Three-dimensional gravity

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The geometry of spacetime is determined by Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

Well, not quite. The geometry is known once the full curvature tensor

$$R^{\mu}{}_{\nu\alpha\beta}$$

is known. The Einstein equations only fix the Ricci 'trace'

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

The part of $R^{\mu}{}_{\nu\alpha\beta}$ not fixed by Einstein equations are the gravitation waves!.

A one slide Lecture...

In three dimensions, the *full* curvature tensor can be expressed in terms of the Einstein tensor,

$$\begin{aligned} R^{\mu\nu}{}_{\alpha\beta} &= \epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}\left(R^{\gamma}_{\ \rho} - \frac{1}{2}R\delta^{\gamma}{}_{\rho}\right), \\ &= \epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}\ 8\pi G\ T^{\gamma}_{\ \rho} \end{aligned}$$

Hence the full Riemann tensor is determined by matter content.

- In vacuum; $T_{\mu\nu} = 0 \quad \Rightarrow \quad R^{\mu}{}_{\nu\alpha\beta} = 0$. The general solution is

$$ds^2 = -dt^2 + dx^2 + dy^2$$

- Cosmological constant; $8\pi G T_{\mu\nu} = \Lambda g_{\mu\nu}$ the general solution is (anti-)de Sitter space,

$$ds^2 = -\left(1 - \Lambda r^2\right) dt^2 + \frac{dr^2}{1 - \Lambda r^2} + r^2 d\phi^2.$$

Again, there is only one solution (up to trivial symmetries).

We have solved the theory, thank you for your attention.

Is this the whole story?

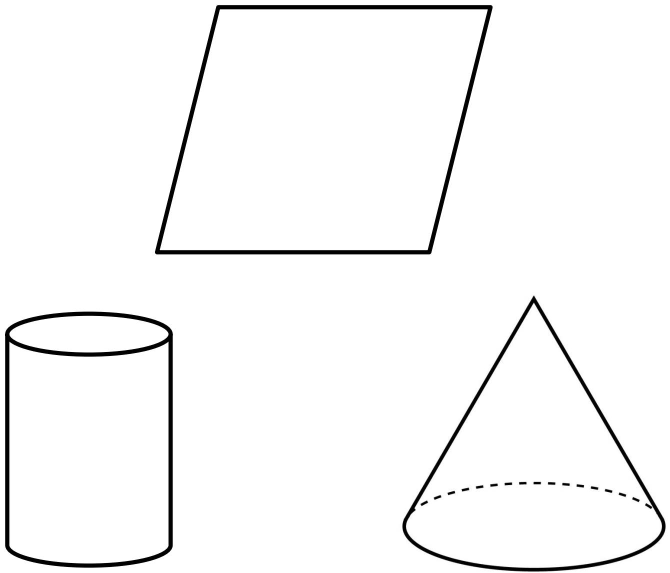
"Two configurations that differ by a gauge transformation are physically the same"

There are (at least) two instances where this statement, while valid, is tricky:

- Non-trivial topological structures. Two configurations may locally be related by gauge transformation but not globally. Example: A plane vs a cylinder.
- Open manifolds with asymptotic conditions. A pure coordinate transformation may actually be non trivial. Example: a black hole with non-zero momentum.

Three dimensional general relativity is locally trivial. However, interesting solutions with "topological excitations", "global charges", or both do exist.

The plane & its descendants, the cylinder, the cone,..



Cylinder, no sources, no fixed points

For example, in vacuum, without attempting a formal classification of solutions, we can write the "groundstate" (universal covering),

$$ds^2 = -dt^2 + dx^2 + dy^2,$$

and the discrete set of "excited states"

- The cylinder: $-\infty < t < \infty, \quad -\infty < x < \infty, \quad 0 \leq y < 2\pi,$
- The torus: $-\infty < t < \infty, \quad 0 \leq x < 2\pi, \quad 0 \leq y < 2\pi,$
- CTC's: $0 < t < 2\pi, \quad 0 \leq x < 2\pi, \quad 0 \leq y < 2\pi,$

These are all locally equivalent, but represent different states, with (very) different weights on a path integral.

