theorem,

$$0 = \frac{d}{dt} \left(B_s(q^i, \delta_s q^i) - \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \right)$$

Hamiltonian & inverse Noether Theorem

Let $Q(p,q)$ be a conserved charge,

$$[Q,H]=0. \qquad \text{(Poisson brackets)}$$

Then, the following transformations define a symmetry:

$$\begin{aligned} \delta q^i &= [q^i,Q]=\frac{\partial Q}{\partial p_i} \\ \delta p_i &= [p_i,Q]=-\frac{\partial Q}{\partial q^i} \end{aligned}$$

Indeed, the variation of $\int dt(p\dot{q}-H)$ is a total derivative:

1. $\delta(p_i\dot{q}^i)=\delta p_i\dot{q}^i-\dot{p}_i\delta q^i+\frac{d}{dt}(p_i\delta q^i)=$
   $-\frac{\partial Q}{\partial q^i}\dot{q}^i-\dot{p}_i\frac{\partial Q}{\partial p_i}+\frac{d}{dt}(p_i\delta q^i)=-\frac{dQ}{dt}+\frac{d}{dt}(p_i\delta q^i)$
2. $\delta H=\frac{\partial H}{\partial q^i}\delta q^i+\frac{\partial H}{\partial p_i}\delta p_i=\frac{\partial H}{\partial q^i}\frac{\partial Q}{\partial p_i}-\frac{\partial H}{\partial p_i}\frac{\partial Q}{\partial q^i}=[H,Q]=0.$

Symmetries

Noether Symmetries

$$\frac{dQ(p,q)}{dt} = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, Q] \\ \delta p_i = [p_i, Q] \end{array}$$

We shall now explore gauge symmetries with the properties:

Gauge Symmetries

$$\phi(q,p) = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, \phi] \\ \delta p_i = [p_i, \phi] \end{array}$$

- It may happen that $\delta A_\mu = \partial_\mu \Lambda$ is generated by a non-zero charge Q. If so, we **do not** call it 'gauge'.
- We reserve the word gauge for transformations generated by constraints.

Important gauge theories are

1. Yang-Mills theories (including QED)

$$I_{YM}[A^a_\mu] = -\frac{1}{4}\int \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}), \qquad \delta A^a_\mu(x) = D_\mu \lambda^a(x)$$

2. Einstein Gravity (and its generalizations, $f(R)$ gravity, Gauss-Bonnet, Chern-Simons...)

$$I_{GR}[g_{\mu\nu}] = \int \sqrt{g}(R - 2\Lambda), \qquad \delta g_{\mu\nu}(x) = \xi^\alpha g_{\mu\nu,\alpha} + \xi^\alpha{}_{,\mu} g_{\alpha\nu} + \xi^\alpha{}_{,\nu} g_{\mu\alpha}$$

3. The string worldsheet action

$$I[X^\mu, h_{\sigma\rho}] = \int \sqrt{h} h^{\sigma\rho}\partial_\sigma X^\mu \partial_\rho X^\nu \eta_{\mu\nu}, \qquad \delta X^\mu = \epsilon^\sigma \partial_\sigma X^\mu$$

4. Chern-Simons

$$I[A] = \int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

Examples of Lagrangians with a gauge symmetry in particle mechanics are:

1. The parameterized non-relativistic point particle (its constraint is Schroedinger equation),

$$I[q(\tau), t(\tau)] = \int \left(\frac{1}{2\dot{t}} \dot{q}^2 - \dot{t} V(q) \right) d\tau$$

2. The relativistic point particle (its constraint is Klein-Gordon's equation)

$$I[x^\mu] = -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau$$

Gauge symmetry implies a Hamiltonian constraint

Chern-Simons : $I[A_\mu] = \int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \qquad (\delta A^a_\mu = D_\mu \lambda^a)$

$$= \int \mathrm{Tr}\left[A\dot{A} - A_0 F\right]$$

Yang-Mills : $I[A^a_\mu] = -\frac{1}{4}\int F^{\mu\nu}_a F^a_{\mu\nu}, \qquad (\delta A^a_\mu = D\lambda^a)$

$$= \int \left[E^i_a \dot{A}^a_i - \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) + A^a_0 \nabla \cdot \vec{E}_a\right]$$

