Exact S-matrices

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Abstract. The aim of these notes is to provide an elementary introduction to some of the basic elements of exact S-matrix theory. This is a large subject, and only the beginnings will be covered here. A particular omission is any serious discussion of the Yang-Baxter equation; instead, the focus will be on questions of analytic structure, and the bootstrap equations. Even then, what I have to say will only be a sketch of the simpler aspects. The hope is to give a hint of the many curious features of scattering theories in 1+1 dimensions.

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1 Introduction – what's so special about 1+1?

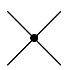
To get things started, I want to describe a particularly simple calculation that can be done in probably the simplest nontrivial quantum field theory imaginable, namely $\lambda\phi^4$ theory in a universe with only one spatial dimension.

The Lagrangian to consider is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ ,$$

resulting in the Feynman rules

$$= \frac{i}{p^2 - m^2 + i\epsilon}$$



$$= -i\lambda$$

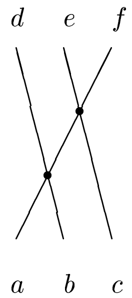
The task is to calculate the connected $2 \to 4$ production amplitude, at tree level. Actually, to keep track of the diagrams it is a little easier to look at the $3 \to 3$ process, leaving implicit the understanding that one of the *out* momenta will be crossed to *in* at the end. I'll label the three *in* particles as a, b, c, and the three *out* particles as d, e, f, and opt to cross c from *in* to *out* later. It also helps to adopt light-cone coordinates from the outset, using

$$(p, \bar{p}) = (p^0 {+} p^1 , p^0 {-} p^1)$$

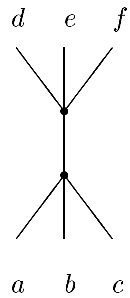
and then solving the mass-shell condition $p\bar{p} = m^2$ by writing the *in* and *out* momenta as

$$p_a = (ma, ma^{-1}) \quad , \quad p_b = (mb, mb^{-1})$$

and so on, with $a, b, \ldots$ real numbers, positive for particles travelling forwards in time. In terms of these variables, the crossing from $3 \to 3$ to $2 \to 4$ amounts to a continuation from c to $-c$. For the $3 \to 3$ amplitude there are just two classes of diagram:



(A)



(B)

The internal momentum in (A) is $p = m(a{+}b{-}d, a^{-1}{+}b^{-1}{-}d^{-1})$, and so its propagator contributes

$$\frac{i}{p^2 - m^2} = \frac{i}{m^2} \frac{1}{(a{+}b{-}d)(a^{-1}{+}b^{-1}{-}d^{-1}) - 1} = \frac{i}{m^2} \frac{-abd}{(a{+}b)(a{-}d)(b{-}d)}$$

to the total scattering amplitude. Given the agreement above that one of the *out* momenta is actually in-going, this propagator is never on-shell, and so forgetting about the $i\epsilon$ does not cause any error. The same remark applies to diagram (B), for which

$$\frac{i}{p^2 - m^2} = \frac{i}{m^2} \frac{1}{(a{+}b{+}c)(a^{-1}{+}b^{-1}{+}c^{-1}) - 1} = \frac{i}{m^2} \frac{abc}{(a{+}b)(a{+}c)(b{+}c)} \ .$$

Adding these together, with a brief pause to check that the diagrams have been counted correctly, yields the full result at tree level:

$$\langle out | in \rangle_{\rm tree} = -\frac{i\lambda^2}{m^2} A_{\rm legs} H(a,b,c,d,e,f)$$

where $A_{\rm legs}$ contains all the factors living on external legs and so on that will be the same for all diagrams, and

$$H(a,b,c,d,e,f) = \sum_{\substack{cycl\{abc\} \\ cycl\{def\}}} \frac{-abd}{(a+b)(a-d)(b-d)} + \frac{abc}{(a+b)(b+c)(c+a)} \ ,$$

with the sum running over all cyclic permutations of $\{a,b,c\}$ and $\{d,e,f\}$.

Now I need the following fact:

$$\text{If} \qquad a+b+c=d+e+f \quad \text{and} \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{d}+\frac{1}{e}+\frac{1}{f}$$
$$\text{then} \qquad H(a,b,c,d,e,f)\equiv -1\ .$$

The two conditions are the so-far ignored conservation of left- and right- lightcone momenta. The formula makes no mention of the signs of the arguments to H, and certainly holds with c negative. The conclusion:

- In 1+1-dimensional $\lambda\phi^4$ theory, the $2\to 4$ amplitude is a *constant* at tree level.

It is now very tempting to cancel this amplitude completely, by adding a term

$$-\frac{1}{6!}\frac{\lambda^2}{m^2}\phi^6$$

to the original Lagrangian. In 1+1 dimensions this does not spoil renormalisability, and gives a theory in which the $2\to 4$ amplitude *vanishes* at tree level. With $\beta^2=\lambda/m^2$, the new Lagrangian is

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2-\frac{m^2}{\beta^2}\left[\frac{1}{2}\beta^2\phi^2+\frac{1}{4!}\beta^4\phi^4+\frac{1}{6!}\beta^6\phi^6\right]\ .$$

This is already curious, but it is possible to go much further. Calculating the $2\to 6$ amplitude (left as an exercise for the energetic reader) should reveal that *this* is now constant, ready to be killed off by a judiciously-chosen ϕ^8 term, and so on. At each stage a residual constant piece can be removed by a (uniquely-determined) higher-order interaction. Keep going, and infinitely-many diagrams later you should find

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2-\frac{m^2}{\beta^2}\left[\cosh(\beta\phi)-1\right]\ ,$$

the sinh-Gordon Lagrangian. Sending β to $i\beta$ converts this into the well-known sine-Gordon model, to which the discussion will return in later lectures.

The claim of uniqueness just made deserves a small caveat. I began the calculation with no ϕ^3 term in the initial Lagrangian, and a discrete $\phi\to-\phi$ symmetry which persisted throughout. But what if I had instead started with a nonzero ϕ^3 term, and tried to play the same game? This is definitely a harder problem, but the final answer can be predicted with a fair degree of confidence:

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{3!}\lambda\phi^3-\frac{1}{4!}\frac{3\lambda^2}{m^2}\phi^4-\dots\\&=\frac{1}{2}(\partial\phi)^2-\frac{m^2}{6\beta^2}\left[e^{2\beta\phi}+2e^{-\beta\phi}-3\right]\ ,\end{aligned}$$

where this time $\beta=\lambda/m^2$. The special properties of this Lagrangian have been been noticed by various authors over the years, the earliest probably being M.Tzitzéica, in an article published in 1910.

Exercise: verify the relationship between the ϕ^3 and ϕ^4 couplings in the Lagrangian just given by means of a tree-level calculation.
Suggestion: consider a $2 \to 3$ process with both *in* momenta equal to $(1,1)$, and one of the *out* momenta equal to $(1{+}\delta,(1{+}\delta)^{-1})$ with δ small. For the desired result it will suffice to demand that the contributions to the amplitude proportional to δ^{-2} cancel once all relevant diagrams have been added together. Even this is a little subtle...

One last comment on the uniqueness question: it is easy to see that all possibilities for a single interacting massive scalar field with no tree-level production have now been exhausted. Starting with a ϕ^3 or ϕ^4 interaction term must, if it works at all, lead to one of the two theories just discussed: the higher ϕ^m couplings are uniquely determined by the need to cancel the constant part of the $2 \to m{-}2$ production amplitude. On the other hand, if both the ϕ^3 and ϕ^4 couplings are set to zero, then the same argument shows that all higher couplings must also be zero, and the theory is free.

To summarise, it appears that in 1+1 dimensions there are some interacting Lagrangians with the remarkable property that the resulting field theories have no tree-level particle production. Araf'eva and Korepin showed, in 1974, that for the sinh-Gordon model this is also true at one loop. The tree-level result was a sign of interesting classical behaviour; that it persists to one loop is evidence that the quantum theory might also be rather special.

Before continuing along this line, I want to return to the $3 \to 3$ amplitude. Should we conclude that its connected part is also zero? Contrary to initial expectations, the answer to this question is a definite *no.* For the $3 \to 3$ process, it is no longer legitimate to forget about the $i\epsilon$'s. For the diagrams of type (A), the intermediate particle can now be on-shell, and when this happens the $i\epsilon$ must be retained until all contributing diagrams have been added together. This is relevant whenever the set of ingoing momenta is equal to the set of outgoing momenta, and in such situations it turns out that the final result is indeed nonzero. Thus the connected part of the $3 \to 3$ amplitude does not vanish, but it does contain an additional delta-function which enforces the equality of the initial and final sets of momenta. We have found a model for which, at least at tree level, the connected $3 \to 3$ amplitude violates at certain points two of the usual assumptions made of an analytic S-matrix:

• it is *not* found by crossing the $2 \to 4$ amplitude;

• it is *not* analytic in the residual momenta once overall momentum conservation has been imposed.[1]

Even more remarkably, the interaction, while nontrivial, affects the participating particles in a minimal way: it does not change their momenta. It is clear that something odd is going on, but it is not so clear quite what, and even less

[1] It should be mentioned that developments in the theory of analytic S-matrices have included a general understanding of phenomena such as these, which are not restricted to 1+1 dimensions. See, for example, Chandler (1969) and Iagolnitzer (1973,1978a). The 1+1 dimensional case is treated in Iagolnitzer (1978b).

clear why. Evaluating yet more Feynman diagrams is unlikely to shed much light on these questions, and besides, an infinite amount of work would be needed before we could be completely sure that any of these properties feature in the full quantum theory. A more sophisticated approach is needed. What could force these amplitudes to vanish, irrespective of the structure of the Feynman diagrams? One possibility is that conservation laws might limit the set of *out* states accessible from any given *in* state. The far-reaching consequences of this idea are the subject of the next lecture.

2 Conserved quantities and factorisability

After the somewhat informal introduction, the time has come to be a little more precise, at least to the extent of pausing to set up some notation.

First, I should allow for more than one particle type, so different masses m_a, m_b and so on make an appearance. A single particle of mass m_a will be on-shell when its light-cone momenta p_a, $\bar{p}_a$ satisfy $p_a\bar{p}_a = m_a^2$. It will be convenient to solve this equation not via the variable $a = p_a/m_a$ used in the last lecture, but rather via a parameter $\theta = \log a$ called the rapidity. Thus,

$$p_a = m_a e^{\theta_a} \quad , \quad \bar{p}_a = m_a e^{-\theta_a} \ .$$

Recall that a was a positive real number for the forward component of the mass shell; this corresponds to θ ranging over the entire real axis. The backwards component of the mass shell, found by negating a, can be parametrised by this same rapidity so long as it is shifted onto the line $\mathrm{Im}\,\theta = \pi$. This will be relevant when discussing the crossing of amplitudes.

An n-particle asymptotic state can now be written as

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)\rangle_{\substack{in\\out}}$$

where the symbol $A_{a_i}(\theta_i)$ denotes a particle of type a_i, travelling with rapidity θ_i. By smearing the momenta a little so as to produce wavepackets, each particle can be assigned an approximate position at each moment. In a massive theory, the only sort of theory I will be bothering with, all interactions are short-ranged and so the state behaves like a collection of free particles except at times when two or more wavepackets overlap. All of this can be made more precise, but not in these lectures.

An *in* state is characterised by there being no further interactions as $t \to -\infty$. This means that the fastest particle must be on the left, the slowest on the right, with all of the others ordered in between. It is convenient to represent this situation by giving the $A_{a_i}(\theta_i)$ a life outside the $|\,\rangle_{in}$ and $|\,\rangle_{out}$ ket vectors, thinking of them as noncommuting symbols with their order on the page reflecting the spatial ordering of the particles that they represent. Thus an *in* state would be written

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)$$

with

$$\theta_1 > \theta_2 > \ldots > \theta_n \ .$$

Similarly, an *out* state has no further interactions as $t \to +\infty$, and so each particle must be to the left of all particles travelling faster than it, and to the right of all particles travelling slower. In terms of the non-commuting symbols, one such state is

$$A_{b_1}(\theta_1)A_{b_2}(\theta_2)\ldots A_{b_n}(\theta_n)$$

now with

$$\theta_1 < \theta_2 < \ldots < \theta_n \ .$$

Products of the symbols with other orderings of the rapidities can be thought of as representing states at other times when all the particles are momentarily well-separated. Asymptotic completeness translates, at least partially, into the claim that any such product can be expanded either as a sum of products in the *in*-state ordering, or as a sum of products in the *out*-state ordering.