Cone, sources, fixed points

Another class of solutions, parameterized by a continuous number α [Deser, Jackiw and t'Hooft (1984)]

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2, \qquad 0 < r < \infty, \quad 0 \leq \theta < 2\pi(1-\alpha)$$

The geometry is now a cone with a singularity at the apex. There is energy momentum and Einstein equations have the form

$$G_{\mu\nu} = \delta^2(r)\, s_{\mu\nu}$$

These solutions represent particles. Multi-particle solutions can also be build. They follow geodesics and have interesting dynamics.

α also represents a global charge, mass, associated to a non-trivial coordinate change.

Quotients of dS

With a positive cosmological constant the "groundstate" is

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2}} + r^2 d\phi^2.$$

having six Killing vectors spanning the group SO(3,1). We can use these Killing vectors to build new solutions starting from dS.

See Deser and Jackiw (1984). ['t Hooft declined participating because the cosmological constant was too "un-physical"]

Quotients of AdS

With a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$ the "groundstate" (universal covering) is

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\phi^2.$$

having six Killing vectors spanning the group SO(2,2). Identifying along Killing vectors, three distinguished cases:

1. Fixed points $\Rightarrow$ particles. Deser and Jackiw (1984).
2. No fixed points $\Rightarrow$ black holes.
3. Self dual: $SO(2,2) = SO(2,1) \times SO(2,1)$. Pick a vector from one SO(2,1) copy. Solutions are **not** asymptotically AdS. Played a (negative) role in an attempt by Witten (arXiv:0706.3359) to understand black hole microstates.

Diffeomorphisms and gauge transformations on spaces with boundary

On manifolds with a boundary, the statement

"All configurations that differ by a gauge transformation are physically indistinguishable"

is not correct. A theory on a manifold with a boundary may contain infinitely many solutions that differ by a 'non-trivial' gauge transformation. It will happen that,

$$A_\mu^{(1)} = A_\mu^{(2)} + \partial_\mu \Lambda$$

and yet, $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are physically distinguishable.

Table of contents

In these Lectures we shall discuss:

The 3d black hole (non-rotating).

Take your favourite symbolic algebra package and check that the metric

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2}} + r^2 d\varphi^2$$

satisfies $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ for any M. It seems that there is a horizon at

$$\frac{r^2}{\ell^2} = M.$$

with area $A = 2\pi\ell\sqrt{M}$. But,

1. Didn't we prove that all solutions at $d = 3$ were just AdS...?
2. Is this a black hole? Is M the mass?
3. What happens for $M < 0$?
4. If I set $M = -1$, we recover AdS_3!

Building the black hole from AdS I

Euclidean case, slightly easier. AdS space in Poincare coordinates:

$$
\begin{aligned}
ds^2 &= \left(1+\frac{r'^2}{\ell^2}\right)dt'^2 + \frac{dr'^2}{1+\frac{r'^2}{\ell^2}} + r'^2 d\varphi'^2 \\
&= \frac{\ell^2}{z^2}(dx^2+dy^2+dz^2) \qquad \text{isometry: } x^\mu \to \lambda x^\mu
\end{aligned}
$$

and this metric can be connected to the black hole via:

$$
\begin{aligned}
x &= \sqrt{1-\frac{M\ell^2}{r^2}}\cos\frac{\sqrt{M}\tau}{\ell}\, e^{\sqrt{M}\varphi} \\
y &= \sqrt{1-\frac{M\ell^2}{r^2}}\sin\frac{\sqrt{M}\tau}{\ell}\, e^{\sqrt{M}\varphi}, \qquad \text{isometry: } x^\mu \to \lambda x^\mu \\
z &= \frac{\sqrt{M}\ell}{r}\, e^{\sqrt{M}\varphi}
\end{aligned}
$$

$$
ds^2 = \left(-M+\frac{r^2}{\ell^2}\right)d\tau^2 + \frac{dr^2}{-M+\frac{r^2}{\ell^2}} + r^2 d\varphi^2
$$