Gravity : $I[g] = \int \sqrt{g}R, \qquad (\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu})$

$$= \int \left[\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_\perp - N^i\mathcal{H}_i\right]$$

General structure of a gauge theory in Hamiltonian form:

$$I[p_i, q^j, \lambda^\alpha] = \int dt \left[\, p_i \dot{q}^i - H_0(p,q) - \lambda^\alpha \phi_\alpha(p,q) \, \right]$$

The λ^α-equations of motion are constraints

$$\phi_\alpha(p,q) = 0.$$

There is a gauge symmetry if ϕ_α are zero at all times,

$$\dot{\phi}_\alpha = \frac{\partial \phi_\alpha}{\partial q^i} \dot{q}^i + \frac{\partial \phi_\alpha}{\partial p_i} \dot{p}_i = [\phi_\alpha, H_0 + \lambda^\beta \phi_\beta] = 0.$$

for all $\lambda^\beta(t)$. This happens if and only if,

$$\begin{aligned} [H_0, \phi_\alpha] &= C^\beta_{\ \alpha} \, \phi_\beta \\ [\phi_\alpha, \phi_\beta] &= f^\gamma_{\ \alpha\beta} \, \phi_\gamma \end{aligned}$$

The constraints ϕ_α are said to satisfy a "a first class algebra".

General proof of gauge invariance

Consider a Hamiltonian action of the form,

$$I[p_i, q^j, \lambda^\alpha] = \int dt \left[p_i \dot{q}^i - H_0(p,q) - \lambda^\alpha \phi_\alpha(p,q) \right].$$

If the constraints are first class, then the following transformation is a gauge symmetry of the action,

$$\begin{aligned} \delta q^i(t) &= [q^i, \phi_\alpha] \epsilon^\alpha(t) \\ \delta p_i(t) &= [p_i, \phi_\alpha] \epsilon^\alpha(t) \\ \delta \lambda^\alpha(t) &= -\dot{\epsilon}^\alpha(t) - C^\alpha_{\ \beta} \epsilon^\beta(t) - f^\alpha_{\ \beta\gamma} \lambda^\beta \epsilon^\gamma(t) \end{aligned}$$

where $\epsilon^\alpha(t)$ is a fully arbitrary function of time.

- In QED, the Lagrange multiplier is A_0. Recall that $\delta A_\mu = \partial_\mu \epsilon$ thus $\delta A_0 = \dot{\epsilon}$, as expected. This is an Abelian theory with $C^\alpha_{\ \beta} = 0 = f^\alpha_{\ \beta\gamma}$.

Proof: For any function $A(p,q)$ of the canonical variables: $\delta A(q^i, p_i) = \frac{\partial A}{\partial p_i}\delta p_i + \frac{\partial A}{\partial q^i}\delta q^i = [A, \phi_\alpha]\epsilon^\alpha$. In particular,

$$\begin{aligned} \delta H_0 &= [H_0, \phi_\alpha]\epsilon^\alpha = C^\beta{}_\alpha \phi_\beta\, \epsilon^\alpha \\ \delta \phi_\gamma &= [\phi_\gamma, \phi_\alpha]\epsilon^\alpha = f^\beta_{\gamma\alpha}\phi_\beta\, \epsilon^\alpha \end{aligned}$$

The variation of the kinetic term is

$$\begin{aligned} \delta(p_i \dot{q}^i) &= \delta p_i \dot{q}^i - \dot{p}_i \delta q^i \qquad + \text{total derivative} \\ &= [p_i, \phi_\alpha]\epsilon^\alpha\, \dot{q}^i - \dot{p}_i [q^i, \phi_\alpha]\epsilon^\alpha \\ &= \epsilon^\alpha \left(\frac{\partial \phi_\alpha}{\partial q^i}\dot{q}^i + \frac{\partial \phi_\alpha}{\partial p_i}\dot{p}_i \right) \\ &= \epsilon^\alpha\, \dot{\phi}_\alpha \\ &= -\dot{\epsilon}^\alpha\, \phi_\alpha \qquad + \text{total derivative} \end{aligned}$$

$\Rightarrow$ The variations of H_0, ϕ_α, $p_i \dot{q}^i$
all give terms proportional to the constraints.