The S-matrix provides the mapping between the *in*-state basis and the *out*-state basis. In the new notation this reads, for a two-particle *in*-state,

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = \sum_{n=2}^{\infty} \sum_{\theta'_1<\ldots<\theta'_n} S_{a_1 a_2}^{b_1 \ldots b_n}(\theta_1,\theta_2;\theta'_1\ldots\theta'_n)A_{b_1}(\theta'_1)\ldots A_{b_n}(\theta'_n) \ ,$$

where $\theta_1 > \theta_2$, a sum on $b_1 \ldots b_n$ is implied, and the sum on the θ'_i will generally involve a number of integrals, with the rapidities appearing additionally constrained by the overall conservation of left- and right- lightcone momenta:

$$m_{a_1}e^{\pm\theta_1} + m_{a_2}e^{\pm\theta_2} = m_{b_1}e^{\pm\theta'_1} + \ldots + m_{b_n}e^{\pm\theta'_n} \ .$$

The notation works because the number of dimensions of space, namely 1, matches the 'dimensionality' of a sequence of symbols in a line of mathematics; it can't be used for higher-dimensional theories. However, at this stage it makes no mention of integrability, and can be set up for any massive quantum field theory in 1+1 dimensions.[2]

Next, to the conserved quantities. One such is energy-momentum, a spin-one operator. In lightcone components this acts on a one-particle state as

$$P|A_a(\theta)\rangle = m_a e^{\theta}|A_a(\theta)\rangle \quad , \quad \bar{P}|A_a(\theta)\rangle = m_a e^{-\theta}|A_a(\theta)\rangle \ .$$

Beyond this, operators can be envisaged transforming in higher representations of the 1+1 dimensional Lorentz group:

$$Q_s|A_a(\theta)\rangle = q_a^{(s)}e^{s\theta}|A_a(\theta)\rangle \ .$$

[2] One caveat, though: in nonintegrable theories amplitudes for the scattering of wavepackets usually depend on impact parameters as well as momenta. Thus in general the notation should not be taken too literally, but rather used as a shorthand for recording momentum space results. In integrable cases we'll see shortly that this dependence goes away, and so I can afford to be a little careless about this point.

The integer s is called the (Lorentz) spin of Q_s. Since $Q_{|s|}$ transforms as s copies of P, and $Q_{-|s|}$ as s copies of $\bar{P}$, it makes sense to think of Q_s and Q_{-s} as rank $|s|$ objects. The simple 'left-right' splitting is special to 1+1 dimensions.

I'll only consider those operators Q_s that come as integrals of local densities, and this has the important consequence that their action on multiparticle wavepackets is additive:

$$Q_s|A_{a_1}(\theta)\dots A_{a_n}(\theta_n)\rangle = (q^{(s)}_{a_1}e^{s\theta_1} + \dots + q^{(s)}_{a_n}e^{s\theta_n})|A_{a_1}(\theta)\dots A_{a_n}(\theta_n)\rangle \ .$$

These are called local conserved charges and they are all in involution (they commute) since, essentially by assumption, they have been simultaneously diagonalised by the basis of asymptotic multiparticle states that I have chosen. This is not inevitable: nonlocal charges, often associated with fractional-spin operators, can be very important. The papers of Lüscher (1978), Zamolodchikov (1989c) and Bernard and Leclair (1991) are good starting-points for those interested in this aspect of the subject.

Even without the more exotic possibilities, the consequences of the extra local conserved charges are profound. In fact, Coleman and Mandula showed in 1967 that in three spatial dimensions the existence of even just one conserved charge transforming as a tensor of second or higher rank forces the S-matrix of the model to be trivial. (For a simple-minded explanation of this fact, see later in this lecture.) This is not true in 1+1 dimensions, but nevertheless the possibilities for the S-matrix are severely limited: it must be consistent with

- no particle production;
- equality of the sets of initial and final momenta;
- factorisability of the $n \to n$ S-matrix into a product of $2 \to 2$ S-matrices.

The first two of these properties sum up the behaviour which had emerged experimentally by the end of the last lecture, and the third is a bonus, rendering the task of finding the full S-matrices of a whole class of 1+1 dimensional models genuinely feasible.

I shall outline a couple of arguments for why these properties should follow from the existence of the conserved charges.

The first simply imposes the conservation of the charges directly. Consider an $n \to m$ amplitude, with ingoing particles $A_{a_1}(\theta_1), \dots, A_{a_n}(\theta_n)$, and outgoing particles $A_{b_1}(\theta'_1), \dots, A_{b_m}(\theta'_m)$. If a charge Q_s is conserved, then an initial eigenstate of Q_s with a given eigenvalue must evolve into a superposition of states all sharing that same eigenvalue. For the amplitude under discussion this implies that

$$q^{(s)}_{a_1}e^{s\theta_1} + \dots + q^{(s)}_{a_n}e^{s\theta_n} = q^{(s)}_{b_1}e^{s\theta'_1} + \dots + q^{(s)}_{b_n}e^{s\theta'_m} \ .$$

Now if conserved charges Q_s exist for infinitely many values of s, then there will be infinitely many such equations, and for generic *in* momenta the only way to satisfy them all will be the trivial one, namely $n = m$ and, perhaps after a reordering of the *out* momenta,

$$\theta_i = \theta'_i \quad ; \quad q^{(s)}_{a_i} = q^{(s)}_{b_i} \qquad i = 1 \dots n \ ,$$

where s runs over the spins of the non-trivial conserved charges with nonzero spin (or over all the nonzero integers, if we agree to set $q^{(s)} \equiv 0$ for those s at which a local conserved charge cannot be defined). This does not quite imply that the outgoing set of labels, $\{b_1, \dots b_n\}$, is equal to the ingoing set $\{a_1, \dots a_n\}$ – they just need to agree about the values of all of the nonzero spin conserved charges. Nevertheless, it is enough to establish the absence of particle production, and the equality of the initial and final sets of momenta, though factorisability is harder to see from this point of view. One caveat should also be mentioned: in many models, it turns out that there are some solutions to the conservation constraints with $n \neq m$. However these are only found for exceptional sets of ingoing momenta, which are unphysical to boot, so this fact does not change the conclusions for the S-matrix. (In fact, they are associated with solutions to the conserved charged bootstrap equations, a topic to be discsussed in a later lecture.) A more severe problem comes with the realisation that this argument hasn't escaped the infinite workload mentioned at the end of the last lecture. Consider, for example, a two-particle collision. As the relative momenta of the incident particles increases, the number of particles permitted energetically in the out state grows without limit. To be absolutely sure that, no matter how fast the two particles are fired at each other, only two particles will come out, infinitely many conservation constraints are needed. This might not matter – practical considerations are always going to limit the relative momenta to which we have access – were we not ambitious enough to hope for an *exact* formula for the S-matrix. This requires an understanding of all energy scales, and so the infinite amount of work appears to be unavoidable.

This should be motivation enough for the second argument, which can be found in a 1980 article by Parke, itself building on an observation which dates back at least to Shankar and Witten (1978). The argument also establishes factorisability and imposes the Yang-Baxter equation on the two-particle S-matrix. The key is to make use of the fact that we're dealing with a local, causal quantum field theory, by considering the effect of the conserved charges on localised wavepackets.

First take a single-particle state, with position space wavefunction

$$\psi(x) \propto \int_{-\infty}^{\infty} dp e^{-a^2(p-p_1)^2} e^{ip(x-x_1)} .$$

This describes a particle with spatial momentum approximately p_1, and position approximately x_1. Act on this with an operator giving a momentum-dependent phase factor $e^{-i\phi(p)}$. The wavefunction becomes

$$\tilde{\psi}(x) \propto \int_{-\infty}^{\infty} dp e^{-a^2(p-p_1)^2} e^{ip(x-x_1)} e^{-i\phi(p)} .$$

Most of the integral comes from $p \approx p_1$, and $\phi(p)$ can be expanded in powers of $(p-p_1)$ to find $\tilde{p}_1$ and $\tilde{x}_1$, the revised values of the momentum and position:

$$\tilde{p}_1 = p_1 \quad , \quad \tilde{x}_1 = x_1 + \phi'(p_1) .$$

For a multiparticle state a product of one-particle wavefunctions will be a good approximation when the particles are well separated, and on such a state $|p_a p_b \ldots\rangle$, the action is to shift the position of particle a by $\phi'(p_a)$, that of b by $\phi'(p_b)$, and so on.

Strictly speaking, for compatibility with the earlier discussions I should now consider the actions of the operators $Q_{|s|}$ and $Q_{-|s|}$, as Parke did in his article. However the essentials of the argument will be conveyed if I instead assume the conservation of operators P_s acting on one-particle and well-separated multiparticle states as $(P_1)^s$, with P_1 the spatial part of the two-momentum operator. Acting with $e^{-i\alpha P_s}$, the phase factor is $\phi_s(p) = \alpha p^s$, so a particle with momentum p_a will have its position shifted by $s\alpha p_a^{s-1}$. The case $s = 1$, momentum itself, just translates every particle by the same amount α. But, crucially, for $s > 1$ particles with different momenta are moved by different amounts.

The argument continues as follows. First consider a $2 \to m$ process, labelled as in figure 1.

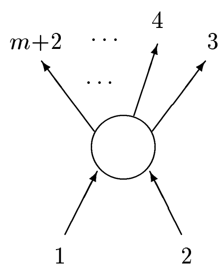


Fig. 1. A $2 \to m$ process

For the amplitude to be non-vanishing, the time when the first two particles collide, call it t_{12}, must precede the time t_{23} when the trajectory of particle 2, the slower incomer, intersects that of particle 3, the fastest outgoer:

$$t_{12} \leq t_{23} \ .$$

Why should this be so? Nothing can happen until the wavepackets of particles 1 and 2 overlap. After this, it suffices to follow the path of the rightmost particle until all have separated in order to establish the inequality. Note that this could be violated on microscopic timescales, but not macroscopically: hence the term 'macrocausality' for this sort of property.

The constraint is rendered vastly more powerful if there is a conserved higher-spin charge P_s in the model. Since it must commute with the S-matrix, we have

$$\langle final|S|initial\rangle = \langle final|e^{i\alpha P_s} S e^{-i\alpha P_s}|initial\rangle$$

and so $e^{i\alpha P_s}$ can be used to rearrange the initial and final configurations without changing the amplitude. All that remains – and for this you should consult Parke's article – is to show that if any of the outgoing rapidities are different from θ_1 or θ_2, then shifting the configurations around in this way will give a pattern of trajectories for which $t_{12} > t_{23}$. By macrocausality the amplitude for this pattern must vanish, and then by the insensitivity of the amplitudes to shifts induced by $e^{i\alpha P_s}$ all of the other amplitudes, including the one initially under consideration, must also vanish. Hence the only possibilities for the two incoming particles are two outgoing particles with the same pair of rapidities as before the interaction, which is the result required for n=2.

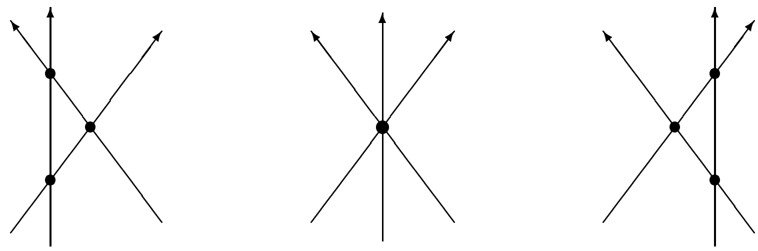
To complete the missing step, Parke actually needed to assume the existence of *two* extra charges of higher spin. However, since a parity-conjugate pair Q_s, Q_{-s} will do, this is scarcely a problem, at least in parity-symmetric theories.

For completeness, I should mention that there is a quicker argument for this $2 \to m$ amplitude, to be found in Polyakov (1977), which revives the previous line of reasoning, though with a slight twist. As previously noted, if the first argument is attempted with the time in figure 1 running up the page, more and more conserved charges will be needed as m increases in order to eliminate all the undesired possibilities for the final configuration. But by T-invariance, the $2 \to m$ amplitude will only be nonvanishing if the same is true of the time-reversed $m \to 2$ amplitude. But now there are just two outgoing momenta, and these are fixed, up to a discrete ambiguity, by energy-momentum conservation and the on-shell condition. After this, *any* extra charge will suffice to eliminate the process. Economical as this argument is, it does not cover the general $m \to n$ amplitude, and factorisability and the Yang-Baxter equation are missed.

One other aside before moving on: however the higher-spin conserved charges are used to reshuffle the positions of an incident pair of particles, if their rapidities differ then their trajectories will still cross somewhere. This is special to 1+1 dimensions: with more than one spatial dimension to play with, conserved higher spin charges can be used to make trajectories miss each other completely, even on macroscopic scales. It is then but a short step to deduce that the S-matrix must be trivial – and this, in admittedly sketchy form, is an argument for the Coleman-Mandula theorem alluded to earlier.