Building the black hole from AdS II. Notations

A circle in $\Re^2$ is defined by the 'surface'

$$x^2 + y^2 = \ell^2$$

1. A good coordinate along the surface is φ with

$$x = \ell \cos\varphi, \qquad y = \ell \sin\varphi$$

2. The induced metric on the 'surface' is

$$ds^2 = dx^2 + dy^2 = \ell^2 d\varphi^2$$

3. The circular symmetry is characterized by the tangent vector

$$\vec{\xi} = x\hat{y} - y\hat{x}, \qquad \vec{\xi}^{\,2} = x^2 + y^2$$

Building the black hole from AdS II. Embedding space

Anti-de Sitter space is the surface on $\Re^4$

$$x^2 + y^2 - u^2 - v^2 = -\ell^2$$

1. A good, global, parametrization to this surface is

$$x = r\cos\varphi, \qquad u = \sqrt{\ell^2 + r^2}\sin t$$
$$y = r\sin\varphi \qquad v = \sqrt{\ell^2 + r^2}\cos t$$

2. Induced metric:

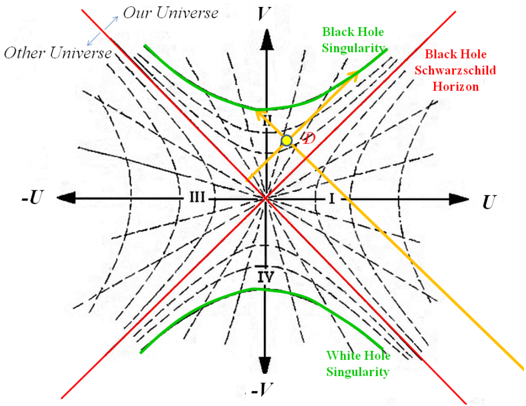
$$\begin{aligned} ds^2 &= dx^2 + dy^2 - du^2 - dv^2, \\ &= -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2 \end{aligned}$$

3. Symmetries (rotations/boosts in 6 planes)

$$\xi_1 = x\hat{y} - y\hat{x} \quad \text{(rotation)}, \qquad \xi_2 = x\hat{u} + u\hat{x} \quad \text{(boost)}, \ldots.$$
$$\xi_1^2 = x^2 + y^2, \qquad \xi_2^2 = x^2 - u^2$$

Let us make a black hole out of this surface!

Kruskal diagram of a black hole



For a black hole, each point is sphere (S_1, in three dimensions)

You can't escape along the orthogonal directions.

Identifications on embedding space

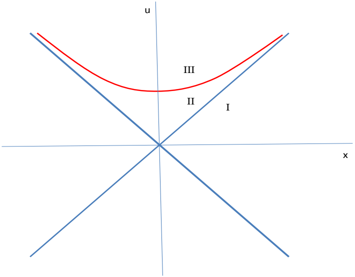
$$v^2 - y^2 = \ell^2 + x^2 - u^2.$$

Pick the Killing vector $\left[\xi^2 = \frac{r_+^2}{\ell^2}(v^2 - y^2),\ r_+ > 0\right]$

$$\vec{\xi} = \frac{r_+}{\ell}(y\hat{v} + v\hat{y}) \quad \Rightarrow \quad u^2 = x^2 + \ell^2\left(1 - \frac{\xi^2}{r_+^2}\right)$$

Split the u, x plane in regions

$$\begin{aligned} \text{I} \quad &: \ \xi^2 > r_+^2 \\ &: \ \xi^2 = r_+^2 \\ \text{II} \quad &: \ 0 < \xi^2 < r_+^2, \\ &: \ \xi^2 = 0 \\ \text{III} \quad &: \ \xi^2 < 0. \end{aligned}$$



Compactify the orbit of ξ. Each point is nowS_1. A black hole!

I: black hole exterior; II: interior; III: degenerated (removed).

The black hole metric

$$-v^2 - u^2 + x^2 + y^2 = -\ell^2$$

and introduce the following coordinates on it:

$$u = \ell \frac{r}{r_+} \cosh \frac{r_+ \varphi}{\ell} \qquad y = \ell \frac{\sqrt{r^2 - r_+^2}}{r_+} \cosh \frac{r_+ t}{\ell^2}$$

$$x = \ell \frac{r}{r_+} \sinh \frac{r_+ \varphi}{\ell} \qquad v = \ell \frac{\sqrt{r^2 - r_+^2}}{r_+} \sinh \frac{r_+ t}{\ell^2}$$

- Why these coordinates? $\xi = \frac{r_+}{\ell}(y\hat{v} + v\hat{y}) = \frac{\partial}{\partial \varphi}$
  The identification is simply:

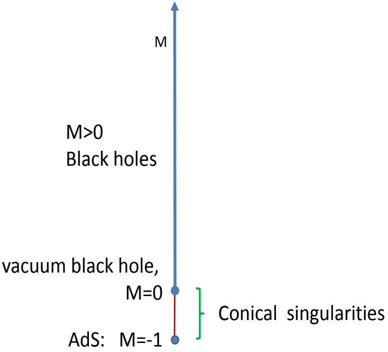
$$\varphi \sim \varphi + 2\pi\, n$$

- The induced metric is the black hole ($M = r_+^2$)

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + d\varphi^2$$

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + d\varphi^2$$

- $M \geq 0$: black holes
- $-1 < M < 0$:
  conical singularities
- $M = -1$: AdS
  the ground state



For $M = -\alpha^2$ there is no horizon. The metric near the origin is conical

$$ds^2 \simeq -\alpha^2 dt^2 + \frac{dr^2}{\alpha^2} + r^2 d\varphi^2$$

A more general Killing vector with two parameters,

$$\xi = r_+ \left(y \frac{\partial}{\partial v} + v \frac{\partial}{\partial y} \right) + r_- \left(x \frac{\partial}{\partial u} + u \frac{\partial}{\partial x} \right)$$

yields the rotating black hole

$$ds^2 = -(-M + r^2)dt^2 + \frac{dr^2}{-M + r^2 + \frac{J^2}{4r^2}} + Jdtd\varphi + r^2 d\varphi^2$$

with mass and angular momentum

$$M = r_+^2 + r_-^2, \qquad J = 2r_+ r_-$$

and two horizons $r_\pm$.

The three-dimensional black hole:

- Has two horizons provided $|J| < M$. It has an ergosphere.
- has mass M and angular momentum J
- Has Hawking temperature $T = \frac{r_+^2 - r_-^2}{2\pi r_+}$ which vanishes if $r_+ = r_-$.
- Has Bekenstein-Hawking entropy $S = \frac{2\pi r_+}{4G}$

Some history:

- This construction is sometimes referred to as BTZ black hole, after the paper MB, C. Teitelboim, and J. Zanelli (1992), Phys.Rev.Lett.69:1849.
- The correct name should be BHTZ. The identifications and understanding of the solution appeared in MB, M. Henneaux, C. Teitelboim and J. Zanelli (1993), Phys.Rev.D48: 1506.
- Actually, BBHZ!

Is M the mass? In what sense?
Is J an angular momentum? In what sense?

To understand the black hole parameters we need to introduce:

1. Gauge theories
2. Constraints
3. 'non-trivial' gauge transformations
4. Noether charges for non-trivial transformations

A quick look at advanced features

- Topological nature of black hole parameters
- Boundary interpretation for M and J

M and J as holonomies

The charges M and J do have a topological meaning [D. Cangemi, M. Leblanc and R. Mann (1993)]. In a Chern-Simons formulation [Achúcarro and Townsend (1986), Witten (1988)],

$$\frac{1}{16\pi G}\int\sqrt{g}\left(R+\frac{2}{\ell^2}\right)=\frac{k}{4\pi}\int\left(AdA+\frac{2}{3}A^3\right)-\frac{k}{2\pi}\int\left(\bar{A}d\bar{A}+\frac{2}{3}\bar{A}^3\right)$$

The black hole manifold has a non-contractible loop.
The holonomies are

$$\mathrm{Tr}e^{\oint A}=e^{2\pi\sqrt{M+J}}\neq 1,\qquad \mathrm{Tr}e^{\oint\bar{A}}=e^{2\pi\sqrt{M-J}}\neq 1$$

Only for anti-de Sitter space,

$$M=-1,\qquad J=0,$$

the holonomies are trivial.