Putting all together, the variation of the full action becomes:

$$\begin{aligned} \delta I &= \int \delta(p_i \dot{q}^i) - \delta H_0 - \lambda^\alpha \delta\phi_\alpha - \delta\lambda^\alpha \phi_\alpha \\ &= -\int \left(\dot{\epsilon}^\alpha + C^\alpha_{\ \beta}\epsilon^\beta + f^\alpha_{\ \gamma\beta}\lambda^\gamma \epsilon^\beta + \delta\lambda^\alpha \right) \phi_\alpha. \end{aligned}$$

We can choose $\delta\lambda^\alpha$ to cancel everything making the action invariant.

$$= 0$$

up to total derivatives.

Gauge theories, equivalent classes of solutions

Why two configurations that differ by a deformation generated by a constraint are physically indistinguishable?

$$\begin{aligned} \dot{p}_i &= [p_i, H_0] + [p_i, \phi_\alpha]\lambda^\alpha \\ \dot{q}^i &= [q^i, H_0] + [q^i, \phi_\alpha]\lambda^\alpha \\ \phi_\alpha &= 0 \end{aligned}$$

- Given the fields at time t, the equations fixed them at time $t + \delta t$, only up to a gauge transformation.

These theories seem inconsistent...? Nop. A clever interpretation is available:

Only combinations that do not see λ^α are physical. [λ^α must be unobservable.] We must mod out by the set of all gauge transformations

- The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields ($F_{\mu\nu}$) in QED.
- Curvature invariants, $g^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu}R_{\mu\nu}$, ... in gravity.
- Wilson loops $Pe^{\oint_\gamma A}$ in Yang-Mills theory.

Comment 1. A gauge theory can also have Noether symmetries:

$$I[A] = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu} \qquad \begin{cases} gauge: & \delta A_\mu = \partial_\mu \epsilon(x^\nu) \\ & \\ Lorentz: & \delta A_\mu = \rho^\nu{}_\mu A_\nu, \quad \rho^{\mu\nu} = -\rho^{\nu\mu} \\ Translations: & \delta A_\mu = \rho^\nu \partial_\nu A_\mu \end{cases}$$

Exercise: The (classical) global group of QED (and Yang-Mills) is much larger,

$$\delta A_\mu = F_{\mu\nu}\rho^\nu(x), \qquad \left(\rho_{\mu,\nu} + \rho_{\nu,\mu} = \frac{1}{2}\rho^\sigma{}_\sigma \eta_{\mu\nu}\right) \qquad (3)$$

1. Prove that (3) is a symmetry of the Maxwell action
2. Prove that (3) contains Lorentz, Translations, but also Dilatations, and Special Conformal Transformations (last two broken in QM)
3. Compute the translational Noether current ($\rho^\mu = a^\mu$)

$$J^\mu = \underbrace{\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\nu\right)}_{T^\mu{}_\nu} a^\nu, \qquad T^{\mu\nu}: \begin{cases} \text{Symmetric} \\ \text{Gauge invariant} \end{cases}$$

Comment 2:

Is the scalar field action

$$I[X] = \int \sqrt{h} h^{\mu\nu} \partial_\mu X \partial_\nu X, \quad \Rightarrow \quad \frac{1}{\sqrt{h}} \partial_\mu \left(\sqrt{h} h^{\mu\nu} \partial_\nu X \right) = 0$$

on a curved, but fixed background $h_{\mu\nu}$, gauge invariant?