To deal with three incoming particles, consider first how the trajectories would look were there no interactions in the model. Figure 2 shows the three distinct possibilities – which one actually occurs depends on the particular spatial positions of the incident wavepackets.

In cases 1 and 3, when we switch the interaction back on again the results just established for two incident particles, together with locality, are enough to see that the pictures do not change in any essential way. Furthermore, as the interaction proceeds by a series of two-body collisions, these amplitudes must *factorise* into products of $2 \to 2$ amplitudes. Case 2 in general would give something new. However, using $e^{-i\alpha P_s}$ in the manner discussed at length above, it can be converted into one of the other cases. Hence there is never any particle production, individual momenta are conserved, and the amplitudes



(1) (2) (3)

Fig. 2. Possibilities for a $3 \to 3$ process

always factorise. In addition, the equality of amplitudes 1 and 3 gives a constraint on the two-body amplitudes, known as the Yang-Baxter equation. More on this in the next lecture, once the necessary notation has been set up.

To go beyond three incoming particles, an inductive argument can be used, showing that a set of n incident particles can always be shuffled around in such a way that the interaction occurs via a sequence of events in which at most $n{-}1$ particles are participating.

The ultimate conclusion is that in any local scattering theory in 1+1 dimensions with a couple of local higher-spin conserved charges (and a parity-conjugate pair $\{Q_s, Q_{-s}\}$, $s > 1$, will certainly do), there is no particle production, the final set of momenta is equal to the initial set, and the $n \to n$ S-matrix factorises into a product of $2 \to 2$ S-matrices. These are the three properties promised earlier, and now they can be established with only a finite amount of work.

Finally, I would like to mention a mild paradox that might at first sight seem troubling. If $\{p'_1 \dots p'_n\} = \{p_1 \dots p_n\}$ for every set of ingoing momenta, then surely $\sum (p_a)^s$ is conserved for all s, resulting in conserved charges at *all* spins, in any model for which the arguments above apply? This reasoning misses a key feature of the objects we are dealing with: for Q_s to qualify as a local conserved charge, it must be possible to write it as the integral of a local conserved density:

$$Q_s = \int_{-\infty}^{\infty} T_{s+1} dx \ .$$

There is no a priori reason why such a density should exist, even if the sums $\sum (p_a)^s$ happen to be conserved. In fact, the set of spins s at which this can be done forms a rather good fingerprint for a model, and turns out to constrain its behaviour in important ways.

3 The two-particle S-matrix

Once the two-particle S-matrix is known, factorisability tells us that the entire S-matrix follows. To find the two-particle S-matrix becomes the main goal. In the algebraic notation of the last lecture, we can write

$$|A_i(\theta_1)A_j(\theta_2)\rangle_{in} = S_{ij}^{kl}(\theta_1 - \theta_2)|A_k(\theta_1)A_l(\theta_2)\rangle_{out}$$

as

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)A_l(\theta_2)A_k(\theta_1) \ ,$$

with $\theta_1 > \theta_2$ to ensure that *in* and *out* states are correctly represented. A sum over k and l is implied, with $k \neq i$ and $l \neq j$ being possible in those situations where some particles are not distinguished by the $Q_{s\neq 0}$ conserved charges. Lorentz boosts shift rapidities by a constant, and so S only depends on the difference $\theta_1 - \theta_2 = \theta_{12}$.

$S_{ij}^{kl}(\theta_1 - \theta_2) \quad =$

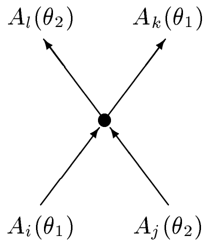


Fig. 3. The two-particle S-matrix

In a theory with r different particle types, knowledge of the r^4 functions $S_{ij}^{kl}(\theta)$ will thus give the full S-matrix. Not all of these functions are independent, and their analytic properties are heavily constrained. Such general features are the subject of this lecture.

First, as just mentioned, in an integrable model the matrix element S_{ij}^{kl} can only be nonzero if A_i and A_k, and A_j and A_l, agree on the values of all of the local conserved charges with nonzero spin (which, in particular, requires $m_i = m_k$ and $m_j = m_l$). Next, the assumptions of P, C and T invariance imply

$$S_{ij}^{kl}(\theta) = S_{ji}^{lk}(\theta) \quad ; \quad S_{ij}^{kl}(\theta) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta) \quad ; \quad S_{ij}^{kl}(\theta) = S_{lk}^{ji}(\theta) \ .$$

Analytic properties of the S-matrix are usually discussed in terms of the Mandelstam variables s, t and u:

$$s = (p_1 + p_2)^2 \ , \quad t = (p_1 - p_3)^2 \ , \quad u = (p_1 - p_4)^2 \ ,$$

with $s+t+u = \sum_{i=1}^{4} m_i^2$. In 1+1 dimensions only one of these is independent, and it is standard to focus on s, the square of the forward-channel momentum. In terms of the rapidity difference $\theta_{12} = \theta_1 - \theta_2$,

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta_{12} \ .$$

For a physical process, θ_{12} is real and so s is real and satisfies $s \geq (m_i + m_j)^2$. But we can consider the continuation of $S(s)$ up into the complex plane. Placing the branch cuts in the traditional way, this results in a function with the following properties:

• S is a singlevalued, meromorphic function on the complex plane with cuts on the portions of the real axis $s \leq (m_i - m_j)^2$ and $s \geq (m_i+m_j)^2$. Physical values of $S(s)$ are found for s just above the right-hand cut. This first sheet of the full Riemann surface for S is called the physical sheet.

• S is real-analytic: it takes complex-conjugate values at complex-conjugate points:

$$S_{ij}^{kl}(s^*) = \left[S_{ij}^{kl}(s)\right]^* .$$

In particular $S(s)$ is real if s is real and $(m_i-m_j)^2 \leq s \leq (m_i+m_j)^2$.

The situation is depicted in figure 4.

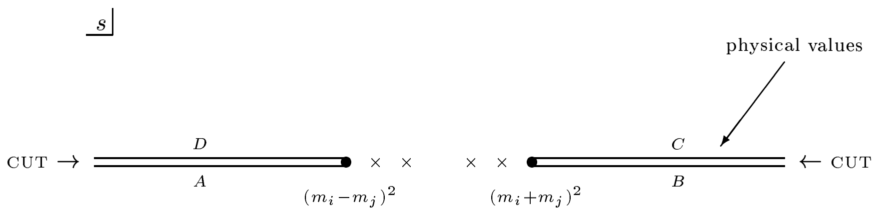


Fig. 4. The physical sheet

Unitarity requires that $S(s^+)S^\dagger(s^+) = 1$ whenever s^+ is a physical value for s, just above the right-hand cut: $s^+ = s + i0$, $s > (m_i + m_j)^2$. This should be understood as a matrix equation, with a sum over a complete set of asymptotic states hiding between S and $S^\dagger$. As s^+ grows, it becomes energetically possible for states with more and more particles to participate in the sum. Generally this brings the $2 \to m$ S-matrix elements into the story with $m = 3, 4, \ldots$, and gives the $2 \to 2$ S-matrix elements a series of branch points along the real axis, located at the $3, 4, \ldots$ particle thresholds. However for an integrable model these production amplitudes should all be zero, and so for all physical s^+ unitarity reads

$$S_{ij}^{kl}(s^+) \left[S_{kl}^{nm}(s^+)\right]^* = \delta_i^n \delta_l^m .$$

With the help of real analyticity this can be rewritten as

$$S_{ij}^{kl}(s^+) S_{kl}^{nm}(s^-) = \delta_i^n \delta_l^m ,$$

with $s^- = s - i0$, just below the right-hand cut. This equation shows the need for a branch cut running rightwards from the two-particle threshold $s = (m_i+m_j)^2$; if we accept that the cut actually starts at this threshold, then it is easy to see that the branch is of square-root type. The argument goes as follows. Let $S_\gamma(s)$ be the function obtained by analytic continuation of $S(s)$ once anticlockwise around the branch point. Unitarity amounts to the requirement that $S(s^+)S_\gamma(s^+) = 1$ for all physical values of s^+. When written in this way, the relation can be analytically continued to all s, so

$$S_\gamma(s) = S^{-1}(s) \ .$$

In particular, if s^- is a point just below the cut, then

$$S_\gamma(s^-) = S^{-1}(s^-) = S(s^+) \ ,$$

the last equality following from a second application of unitarity. Now $S_\gamma(s^-)$ is just the analytic continuation of $S(s^+)$ twice around $(m_i{+}m_j)^2$. Therefore, twice round the branch point gets us back to where we started, and the singularity is indeed a square root.

So much for the right-hand cut. The left-hand half of the figure, containing the second cut running in the opposite direction, can be understood via the fundamentally relativistic property of crossing. If one of the incoming particles, say j, is crossed to become outgoing while simultaneously one of the outgoers, say l, crosses in the opposite sense and becomes ingoing, then the amplitude for another physical two-particle scattering process results. For this new amplitude the incomers are i and $\bar{l}$, and the outgoers k and $\bar{\jmath}$, where an overbar has been introduced to denote the (possibly trivial) operation of conjugation on particle labels. All of this amounts to looking at figure 3 from the side, with the forward-channel momentum now not s but rather $t = (p_1 - p_3)^2$. In this particular case $p_3 = p_2$, and the relation between t and s is very simple:

$$t = (p_1 - p_2)^2 = 2p_1^2 + 2p_2^2 - (p_1 + p_2)^2 = 2m_i^2 + 2m_j^2 - s \ .$$

Crossing symmetry states that the amplitude for this process can be obtained by analytic continuation of the previous amplitude into a region of the s plane where t becomes physical, that is $t \in \mathbb{R}$ and $t \geq (m_i{+}m_j)^2$. Physical amplitudes correspond to approaching this line segment from above in the t plane, and hence from below in the s plane. Thus the amplitudes are on the lower edge of the left-hand cut, marked A on figure 4. In equations:

$$\underset{\text{(on } C)}{\underset{\uparrow}{S_{ij}^{kl}(s^+)}} = \underset{\text{(on } A)}{\underset{\uparrow}{S_{i\bar{l}}^{k\bar{\jmath}}(2m_i^2 + 2m_j^2 - s^+)}} \ .$$

Clearly the cross-channel branch point at $(m_i - m_j)^2$ must also be a square root, but this does *not* mean that the Riemann surface for $S(s)$ has just two sheets. Continuing through the left-hand cut can, and usually does, connect with a different sheet from that found through the right-hand cut. Stepping up and down to left and right, the typical $S(s)$, even for an integrable model, lives on an infinite cover of the physical sheet.

This looks rather complicated, but simplifies considerably if, following Zamolodchikov, attention is switched from the Mandelstam variable s to the rapidity difference θ. The transformation is

$$\begin{aligned}\theta &= \cosh^{-1}\left(\frac{s-m_i^2-m_j^2}{2m_im_j}\right)\\ &= \log\left[\frac{1}{2m_im_j}\left(s-m_i^2-m_j^2+\sqrt{(s-(m_i+m_j)^2)(s-(m_i-m_j)^2)}\right)\right]\end{aligned}$$

and it maps the physical sheet into the region

$$0 \leq \operatorname{Im}\theta \leq \pi$$

of the θ plane called the physical strip. Most importantly, the cuts are opened up, so that $S(\theta)$ is analytic at the images 0 and $i\pi$ of the two physical-sheet branch points, and also at the images $in\pi$ of the branch points on all of the other, unphysical, sheets. Since, by integrability, these are expected to be the only branch points, S is a meromorphic function of θ. The other sheets are mapped onto a succession of strips

$$n\pi \leq \operatorname{Im}\theta \leq (n+1)\theta\ .$$

The new image of the Riemann surface is shown in figure 5.

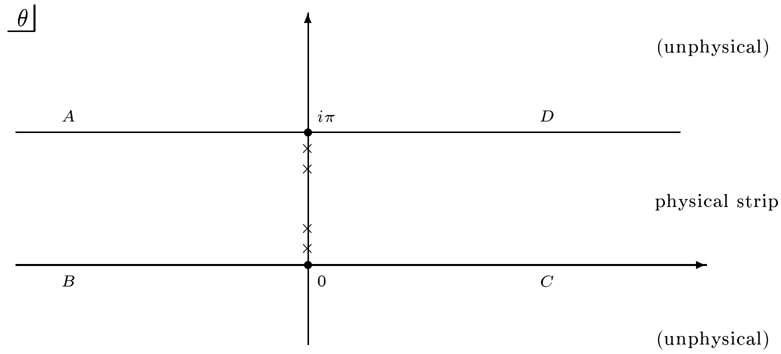


Fig. 5. The θ plane

The previous relations can now be translated to give a list of constraints on $S(\theta)$ to be carried forward into later lectures:

- Real analyticity: $S(\theta)$ is real for θ purely imaginary;
- Unitarity: $S_{ij}^{nm}(\theta)S_{nm}^{kl}(-\theta) = \delta_i^k\delta_j^l$;
- Crossing: $S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi-\theta)$.