A quick derivation the Brown-Henneaux conformal symmetry

Start with AdS_3 in “Poincare” complex coordinates:

$$ds^2 = e^{2\rho} dz d\bar{z} + d\rho^2$$

In these coordinates the (asymptotics) black hole is

$$ds^2 = e^{2\rho} dz d\bar{z} + d\rho^2 + (M + J) dz^2 + (M - J) d\bar{z}^2 + \cdots$$

Brown and Henneaux discovered that this metric can be upgraded to (while still being a solution)

$$ds^2 = e^{2\rho} dz d\bar{z} + d\rho^2 + T(z) dz^2 + \bar{T}(\bar{z}) d\bar{z}^2 + \cdots$$

where $T(z)$ and $\bar{T}(\bar{z})$ are arbitrary functions of their arguments.

A coordinate transformation with lots of interpretation

$$ds^2 = e^{2\rho} dz d\bar{z} + d\rho^2 + T(z) dz^2$$

This $\uparrow$ is an exact solution for any $T(z)$

$$z = f(z') \quad \to \quad = e^{2\rho}\partial' f \, dz' d\bar{z} + d\rho^2 + T(z)(\partial' f)^2 dz'^2$$

We can eliminate the Jacobian in the first term by doing

$$e^{2\rho}\partial' f = e^{2\rho'}, \qquad \bar{z} = \bar{z}' - \frac{1}{2} e^{-2\rho'} \frac{\partial'^2 f}{\partial' f}.$$

and go back to exactly the metric we started from

$$ds^2 = e^{2\rho'} dz' d\bar{z}' + d\rho'^2 + T'(z') dz'^2,$$

where –note the Schwarz derivative(!)–

$$T'(z') = T(z)\left(\partial' f\right)^2 - \frac{c}{12}\{f, z'\}.$$

This is just a coordinate change.... but has a nice interpretation!
(We have also included c here. See later for explanation.)

Diffeomorphisms Farm

Coordinate transformations in GR are (Hamiltonian) generated by

$$H[\epsilon] = \int_\Sigma \epsilon^\mu \mathcal{H}_\mu + \int_{\partial\Sigma} \epsilon^\mu q_\mu; \qquad \mathcal{H}_\mu = 0, \quad q_\mu \neq 0$$

The boundary integral absorbs pieces from the bulk when computing $[H, X]$. Two classes of coordinate transformations [T.Regge and C.Teitelboim (1974)]

1. If $\epsilon(x)$ is such that $\int_{\partial\Sigma} \epsilon^\mu q_\mu = 0$. These are true "gauge transformations". They do not change the physical state
2. If $\epsilon(x)$ is such that $\int_{\partial\Sigma} \epsilon^\mu q_\mu \neq 0$. These are not "gauge transformations" but a global symmetry with a conserved charge $H \neq 0$. They do change the physical state: $H|\Psi\rangle \neq 0$.

The conformal transformations discussed above precisely fall on that class.

- The asymptotic symmetry group of three dimensional gravity is the conformal group with a non-zero central charge,

$$c = \frac{3\ell}{2G}.$$

- For **any** unitary modular invariant 2d CFT, the number of states consistent with given (large) $L_0, \bar{L}_0$ is

$$\rho(T, \bar{T}) = e^{2\pi\sqrt{\frac{c}{6}L_0}+2\pi\sqrt{\frac{c}{6}\bar{L}_0}}.$$

- Plug the black hole values for $L_0, \bar{L}_0$ and obtain exactly Bekenstein-Hawking entropy [Strominger (1997)]

$$\rho(M, J) = e^{\frac{2\pi r_+}{4G}}.$$

- Note the crucial role of the central charge c.

4d playground

The following metrics are solutions to Einstein equation in four dimensions [Aminneborg, Bengtsson, Holst, Peldan (1996);Vanzo (1997)]

$$g = 0: \quad ds^2 = -\left(1 - \frac{2m}{r} + \frac{r^2}{l^2}\right) + \frac{dr^2}{1 - \frac{2m}{r} + \frac{r^2}{l^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$g = 1: \quad ds^2 = -\left(0 - \frac{2m}{r} + \frac{r^2}{l^2}\right) + \frac{dr^2}{0 - \frac{2m}{r} + \frac{r^2}{l^2}} + r^2(dx^2 + dy^2)$$

$$g: \quad ds^2 = -\left(-1 - \frac{2m}{r} + \frac{r^2}{l^2}\right) + \frac{dr^2}{-1 - \frac{2m}{r} + \frac{r^2}{l^2}} + r^2(d\theta^2 + \sinh^2\theta d\varphi^2)$$