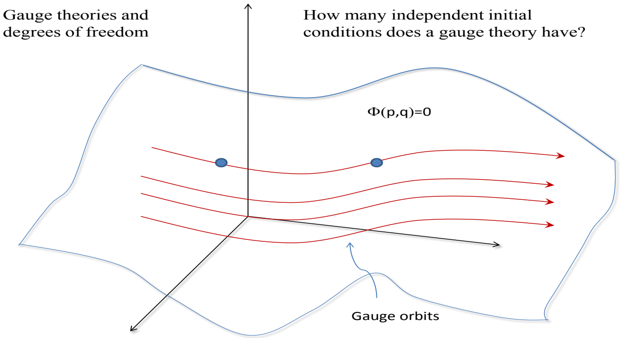
No. There are no constraints, no Lagrange multipliers. No gauge symmetry.

If the metric is dynamical (string worldsheet action)

$$I[X, h_{\mu\nu}] = \int \sqrt{h} h^{\mu\nu} \partial_\mu X \partial_\nu X \Rightarrow \begin{cases} \frac{1}{\sqrt{h}} \partial_\mu \left(\sqrt{h} h^{\mu\nu} \partial_\nu X \right) = 0 \\ \partial_\mu X \partial_\nu X - \frac{1}{2} h_{\mu\nu} h^{\alpha\beta} \partial_\alpha X \partial_\beta X = 0 \\ \quad \upuparrows \quad \text{(Virasoro constraints)} \end{cases}$$

the action is gauge invariant. Varying $h_{\mu\nu}$ yields constraints, and $h_{0\mu}$ are the Lagrange multipliers.

Number of degree of freedom



$2N$ $\{p_i, q^i\}$'s with first order equations: "$2N$ initial conditions."

- g constraints $\phi_\alpha(p, q) = 0$ on initial conditions.
- Two initial conditions related by a gauge are the same.

$$\text{Number of degrees of freedom} \equiv \frac{1}{2}(2N - g - g) = N - g.$$

Examples

- 4d Gravity: $g_{ij} = 6$ functions - 4 symmetries $= 2$
- QED : $A_i = 3$ functions - 1 symmetry $= 2$
- d-dimensional Gravity: $g_{ij} = \frac{(d-1)d}{2}$ functions - d symmetries $= \frac{d(d-3)}{2}$.
- d-dimensional Yang-Mills: $A_i^a = (d-1)N$ fields - N symmetries $= N(d-2)$

“Old” Dirac quantization condition

Quantize q^i, p_j. What is the role of the constraints

$$\hat{\phi}_\alpha = \phi_\alpha(\hat{q}, \hat{p}) \quad ?$$

For example, in particle quantum mechanics, rotations are generated by $\vec{L} = \vec{r} \times \vec{p}$. A rotated state is $\delta|\Psi\rangle = i\vec{\alpha} \cdot \vec{L}|\Psi\rangle$

In a gauge theory, the symmetry is generated by $\hat{\phi}_\alpha$ and the “rotated” state will be

$$\delta|\Psi\rangle = \epsilon^\alpha \hat{\phi}_\alpha |\Psi\rangle$$

But gauge transformations are not observable. States must be invariant. Dirac imposed,

$$\delta|\Psi\rangle = 0 \quad \Rightarrow \quad \hat{\phi}_\alpha |\Psi\rangle = 0$$

Examples of Dirac quantization: Free relativistic particle

$$I[X^\mu(\tau)] = -m\int d\tau\sqrt{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$
$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -m\frac{\dot{X}_\mu}{\sqrt{\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}}$$

$$\left[\begin{array}{c}\text{Invariant under}\\ \tau\to\tau' = f(\tau)\end{array}\right]$$

and it follows directly that

$$p_\mu p^\mu + m^2 = 0,$$

We now quantize $\hat{p}^\mu = i\frac{\partial}{\partial X^\mu}$ and Dirac condition becomes:

$$(-\Box + m^2)\psi = 0, \qquad \text{Kein-Gordon equation}$$

Parametrized non-relativistic particle

(time $t(\tau)$ as a canonical variable)

$$I[\vec{r}(t)] = \int dt \left(\frac{m}{2} \left(\frac{d\vec{r}}{dt} \right)^2 - V(\vec{r}) \right)$$