A couple of remarks: first, both the unitarity and the crossing equations can now be analytically continued, and apply to the whole of the θ plane, not just along the line segments of physical values. Second, the unitarity constraint means that it is consistent to extend the algebraic relation

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1-\theta_2)A_l(\theta_2)A_k(\theta_1)$$

to $\theta_1 < \theta_2$. Unitarity then becomes a consequence of the algebra, and the single-valued nature of products of the non-commuting symbols.

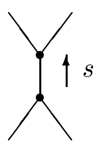
Finally to some unfinished business from the previous lecture. Shifting trajectories showed that the amplitudes (1) and (3) of figure 2 must be equal. If the two-particle S-matrix is not completely diagonal, this equality is not automatic but instead results in the following consistency condition:

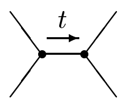
$$S_{ij}^{\beta\alpha}(\theta_{12})S_{\beta k}^{n\gamma}(\theta_{13})S_{\alpha\gamma}^{ml}(\theta_{23}) = S_{jk}^{\beta\gamma}(\theta_{23})S_{i\gamma}^{\alpha l}(\theta_{13})S_{\alpha\beta}^{nm}(\theta_{12})\ ,$$

where $\theta_{ab}=\theta_a-\theta_b$, and θ_1, θ_2 and θ_3 are the rapidities of particles i, j and k. This is the Yang-Baxter equation, forced by the ability of the conserved charges to shift particle trajectories around. In theories where particles appear in multiplets transforming under some symmetry group, this equation together with some minimality assumptions is often enough to conjecture the complete functional form of S. The equation is equivalent to associativity for the algebra of the $A_i(\theta)$'s: moving from $A_i(\theta_1)A_j(\theta_2)A_k(\theta_3)$ to a sum of products $A_l(\theta_3)A_m(\theta_2)A_n(\theta_1)$, the result is independent of the order of the pair transpositions, if and only if the Yang-Baxter equation holds for the two-particle S-matrix elements.

4 Pole structure and bound states

The remaining features of figures 4 and 5 are the crosses marked between the two thresholds. The first things one might expect to find in these locations are simple poles corresponding to stable bound states, appearing either in the forward (s) or the crossed (t) channel:





This is potentially important – for example, it might signal the presence of hitherto unsuspected particles in the spectrum of the model. Most of the remaining lectures will be spent on this point. I'll start by recalling a selection of reasons why the association between simple poles in an S-matrix and bound states is natural:

• Potential scattering: in quantum mechanics, if the S-matrix for the scattering of a particle off a potential has a pole – which, as it happens, is always simple

– then it is possible to use it to *construct* a wavefunction for the particle bound to the potential;

- tree-level Feynman diagrams;
- an 'axiom', justified if by nothing else by experience in 3+1 dimensions.

It turns out that life isn't so simple in 1+1 dimensions. To explain this I'll use the grandparent of all integrable field theories, the sine-Gordon model. Take the sinh-Gordon Lagrangian introduced in the first lecture, and replace β by $i\beta$. Re-zeroing the energy of the classical ground state, you will find

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

with

$$V(\phi) = \frac{m^2}{\beta^2}\left[1 - \cos(\beta\phi)\right] \ .$$

There is extra structure here, as compared to the sinh-Gordon model, since there are infinitely-many classical vacua, $\phi(x) = 2n\pi/\beta$, $n \in \mathbb{Z}$. There is a conserved, spin-zero topological charge Q_0:

$$Q_0 = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \partial_x\phi dx$$

which is non-zero for configurations which interpolate between different vacua. (Formally the charge can also be defined, and is conserved, for the sinh-Gordon model – the only problem is that it is identically zero for all finite-energy configurations.)

Classically, the model has a soliton s with $Q_0 = +1$, and an antisoliton $\bar{s}$ with $Q_0 = -1$, both with mass M, say, and both interpolating between neighbouring vacua. There are no classically stable solutions with $|Q_0| > 1$. Solitons repel solitons, antisolitons repel antisolitons, but solitons and antisolitons attract. Therefore the classical theory additionally sees a continuous family of so-called breather solutions, which are $s\bar{s}$ bound states. Although not static, they are periodic in time and in most respects behave just like further particle states. Their 'masses' range from 0 (tightly-bound) to $2M$ (almost unbound).

In the quantum theory, the breather spectrum becomes discrete, just as would be expected from quantum mechanics. If s and $\bar{s}$ have mass M, then the breather masses are

$$M_k = 2M\sin\frac{\pi k}{h} \ , \qquad k = 1, 2, \ldots < \frac{8\pi}{\beta^2} - 1$$

where

$$h = \frac{16\pi}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right) \ .$$

This was found by Dashen, Hasslacher and Neveu in 1975 via a semiclassical quantisation of the two-soliton solution, and is thought to be exact. Notice that as $\beta \to 0$, corresponding to the classical limit, the continuous breather spectrum is recovered.

The S-matrix elements of the solitons provide an illustration of the notational technology set up earlier. The model turns out to possess higher-spin conserved charges, and so all of the previous discussions apply. However at generic values of β^2 none of them breaks the $\phi \to -\phi$ symmetry of the original Lagrangian, and so none can be used to distinguish the soliton from the antisoliton. That leaves Q_0, which makes a fine job of distinguishing a single soliton from a single antisoliton but, as we shall now see, is not quite powerful enough when acting on two-particle states to rule out nondiagonal scattering.

Consider a general two-particle in-state $|A(\theta_1)_{s,\bar{s}}A(\theta_2)_{s,\bar{s}}\rangle_{in}$, each particle either a soliton or an antisoliton. The higher-spin charges can be used in the ways explained earlier to show that any out-state into which this state evolves must again contain two particles with rapidities θ_1 and θ_2, each either a soliton or an antisoliton. Thus before recourse is made to the spin-zero charge, a four-dimensional space of out-states is available. The topological charge Q_0 acts in this space as follows:

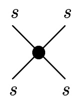
$$Q_0 \begin{pmatrix} |A_s(\theta_1)A_s(\theta_2)\rangle \\ |A_s(\theta_1)A_{\bar{s}}(\theta_2)\rangle \\ |A_{\bar{s}}(\theta_1)A_s(\theta_2)\rangle \\ |A_{\bar{s}}(\theta_1)A_{\bar{s}}(\theta_2)\rangle \end{pmatrix} = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix} \begin{pmatrix} |A_s(\theta_1)A_s(\theta_2)\rangle \\ |A_s(\theta_1)A_{\bar{s}}(\theta_2)\rangle \\ |A_{\bar{s}}(\theta_1)A_s(\theta_2)\rangle \\ |A_{\bar{s}}(\theta_1)A_{\bar{s}}(\theta_2)\rangle \end{pmatrix} .$$

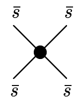
The soliton-soliton and antisoliton-antisoliton states are picked out uniquely, and therefore must scatter diagonally. The same cannot be said for the remaining two states, and it is through this loophole that nondiagonal scattering enters the story.

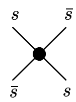
Taking charge conjugation symmetry into account, there are just three independent amplitudes to be determined. With $\theta = \theta_1 - \theta_2$, these can be written as:

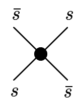
$$\begin{pmatrix} |A_s(\theta_1)A_s(\theta_2)\rangle_{in} \\ |A_s(\theta_1)A_{\bar{s}}(\theta_2)\rangle_{in} \\ |A_{\bar{s}}(\theta_1)A_s(\theta_2)\rangle_{in} \\ |A_{\bar{s}}(\theta_1)A_{\bar{s}}(\theta_2)\rangle_{in} \end{pmatrix} = \begin{pmatrix} S(\theta) & & & \\ & S_T(\theta) & S_R(\theta) & \\ & S_R(\theta) & S_T(\theta) & \\ & & & S(\theta) \end{pmatrix} \begin{pmatrix} |A_s(\theta_1)A_s(\theta_2)\rangle_{out} \\ |A_s(\theta_1)A_{\bar{s}}(\theta_2)\rangle_{out} \\ |A_{\bar{s}}(\theta_1)A_s(\theta_2)\rangle_{out} \\ |A_{\bar{s}}(\theta_1)A_{\bar{s}}(\theta_2)\rangle_{out} \end{pmatrix} .$$

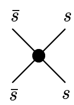
The same information can be given pictorially:

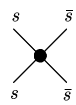
$S(\theta) =$ 

$=$ 

$S_T(\theta) =$ 

$=$ 

$S_R(\theta) =$ 

$=$ 

and also using the noncommuting symbols:

$$A_s(\theta_1)A_s(\theta_2) = S(\theta)A_s(\theta_2)A_s(\theta_1)$$
$$A_s(\theta_1)A_{\bar{s}}(\theta_2) = S_T(\theta)A_{\bar{s}}(\theta_2)A_s(\theta_1) + S_R(\theta)A_s(\theta_2)A_{\bar{s}}(\theta_1)\ .$$

Unitarity and crossing constrain these amplitudes. As a simple exercise, it is worthwhile to check that unitarity amounts to

$$S(\theta)S(-\theta) = 1$$
$$S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta) = 1$$
$$S_T(\theta)S_R(-\theta) + S_R(\theta)S_T(-\theta) = 0$$

while crossing is

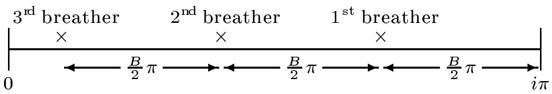
$$S(i\pi - \theta) = S_T(\theta)$$
$$S_R(i\pi - \theta) = S_R(\theta)$$

In 1977, Zamolodchikov was able to build on the earlier proposal of Korepin and Faddeev (1975) for the special points $h = 2n$, $n \in \mathbb{N}$ (at which $S_R(\theta)$ vanishes), to conjecture an exact formula for the S-matrix. Subsequent derivations made use of the Yang-Baxter equation, but in any event I only want to quote the physical pole structure here. (A pole is called 'physical' if, like the crosses on figure 5, it lies on the physical strip.) A moment's thought about the ways that the vacua fit together shows that:

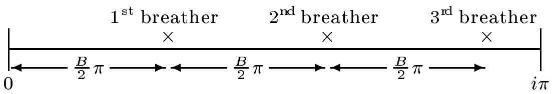
- S_T can only form breathers in the forward channel;
- S can only form breathers in the crossed channel;
- S_R can form both.

This is precisely matched by Zamolodchikov's S-matrix: in terms of $B(\beta) = 2\beta^2/(8\pi - \beta^2) = 4/h$, the poles of S_T, S and S_R in the physical strip are found at the following points:

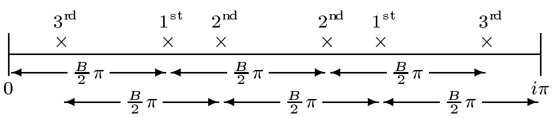
- S_T: $(1 - k\frac{B}{2})\pi i$, $k = 1, 2, \ldots$:



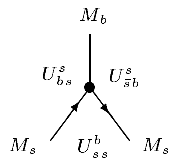
- S: $k\frac{B}{2}\pi i$, $k = 1, 2, \ldots$:



- S_R: $(1-k\frac{B}{2})\pi i$, $k\frac{B}{2}\pi i$, $k = 1, 2, \ldots$:



(Beyond the physical strip, S_T, S and S_R have a proliferating set of unphysical poles, there to fix up crossing and unitarity, but this aspect will not be important below.) In the illustrations, the particles responsible for the poles have also been indicated. To check that these have been placed correctly, all that is needed is some elementary kinematics. Suppose that a soliton s and an antisoliton $\bar{s}$, of masses $M_s = M_{\bar{s}} = M$ and moving with respective rapidities θ_1 and $\theta_2 = -\theta_1$, fuse to form a (stationary) breather of mass M_b. The relative rapidity of the two particles is $\theta_{12} = 2\theta_1$, and the S-matrix will normally have a simple, forward-channel pole at exactly this point. Conservation of energy dictates that $M_b = 2M\cosh(\theta_{12}/2)$. It will be convenient to write this special value of θ_{12} as $iU^b_{s\bar{s}}$, where $U^b_{s\bar{s}}$ is called the fusing angle for the fusing $s\,\bar{s} \to b$:



By convention, an arrow pointing forwards in time marks a soliton, and an arrow pointing backwards an antisoliton; lines without arrows are breathers of some sort. Rotating the diagram by $\pm 2\pi/3$ gives pictures of $b\,s$ and $\bar{s}\,b$ scattering, and the corresponding fusing angles have also been indicated. If all of the poles in S_T are forward-channel, then the values of the fusing angles follow from the positions of these poles:

$$U^b_{s\bar{s}} = \left(1 - k\frac{B}{2}\right)\pi \quad , \qquad U^s_{bs} = U^{\bar{s}}_{\bar{s}b} = \left(\frac{1}{2} + k\frac{B}{4}\right)\pi \,.$$

The angles are all real, reflecting the fact that the bound states are below threshold and the relative rapidities at which they are formed purely imaginary. The masses of the corresponding bound states are therefore

$$M_b = 2M\cos\left(\frac{\pi}{2} - k\frac{B}{4}\pi\right) = 2M\sin\left(\frac{k\pi}{h}\right) \,,$$

and these match the spectrum of breather masses.

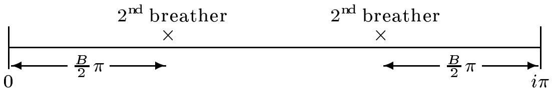
For later use, the precise relationship between S_T, S and S_R is:

$$S_T(iu) = S(i\pi - iu) = \frac{\sin(\frac{2}{B}u)}{\sin(\frac{2}{B}\pi)} S_R(iu) \,.$$

The first equality is merely crossing symmetry, whilst the factor of $\sin(\frac{2}{B}u)$ multiplying S_R is there to exclude the crossed-channel poles in S_R from S_T, and the forward-channel poles in S_R from S.

There are also S-matrix elements involving the breathers. These can be deduced using bootstrap equations, to be described a little later, but for now the focus is elsewhere and so I'll just quote the required result, concerning the scattering of two copies of the first breather:

• S_{11} has poles at $i\frac{B}{2}\pi$ and $i\pi - i\frac{B}{2}\pi$:

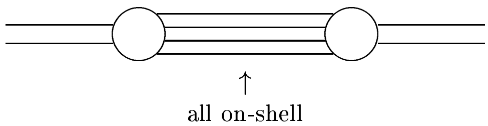


This looks fine: it is easy to check that the pole at $i\frac{B}{2}\pi$ can be blamed on a copy of the second breather as a forward-channel bound state, and the other one on the same particle appearing in the crossed channel. But now consider what happens as β, and hence $B/2$, increases. Each time $B/2$ passes an inverse integer, a pole in S_T leaves the physical strip and the corresponding breather leaves the spectrum of the model. Finally, when $B/2$ passes $1/2$, the second breather drops out. The theory, now well into the quantum regime, has just the soliton, the antisoliton, and the first breather in its spectrum. And this is problematical: S_{11} still has a pair of simple poles. How can this be, if the particle previously invoked to explain them is no longer there?

The answer was found by Coleman and Thun in 1978, and requires a preliminary diversion into the subject of anomalous threshold singularities. These are most simply understood by asking how an individual Feynman diagram might become singular. If the external momenta are such that a number of internal propagators can find themselves simultaneously on-shell, then it turns out that the loop integrals give rise to a singularity in the amplitude. Apart from the somewhat trivial examples provided by tree-level diagrams, these singularities are always branch points in spacetimes of dimension higher than two; but in 1+1 dimensions, they can give rise to poles instead.

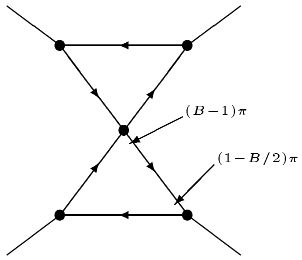
Once this is known, the problem of identifying the positions of such singularities becomes a geometrical exercise in gluing together a collection of on-shell vertices so as to make a pattern that closes. For three-point vertices, the on-shell requirement simply forces the relative Minkowski momenta to be equal to i times the fusing angles. If all couplings are below threshold, then all fusing angles are real and the resulting patterns can be drawn as figures in two Euclidean dimensions. These pictures are known as Landau, or on-shell, diagrams.

In fact, the characterisation as so far given also encompasses the more usual multiparticle thresholds, which are associated in perturbation theory with on-shell diagrams of the following type:

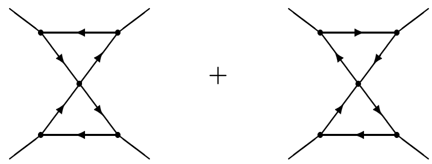


(Exceptionally, time is running sideways in this picture.) Here, the on-shell particles are all in the same channel, and the value of s at which the singularity is found is simply the square of the sum of the masses of the intermediate on-shell particles. To qualify as 'anomalous', something more exotic should be going on, and the position of the singularity will no longer have such a straightforward relationship with the mass spectrum of the model.

The moral is that when we come to analyse the pole structure of an S-matrix in 1+1 dimensions there are more things to worry about than just the tree-level processes discussed so far. Returning to the sine-Gordon model, as the point $B/2 = 1/2$ is passed, an on-shell diagram does indeed enter the game as far as the scattering of two of the first breathers is concerned:



This diagram only invokes the solitons and antisolitons on the internal lines, which are present in the spectrum whatever the coupling. A couple of internal angles are marked, making it clear that the figure will only close if $B/2 \geq 1/2$. However we are not quite out of the woods yet: diagrams of this sort are expected to yield double poles when evaluated in 1+1 dimensions, and not the single poles that we are after as soon as $B/2$ passes $1/2$. (Actually at $B/2 = 1/2$, S_{11} does indeed have a double pole, but the understanding of a single extra point is scarcely major progress.) The final ingredient is to notice that for $B/2 > 1/2$, two of the internal lines must inevitably cross over. When a soliton and an antisoliton meet we should allow for reflection as well as transmission, since we have already seen that both amplitudes are generally nonzero. Thus not one but four diagrams are relevant to the amplitude near to the value of θ_{12} of interest. The full story is given by the diagrams



together with their overall conjugates, in which all arrows are reversed. Individually each diagram contributes a double pole, but these must be added together with the correct relative weights. The only difference between the two diagrams shown is that the central blob on the first carries with it a factor of $S_T(\theta)$, and the central blob of the second a factor of $S_R(\theta)$. At the pole position, the value of θ is fixed by the on-shell requirement to be equal to iu, with $u = (B{-}1)\pi$. Referring back to the earlier expression relating S_T to S_R, we have

$$\begin{aligned} S_T((B{-}1)\pi i) &= \frac{\sin(\frac{2}{B}(B{-}1)\pi)}{\sin(\frac{2}{B}\pi)} S_R((B{-}1)\pi i) \\ &= -S_R((B{-}1)\pi i) \end{aligned}$$

Thus exactly when the individual diagrams have a double pole, $S_T + S_R = 0$, a cancellation occurs, and the field-theoretic prediction is for a simple pole, exactly as seen in the S-matrix. Coleman and Thun dubbed explanations of this sort 'prosaic', since they do not rely on properties special to integrable field theories – a non-integrable (albeit very finely-tuned) theory would be perfectly capable of exhibiting the same behaviour. Nonetheless, there is a certain miraculous quality about the result. The cancellation between S_T and S_R is very delicate: S_T describes a classically-allowed process, while S_R does not (there is no classical reflection of solitons). It is also noteworthy that the Landau diagrams expose intrinsically field-theoretical aspects of the theory, since loops are involved. Their relevance tells us that quantum mechanical intuitions about bound states and pole structure may occasionally be misleading.
Some general lessons can be drawn from all of this:

• The S-matrix can have poles between $\theta = 0$ and $\theta = i\pi$;

• these can be first order, second order, or in fact much higher order (examples up to 12^{th} order are found in the affine Toda field theories);

• even for a first-order pole, a direct interpretation in terms of a bound state is not inevitable;

• but there is always some (prosaic$^{\text{TM}}$) explanation in terms of standard field theory.

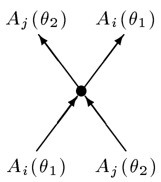
5 Bootstrap equations

If we decide that our theory does contain a bound state, then the next task is to find the S-matrix elements involving this new particle, and then to look for evidence of further bound states in these, and so on. Rather than continuing with the sine-Gordon example, which showed how complicated the story can become, I will make a tactical retreat at this point to a class of models where the behaviour is rather simpler, and the workings of the bound states can be seen more cleanly. The structure is still rewardingly rich, so this won't be too great a sacrifice.

The key concession is to assume that there are no degeneracies among the one-particle states once all of the non-zero spin conserved charges have been specified. This closes off the loophole exploited by the sine-Gordon model, and forces the scattering to be diagonal.

The S-matrix now only needs two indices:

$S_{ij}(\theta_1 - \theta_2) \quad =$



Two of the previous constraints on the two-particle S-matrix elements can therefore be simplified:

• Unitarity: $S_{ij}(\theta)S_{ij}(-\theta)=1$;

• Crossing: $S_{ij}(\theta)=S_{i\bar{j}}(i\pi-\theta)$.

(In contrast to its previous incarnation, there is no sum on repeated indices in the unitarity equation.) Combining these two reveals the important fact that

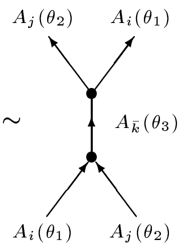
$$S_{ij}(\theta{+}2\pi i)=S_{ij}(\theta)\,,$$

so that for diagonal scattering the Riemann surface for the S-matrix really is just a double cover of the complex plane – whether you go round the left or the right branch point in figure 4, you always land up on the same unphysical sheet.

The simplification is even more drastic for the third constraint: the loss of matrix structure, already evident in the revised unitarity equation, means that the Yang-Baxter equation is trivially satisfied for any $S_{ij}(\theta)$ whatsoever.

Fortunately, a vestige of algebraic structure does remain, in the guise of the pattern of bound states. Suppose that $S_{ij}(\theta_{12})$ has a simple pole, at $\theta_{12}=iU_{ij}^k$ say, which really is due to the formation of a forward-channel bound state. Note that, in a unitary theory, forward and crossed channel poles can be distinguished by the fact that the residues are positive-real multiples of i in the forward channel, and negative-real multiples in the crossed channel. The previous picture of the scattering process can be 'expanded' near to the pole:

$$S_{ij}(\theta_{12}\approx iU_{ij}^k)\qquad\sim$$



(The intermediate particle is labelled $\bar{k}$ for convenience, in anticipation of a convention that all indices on a three-point coupling will be ingoing.) There are a number of immediate consequences:

(1) The quantum coupling C^{ijk} is nonzero at the point where particles i, j and k are all on shell.

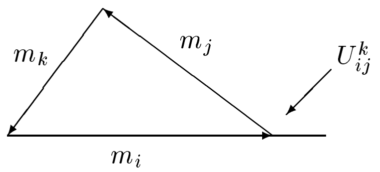
(2) At the rapidity difference $\theta_{12}=iU_{ij}^k$, the intermediate particle $A_{\bar{k}}(\theta_3)$ is on shell and survives for macroscopic times. On general grounds (the 'bootstrap principle', or 'nuclear democracy'), $A_{\bar{k}}$ is expected to be one of the other asymptotic one-particle states of the model.

(3) Since $s=m_k^2$ when this happens, we have

$$m_k^2=m_i^2+m_j^2+2m_im_j\cos U_{ij}^k\,.$$

Of course, the U_{ij}^k are just the fusing angles already seen in the last lecture.

For a more geometrical characterisation of the fusing angles, observe that the formula just given is familiar from elementary trigonometry, and implies that U_{ij}^k is the outside angle of a 'mass triangle' of sides m_i, m_j and m_k :



$(\, C^{ijk} \neq 0 \,)$

With $C^{ijk} \neq 0$, poles are also present in S_{jk} and S_{ki} . From the triangle just drawn, the three fusing angles involved satisfy

$$U_{ij}^k + U_{jk}^i + U_{ki}^j = 2\pi \ .$$

Concrete examples, the nonzero quantum couplings $C^{bs\bar{s}}$ in the sine-Gordon model, were mentioned in the last lecture.