$$I[\vec{r}(\tau), t(\tau)] = \int d\tau \left(\frac{m}{2} \frac{\dot{\vec{r}}^2}{\dot{t}} - \dot{t} V(\vec{r}) \right), \qquad \left[\begin{array}{c} \text{Invariant under} \\ \tau \to \tau' = f(\tau) \end{array} \right]$$

$$\left. \begin{array}{ll} \vec{p} & = \frac{\partial L}{\partial \dot{\vec{r}}} = m \frac{\dot{\vec{r}}}{\dot{t}}, \\ p_t & = \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2} \frac{\dot{\vec{r}}^2}{\dot{t}^2} - V(\vec{r}). \end{array} \right\} \qquad \phi = p_t + \frac{1}{2m} \vec{p}^2 + V(\vec{r}) = 0$$

Quantize

$$p_t = -i \frac{\partial}{\partial t}, \qquad \vec{p} = i \nabla$$

and the Dirac condition becomes:

$$i \frac{\partial \Psi}{\partial t} = \left(-\frac{1}{2m} \nabla^2 + V \right) \Psi, \qquad \text{Schroedinger equation}$$

The string worldsheet action

...infinitely many degrees of freedom, it requires a detailed analysis:

$$I[X, h_{\mu\nu}] = \int \sqrt{h} h^{\mu\nu} \partial_\mu X \partial_\nu X \;\Rightarrow\; \begin{cases} \frac{1}{\sqrt{h}} \partial_\mu \left(\sqrt{h} h^{\mu\nu} \partial_\nu X \right) = 0 \\ \partial_\mu X \partial_\nu X - \frac{1}{2} h_{\mu\nu} h^{\alpha\beta} \partial_\alpha X \partial_\beta X = 0 \\ \Uparrow \quad \text{(Virasoro constraints)} \end{cases}$$

$$: \partial_\mu X \partial_\nu X - \frac{1}{2} h_{\mu\nu} h^{\alpha\beta} \partial_\alpha X \partial_\beta X : \;\Rightarrow\; \hat{L}_n, \; \hat{\bar{L}}_n$$

Dirac "improved condition" becomes

$$L_n |\Psi\rangle = 0, \qquad n > 0$$

Are gauge and Noether symmetries really disconnected?

- "Global" symmetries, generated by non-zero charges Q_n. Their action change states. $Q_n(\hat{p}, \hat{q})|\Phi\rangle = |\Phi'\rangle$.
- Gauge symmetries, generated by constraints $\phi_\alpha = 0$. Their action do not change physical states. $\phi_\alpha(\hat{p}, \hat{q})|\Phi\rangle = 0$.

However, as we shall see, some "gauge" symmetries on manifolds with boundary are generated by a combination!

$$G(\lambda) = \int d^3x \, \lambda^a \, \phi_a \; + \; Q[\lambda]$$

And they cannot be disentangle. We conclude:

"Gauge Farm"

All gauge transformations have the same interpretation,

but some have more interpretation than others.

AdS_3/CFT_2

The coordinate transformation (see previous Lecture) that maps

$$ds^2 = e^{2\rho} dz d\bar{z} + d\rho^2 + \frac{6}{c}\, T(z) dz^2$$

into

$$ds^2 = e^{2\rho'} dz' d\bar{z}' + d\rho'^2 + \frac{6}{c}\, T'(z') dz^2$$

with

$$T'(z') = T(z) \left(\partial' f\right)^2 - \frac{c}{12}\{f, z'\}.$$

falls precisely in this category.

- Metrics with different values of $T(z)$ do represent different states.
- This explains how AdS_3 –which naively has no degrees of freedom– can be dual to a CFT_2 with infinitely many degrees of freedom.

Stop talking and calculate!

We shall exhibit the "Regge-Teitelboim effect" with the example of Chern-Simons theory which is:

- simple
- yet not trivial (like QCD)
- modern and fun mathematically
- lots of applications
- ...I understand it well

See you tomorrow.