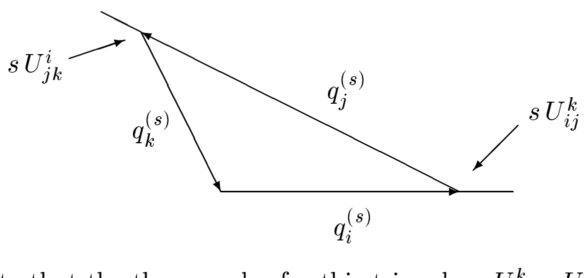
Since the $\bar{k}$ is supposed to be long-lived when $\theta_{12} = iU_{jk}^k$, it should be possible to evaluate a conserved charge Q_s after the fusing of i and j into $\bar{k}$, as well as before. The action of Q_s on $|A_{\bar{k}}(\theta_3)\rangle$ and $|A_i(\theta_1)A_j(\theta_2)\rangle$ was given at the beginning of the second lecture; equating the two at the relevant rapidities gives a constraint on the numbers $q_i^{(s)}$ which characterise Q_s:

$$C^{ijk} \neq 0 \quad \Rightarrow \qquad q_{\bar{k}}^{(s)} = q_i^{(s)} e^{is\bar{U}_{ki}^j} + q_j^{(s)} e^{-is\bar{U}_{kj}^i} \,,$$

where $\bar{U} = \pi - U$. (To see this, switch to the frame where $\bar{k}$ is stationary. Then $\theta_1 = i\bar{U}_{ki}^j$ and $\theta_2 = -i\bar{U}_{kj}^i$.) The relations $q_{\bar{k}}^{(s)} = (-1)^{s+1} q_k^{(s)}$ and $\bar{U} + \bar{U} + \bar{U} = \pi$ can be employed to put this into a more symmetrical form:

$$C^{ijk} \neq 0 \quad \Rightarrow \qquad q_i^{(s)} + q_j^{(s)} e^{isU_{ij}^k} + q_k^{(s)} e^{is(U_{ij}^k + U_{jk}^i)} = 0 \,.$$

Drawing this equation in the complex plane shows that it has a nice interpretation as a closure condition for a 'generalised mass triangle':



$(\, C^{ijk} \neq 0 \,)$

(Note that the three angles for this triangle, $s\,U_{ij}^k$, $s\,U_{jk}^i$ and $s\,U_{ki}^j$, now add up to $2\pi s$ instead of just 2π.)

This set of equations constitutes the conserved charge bootstrap. Given a set of masses and three-point couplings, the fusing angles can be determined by the mass triangles. The angles in the higher-spin triangles are then fixed, and at any given spin s the demand that all of the triangles at that spin should close provides an overdetermined set of conditions on the values of the $q_i^{(s)}$. Should the only solution be the trivial one, $q_i^{(s)} = 0$ for all i, then we can conclude that $Q_s \equiv 0$ and there is no conserved charge of that spin. The surprise is that there should be *any* choice of the initial masses and couplings such that the higher-spin triangles can be made to close for at least an infinite subset of spins. However, when this does happen, the set of spins at which the triangles do close gives access to the fingerprint of spins mentioned earlier, without the need to find the local conserved densities explicitly.

A fair amount of physical intuition has been used to arrive at these conclusions, and one particular point deserves mention. Given that the fusing angles are always real in applications of interest, most if not all of the momenta entering the discussion are complex, and this might cast doubt on the spacetime language that has been used throughout. I have implicitly assumed that the states, their fusings, and the action on them of the conserved charges, continue to behave in the expected manner after the necessary analytic continuations have been made.

For the S-matrix we can use a similar argument. Consider another particle l, which might interact either before or after particles i and j fuse to form $\bar{k}$. Which depends on the impact parameter, but in an integrable model this should be irrelevant. Translating this into equations,

$$C^{ijk} \neq 0 \quad \Rightarrow \qquad S_{l\bar{k}}(\theta) = S_{li}(\theta - i\bar{U}_{ki}^{j}) S_{lj}(\theta + \bar{U}_{jk}^{i}) \,,$$

and this is the S-matrix bootstrap equation. It can be given a more symmetrical appearance using crossing symmetry and unitarity, becoming:

$$C^{ijk} \neq 0 \quad \Rightarrow \qquad S_{li}(\theta) S_{lj}(\theta + iU_{ij}^{k}) S_{lk}(\theta + i(U_{ij}^{k} + U_{jk}^{i})) = 1 \,.$$

Imposing these relations for each non-vanishing three-point coupling provides an overdetermined set of functional equations, and again it is rather surprising that there are any solutions. In fact the two bootstraps are rather directly related:

Exercise: with the help of a logarithmic derivative and a Fourier expansion, show that each solution to the S-matrix bootstrap contains within it a solution to the corresponding conserved charge bootstrap.

Starting from an initial guess of a single S-matrix element, we can now search for poles, infer some three-point couplings, apply the bootstrap to deduce further S-matrix elements, and then iterate away. If the process closes on a finite set of particles, then we can chalk up a success and go on to another problem; if not, then the initial guess should probably be revised. This is precisely the approach that A.B.Zamolodchikov took in his pioneering work, (1989a) and (1989b), on perturbed conformal field theories. The next lecture is devoted to a particularly interesting example of this procedure which relates to the behaviour of the $T = T_c$ Ising model, in a small magnetic field.

6 Zamolodchikov's E_8-related S-matrix

The critical Ising model is found at zero magnetic field, with the temperature carefully adjusted to the critical value T_c. If the continuum limit is taken at this point, the result is well-known to be described by the $c = \frac{1}{2}$ conformal field theory, a very well-understood object. In the papers (1989a) and (1989b), Zamolodchikov probed nearby points by considering the actions

$$S_{\text{pert}} = S_{\text{CFT}} + \lambda \int d^2x \phi(x) \,,$$

where S_{CFT} is a notional action for the $c = \frac{1}{2}$ conformal field theory, inside of which ϕ sits as one of the spinless, relevant fields. There are just two of these for the Ising model, and one of them, usually labelled σ, can be identified with the scaling limit of the local magnetisations (spins) on the lattice. Thus perturbing by σ corresponds to switching on a magnetic field. The game now is to exploit the great control that we have of the unperturbed situation to divine some information about the perturbed model. In this particular case, Zamolodchikov used an ingenious argument, based on the counting of dimensions in Virasoro representations, to establish that the perturbed model supported at the very least local conserved charges with the following spins:

$$s \;=\; 1,\ 7,\ 11,\ 13,\ 17,\ 19.$$

This tells us two things. First, there are certainly enough charges here to employ Parke's argument, and so the perturbed model, if massive, possesses a factorisable S-matrix (and the model must be massive, since the c-theorem tells us that the central charge of any conformal infrared limit would be less than $\frac{1}{2}$, and there is no such unitary conformal field theory). Second, we now have the first part of the fingerprint of conserved spins, and can hope to use this information to build a bridge between the ultraviolet information residing in the characterisation of the model as a perturbed conformal field theory, and the infrared information that would be revealed if we knew its S-matrix.

To commence the search for this S-matrix, suppose that the massive theory possesses a particle of mass m_1, say. In addition, assume for the time being that the model falls into the simplest class, that of diagonal scattering theories. The magnetic field breaks the $\mathbb{Z}_2$ symmetry of the unperturbed model, and so there is no reason to exclude an interaction of ϕ^3 type from the effective Lagrangian of the perturbed theory. This places the model in the same general class as the second example discussed in the first lecture, and makes it natural for C^{111} to be nonzero. For this coupling the mass triangle is equilateral, and the fusing angles are therefore all equal to $2\pi/3$. The conserved charge bootstrap equation is

$$C^{111} \neq 0 \quad \Rightarrow \qquad q_1^{(s)} + q_1^{(s)} e^{2\pi i s/3} + q_1^{(s)} e^{4\pi i s/3} = 0\,.$$

This equation has a nontrivial solution whenever s has no common divisor with 6 :

$$s \;=\; 1,\ 5,\ 7,\ 11,\ 13,\ 17\ldots\,.$$

This is too much of a good thing: the fingerprint contains rather too many spins for comfort. Whilst the unwanted charges might vanish for other reasons, it would be more satisfying if the cast of particles could be enlarged a little, so as to restrict the set of conserved spins a bit more. Besides, earlier work described in McCoy and Wu (1978) had led Zamolodchikov to suspect the presence of at least a couple of further masses in the particle spectrum. Taking things one step at a time, he first enlarged the spectrum by adding just one more particle type, with mass m_2, and supposed that both C^{112} and C^{221} were nonzero. The fusing angles are not so easily determined now, but if the ignorance is encoded in the pair of numbers $y_1 = \exp(iU^1_{21})$ and $y_2 = \exp(iU^2_{12})$, then two of the bootstrap equations are:

$$\begin{aligned} C^{121} \neq 0 \quad &\Rightarrow \qquad q^{(s)}_1 + q^{(s)}_2 (y_1)^s + q^{(s)}_1 (y_1)^{2s} = 0\,; \\ C^{212} \neq 0 \quad &\Rightarrow \qquad q^{(s)}_2 + q^{(s)}_1 (y_2)^s + q^{(s)}_2 (y_2)^{2s} = 0\,. \end{aligned}$$

Eliminating $q^{(s)}_1$ and $q^{(s)}_2$,

$$(y_1^s + y_1^{-s})(y_2^s + y_2^{-s}) = 1\,,$$

at least at those values of s for which there is a nontrivial conserved charge. If there are more than a couple of these, then the system is overdetermined; nevertheless, if $y_1 = \exp(4\pi i/5)$ and $y_2 = \exp(3\pi i/5)$ then there is a solution for every odd s which is not a multiple of 5. This yields the following set of fusing angles:

$$\begin{aligned} U^1_{12} = U^1_{21} = 4\pi/5\,, \quad U^2_{11} = 2\pi/5\,; \\ U^2_{21} = U^2_{12} = 3\pi/5\,, \quad U^1_{22} = 4\pi/5\,, \end{aligned}$$

and the golden mass ratio

$$\frac{m_2}{m_1} = 2\cos\frac{\pi}{5}\,.$$

(There are other solutions such as $(-y_1, -y_2)$ or (y_2, y_1), but the choice taken is the only one which yields sensible fusing angles and $m_1 < m_2$.)

This is very promising: the multiples of 5 were exactly the values of s that had to be eliminated in order to match the sets of conserved spins. Thus encouraged, we can start to think about the S-matrix.

It has been assumed that all the particles are self-conjugate (the absence of any even spins from the fingerprint is a good hint that this assumption is correct) and so each S-matrix element S_{ij} must be individually crossing-symmetric, $S_{ij}(\theta) = S_{ij}(i\pi{-}\theta)$, as well as unitary. It is convenient to construct these as products of a basic 'building block' $(x)(\theta)$, where

$$(x)(\theta) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{60}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{60}\right)}\,.$$

(The 60 in the denominators has been chosen with advance knowledge of the final answer, so that all of the arguments x will turn out to be integers.) Unitarity is built into these blocks, whilst the crossing symmetry just mentioned is assured if each block (x) is always accompanied by $(30{-}x)$. The block (x) has a single physical-strip pole at $i\pi x/30$, and no physical-strip zeroes.

Consider first $S_{11}(\theta)$. The nonzero couplings C^{111} and C^{112} imply forward-channel poles at $iU^1_{11}=2\pi i/3$ and $iU^2_{11}=2\pi i/5$. Incorporating these and their crossed partners into a first guess for the S-matrix element gives

$$S_{11}=(10)(12)(18)(20)\ .$$

However this can't be the whole story. The S-matrix bootstrap equation for the ϕ^3 coupling C^{111} requires that $S_{11}(\theta{-}i\pi/3)S_{11}(\theta{+}i\pi/3)$ should be equal to $S_{11}(\theta)$. But it is easy to check that for the guess just given,

$$S_{11}(\theta{-}i\pi/3)S_{11}(\theta{+}i\pi/3)=-(2)(8)(10)(20)(22)(28)\ ,$$

which is not the desired answer. Stare at the equations long enough, though, and you might just spot that all will be well if the initial guess is multiplied by the factor $-(2)(28)$. Thus the minimal solution to the constraints imposed so far is

$$S_{11}=-(2)(10)(12)(18)(20)(28)\ .$$

The bootstrap equations have forced the addition of two extra poles, and the simplest option is to suppose that these are the forward- and crossed-channel signals of a further particle, with mass $m_3=2m_1\cos(\pi/30)$, and a nonzero coupling C^{113}. Of course, the story is not over yet. Using the bootstrap for the fusing $1\,1\to 2$ allows S_{12} to be obtained from the provisional S_{11}:

$$S_{12}=(6)(8)(12)(14)(16)(18)(22)(24)\,.$$

The poles from the blocks (14), (18) and (24) are correctly-placed to match forward-channel copies of particles the 3, 2 and 1 respectively, those in (6), (12) and (16) can then be blamed on the same particles in the crossed channel, but the blocks (8) and (22) are not so easily dismissed, and require the addition of yet another particle, of mass m_4 say. (Consideration of the signs of the residues shows that the forward-channel pole is at $8\pi i/30$.) Next, the bootstrap for $1\,1\to 3$ predicts

$$S_{13}=(1)(3)(9)(11)^2(13)(17)(19)^2(21)(27)(29)\,.$$

(Exercise: check at least one of these claims.)

Apart from the double poles, which should not be too alarming after the earlier investigations of the sine-Gordon model, there is one more pair of simple poles here which cannot be explained in terms of the spectrum seen so far, and so a further mass, m_5 say, is revealed.

There is nothing to stop the mythical energetic reader from continuing with all this, and it turns out that no further backtracking is required – with just one correction to the initial guess, Zamolodchikov had arrived at a consistent conjecture for $S_{11}(\theta)$. Furthermore, the final answer turned out to have a number of intriguing properties. These can be summarised in a list of what might be called 'S-matrix data':

• 8 particle types $A_1, \ldots A_8$;

• 8 masses m_i, $i = 1, \ldots 8$, which together form an eigenvector of the Cartan matrix of the Lie algebra E_8:

$$C_{ij}^{[E_8]} m_j = (2 - 2\cos\frac{\pi}{30}) m_i \ ;$$

(This allows each particle type to be attached to a spot on the E_8 Dynkin diagram – more on this later.)

• solutions to the conserved-charge bootstrap found at

$$s \ = \ 1,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29\ \ldots$$

thus fitting the fingerprint found from perturbed conformal field theory (and also the exponents of E_8, repeated modulo 30);

• 'charges' associated with these solutions which form further eigenvectors of $C_{ij}^{[E_8]}$:

$$C_{ij}^{[E_8]} q_j^{(s)} = (2 - 2\cos\frac{\pi s}{30}) q_i^{(s)} \ ;$$

• a full two-particle S-matrix which is a collection of complicated but elementary functions, with poles at integer multiples of $i\pi/30$, all products of the elementary building blocks introduced earlier.

There is no space to record the full S-matrix here, but a complete table can be found in, for example, Braden et al. (1990).

One note of caution: elegant though it might be, it is not completely clear that this is the answer to the question originally posed, given the number of assumptions that were made along the way. Probably the most convincing reassurance comes on recalculating the central charge of the unperturbed model from the conjectured S-matrix, using a technique called the thermodynamic Bethe ansatz. Its use in this context was first advocated by Al.B.Zamolodchikov (1990), and the specific calculation for the E_8-related S-matrix can be found in Klassen and Melzer (1990).

7 Coxeter geometry

It is a finite though lengthy task to check all of the bootstrap equations for Zamolodchikov's S-matrix, and to verify the properties listed at the end of the last lecture. However there is something not completely satisfactory about this, and a feeling that an underlying structure remains to be discovered, a structure that might help to explain quite why such an elegant solution to the bootstrap should exist at all. The purpose of this short section, something of an aside from the main development, is to show that at least some parts of this question can be answered.

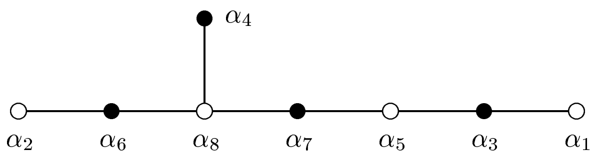
One mathematical preliminary is required, a quick recap on the Weyl group of E_8. Imagine a hedgehog Φ of 240 vectors, or 'roots', sitting in eight dimensions. They all have equal length, and together they make up the root system of E_8. Each root can be written as an integer combination of the simple roots $\{\alpha_1, \ldots \alpha_8\}$:

$$\alpha \in \Phi \quad \Rightarrow \quad \alpha = \sum_{i=1}^{8} m_i \alpha_i$$

with $m_i \in \mathbb{Z}$, and the m_i either all non-negative, or all non-positive. Actually, there are 240 different eight-element subsets of Φ which could serve as the simple roots, but their geometrical properties are all identical, and can be summarised by giving the set of their mutual inner products, as encoded either in the Cartan matrix

$$C_{ij}^{[E_8]} = 2\frac{\alpha_i . \alpha_j}{\alpha_j^2} \ ,$$

or the Dynkin diagram



Pairs of simple roots joined by a line have inner product -1, and all other pairs are orthogonal. In particular this means that the black-coloured roots are mutually orthogonal, as are the white-coloured roots. The labelling might look random, but recall from the last lecture that the vector of masses formed an eigenvector of the Cartan matrix, so that each particle type in Zamolodchikov's S-matrix can be assigned to a point on the Dynkin diagram. The labels used here correspond to these particle labels, with $m_1 < m_2 < \ldots < m_8$.

For each $\alpha \in \Phi$, define the Weyl reflection r_α to be the reflection in the 7-dimensional hyperplane orthogonal to α :

$$r_\alpha \ : \quad x \mapsto x - 2\frac{\alpha . x}{\alpha^2}\alpha \, .$$

The products of the Weyl reflections in any order and of any length together make up W, the Weyl group of E_8. This group maps Φ to itself, and is *finite*: $|W| < \infty$. (Note that W is therefore a finite reflection group, a much simpler object than the Lie group or algebra with which it is associated.) One more fact: to generate W, it's enough to start with the set of simple reflections $\{r_{\alpha_1}, \ldots r_{\alpha_8}\}$, and I will write these as $\{r_1, \ldots r_8\}$.

Now I want to study the properties of one particular element $w \in W$. It is a Coxeter element, meaning that it is a product in some order of a set of simple reflections. Although the ordering is not crucial, the result I'm after is most transparent if I pick

$$w = r_3 r_4 r_6 r_7 r_1 r_2 r_5 r_8 \ .$$

This is a Steinberg ordering: reflections of one colour act first, followed by those of the other. The ordering amongst the reflections of like colour is immaterial – they all commute, since the corresponding simple roots are orthogonal. The project is to see how w^{-1} acts on Φ, and as a start we can examine the orbit of α_1 under w^{-1}. Noting that

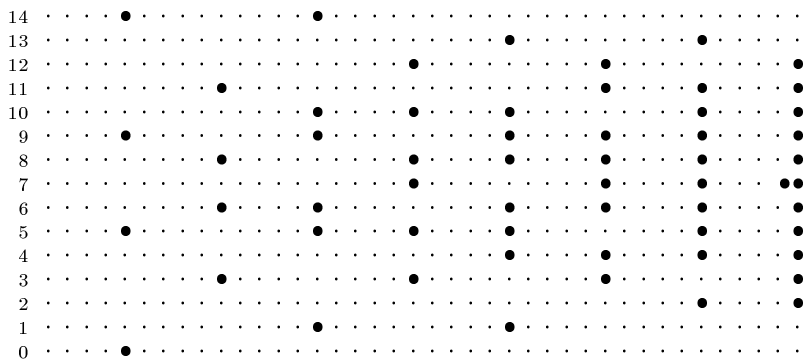
$$r_i(\alpha_j) = \alpha_j - C^{[E_8]}_{ji}\alpha_i \ , \qquad \text{(no sum on } j\text{)}$$

the individual simple reflection r_i negates α_i, adds α_i to all roots α_j joined to α_i by a line on the Dynkin diagram, and leaves the others alone. With this information it doesn't take too long to compute that

$$w^{-1}(\alpha_1) = r_8 r_5 r_2 r_1 r_7 r_6 r_4 r_3(\alpha_1) = \alpha_3 + \alpha_5 \ .$$

(Exercise: check this!)

To continue is easy if a little tedious, acting repeatedly with w^{-1} to find $w^{-2}(\alpha_1)$, $w^{-3}(\alpha_1)$, and so on. After 30 steps, you should find yourself back at α_1 (the number 30 might just be familiar from the list at the end of the last lecture). The story for the first 14 of these steps is contained in the following table:



Images of α_1 under w^{-1}

The coefficient of α_i in the expansion of $w^{-p}(\alpha_1)$ is given by the number of blobs ($\bullet$) in the i^{th} position of the p^{th} row. For the E_8 Weyl group, $w^{15} = -1$ and so the rest of the table, rows 15 to 29, can be omitted.

All of this might seem a long way from exact S-matrices, but in fact the Weyl group computation just performed and the earlier bootstrap manipulations are in some senses one and the same calculation, just looked at from orthogonal directions. To explain this somewhat delphic remark, I will first rewrite the S-matrix elements already seen in a new and slightly more compact notation. Observe that, apart from the blocks (2) and (28) in the formula for S_{11}, every block (x) in S_{11}, S_{12} and S_{13} can be paired off with either $(x{-}2)$ or $(x{+}2)$. Noticing that $(0) = 1$ and $(30) = -1$, this pairing can be extended to the recalcitrant S_{11} as well, and in fact works for all of the other S-matrix elements too. Thus we can at least save some ink if we define a larger building block

$$\{x\} = (x-1)(x+1) \ .$$

and rewrite the S-matrix elements found previously as

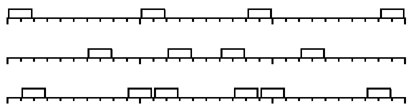
$$\begin{aligned} S_{11} &= \{1\}\{11\}\{19\}\{29\} \\ S_{12} &= \{7\}\{13\}\{17\}\{23\} \\ S_{13} &= \{2\}\{10\}\{12\}\{18\}\{20\}\{28\} \end{aligned}$$

The next step is to introduce a pictorial representation of these formulae. Start by drawing a line segment to represent the interval from 0 to $i\pi$ on which the physical-strip poles are found. Then for each block $\{x\}$ in the S-matrix element, place a small brick on the line segment, running from $i(x-1)\pi/30$ to $i(x+1)\pi/30$. (Thus, the poles are located at the ends of the bricks.) The formulae just given become

S_{11} =

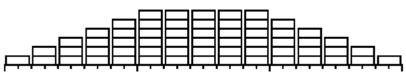
S_{12} =

S_{13} =



Rotate these three by 90 degrees and you should observe a neat match with the first three columns of the table on the last page, of images of α_1 under w^{-1}.

This is a glimpse of a general construction, which allows a diagonal scattering theory to be associated with every simply-laced Weyl group. Further details can be found in Dorey (1991,1992a); see also Fring and Olive (1992) and Dorey (1992b). All of these scattering theories were in fact already around in the literature: in addition to the articles by Zamolodchikov already cited, some relevant references are Köberle and Swieca (1979), Sotkov and Zhu (1989), Fateev and Zamolodchikov (1990), Christe and Mussardo (1990a,b), Braden et al. (1990), and Klassen and Melzer (1990). Why then worry about Weyl groups? This is ultimately a matter of taste, but it should be mentioned that the construction goes rather deeper than the curious coincidences described so far. The geometry of finite reflection groups appears to replace the rather more complicated Lie algebraic concepts that might have been a first guess as to the underlying mathematical structure. Features such as the coupling data and the pole structure can be related to simple properties of root systems, and this allows the bootstrap equations both for the conserved currents and for the S-matrices to be proved in a uniform way.



8 Affine Toda field theory

Zamolodchikov's E_8-related S-matrix is an example of a diagonal S-matrix with few of the subtleties that made the treatment of the sine-Gordon model so delicate. Whilst higher poles are certainly present, their orders are always just as would be predicted from an initial glance at the possible Landau diagrams. In particular, simple poles are always associated with bound states. Since the cancellations which complicated the sine-Gordon case relied on the non-diagonal nature of its S-matrix, one might suppose that diagonal scattering theories would always behave in a straightforward manner. Curiously enough, this turns out not to be true. The affine Toda field theories, the subject of this lecture, provide a number of elegant counterexamples.

The study of these models begins with a standard, albeit non-polynomial, scalar Lagrangian in 1+1 dimensions:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\sum_{a=0}^{r} n_a e^{\beta\alpha_a\cdot\phi}.$$

This describes the interaction of r scalar fields, gathered together into the vector $\phi \in \mathbb{R}^r$. The set $\{\alpha_0 \dots \alpha_r\}$ is a collection of $r{+}1$ further vectors in $\mathbb{R}^r$, which must be carefully picked if the model is to be integrable. It turns out that there is a classically acceptable choice for every (untwisted or twisted) affine Dynkin diagram $g^{(k)}$, thought of as encoding the mutual inner products of the α_a. By convention α_0 corresponds to the 'extra' spot on the affine diagram, and the integers n_a satisfy $n_0 = 1$ and $\sum_{a=0}^{r} n_a\alpha_a = 0$. The real constant m sets a mass scale, while β governs the strength of the interactions. When β is also real, the models generalise sinh-Gordon rather than sine-Gordon and there is no topology to worry about. In fact once we go beyond the sinh-Gordon example making β imaginary is no longer an innocent operation, since in all other cases the manifest reality of the Lagrangian is promptly lost. Despite these problems the models with β purely imaginary have received a fair amount of attention, starting with the work of Hollowood (1992). However, in this lecture I will stick to the cases where β is real.

As classical field theories, these models are all integrable, and exhibit conserved quantities at spins given by the exponents of $g^{(k)}$, repeated modulo a quantity called the k^{th} Coxeter number, $h^{(k)}$:

$$h^{(k)} = k\sum_{a=0}^{k} n_a \ .$$

(For the untwisted diagrams, $k{=}1$ and $h^{(k)}$ is the same as the usual Coxeter number h.)

However when we turn to the quantum theory, none of the elegant classical apparatus, as described in, for example, Mikhailov et al. (1981), Wilson (1981), and Olive and Turok (1985), is immediately applicable. A more elementary approach is appropriate, studying the models with the standard perturbative tools of quantum field theory before proceeding to some exact conjectures. Arinshtein et al. were the first to try this, for the $a_n^{(1)}$ theories, in 1979. Interest in the subject was renewed following Zamolodchikov's work on perturbed conformal field theories, and the fact that in the meantime the other classically-integrable possibilities, related to the other affine Dynkin diagrams, had been uncovered. Initially, only the so-called self-dual models were understood, and elements of this story can be found in Christe and Mussardo (1990a,b) and Braden et al. (1990,1991). The other, non self-dual, cases were more tricky, since they turned out to fall into the class of less straightforward scattering theories for which simple poles do not always have simple explanations. The crucial step was made by Delius et al. (1992), and the papers by Corrigan et al. (1993) and Dorey (1993) can be consulted for the few cases not covered in their work. In the remainder of this lecture I will outline some aspects of these quantum considerations, but the discussion will perforce be very sketchy. In addition to the references just cited, the review by Corrigan (1994) is a good place to start for those interested in delving deeper into this subject. That the field is still developing is evinced by an article by Oota (1997) which appeared as these notes were being written up, indicating that the ideas discussed in the last lecture may also be relevant, if suitably q-deformed, to the non self-dual theories that had previously resisted any geometrical interpretation.

If we are to treat these models as ordinary quantum field theories, then the first step must be to find out what the multipoint couplings are. To this end, the potential term in the Lagrangian can be expanded as follows:

$$
\begin{aligned}
V(\phi) &\equiv \frac{m^2}{\beta^2}\sum_{a=0}^{r} n_a e^{\beta\alpha_a\cdot\phi} \\
&= \frac{m^2}{\beta^2}\sum_{a=0}^{r} n_a + \frac{1}{2}(M^2)^{ij}\phi^i\phi^j + \frac{1}{3!}C^{ijk}\phi^i\phi^j\phi^k + \ldots
\end{aligned}
$$

where summations on the repeated indices i, j and k running from 1 to r are implied, and the two and three index objects

$$(M^2)^{ij} = m^2\sum_{a=0}^{r} n_a\alpha_a^i\alpha_a^j$$

and

$$C^{ijk} = m^2\beta\sum_{a=0}^{r} n_a\alpha_a^i\alpha_a^j\alpha_a^k$$

can be thought of as the mass^2 matrix and the set of three-point couplings, at least classically. Much as for theories discussed in the first lecture, it is possible to view the C^{ijk} as containing the 'bones' of the model, with the higher couplings, hidden as '$+\ \ldots$', there just to tidy away any residual production amplitude backgrounds that would otherwise spoil integrability. Now the general idea is the following: first diagonalise M^2 to find the classical particle masses $m_1 \ldots m_r$, and then compute the C^{ijk} in the eigenbasis of M^2 to find the classical three-point couplings between the corresponding one-particle states. At this level there are already some surprises: for example, it turns out that in all of the untwisted cases, the set of masses form the eigenvector, with lowest eigenvalue, of the corresponding non-affine Cartan matrix. This was initially noticed on a case-by-case basis, before being proved in a general way by Freeman (1991). The Coxeter element, described in the last lecture, turns out to be crucial in this discussion. This work was further elaborated by Fring et al. (1991), elucidating in particular earlier observations about the three-point couplings.

However it has been obtained, once the classical data is known two things can be done: on the one hand the masses and three-point couplings can be fed into the bootstrap to make some initial conjectures as to the full quantum S-matrices, and on the other perturbation theory can be attempted in order to check these conjectures. As hinted above, the affine Toda field theories split into two classes when this programme is attempted: 'straightforward' and 'not straightforward'. To make this distinction more precise, define a duality operation on the set of all affine Dynkin diagrams by

$$\{\alpha_0 \ldots \alpha_r\} \leftrightarrow \{\alpha_0^\vee \ldots \alpha_r^\vee\}$$

where

$$\alpha_a^\vee \equiv \frac{2}{\alpha_a^2}\alpha_a \,.$$

(This is sometimes called Langlands duality.)

When appropriately normalised, the sets of vectors associated with the $a_n^{(1)}$, $d_n^{(1)}$, $e_n^{(1)}$ and $a_{2n}^{(2)}$ affine Dynkin diagrams are self-dual in this sense, and the corresponding affine Toda field theories are also called self-dual. These are the 'straightforward' cases: conjectures based on the classical data lead to self-consistent quantum S-matrices, and to date these have passed all perturbative checks to which they have been subjected. For example, the mass ratios, and hence the fusing angles, are preserved at one loop. As a result, the bootstrap structure is essentially blind to the value of the coupling β, which enters into the S-matrices via a function

$$B(\beta) = \frac{1}{2\pi}\frac{\beta^2}{1+\beta^2/4\pi} \,.$$

There is a simple relationship between the S-matrices for certain perturbed conformal field theories and the S-matrices for the self-dual affine Toda models: all that has to be done is to replace building blocks of the type seen in the last lecture

$$\{x\}_{\mathrm{PCFT}} \equiv (x{-}1)(x{+}1)$$

by the slightly more elaborate blocks

$$\{x\}_{\mathrm{toda}} \equiv \frac{(x{-}1)(x{+}1)}{(x{-}1{+}B)(x{+}1{-}B)} \,.$$

Zamolodchikov's E_8-related S-matrix is related in this way to the S-matrix of the $e_8^{(1)}$ affine Toda field theory; more generally, for $g \in \{a, d, e\}$ the correspondence is between the g affine Toda field theory and a perturbation of the $g^1 \times g^1 / g^2$ coset model, while $a_{2n}^{(2)}$ turns into a perturbation of the nonunitary minimal model $\mathcal{M}(2, 2n{+}3)$. (Note though that the factors of 60 appearing in the earlier definition of (x) should be replaced by $2h$, with h the relevant Coxeter number). For every self-dual affine Toda S-matrix, there is thus a companion 'minimal' S-matrix, sharing the same physical pole structure but lacking the coupling-constant dependent physical strip zeroes, which in the Toda theories serve to cancel the poles in the $\beta \to 0$ limit. Note also that replacing β by $4\pi/\beta$ sends B to $2{-}B$ and leaves the Toda blocks unchanged – a strong-weak coupling duality.

The remaining, non self-dual, models behave in a much more complicated way. The classical data is still elegant, but there is quantum trouble: conjectures based on the raw classical data are no longer self-consistent, and perturbative checks show varying mass ratios, causing the fusing angles to depend on the value of the coupling constant. These perturbative results are reinforced by the results of Kausch and Watts (1992) and Feigin and Frenkel (1993), which indicate that the correct general implementation of strong-weak coupling duality is not only to replace β by $4\pi/\beta$, but also to replace each α_a by its dual, $\alpha_a^\vee$. (Of course, for the self-dual theories this latter operation has no effect and so the earlier statement of duality remains correct for these cases.) This means that for each dual pair of classical affine Toda field theories, there should be just one quantum theory – there are 'fewer' genuinely distinct quantum theories than expected, and the different classical theories can be recovered by taking strong or weak coupling limits. The predicted dualities are:

$$\begin{array}{ll} b_n^{(1)} \leftrightarrow a_{2n-1}^{(2)} \qquad & g_2^{(1)} \leftrightarrow d_4^{(3)} \\ c_n^{(1)} \leftrightarrow d_{n+1}^{(2)} \qquad & f_4^{(1)} \leftrightarrow e_6^{(2)} \end{array}$$

(Exercise: compare and contrast the classical conserved charge fingerprints for these models. Are they compatible with duality?)

If this picture is correct, then the mass ratios have no option but to vary: if a model is non self-dual, then (as can be checked case-by-case) its classical mass spectrum is always different from that of its dual. One spectrum is found at $\beta \to 0$, the other at $\beta \to \infty$, and the quantum theory, if it exists at all, must find some way of interpolating between the two. Given the apparent rigidity of the bootstrap equations, this looks to be rather a tall order. Nevertheless, Delius et al. (1992) decided to take the perturbatively-calculated shifts in the mass ratios seriously, and were led to a set of conjectures for most of the models in the above list (the only cases that remained were $d_4^{(3)}$, $e_6^{(2)}$ and $f_4^{(1)}$, and these were subsequently found to behave in just the same way). Whenever conjectures existed for both halves of a dual pair, they swapped over under $\beta \to 4\pi/\beta$. In fact, Delius et al. made their proposals independently of any expectations of duality; that it emerged anyway from their calculations can be seen in retrospect as strong evidence that they were on the right track. Further support came from the numerical results of Watts and Weston (1992), who examined the coupling-dependence of the single mass ratio found in the $g_2^{(1)}/d_4^{(3)}$ dual pair.

With the mass ratios depending on the coupling, it is no longer possible to use the simple building blocks $\{x\}_{\rm toda}$ introduced earlier. A slightly more elaborate two-index block $\{x,y\}$ can be found in Dorey (1993), and is probably the most direct generalisation of the self-dual blocks. One point to note is that the natural way for the coupling to enter these blocks requires the previous definition of the function $B(\beta)$ to be slightly modified, so as to encompass the non self-dual theories as well:

$$B(\beta)=\frac{1}{2\pi}\frac{\beta^2}{h/h^\vee+\beta^2/4\pi}\ .$$

Here h is the Coxeter number of the relevant affine Dynkin diagram, and $h^\vee$ that of its (Langlands) dual.

There are, however, a couple of features of the non self-dual S-matrices which give pause for thought. Some simple poles, expected on the basis of the nonzero classical couplings and quantum mass ratios, turn out to be absent, whilst other simple poles, which are present in the quantum S-matrices, are not at locations which match any of the particle masses.

The resolution of the first problem appears to be that quantum corrections exactly cancel some of the classical three-point couplings, when evaluated on shell. This means that some quantum couplings C^{ijk} vanish even though their classical counterparts do not. The result is very delicate and has only been checked to one loop, but is probably necessary if duality is to hold, for the simple reason that the set of classical three-point couplings in a theory and its dual do not in general coincide. Those couplings which show signs of vanishing once quantum effects are taken into account are precisely those which are anyway absent at the classical level in the dual model.

As for the second problem, the mechanism is not too far removed from that operating in the sine-Gordon model. However, as mentioned at the beginning of this lecture, the fact that the scattering is diagonal means that we can no longer hope to generate simple poles through cancellations between competing Landau diagrams. Fortunately there is a compensating feature of the affine Toda S-matrices which allows the basic idea to be saved: they all exhibit zeroes as well as poles on the physical strip. These were already visible in the self-dual blocks $\{x\}_{\rm toda}$ defined earlier, and are equally present in the more general blocks $\{x,y\}$. In the self-dual cases the zeroes are merely spectators, but in the non self-dual theories they come to play a much more central role in the pole analysis. A detailed discussion can be found in Corrigan et al. (1993), and it turns out that for every 'anomalous' simple pole in the non self-dual affine Toda S-matrices, Landau diagrams can be drawn in which some internal lines cross. Just as for sine-Gordon, the S-matrix elements for these internal crossings must be factored into the calculation before the overall order of any pole can be predicted. This time, these factors vanish individually as the diagrams are put on shell, and thereby manage to demote ostensibly higher poles into the simple poles that are required in order to match the quantum S-matrices.

Further reading, and acknowledgements

As promised, these lectures have only skimmed the surface of a large subject. In addition to the references mentioned in the main text, the review articles by Zamolodchikov and Zamolodchikov (1979), Zamolodchikov (1980) and Mussardo (1992) are recommended, as is the discussion of the sine-Gordon model given by Goebel (1986). (The opening section the first lecture was in fact inspired by a remark in this article.)

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